

MATHEMATICS COMPETITIONS

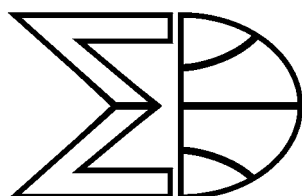
JOURNAL OF THE
WORLD FEDERATION OF NATIONAL
MATHEMATICS COMPETITIONS



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From the President

Dear readers,

Before anything else, it is my sad duty to inform you of the passing of a dear friend and valued contributor to the WFNMC, George Berzsenyi. Among his many prizes, his contributions to the competition world were acknowledged with an Erdős award in 1996. He will be sorely missed by all of us that knew him.

You may have noticed that this issue of *Mathematics Competitions* is being published a little bit later than usual. This is because we have been waiting to include several articles resulting from talks at the tenth Congress of the WFNMC, which was held in Kuala Lumpur, Malaysia from July 22nd to 28th, 2026. The congress was a great success, and I would like to thank everyone who helped to make it possible.

Most important of all, a huge thank you must go to Suhaimi Ramly and his team for organising the event in such a wonderful fashion, and also to Krzysztof Ciesielski for all his work coordinating the academic program. Thank you to both especially for the excellent communication in the months leading up to the event.

Thanks also to our Key-Note Speakers Richard Elwes, Po-Shen Lo, Alexander Soifer, and Lê Anh Vinh for their inspiring talks and to the Chairs of the Topic Groups, Alexander Soifer, Hayk Sedrakyan, Paul Vaderlind and Lukas Donner.

Finally, a huge Thank You to all participants for making the congress what it was! I hope we will all be able to get together again at the next WFNMC mini-conference in Prague in 2028 and at our next full Congress in 2030!

Robert Geretschläger

Editor's Page

Dear Competitions enthusiasts, readers of our *Mathematics Competitions* journal!

Mathematics Competitions is the right place for you to publish and read the different activities about competitions in Mathematics from around the world. For those of us who have spent a great part of our life encouraging students to enjoy mathematics and the different challenges surrounding its study and development, the journal can offer a platform to exhibit our results as well as a place to find new inspiration in the ways others have motivated young students to explore and learn mathematics through competitions. In a way, this learning from others is one of the better benefits of the competitions environment.

Following the example of previous editors, I invite you to submit to our journal *Mathematics Competitions* your creative essays on a variety of topics related to creating original problems, working with students and teachers, organizing and running mathematics competitions, historical and philosophical views on mathematics and closely related fields, and even your original literary works related to mathematics.

Just be original, creative, and inspirational. Share your ideas, problems, conjectures, and solutions with all your colleagues by publishing them here. We have formalized the submission format to establish uniformity in our journal.

Submission Format

FORMAT: should be LaTeX, TeX, or for only text articles in Microsoft Word, accompanied by another copy in pdf. However, the authors are strongly recommended to send article in TeX or LaTeX format. This is because the whole journal will be compiled in LaTeX. Thus your Word document will be typeset again. Texts in Word, if sent, should mainly contain non-mathematical text and any images used should be sent separately.

START: with the title centered in Large format (roughly 14 pt), followed on the next line by the author(s)' name(s) in italic 12 pt.

MAIN TEXT: Use a font from the Times New Roman family or 12 pt in LaTeX.

END: with your name, address, email and your website (if applicable).

INCLUDE: your high-resolution small photo and a concise professional summary of your works and titles.

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The resolution of your illustrations must be at least 300 dpi, or, preferably, done as vector illustrations.

If a text is embedded in illustrations, use a font from the Times New Roman family in 11 pt.

In figures, the letters used in labeling points, distances, etc. must be written in the same font as in the text that refers to these figures. Note that if the manuscript is prepared in Word, then mathematical symbols will be compiled with the use of the math mode in LaTeX.

Furthermore, a version of the figure in a LaTeX-compatible graphic language (such as TikZ, PSTricks or Asymptote) must be included with the submission to insure compatibility with the text.

