CLASSIFYING METHODS OF PROBLEM SOLVING - AND MY FAVOURITES

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The author, now retired after many years working with Australian and international mathematics competitions, had intended to write in retirement an ambitious book detailing methods of problem solving, pedagogy in each case, and of course the usual examples and exercises. With Bulgarian colleague Jordan Tabov, he had already co-authored three books on the subject (Tabov and Taylor [2], Tabov and Taylor [3] and Tabov, Kolev and Taylor [4]). These books tend to describe some topics at a technical and advanced level. Here, the author wanted to start at the school syllabus level and describe interesting methods which a motivated school student can pick up easily and lead them to further experiences, maybe even the mathematics Olympiads. A brief introduction to the idea was in chapter 1.4 of Barbeau and Taylor [1], the final report of ICMI Study 16 'Challenging Mathematics in and beyond the Classroom'. However the author has finally decided not to write a book, which once printed remains unchanged until a second printing possibly develops. Instead he has decided to set up a freely accessible web site of mathematical content which supplements his history of the Australian Mathematics Trust, which can be found at https://sites.google.com/site/pjt154/.

At present the mathematics part of site has 15 chapters, none of them very long, with pedagogy when appropriate, and examples. It might be expanded or edited at times, but has given the author the opportunity to classify problems based on his own experience. It has also given him a chance to expose some of his favourite problems. Favourite problems tend to be ones which are easy to state, have a potentially interesting result, and a neat structural solution, with no scope for difficulty caused by computation. This brief paper scopes this site, not always in the same order, and with only a small number of examples (there are more on the web site). There is no attempt or intent here of attempting an Olympiad classification or syllabus, which would be far more comprehensive and technical. Of course other mathematicians will have their own ways of putting these things together. The author hopes this paper might help those teachers who are starting mathematics circles. Note that most of the problems below come from competitions of which the Australian Mathematics Trust has complete material in books which can be purchased from its web site www.amt.edu.au .

1. Diophantine Equations

In the Australian Mathematics Competition we have always felt free to pose Diophantine problems, even though they are not explicitly part of a normal classroom experience. They are very intuitive, but it is nice to visit the subject and look for systematic ways of finding certain integer solutions of linear equations.

2. Pigeonhole Principle

This elementary idea, thought to have been first articulated as such by Dirichlet and often known as Dirichlet's principle, is, as is well known, simple but powerful, helping to tighten up a solution once the principle is understood. The following is an example (taken from International Mathematics Tournament of Towns) of an accessible problem whose solution is best wrapped up using this idea.

Example 1

Ten friends send greeting cards to each other, each sending 5 cards to different people. Prove that at least two of them sent cards to each other.

Strategy

The words 'at least' are the ones which give the experienced student the clue that the pigeonhole principle will be useful here. The question is going to require a count of how many routes there are, and how many different routes are taken. Since routes occur in pairs (one for each direction) the objective of the proof will be to find that more than half the routes must be used, as then the pigeon hole principle will require that at least two are such a pair.

Solution 1

Since each of the ten friends can send to nine others, there are 90 available routes. However, each pair of friends is involved in 2 routes, so that there are 45 pairs. If more than 45 cards are sent, then by the pigeonhole principle, two of the transmissions must be on the same route in opposite directions. In this case since each student actually sends 5 cards, there are 10 times 5, or 50 (≥ 45) transmissions altogether and thus there are two friends who send cards to each other.

Note

Discussion on such challenges, working out what might be the pigeons, and what might be the pigeonhole, can be very rewarding in mathematical circles.

3. Discrete Optimisation

Discrete optimisation is quite a different skill than that found in calculus, where the variables are real (not integers), and situations are continuous. Here the variables are integer, and the standard method involves two steps,

1. Show optimality, that is, give an argument to show that the proposed solution cannot be exceeded.

2. Show that this demonstrated optimum exists.

The first part is usually a challenging mathematical argument, while the second requires no more than production of an optimum. Other than the fact that the variables are integer, the method is usually given away by the openness of a statement requiring an optimum (maximum or minimum).

