World Youth Mathematics Intercity Competition

Simon Chua, Andy Liu & Bin Xiong

Simon Chua is the founder and president of the Mathematics Trainers' Guild, Philippines. The first Filipino to win the Paul Erdös National Award *from the World Federation of National Mathematics Competitions for his distinguished and sustained contribution to the enrichment of mathematics education in the Philippines. His ambition is to transform all mathematics teachers in the Philippines to good mathematics educators, so that more Filipino kids would achieve international recognition in the field of mathematics.*

Andy Liu is a professor of mathematics at the University of Alberta in Canada. His research interests spans discrete mathematics, geometry, mathematics education and mathematics recreations. He edits the Problem Corner of the MAA's magazine Math Horizons. He was the Chair of the Problem Committee in the 1995 IMO in Canada. His contribution to the 1994 IMO in Hong Kong was a main reason for his being awarded a David Hilbert International Award by the World Federation of National Mathematics Competitions. He has trained students in all six continents.

Bin Xiong is a professor of mathematics at the East China Normal University. His research interest is in problem solving and gifted education. He is a member of the Chinese Mathematical Olympiad committee and the Problem Subcommittee. He had served as the leader of the national team at the 2005 International Mathematical Olympiad in Mexico. He had been a trainer at the national camp many times. He is involved in the National Junior High School Competition, the National High School Mathematics Competition, the Western China Mathematical Olympiad and the Girls' Mathematical Olympiad.

The World Youth Mathematics Intercity Competition was founded in 1999 by Prof. Hsin Leou of the Kaohsiung National Normal University in Taiwan. It was designed as an International Mathematical Olympiad for junior high school students. Each team consists of a leader, a deputy leader and four contestants. They represent their city rather than their country, downplaying politics.

For the first two years, the contest was actually held in Kaohsiung, and the participating teams were all from South East Asia. The hosts in 2001 and 2002 were the Philippines and India, respectively. It was not held in 2003 during the height of the SARS scare. In 2004, the host was Macau.

In 2005, the contest returned to Kaohsiung. Much progress has been made since the beginning. China had joined the rank, and in fact the Chinese city of Wen Zhou hosted the 2006 contest. In 2007, when the contest was held in Chang Chun, China, there were sixty-five teams, including those from Canada, Iran, South Africa and the United States of America.

There is now a permanent WYMIC board. The president is Mr. Wen-Hsien Sun of Chiu Chang Mathematics Foundation in Taipei, and the secretary is Mr. Simon Chua of the Mathematics Trainers' Guild in

Zamboanga. Added to the board each year are the local organizing committee and the local problem committee. Prof. Zonghu Qiu of Academia Sinica in Beijing is the advisor for the former, and Prof. Andy Liu and Prof. Bin Xiong are advisors for the latter.

The event typically lasts five days in late July. Day 1 is for arrival. Day 2 is taken up with registration, the opening ceremony and team leader meetings. Day 3 is the actual competition, with an individual contest in the morning and a team contest in the afternoon. Day 4 is set aside for excursion, with a banquet in the evening. Day 5 is for departure after the closing ceremony. Slight variation occurs from year to year.

Gold, silver and bronze medals are awarded for the individual contest, as well as honorable mentions. The teams are drawn into a number of groups. Within each group, gold, silver and bronze medals are awarded for the team contest. A second set of medals are awarded on the basis of the sum of the best three scores in the individual contest. Finally, a team is declared the grand champion, based on the sum of these two team scores. The next two teams also received trophies. In addition, there are two non-academic team awards, one for the best behaviour and one for popularity.

A most special feature of the WYMIC is the Cultural Evening. Each team must perform on stage for about five minutes, to showcase their ethnic identities. The performances vary from instrumental and choral music to power-point presentations of scenery. This cements international friendship and fosters mutual understanding. Two more team awards are handed out, one for the best performance and one for innovation

In 2009, the contest will be hosted by South Africa in Durban. It is a golden opportunity for European and other African teams to join in. For further information, contact Prof. Gwen Williams at gwilliams@telkomsa.net. To conclude this paper, we append the problems used in the 2007 contest.

1 Individual Contest

Time limit: 120 minutes 2007/7/23 Changchun, China

Section I

In this section, there are 12 questions, fill in the correct answers in the spaces provided at the end of each question. Each correct answer is worth 5 points.

1. Let A_n be the average of the multiples of n between 1 and 101. Which is the largest among A_2, A_3, A_4, A_5 and A_6?

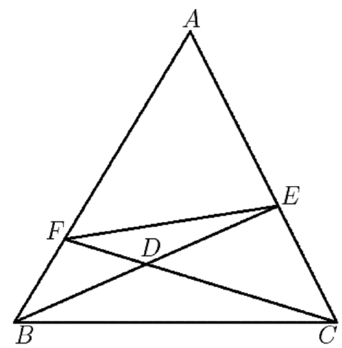
Solution. The smallest multiple of n is of course n. Denote by an the largest multiple of n not exceeding 101. Then $A_n = (n+a_n)/2$. Hence $A_2 = A_3 = 51$, $A_4 = 52$, $A_5 = 52.5$ and $A_6 = 51$, and the largest one is A_5.