REFERENCES: Every reference must be cited in the paper. References should be ordered alphabetically by author. If any webpages are mentioned as webpages, not as sources of articles, they should be in the end of the reference list. Any cited webpage should include a retrieval date. At the present we are using BibLaTeX and Biber for compiling the Reference section. In order to follow homogeneously our guidelines, please hand in a separate Reference file. A recommended citation in this file is the following:

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title = The New Mathematical Book: Mathematics of Coloring and the Colorful Life of Its Creators,
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author = Geretschläger, Robert and Donner, Lukas,
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journal = ZDM–Mathematics Education,
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Please submit your manuscripts to María Elizabeth Losada at
`director.olimpiadas@uan.edu.co`

We are counting on receiving your contributions, informative, inspired and creative. Best wishes,

Maria Elizabeth Losada
EDITOR

**Chromatic Number of the Plane Problem Yesterday, Today,
Tomorrow, and Mathematical Olympiads¹**
30 years without Paul Erdős (1913 – 1996), In loving memory

Alexander Soifer



Professor Soifer is a mathematics professor at the University of Colorado, Colorado Springs (UCCS). In addition to mathematics, he also teaches art history and European cinema. He served on the Jury of The Soviet Union Mathematical Olympiad (1970 – 1973) and on The USA Mathematics Olympiad Subcommittee (1996 – 2005). Soifer is the founder and the Chair of The Colorado Mathematical Olympiad (1984 – present), now known as The Soifer Mathematical Olympiad. He served as the President of WFNMC (2012-2018) and is the recipient of the WFNMC Paul Erdős Award (2006). Since 1991, he has been publishing research quarterly in “Geombinatorics”, dedicated to open problems of combinatorial geometry (141 issues to date), and has lectured at most WFNMC congresses. Prof. Soifer published over 400 articles and 14 books, including “The New Mathematical Coloring Book” (Springer 2024) and “The Scholar and the State: In Search for Van der Waerden” (Birkhäuser 2015).



Our last visit in person, Baton Rouge, Louisiana, February 22, 1996

[I] can't offer money for nice problems of other people because then I will really go broke... It is a very nice problem. If it were mine, I would offer \$250 for it.

– Paul Erdős Boca Raton, February 6, 1992

The unit distance graph in the plane ... is simple enough to describe to a nonmathematician, and so enigmatic that finding its chromatic number is a new four-color map problem for graph theorists.

– Ronald L. Graham and Eric Tressler

¹First presented as a Plenary talk at the 10th Congress of the World Federation of National Mathematics Competition, Kuala Lumpur, Malaysia, July 2026.

If Problem 8 [the chromatic number of the plane] takes that long to settle [as the Four-Color problem], we should know the answer by the year 2084.

– Victor Klee and Stan Wagon

Yesterday

Wouldn't you like to learn how the famous problem of finding the chromatic number of the plane was born? It was inspired by the famous Four-Color Conjecture!

Sometime between 1850 and 1852, teenager Francis Guthrie conjectured that four colors suffice for coloring any map under basic conditions (4CC). The problem naturally translates into the language of Graph Theory: Vertices of a planar graph can be 4-colored without creating adjacent monochromatic vertices. In 1976, Appel-Haken-Koch-IBM370 proved Guthrie was correct, thus 4CT was achieved. This result created the Great Debate. Martin Gardner called it “extraordinary” and “deeply unsatisfactory” (which was it, Martin?). We will briefly glance at this debate.

“20 Years After,” as Alexandre Dumas named his sequel to “The Three Musketeers,” actually, The Four if you count D’Artagnan, a fantastic new proof was produced by The Four Modern Musketeers: Robertson, Sanders, Seymour, and Thomas. Naturally, it took The Four to handle Four Colors. :-)

In 1950, another teenager posed a new problem. Eddie Nelson, 18, tried to prove 4CT. On October 5, 1991, Princeton Professor Nelson shared the genesis with me: “Dear Professor Soifer: In the autumn of 1950, I was a student at the University of Chicago and among other things was interested in the Four-Color problem, the problem of coloring graphs topologically embedded in the plane. These graphs are visualizable as nodes connected by wires. I asked myself whether a sufficiently rich class of such graphs might possibly be subgraphs of one big graph whose coloring could be established once and for all, for example, the graph of all points in the plane with the relation of being unit distance apart (so that the wires become rigid, straight, of the same length, but may cross). The idea did not hold up, but the other problem was interesting in its own right and I mentioned it to several people.”