The following example, from the International Mathematics Tournament of Towns, is one in which there is also a nice use of Eulerian graph theory, which is such a useful tool in networking problems.

Example 2

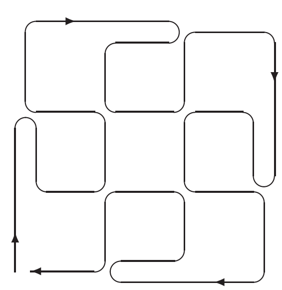
A village is constructed in the form of a square, consisting of 9 blocks, each of side length l, in a 3×3 formation. Each block is bounded by a bitumen road. If we commence at a corner of the village, what is the smallest distance we must travel along bitumen roads, if we are to pass along each section of bitumen road at least once and finish at the same corner?

Strategy

This problem is also an excellent interactive classroom problem. Students can try for some time to improve their first results until everyone is convinced they have a result which cannot be beaten.

Solution 2 (due to Andy Liu)

The diagram shows a closed tour of length 28 and we claim this to be a minimum.



Each of the four corners is incident with two roads and requires at least one visit. Each of the remaining twelve intersections is incident with three or four roads and requires at least two visits. Hence the minimum is at least $4 + 12 \times 2 = 28$.

4. Proof by Cases

Quite often experimentation with a situation leads to a conclusion that a result can only be established after an exhaustive consideration of a mutually exclusive set of cases, leading to the method usually known as proving by cases. The challenge is two-fold. First, one needs to identify the cases that might apply and to describe them in a way that is clear, efficient and non-overlapping. Secondly, one needs to ensure that the cases are exhaustive, that nothing is left out.

The method can be illustrated by the following problem, one of the more challenging problems from the Australian Mathematics Competition.

Example 3

The sum of n positive integers is 19. What is the maximum possible product of these n numbers?

Strategy

This problem is also excellent for classroom interaction. Students can try to obtain maximum products with various selections but soon discover that high numbers adding to 19 don't seem to help, while at the other extreme the number one is also useless. Normally discussions converge on the full solution.

Solution 3

We are looking for a maximum product

$$n_1 \times n_2 \times n_3 \times \cdots \times n_k$$

where $n_1 + n_2 + n_3 + \cdots + n_k = 19$.

If any factor n_i is ≥ 5, it can be replaced with $2 \times (n_i - 2)$ for a larger product, since $2(x-2) > x$ for $x \geq 5$; so every factor is ≤ 4.
If any factor n_i is equal to 4, it can be replaced by 2×2 with no change to the product, so we shall do this and then every factor is ≤ 3.
If any factor is 1, it can be combined with another factor, replacing $1 \times n_i$ by $(n_i + 1)$ which increases the product, so now all factors are 2 or 3.
If there are three or more 2s, $2 \times 2 \times 2$ can be replaced by 3×3 to increase the product. So there are at most two 2s.
There is only one way that 19 can be written as such a sum: there are five 3s and two 2s.
So the maximum product is $3^5 \times 2^2 = 972$.

Note

An extension could be to ask the student what happens if the number 19 is replaced by any other.

A further challenge for a more senior student studying calculus is to decide whether there is a continuous version of the problem and formulate it exactly.

5. Proof by Contradiction

Some of the most famous proofs in mathematics are constructed by contradiction and are accessible from school mathematics, even if not in a formal syllabus.

The following problem, taken from the International Mathematics Tournament of Towns, is most easily solved by contradiction.

Example 4

There are 2000 apples, contained in several baskets. One can remove baskets and/or remove any number of apples from any number of baskets. Prove that it is possible to have an equal number of apples in each of the remaining baskets, with the total number of apples being at least 100.