2. It is a dark and stormy night. Four people must evacuate from an island to the mainland. The only link is a narrow bridge which allows passage of two people at a time. Moreover, the bridge must be illuminated, and the four people have only one lantern among them. After each passage to the mainland, if there are still people on the island, someone must bring the lantern back. Crossing the bridge individually, the four people take 2, 4, 8 and 16 minutes respectively. Crossing the bridge in pairs, the slower speed is used. What is the minimum time for the whole evacuation?

Solution. Exactly five passages are required, three pairs to the mainland and two individuals back to the island. Let the fastest two people cross first. One of them brings back the lantern. Then the slowest two people cross, and the fastest people on the mainland brings back the lantern, The final passage is the same as the first. The total time is $4+2+16+4+4 = 30$ minutes. To show that this is minimum, note that the three passages in pairs take at least $16 + 4 + 4 = 24$ minutes, and the two passages individually take at least $4 + 2 = 6$ minutes.

3. In triangle ABC, E is a point on AC and F is a point on AB. BE and CF intersect at D. If the areas of triangles BDF, BCD

and CDE are 3, 7 and 7 respectively, what is the area of the quadrilateral $AEDF$?



Solution. Since triangles BCD and CDE have equal areas, $BD{=}DE$. Hence the area of triangle DEF is also 3. Let the area of triangle EFA be x. Then $\frac{x}{6} = \frac{AF}{BF} = \frac{x+3+7}{3+7}$. It follows that $10x = 6x + 60$ so that $x = 15$. The area of the quadrilateral $AEDF$ is $15 + 3 = 18$.

4. A regiment had 48 soldiers but only half of them had uniforms. During inspection, they form a 6×8 rectangle, and it was just enough to conceal in its interior everyone without a uniform. Later, some new soldiers joined the regiment, but again only half of them had uniforms. During the next inspection, they used a different rectangular formation, again just enough to conceal in its interior everyone without a uniform. How many new soldiers joined the regiment?

Solution. Let the dimensions of the rectangle be x by y, with $x \leqq y$. Then the number of soldiers on the outside is $2x + 2y - 4$ while the number of those in the interior is $(x - 2)(y - 2)$. From $xy - 2x - 2y + 4 = 2x + 2y - 4$, we have $(x - 4)(y - 4) = xy - 4x - 4y + 16 = 8$. If $x - 4 = 2$ and $y - 4 = 4$, we obtain the original 6×8 rectangle. If $x - 4 = 1$ and $y - 4 = 8$, we obtain the new 5×12 rectangle. Thus the number of new soldiers is $5 \times 12 - 6 \times 8 = 12$.

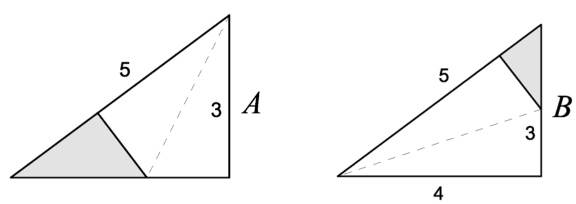
5. The sum of 2008 consecutive positive integers is a perfect square. What is the minimum value of the largest of these integers?

Solution. Let a be the smallest of these integers. Then

$$a + (a + 1) + (a + 2) + \ldots + (a + 2007) = 251 \cdot (2a + 2007) \cdot 2^2.$$

In order for this to be a perfect square, we must have $2a + 2007 = 251n^2$ for some positive integer n. For $n = 1$ or 2, a is negative. For $n = 3$, we have $a = 126$ so that $a + 2007 = 2133$ is the desired minimum value.

6. The diagram shows two identical triangular pieces of paper A and B. The side lengths of each triangle are 3, 4 and 5. Each triangle is folded along a line through a vertex, so that the two sides meeting at this vertex coincide. The regions not covered by the folded parts have respective areas S_A and S_B. If $S_A + S_B = 39$ square centimetres, find the area of the original triangular piece of paper.



Solution. In the first diagram, the ratio of the areas of the shaded triangle and one of the unshaded triangle is $(5 - 3) : 3$ so that S_A is one quarter of the area of the whole triangle. In the second diagram, the ratio of the areas of the shaded triangle and one of the unshaded triangle is $(5 - 4) : 4$ so that S_B is one ninth of the area of the whole triangle. Now $\frac{1}{4} + \frac{1}{9} = \frac{13}{36}$. Hence the area of the whole triangle is $\frac{36}{13}39 = 108$ square centimetres.