Color all points of the plane; the minimum number of colors $\chi(E^2)$ forbidding unit distances between points of the same color is called the *chromatic number of the plane*. Its simple bounds were used in Olympiads:

Two-Colored Plane. Any 2-colored plane contains a monochromatic pair of points at a distance 1 apart. $\chi(E^2) \geq 3$. (Hint: toss on the 2-colored plane a unit equilateral triangle.)

Three-Colored Plane. Any 3-colored plane contains a monochromatic pair of points at a distance 1 apart. $\chi(E^2) \geq 4$.

Solution by the Canadian geometers, brothers Leo and William Moser, (1961). Toss on the given 3-colored plane what we now call *The Mosers Spindle* (Figure 1). Every edge in the spindle has length 1.

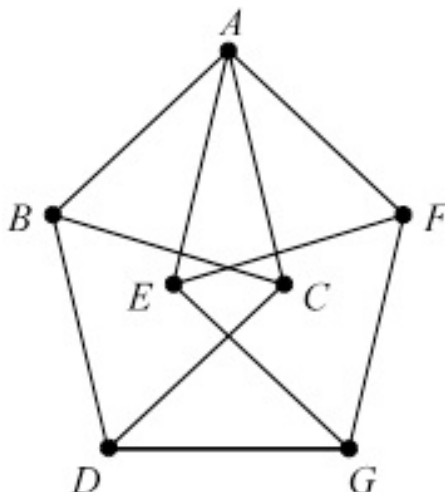


Figure 1: The Mosers Spindle

Assume that the seven vertices of the spindle do not contain a monochromatic segment of length 1. Call the colors used to color the plane red, white, and blue. The solution now will faithfully follow the children’s song: “*ABCDEFG...*”.

Let point A be red, then B and C must be one white and one blue, therefore D is red. Similarly, E and F must be one white and one blue, therefore G is red. We got a monochromatic segment DG of length 1 in contradiction to our assumption. ■

Observe: The Mosers Spindle has worked for us precisely because any 3 points of the spindle contain two points a distance 1 apart. This implies that *in a Mosers spindle that forbids monochromatic distance 1, at most 2 points can be of the same color*. Remember this observation, for we will need it later.

7 colors suffice to forbid monochromatic distance 1 pairs of points. $\chi(E^2) \leq 7$.

All that was proved by Edie Nelson and John Isbell in 1950.

SMO stands for the Soifer (formerly Colorado) Mathematical Olympiad. The 42nd SMO will take place on October 2, 2026. The problems are ready, but as you understand, I may not share them with you.

SMO 7 problem 5 (A. Soifer, 1990). Prove that no matter how the plane is colored in two colors, it contains:

- (A) a triangle with the smallest side 1, angles in the ratio 1: 2: 3, and all vertices colored in the same color;
- (B) a triangle with the smallest side 1, angles in the ratio 1: 2: 4, and all vertices colored in the same color.

Proof of B. Assume that such a triangle does not exist in a 2-colored red and blue plane. Toss a regular 7-gon H of side length one on the plane (see Figure 1). Observe that many

triples of vertices of H , for example, ABD, BAF, DFC, DFG, GAC form a triangle of precisely the shape described in the problem: a triangle with the smallest side 1 and angles in the ratio 1:2:4.

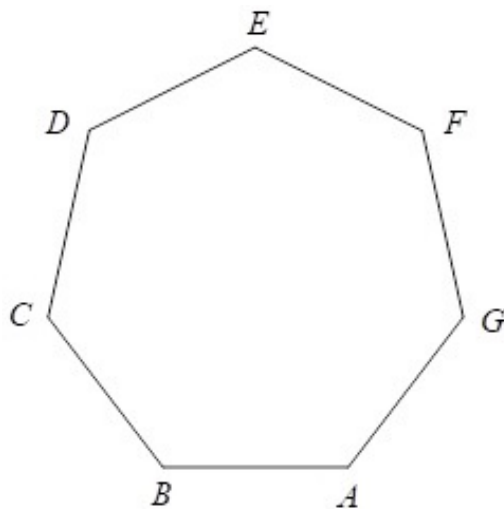


Figure 2

SMO 9, problem 2 (A. Soifer, 1991).

Prove that no matter how the vertices of a regular 1991-gon are colored in two colors, there are three vertices of the same color that form an isosceles triangle.