Strategy

This is hardly in the form that a student might encounter at school, and unlike other examples in this paper the initial challenge is to figure out exactly what is being asked. The indeterminacy of the situation and the variety of possibilities for removal of apples and baskets boggles the mind. An efficient way to control the situation is to suppose that the result is false. As the reader will see in the solution, it is not so difficult in this direction to find a contradiction.

Solution 4

Assume the opposite. Then the total number of baskets remaining is not more than 99 (otherwise we could leave 1 apple in each of 100 baskets and remove the rest). Furthermore, the total number of baskets with at least two apples is not more than 49, the total number of baskets with at least three apples is not more than 33, etc. So the total number of apples is not more than $99+49+33+24+\cdots < 100(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{99}) < 100(1+\frac{1}{2}+\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{64}) < 100(7) = 700 < 2000$.

We thus have a contradiction.

6. Counting by Exhaustion

Combinatorial problems are popular in challenges because they can be less dependent on classroom knowledge and therefore be fair ways of identifying potential problem solvers, where they can use intuitive methods. Counting, or enumeration is a popular source of such problems. Counting problems, properly set, can be solved in the time allocated and they have the advantage of challenging the student later to try to generalise, to enable similar problems to be solved from an algorithm.

The following problem, composed by Australian National University mathematician Bob Bryce for the Australian Mathematics Competition, is an excellent example.

Example 5

I have four pairs of socks to be hung out side by side on a straight clothes line. The socks in each pair are identical but the pairs themselves have different colours. How many different colour patterns can be made if no sock is allowed to be next to its mate?

Discussion

The following solution lists and counts all the cases.

Solution 5

Call the socks *aa*, *bb*, *cc*, *dd*.

There are $4 \times 3 \times 2 = 24$ ways of selecting the first three as *abc* (i.e. the first three different).

These can be arranged as shown:

abc	*a*	*bdcd* *cdbd* *dbcd* *dbdc* *dcbd* *dcdb*	6	
	b	symmetric with *a*	6	
	da	*bcd* *bdc* *cbd* *cdb* *dbc* *dcb*	6	
	db	symmetric with *da*	6	
	dc	symmetric with *da*	6	30

So there are $24 \times 30 = 720$ patterns commencing with *abc*.

There are $4 \times 3 = 12$ ways of selecting the first three as *aba*.

These can be arranged as shown:

aba	*b*	*cdcd*		
		dcdc	2	
	c	*bdcd*		
		dbcd		
		dbdc		
		dcbd		
		dcdb	5	
	d	symmetric		
		with *c*	5	12

So there are $12 \times 12 = 144$ patterns commencing with *aba*.

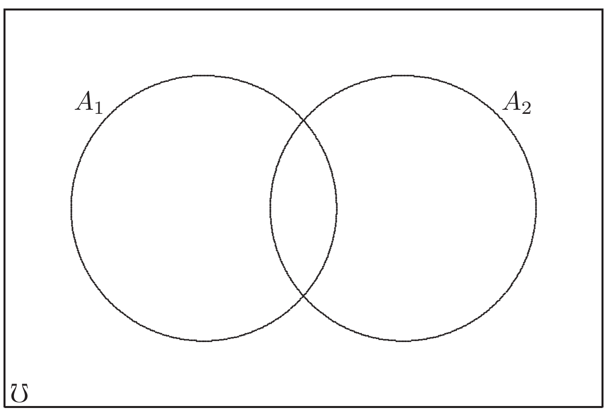
Finally, $720 + 144 = 864$.

7. Counting Systematically

Problems such as the socks question motivate one to generalise to other problems, e.g. 5, 6 or more pairs of socks. Exhaustive counting quickly becomes just that, exhausting. The following alternative solution, due to ANU mathematician Mike Newman, using the the inclusion-exclusion principle, achieves generalisation.

Alternative Solution 5

Consider the problem for 2 pairs of socks, as illustrated in the Venn diagram.



The set A_1, for instance, is the set of arrangements in which pair 1 is together.