7. Find the largest positive integer n such that $3^{1024} - 1$ is divisible by 2^n.

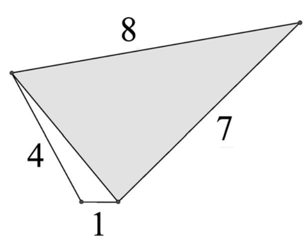
Solution. Note that

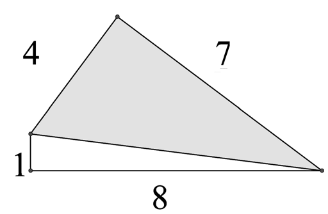
$$3^{1024} - 1 = (3^{512} + 1)(3^{256} + 1)(3^{128} + 1) \ldots (3 + 1)(3 - 1).$$

All 11 factors are even, and 3+1 is a multiple of 4. Clearly $3-1$ is not divisible by 4. We claim that neither is any of the other 9. When the square of an odd number is divided by 4, the remainder is always 1. Adding 1 makes the remainder 2, justifying the claim. Hence the maximum value of n is 12.

8. A farmer has four straight fences, with respective lengths 1, 4, 7 and 8 metres. What is the maximum area of the quadrilateral the farmer can enclose?

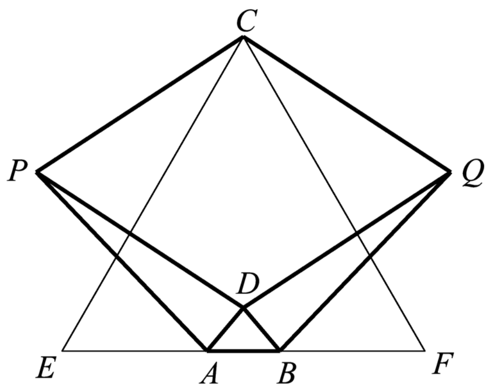
Solution. We may assume that the sides of lengths 1 and 8 are adjacent sides of the quadrilateral, as otherwise we can flip over the shaded triangle in the first diagram. Now the quadrilateral may be divided into two triangles as shown in the second diagram. In each triangle, two sides have fixed length. Hence its area is maximum if these two sides are perpendicular to each other. Since $12+82 = 42+72$, both maxima can be achieved simultaneously. In that case, the area of the unshaded triangle is 4 and the area of the shaded triangle is 14. Hence the maximum area of the quadrilateral is 18.





9. In the diagram (on the next page), $PA = QB = PC = QC = PD = QD = 1$, $CE = CF = EF$ and $EA = BF = 2AB$. Determine BD.

Solution. Let M be the midpoint of EF. By symmetry, D lies on CM. Let $BM = x$. Then $FM = 5x$, $CF = 10x$, $CM = 5\sqrt{5}x$ and $BC = 2\sqrt{19}x$. It follows that $\frac{AB}{BC} = \frac{1}{\sqrt{19}}$. Now Q is the circumcentre of triangle BCD. Hence $\angle BQD = 2\angle BCD = \angle BCA$. Since both QDB and CAB are isosceles triangles, they are similar to each other. It follows that $\frac{BD}{QB} = \frac{AB}{BC} = \frac{1}{\sqrt{19}}$, so that



$BD = \frac{1}{\sqrt{19}}$.

10. Each of the numbers 2, 3, 4, 5, 6, 7, 8 and 9 is used once to fill in one of the boxes in the equation below to make it correct. Of the three fractions being added, what is the value of the largest one?

$$\frac{1}{\square \times \square} + \frac{\square}{\square \times \square} + \frac{\square}{\square \times \square} = 1$$

Solution. We may assume that the second numerator is 5 and the third 7. If either 5 or 7 appears in a denominator, it can never be neutralized. Since the least common multiple of the two remaining numbers is $8 \times 9 = 72$, we use $\frac{1}{72}$ as the unit of measurement. Now one of the three fractions must be close to 1. This can only be $\frac{5}{2\times 3}$ or $\frac{7}{2\times 4}$. In the first case, we are short 12 units. Of this, 7 must come from the third fraction so that 5 must come from the first fraction. This is impossible because the first fraction has numerator 1 and 5 does not divide 72. In the second case, we are short 9 units. Of this, 5 must come from the second fraction so that 4 must come from the third. This can be achieved as shown in the equation below. Hence the largest of the three fractions has value $\frac{7}{8}$.

$$\frac{1}{\boxed{3}\times\boxed{6}}+\frac{\boxed{5}}{\boxed{8}\times\boxed{9}}+\frac{\boxed{7}}{\boxed{2}\times\boxed{4}}=1$$

11. Let x be a positive number. Denote by $[x]$ the integer part of x and by $\{x\}$ the decimal part of x. Find the sum of all positive numbers satisfying $5\{x\}+0.2[x]=25$.

Solution. The given equation may be rewritten as $\{x\}=\frac{125-[x]}{25}$. From $0\leqq\{x\}<1$, we have $100<[x]\leqq 125$. For each solution x, $x=[x]+\{x\}=5+\frac{24}{25}[x]$. It follows that the desired sum is $5(25)+(24/25)(101+102+103+\ldots+125)=2837$.