We wish to compute

$$N(\overline{A_1 \cup A_2}) = N(\mho) - \sum_{i=1}^{2} N(A_i) + N(A_1 \cap A_2).$$

In this problem
$$N(\mho) = \frac{4!}{2^2}.$$
(We divide by 2^2 because the socks within each pair can be changed without changing the arrangement.)
Now
$$N(A_1) = N(A_2) = \frac{3!}{2^1}$$
and
$$N(A_1 \cap A_2) = \frac{2!}{2^0},$$
so the solution is
$$\frac{4!}{4} - 2\frac{3!}{2} + 2 = 6 - 6 + 2 = 2.$$

The inclusion-exclusion principle generalises to give, in the $n-$pair case
$$\begin{aligned} N\left(\overline{\bigcup_{i=1}^{n} A_i}\right) &= N(\mho) - \sum_{i=1}^{n} N(A_i) + \sum_{i \neq j} N(A_i \cap A_j) \\ &\quad - \sum_{i \neq j \neq k} N(A_i \cap A_j \cap A_k) \\ &\quad + \cdots + (-1)^n N\left(\bigcap_{i=1}^{n} A_i\right). \end{aligned}$$

In the case of 3 pairs, this gives
$$\frac{6!}{2^3} - 3\frac{5!}{2^2} + 3\frac{4!}{2^1} - 3! = 90 - 90 + 36 - 6 = 30.$$

In the case of 4 pairs, this gives
$$\begin{aligned} \frac{8!}{2^4} - 4\frac{7!}{2^3} + 6\frac{6!}{2^2} - 4\frac{5!}{2} + 4! &= 2520 - 2520 + 1080 - 240 + 24 \\ &= 864, \end{aligned}$$

hence (D).

In the case of n pairs this gives
$$\binom{n}{0}\frac{(2n)!}{2^n} - \binom{n}{1}\frac{(2n-1)!}{2^{n-1}} + \binom{n}{2}\frac{(2n-2)!}{2^{n-2}} + \cdots + (-1)^n \binom{n}{n}\frac{n!}{2^n}.$$

Note that the first two terms always cancel each other in this particular problem.

The Mathematical Content section of the AMT History site referred to in the abstract to this paper shows generalisations to other types of counting methods, including derangements, partial derangements and the Polya Necklace Method.

8. Inverse Thinking

Sometimes there can be useful challenges involved by thinking in the inverse direction (the method below can also be described as "working backwards"). Here is a problem from the Mathematics Challenge for Young Australians.

Example 6

Let us define a Fibonacci sequence as one in which each term is the sum of the two preceding terms. The first two terms can be any positive integers. An example of a Fibonacci sequence is $15, 11, 26, 37, 63, 100, 163, \ldots$.

1. Find a Fibonacci sequence which has 2000 as its fifth term.
2. Find a Fibonacci sequence which has 2000 as its eighth term.
3. Find the greatest value of n such that 2000 is the nth term of a Fibonacci sequence.

Discussion

Generally one thinks of a Fibonacci sequence in the forward direction. Here, as is common in an inverse thinking scenario, instead of being given the data and then finding the results, we are given the results and are asked to find the data. It is a challenge for students to think this way.

The student can do this by searching through various second-last terms and working back. In doing so, depending on which term they choose, they can work back uniquely but some choices will not go back far. If the second last term is less than 1000, the third last term is greater than 1000 and that is as far as we can go, as the next term would be negative. We do not do much better if the second last term is too high.

The student can eventually focus in on a small range of values for which the sequence can be traced back a few terms, and then finally the one which goes back optimally.

Solution 6

1. Note that the fifth term in the standard Fibonacci series is 5, a factor of 2000. So multiplying the first five terms by 400 yields 400, 400, 800, 1200, 2000. Many other sequences are possible and can easily be found by trial and error.

   Alternative Method: Systematic trialling of possible numbers for the fourth term eventually shows that if the fourth term is 1333, the previous three terms are 667, 666 and 1, giving 1, 666, 667, 1333, 2000. Note that if the fourth term is 1334 or more, the third term is 666 or less, the second term is 668 or more, and the first term must be negative, giving an invalid sequence.

Similarly, if the fourth term is 1001, the previous terms are 999, 2 and 997. Note that if the fourth term is 1000 or less, then the third term is 1000 or more, the second term is 0 or negative, giving an invalid sequence.

It follows that the selection for the fourth term of any integer between 1001 and 1333 inclusive will lead to a valid sequence with 2000 as the fifth term.

Selection for the fourth term of any integer outside this range will lead to an invalid sequence. There are 333 valid sequences.

2. Systematic trialing of possible numbers for the seventh term shows that the selection of any integer between 1231 and 1249 inclusive will lead to a valid sequence with 2000 as the eighth term. Note that if the seventh term is 1230 or less, then the sixth term is 770 or more, the fifth term is 460 or less, the fourth term is 310 or more, the third term is 150 or less, the second term is 160 or more and the first term is negative, giving an invalid sequence.

Similarly, if the seventh term is 1250 or more, the sixth term is 750 or less, the fifth term is 500 or more, the fourth term is 250 or less, the third term is 250 or more and the second term is 0 or negative, giving an invalid sequence.

Selection for the seventh term of any integer outside the range 1231 and 1249 inclusive will, as shown, not lead to a valid sequence.

There are 19 valid sequences. For example, a seventh term of 1231 yields 3, 152, 155, 307, 462, 769, 1231, 2000.

3. With similar reasoning to that in **2.**, systematic trialing of possible numbers for the term preceding 2000 shows that the selection of any integer between 1236 and 1238 inclusive will lead to a valid sequence with 2000 as the tenth term.

Writing each of these sequences in reverse for ten terms yields:
2000, 1236, 764, 472, 292, 180, 112, 68, 44, 24.
2000, 1237, 763, 474, 289, 185, 104, 81, 23, 58.
2000, 1238, 762, 476, 286, 190, 96, 94, 2, 92.

Note that the latter two sequences cannot be extended further, since an extra term will be negative. However the first sequence can be extended to three more terms: 20, 4, 16.

It follows that there is exactly one sequence of 13 terms:
2000, 1236, 764, 472, 292, 180, 112, 68, 44, 24, 20, 4, 16.

This 13-term sequence is the one of maximum length.

Further Discussion

The Golden Ratio can be used in extended thinking of this problem.

As is well known Fibonacci sequences are generated via what are known as recurrence relations of the form $x_{n+2} = x_{n+1} + x_n$, and the ratio x_{n+1}/x_n of successive terms gets closer and closer to what is known as the Golden Ratio, whose value is $(1 + \sqrt{5})/2$, which equals 1.61803398875... (A student should try checking this by calculating the ratio of some successive ratios for higher n.)

So if we are looking for a lengthy sequence ending in 2000 we might expect the second last term to be approximately 2000/1.61803398875..., which is 1236.0679775379..., closest to 1236, which was the actual second-last term in the longest sequence, as we discovered.

9. Invariance

Discovering an invariant in a problem can lead to a simple resolution of an otherwise intractable problem. The method of invariance applies in a situation where a system changes from state to state according to various rules, and some property which is important to the statement of the problem remains unchanged in each transition. The property which doesn't change is known as the invariant.

Looking out for invariants can be rewarding. Question 3 of the 2000 IMO involved an if and only if situation which was very difficult in one of the directions. An Australian student became one of only 14 students to solve the problem by spotting an invariant and bypassing the difficulty, a situation not anticipated by the jury. This helped him win a Silver Medal.

This method is very well illustrated by the following famous problem from the International Mathematics Tournament of Towns, not just for its mathematical properties, but for other various associated aesthetic features.