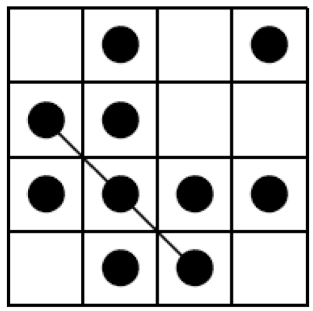
12. A positive integer n is said to be good if there exists a perfect square whose sum of digits in base 10 is equal to n. For instance, 13 is good because $7^2=49$ and $4+9=13$. How many good numbers are among $1,2,3,\ldots,2007$?

Solution. If a positive integer is a multiple of 3, then its square is a multiple of 9, and so is the sum of the digits of the square. If a positive integer is not a multiple of 3, then its square is 1 more than a multiple of 3, and so is the sum of the digits of the square. Now the square of $9\ldots9$ with m 9s is $9\ldots980\ldots01$, with $m-1$ 9s and 0s. Its digit sum is $9m$. Hence all multiples of 9 are good, and there are $\frac{2007}{9}=223$ of them not exceeding 2007. On the other hand, the square of $3\ldots35$ with m 3s is 1212..1225 with m sets of 12. Its digit sum is $3m+7$. Since 1 and 4 are also good, all numbers 1 more than a multiple are good, and there are $\frac{2007}{3}=669$ of them. Hence there are altogether $223+669=992$ good numbers not exceeding 2007.

Section II

Answer the following 3 questions, and show your detailed solution in the space provided after each question. Each question is worth 20 points.

1. A 4×4 table has 18 lines, consisting of the 4 rows, the 4 columns, 5 diagonals running from southwest to northeast, and 5 diagonals running from northwest to southeast. A diagonal may have 2, 3 or 4 squares. Ten counters are to be placed, one in each of ten of the sixteen cells. Each line which contains an even number of counters scores a point. What is the largest possible score?



The maximum score is 17, as shown in the placement in the diagram below. The only line not scoring a point is marked.

We now prove that a perfect score of 18 points leads to a contradiction. Note that the five diagonals in the same direction cover all but two opposite corner cells. These two cells must either be both vacant or both occupied. Note also that we must have a completely filled row, and a completely filled column. We consider three cases.

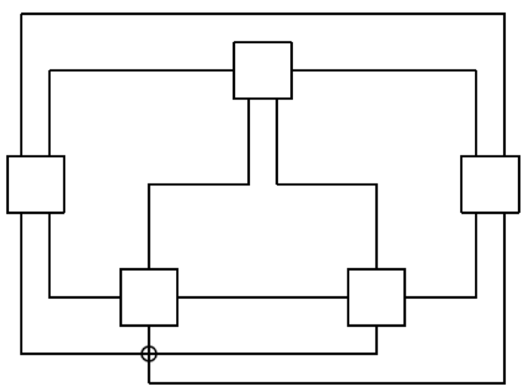
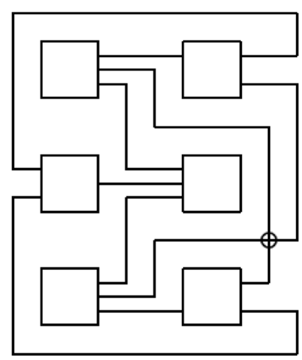
Case 1. All four corner cells are vacant.
We may assume by symmetry that the second row and the second column are completely filled. Then we must fill the remaining inner cells of the first row, the fourth row, the first column and the fourth column. These requires eleven counters.

Case 2. Exactly two opposite corner cells are vacant.
By symmetry, we may assume that one of them is on the first row and first column, and the other is on the fourth row and fourth column. Then we must have exactly one more occupied inner cell on each of the first row, the first column, the fourth row and the fourth column. This means that all four cells in the interior of the table are filled. By symmetry, we may assume that the completely

filled row is the second. It is impossible to score both the diagonals of length 2 which intersect the second row.

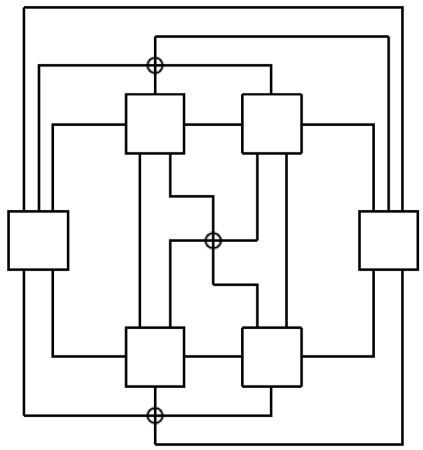
Case 3. All four corner cells are occupied.
We claim that the completely filled row must be either the first or the fourth. Suppose to the contrary it is the second. Then we must fill the first column and the fourth column, thus using up all ten counters. Now there are several diagonals which do not yield scores. This justifies the claim. By symmetry, we may assume that the first row and the first column are completely filled. To score all rows and columns, the remaining two counters must be in the four interior cells. Again, some of the diagonals will not yield scores.

2. There are ten roads linking all possible pairs of five cities. It is known that there is at least one crossing of two roads, as illustrated in the diagram below on the left. There are nine roads linking each of three cities to each of three towns. It is known that there is also at least one crossing of two roads, as illustrated in the diagram below on the right. Of the fifteen roads linking all possible pairs of six cities, what is the minimum number of crossings of two roads?

Solution. The minimum number of crossing of two roads is three, as illustrated in the diagram below.

Suppose at most two crossings of two roads are needed. If we close one road from each crossing, the remaining ones can be drawn without any crossing. We consider two cases.



Case 1. The two roads closed meet at a city.
Consider the other five cities linked pairwise by ten roads, none of which has been closed. It is given that there must be a crossing of two roads, which is a contradiction.

Case 2. The two roads closed do not meet at a city.
Choose the two cities linked by one of the closed roads, and a third city not served by the other closed road. Call these three cities towns. Each is linked to each of the remaining three cities by a road. It is given that there must be a crossing of two roads, which is a contradiction.

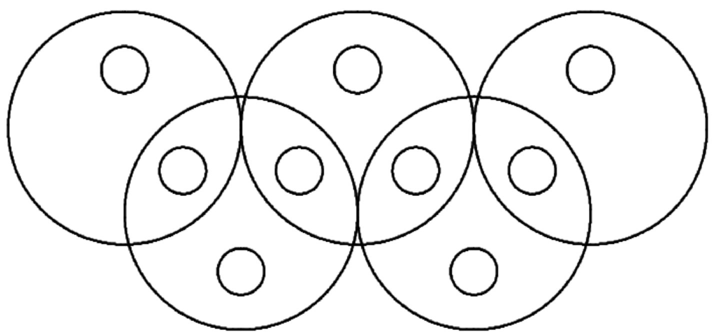
3. A prime number is called an absolute prime if every permutation of its digits in base 10 is also a prime number. For example: 2, 3, 5, 7, 11, 13 (31), 17 (71), 37 (73) 79 (97), 113 (131, 311), 199 (919, 991) and 337 (373, 733) are absolute primes. Prove that no absolute prime contains all of the digits 1, 3, 7 and 9 in base 10.

Solution. Let N be an absolute prime which contains all of the digits 1, 3, 7 and 9 in base 10. Let L be any number formed from the remaining digits. Consider the following seven permutations of N: $10000L + 7931$, $10000L + 1793$, $10000L + 9137$, $10000L + 7913$, $10000L + 7193$, $10000L + 1973$ and $10000L + 7139$. They have different remainders when divided by 7. Therefore one of them is a multiple of 7, and is not a prime. Hence N is not an absolute prime.

2 Team Contest

2007/7/23 Changchun, China

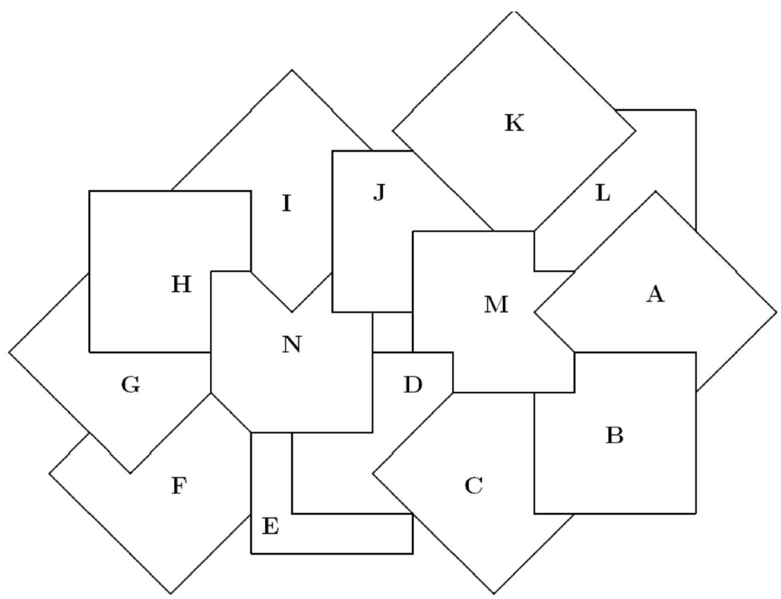
1. Use each of the numbers 1, 2, 3, 4, 5, 6, 7, 8 and 9 exactly once to fill in the nine small circles in the Olympic symbol below, so that the numbers inside each large circle is 14.