Example 7

On the island of Camelot live 13 grey, 15 brown and 17 crimson chameleons. If two chameleons of different colours meet, they both simultaneously change colour to the third color (e.g. if a grey and brown chameleon meet they both become crimson). Is it possible they will all eventually be the same colour?

Strategy

At first sight this problem looks very difficult, and the imagining of two chameleons with dull colours touching noses becoming a bright colour is also a distraction. With problems like this one should either try to solve a simpler version first, or play around with it by going through a few phases looking at what happens in various phases looking for a successful solution. Doing this can cause frustration after a while as the answer is no and one can go for some time not being able to find a successful pathway.

This is also a case where I found working backwards to help. Since there are 45 chameleons altogether, a successful answer would be 45 of one colour, none of the other two. Working backwards in one step gives a configuration 1, 1, 43 (it doesnt matter which are which colours), then maybe 2, 2, 41, and others might be 3, 3, 39 or 1, 4, 40. Eventually one frequently often recognises a case in which all are multiples of 3. After further exploration, working backwards, one eventually notices that all configurations working backwards have all three the same value modulo 3, whereas in our case our starting configuration has the three different values modulo 3. One then notices that during one nose-touch one colour increases by two (the new colour of the two chameleons touching) whereas each of their former colours has decreased by one, and that whatever happens, in one nose-touch the population of each colour will always involve each of the three numbers modulo 3, so the outcome with all three being zero modulo 3 is not possible.

Solution 7

The starting configuration has populations 0, 1, and 2 modulo 3. This situation remains invariant no matter which two chameleons touch noses. So the desired configuration, which has all three equal to zero modulo 3 is not possible.

10. Colouring

There is a beautiful class of problems which involve colourings. They can be illustrated by the following very nice problem from the International Mathematics Tournament of Towns.

Example 8

A 7 by 7 square is made up of sixteen 1 by 3 tiles and one 1 by 1 tile. Prove that the 1 by 1 tile lies either at the centre of the square or adjoins one of its boundaries.

Discussion

This problem has a rather surprising result and at first sight, with all the combinations possible, seems almost impossible to prove. But colouring with 3 colours and looking at the resultant way in which a 3 by 1 domino, or should I say triomino, might cover squares of the board makes the problem accessible. The solution was devised by my colleague John Campbell.

Solution 8

Label each of the squares as follows.

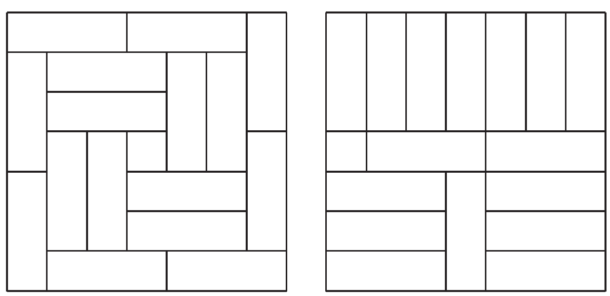
A	B	C	A	B	C	A
B	C	A	B	C	A	B
C	A	B	C	A	B	C
A	B	C	A	B	C	A
B	C	A	B	C	A	B
C	A	B	C	A	B	C
A	B	C	A	B	C	A

Each 1×3 tile must cover squares labeled A, B and C, because they would be filling adjoining squares. Since there are 17 As and only 16 Bs and 16 Cs, the 1×1 square must therefore occupy a square marked A.

However, because the orientation of the square is not relevant, we should eliminate all those occurrences of As which do not remain as As on rotation by, say multiples of $90°$. This leaves only those in the corners, the four along the mid-points of each outer edge, and the one in the centre. This completes the proof.

Note

The following diagram show two such tilings, one with the 1×1 square in the centre, and the other one with it on the edge.



In fact, by rotating the bottom left 4×4 square in the diagram on the right, all three possible positions of the monomino can be achieved (centre side, centre and corner).