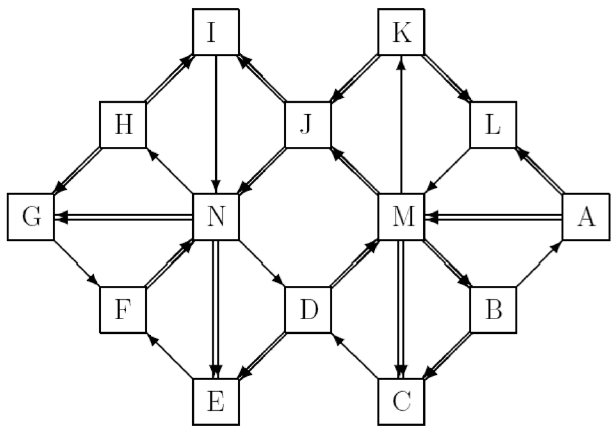
Solution. The sum of the nine numbers is $1+2+3+4+5+6+7+8+9=45$. The total sum of the numbers in the five large circles is $14\cdot 5=70$. The difference $70-45=25$ is the sum of the four numbers in the middle row, because each appears in two large circles. The two numbers at one end must be 9 and 5 while the two numbers at the other end must be 8 and 6. Consider the two numbers at the end of the middle row. Clearly they cannot be 5 and 6. If they are 5 and 8, the other two numbers must sum to 12. With 5 and 8 gone, the only possibility is 9 and 3, but 9 cannot be in the inner part of the middle row. If they are 9 and 8, the other two numbers must sum to 8. Since neither 5 nor 6 can appear in the inner part of the middle row, the only possibility is 7 and 1. However, 7 cannot be in the same large circle with either 9 or 8. It follows that the two numbers at the end of the middle row are 9 and 6, and the other two numbers sum to 10. The only possibility is 7 and 3, and 7 must be in the same large circle with 6.

The remaining numbers can now be filled in easily, and the 9-digit number formed is 861743295.

2. The diagram below shows fourteen pieces of paper stacked on top of one another. Beginning on the pieces marked B, move from piece to adjacent piece in order to finish at the piece marked F. The path must alternately climb up to a piece of paper stacked higher and come down to a piece of paper stacked lower. The same piece may be visited more than once, and it is not necessary to visit every piece. List the pieces of paper in the order visited.



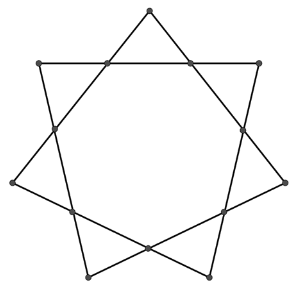
Solution. We construct below a diagram which is easier to use. An arrow from one piece of paper to another represents coming down from the first to the second. Note that each of M and N is connected to 7 other pieces, each of D and J is connected to 4 other pieces, while each of the others is connected to 3 other pieces. The path we seek consists of alternately going along with the arrow and going against it. Of the three arrows at A, the one from B cannot be used as otherwise we would be stuck at A. Equally useless are the arrows from L to M, from M to K, from I to N, from N to H, from G to F, from E to F, and from C to D. In the diagram below, they are drawn as single arrows while the useable ones are drawn as double arrows.



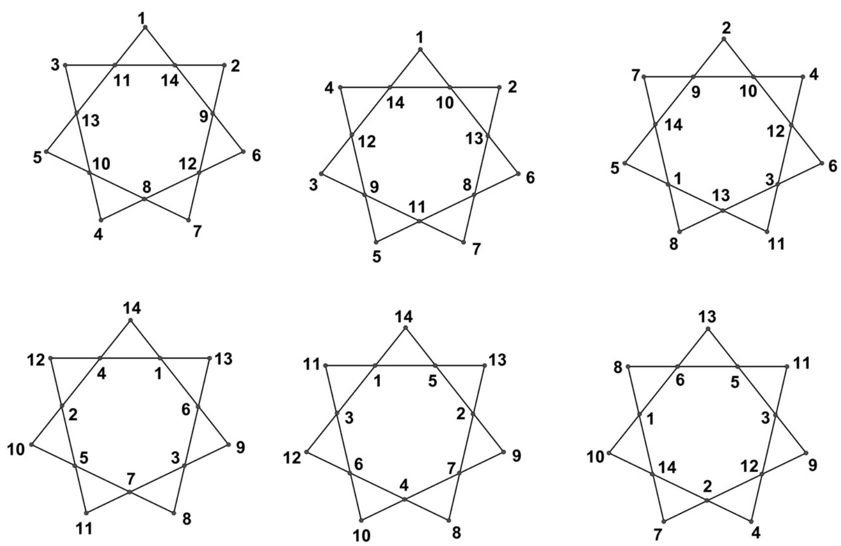
From B, if we climb up to C, then we will go down to M, but we could have gone down to M directly. Once in M, we have a choice of C or J, but C leads back to B, and we will get stuck there. From J, we have to move onto K, L, A, M, D, E, N, G, H, I, J, N and F. Thus the path is $BMJKLAMDENGHIJNF$.

3. There are 14 points of intersection in the seven-pointed star in the diagram (on the next page). Label these points with the numbers $1, 2, 3, \ldots, 14$ such that the sum of the labels of the four points on each line is the same. Give one of solution.