11. Geometry

I will not add much on Geometry, the oldest of the formal mathematics disciplines. It is still the strongest of the IMO disciplines, where normally two of the six problems are generally geometric, with Euclidean proof normally required (although if a coordinate proof is complete it can attract 7 rather than otherwise 0 points). I do not recall three dimensional Euclidean geometry in the IMO in recent years.

Geometry has suffered massive cuts in the syllabi of various countries since the 1980s. This has happened to the extent that the Australian Mathematics Competition is no longer able to set circle geometry problems for years 9 and 10 because it has ceased to be taught there.

Teachers who supported geometry's downgrade said the reason was simple – people do not use geometry in later life. This overlooks the fact that geometry, with its theorems, logic and structure were the main branch of mathematics for developing logical reasoning, a vital skill in later life.

I think geometry has also suffered because the textbooks have been very dry. Old fashioned, theorem, proof, theorem proof, exercise, etc but no motivating discussion. It doesn't have to be like that. We set an Australian Mathematics Competition problem in 1983 (to be found on my web site) where we asked for the most northerly point where one can see the Southern Cross. It was well known that the Southern Cross can be seen in some northern latitudes as French Aviation pioneers used it for navigation when crossing the Sahara in establishing routes to South America. The solution involved a nice use of circle geometry with a cyclic quadrilateral.

12. Logic

It was a standard procedure, particularly in the early days, for us to put in the Australian Mathematics Competition paper each year what we would call a logic problem. The most famous, and quite typical, was the following, from the 1984 paper.

Example 9

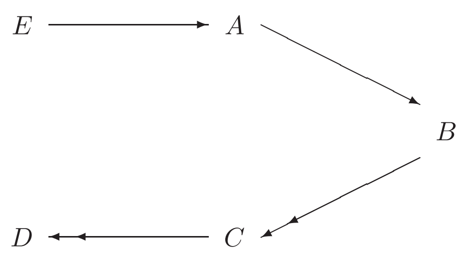
Albert, Bernard, Charles, Daniel and Ellie play a game in which each is a frog or kangaroo. Frogs' statements are always false while kangaroos' statements are always true.

Albert says that Bernard is a kangaroo.
Charles says that Daniel is a frog.
Ellie says that Albert is not a frog.
Bernard says that Charles is not a kangaroo.
Daniel says that Ellie and Albert are different kinds of animals.

How many frogs are there?

Solution 9

Let a single-headed arrow represent '... says ... is a kangaroo' and a double-headed arrow represent '... says ... is a frog'. Then we have



Assume Ellie is a kangaroo, and hence that his statement is true.
Thus: Albert is a kangaroo,
Bernard is a kangaroo,
Charles is a frog and
Daniel is a kangaroo.

But this is not possible since Ellie and Albert are then both kangaroos, contrary to Daniel's statement. This proves that Ellie is not a kangaroo, but a frog instead. Being a frog, Ellie's statement is false.
Thus: Albert is a frog,
Bernard is a frog,
Charles is a kangaroo and
Daniel is a frog.

There are 4 frogs.

Comment

We had some fun with our French colleagues and combined with them to arrange for the kangaroos to tell lies and frogs to tell the truth in the French translation used in New Caledonia and French Polynesia.

13. Graphical Methods

A very useful method if life gets difficult is to attempt graphical thinking. This can particularly help when a student confronts a formula or equation in which there are structures such as modulus signs, fractional powers, rational functions, greatest integer functions and similar. The following example illustrates a standard low-level approach useful in the early stages of problem solving, but could be systematised more in teaching.

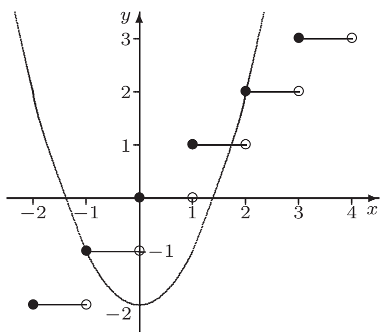
Example 10

Find all the solutions of $x^2 - [x] - 2 = 0$.