Solution. Each of the points of intersection lies on exactly two lines. Hence the common sum is given by $2(1+2+3+\ldots+14)/7 = 30$. We claim that the smallest label on the same line with 14 is 1 or 2. Otherwise, the sum of the labels of the two lines is at least $14 + 14 + 3 + 4 + 5 + 6 + 7 + 8 = 61 > 30 + 30$, which is a contradiction. Similarly, the smallest label on the same line with 13 is 1, 2 or 3. In view of these observations, we put 14 on a line with 1 and on another line with 2, and 13 on a line with 1 and on another line with 2 or 3. This leads to the three labeling on



the top row. The labeling on the bottom row are the complements of the corresponding ones in the top row, that is, each label k is replaced by $15 - k$.



4. Mary found a 3-digit number that, when multiplied by itself, produced a number which ended in her 3-digit number. What is the sum of the numbers which have this property?

Solution. Since $1 \times 1 = 1$, $5 \times 5 = 25$ and $6 \times 6 = 36$, the last digit of the 3-digit number must be either 1, 5 or 6.

No 2-digit number with units digit 1 whose square ends with that number.

There is only one 2-digit number with units digit 5 whose square ends with that number and that is 25.

There is only one 2-digit number with units digit 6 whose square ends with that number and that is 76.

There is only one 3-digit number which ends in 25 whose square ends with that number and that is 625.

There is only one 3-digit number which ends in 76 whose square ends with that number and that is 376.

The sum of these two numbers is 625+376=1001.

5. Determine all positive integers m and$n$ such that m^2+1 is a prime number and $10(m^2+1) = n^2+1$.

Solution. From the given condition, $9(m^2+1) = (n+m)(n-m)$. Note that m^2+1 is a prime number not equal to 3. Hence there are four cases.

a) $n-m = 1$, $n+m = 9(m^2+1)$.
Subtraction yields $9m^2+8 = 2m$, which is impossible.

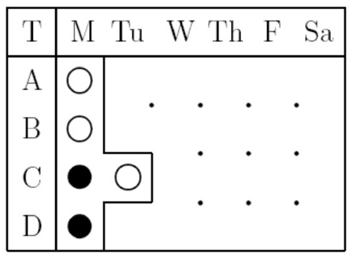
b) $n-m = 3$, $n+m = 3(m^2+1)$.
Subtraction yields $3m^2 = 2m$, which is impossible.

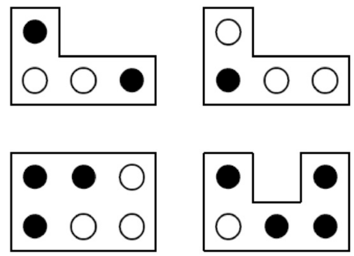
c) $n-m = 9$, $n+m = m^2+1$.
Subtraction yields $2m = m^2-8$, so that $m = 4$ and $n = 13$.
Note that $m^2+1 = 17$ is indeed a prime number.

d) $n-m = m^2+1$, $n+m = 9$.
Subtraction yields $-2m = m^2-8$, so that $m = 2$ and $n = 7$.
Note that $m^2+1 = 5$ is indeed a prime number.

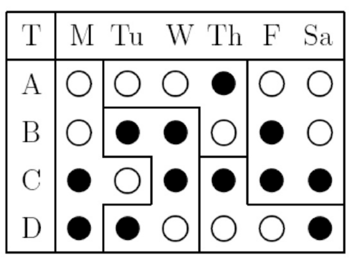
In summary, there are two solutions, $(m,n) = (2,7)$ or $(4,13)$.

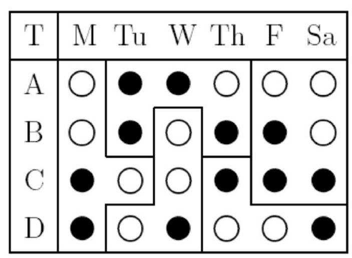
6. Four teams take part in a week-long tournament in which every team plays every other team twice, and each team plays one game per day. The diagram below on the left shows the final scoreboard, part of which has broken off into four pieces, as shown on the diagram below on the right. These pieces are printed only on one side. A black circle indicates a victory and a white circle indicates a defeat. Which team wins the tournament?





Solution. When reconstructing the broken scoreboard, there are two positions where the U-shaped piece can be placed. So as to leave room for the 3×2 rectangle. Once it is in place, the positions for the remaining pieces are determined. They are placed so that there are two black circles and two white circles in each column. There are two possibilities, as shown in the diagrams below, but in either case, the winner of the tournament is Team C.





7. Let A be a 3 by 3 array consisting of the numbers $1, 2, 3, \ldots, 9$. Compute the sum of the three numbers on the i-th row of A and the sum of the three numbers on the j-th column of A. The number at the intersection of the i-th row and the j-th column of a 3 by 3 array B is equal to the absolute difference of these two sums. For example, $b_{12} = |(a_{11} + a_{12} + a_{13}) - (a_{12} + a_{22} + a_{32})|$.