Solution 10 and comment

In questions like this one can go straight into a graphical approach. But one needs to be careful. A common method is to try to develop the left hand side function in stages. In this case the left hand side becomes a little messy and the author's experience is that not all people find all the solutions to be obvious. It is easy to draw qualitatively the function $x^2 - 2$ for example, but then subtracting $[x]$ is messy.

A better strategy in this case is to note that the equation is equivalent to $x^2 - 2 = [x]$ and to draw the two simpler curves of each side, as below.



This leads to a clear discovery that there are three solutions, two of which are integer, $(-1, -1)$ and (2,2) and the one where $y = 1$ and it turns out using symbolic calculation that this root is at $(\sqrt{3}, 1)$.

14. Probability

Probability is not a topic as such in the International Mathematical Olympiad, nor major international competitions generally. In the Australian Mathematics Competition we did try to set however an interesting probability problem each year, mainly because we found probability the most interesting branch of statistics, and also because, like combinatorics, which we embraced, we felt we could set questions which could be solved without formal knowledge of the theory. We even felt we could work round simple Bayesian problems in such a way. Statistics did become part of the Australian syllabus from about the time the Australian Mathematics Competition started, but its syllabus focus would be on aspects such as data analysis. We did frequently set questions on aspects of mean, in addition to probability.

It is not so interesting for me to illustrate with a standard problem, but having just discussed graphical methods, the following former Australian Mathematics Competition problem I find very interesting.

Example 11

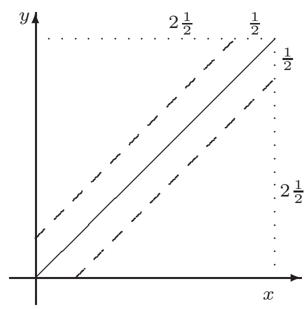
Two forgetful friends agree to meet in a coffee shop one afternoon but each has forgotten the agreed time. Each remembers that the time was somewhere between 2 pm and 5 pm. Each decides to go to the coffee shop at a random time between 2 pm and 5 pm, wait half an hour, and leave if the other doesn't arrive. What is the probability that they meet?

Solution 11

This is a very difficult problem to solve, even for an experienced solver, if trying by any sort of symbolic method. In grappling with it one soon understands the situation much better if looking for a graphical-based solution, as discussed above.

We can do this as follows:

Let one friend's arrival time be x and the other be y. They can both take random values, say, between 0 and 3. They will meet if $|x - y| \leq \frac{1}{2}$.



The probability of their meeting is then

$$\frac{3^2 - (2\frac{1}{2})^2}{3^2} = \frac{9 - 6\frac{1}{4}}{9} = \frac{2\frac{3}{4}}{9} = \frac{11}{36}.$$

15. Problem Solving by reviving past ideas and evolving new ones

Problem solving is an exciting part of mathematics, sometimes at the cutting edge of knowledge, and I have been on problem-setting committees in which we have discovered new results and published refereed papers.

It can be quite exciting to discover forgotten techniques which can be used to solve modern problems. On my web site I cite the rediscovery (in the West anyway) of barycentric coordinates as a useful technique, and that a recent member of the UK IMO team, probably to the chagrin of his leader, who had to mark his work, solved many geometry problems using them. They are particularly useful in certain types of problem, for example proving collinearity of three lines.

Then there is the evolution of new techniques. The web site also shows how the method of the moving parallel evolved through papers in the West and discussion in the Russian student journal *Kvant*, enabling the solution of some nice polygon dissection problems.

CONCLUSION

I have attempted a classification of what I would call immediate beyond-school mathematics problems, and in doing so have used the opportunity to discuss some of my favourite problems, on occasions alluding to pedagogy and describing how some can be used in enrichment classes. All of the above methods are ones I have found to work well in the mathematics circle I used to work with in Canberra.

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