Is it possible to arrange the numbers in A so that the numbers in B are also $1, 2, 3, \ldots, 9$?

a_{11}	a_{12}	a_{13}
a_{21}	a_{22}	a_{23}
a_{31}	a_{32}	a_{33}

A

b_{11}	b_{12}	b_{13}
b_{21}	b_{22}	b_{23}
b_{31}	b_{32}	b_{33}

B

Solution. Let C be defined just like B, except that we use the actual difference instead of the absolute difference. Compute the sum of the nine numbers in C. Each number in A appears twice in this sum, once with a positive sign and once with a negative sign. Hence this sum is 0. It follows that among the nine numbers in C, the number of those which are odd is even. The same is true of the nine numbers in B, since taking the absolute value does not affect parity. Thus it is not possible for the nine numbers in B be $1, 2, 3, \ldots, 9$.

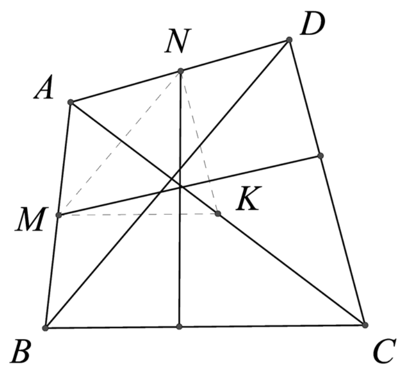
c_{11}	c_{12}	c_{13}
c_{21}	c_{22}	c_{23}
c_{31}	c_{32}	c_{33}

C

8. The diagonals AC and BD of a convex quadrilateral are perpendicular to each other. Draw a line that passes through point M, the midpoint of AB and perpendicular to CD, draw another line through point N, the midpoint of AD and perpendicular to CB. Prove that the point of intersection of these two lines lies on the line AC.

Solution. Let M, K and N be the respective midpoints of AB, AC and AD. Then MN is parallel to BD, MK is parallel to BC and NK is parallel to CD. Hence AC and the two lines in question are the altitudes of triangle MNK, and are therefore concurrent.

9. The positive integers from 1 to n (where $n > 1$ are arranged in a



line such that the sum of any two adjacent numbers is a square. What is the minimum value of n?

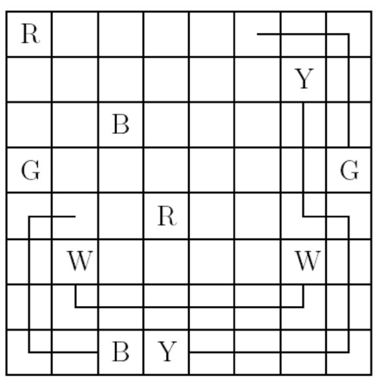
Solution. The minimum value of n is 15. Since $n > 1$, we must include 2, so that $n \geqq 7$ because $2+7 = 9$. For $n = 7$, we have three separate lines $(1, 3, 6)$, $(2, 7)$ and $(4, 5)$. Adding 8 only lengthen the first to $(8, 1, 3, 6)$. Adding 9 now only lengths the second to $(2, 7, 9)$. Hence $n \geqq 10$. Now 8, 9 and 10 all have to be at the end if we have a single line, because we can only have $8 + 1 = 9$, $9 + 7 = 16$ and $10 + 6 = 16$. The next options are $8 + 17 = 25$, $9 + 16 = 25$ and $10 + 15 = 25$. Hence $n \geqq 15$. For $n = 15$, we have the arrangement 8, 1, 15, 10, 6, 3, 13, 12, 4, 5, 11, 14, 2, 7 and 9.

10. Use one of the five colours (R represent red, Y represent yellow, B represent blue, G represent green and W represent white) to paint each square of an 8×8 chessboard, as shown in the diagram below. Then paint the rest of the squares so that all the squares of the same colour are connected to one another edge to edge. What is the largest number of squares of the same colour as compare to the other colours?

Solution. While it may tempting to colour the entire fourth row green, this will divide the red squares, the yellow squares and the blue squares into two disconnected parts. Obviously, the northeast corner is to be used to allow the green path to get around the yellow path. Similarly, the southwest corner is to be used to allow the blue path to get around the white path, and the southeast corner is to

R							
						Y	
		B					
G							G
			R				
	W					W	
		B	Y				

be used to allow the yellow path to get around the white path. In fact, we can complete the entire yellow path. Also, the white path may as well make full use of the seventh row. This brings us to the configuration as shown in the diagram below on the left. It is now not hard to complete the entire configuration, which is shown in the diagram below on the right. The longest path is the green one, and the number of green squares is 24.



R	R	R	R	R	G	G	G
G	G	G	G	R	G	Y	G
G	B	B	G	R	G	Y	G
G	B	G	G	R	G	Y	G
B	B	G	R	R	G	Y	Y
B	W	G	G	G	G	W	Y
B	W	W	W	W	W	W	Y
B	B	B	Y	Y	Y	Y	Y

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