Cecil Rousseau created problem 1 for **1976 USAMO**. He was coloring unit squares of the 4×7 square grid. I translated his problem into the language of nodes of the square grid 3×6 , and then strengthened the result, thus creating the following A-B problem.

SMO 8, problem 5 (A. Soifer, 1991).

- (A) Prove that no matter how the plane is colored in two colors, it contains an $m \times n$ rectangle with all four vertices in points of the same color, such that $m = 1$ or 2, and n is a positive integer not greater than 6.
- (B) Prove the same statement, with n not greater than 5.

Just like Augustus De Morgan kept the flaming torch of 4CC alive, so did Paul Erdős for CNP (chromatic number of the plane).

Today

Paul Erdős posed a problem of finding, what I termed polychromatic number of the plane, PCN.

Find the smallest number of colors $\chi_p(E^2)$ for coloring the plane, so that no color realizes all distances. (Each color may, but not required, to have its own forbidden distance for monochromatic pairs of points). Two high schoolers, Dmitry Raiskii and Alexei Merkov, independently proved the following result in a style reminiscent of best Olympiad solutions.

The Raiskii Theorem. $\chi_p(E^2) \geq 4$.

Proof by Alexei Merkov. Assume the plane is colored in three colors, red, white and blue, but each color forbids a distance: r, w , and b respectively. Equip the 3-colored plane with the Cartesian coordinates with the origin O , and construct in the plane three 7-point sets S_r, S_w and S_b each being the Mosers Spindle (see figure 3), such that all spindles share O as one of their seven vertices, and have edges all equal to r, w , and b respectively. This construction defines 6 “red” vectors $v_1, \dots, v_6$ from the origin O to each remaining point of S_r ; 6 “white” vectors $v_7, \dots, v_{12}$ from O to the points of S_w ; and 6 “blue” vectors $v_{13}, \dots, v_{18}$ from O to the points of S_b – 18 vectors in all.

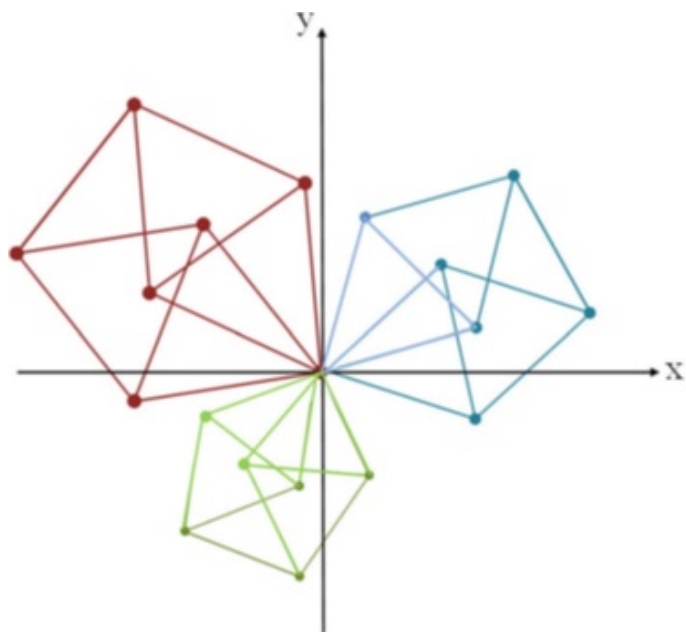


Figure 3: The Cartesian plane with three Mosers Spindles.

Introduce now the 18-dimensional Euclidean space $\mathbb{R}^{18}$ and a function M from $\mathbb{R}^{18}$ to the plane $\mathbb{R}^2$ naturally defined as follows: $(a_1, \dots, a_{18}) \mapsto a_1v_1 + \dots + a_{18}v_{18}$. This function induces a 3-coloring of $\mathbb{R}^{18}$ by assigning a point of $\mathbb{R}^{18}$ the color of the corresponding point of the plane. The first six axes of $\mathbb{R}^{18}$ we will call “red”, the next 6 axes “white”, and the last 6 axes “blue.”

Define by W the subset in $\mathbb{R}^{18}$ of all points whose coordinates include at most one coordinate equal to 1 for each of the three colors of the axes, and the rest (15 or more) coordinates 0. It is easy to verify (do) that W consists of 73 points. For any fixed array of allowable in W coordinates on white and blue axes, we get the 7-element set A of points in W having these fixed coordinates on white and blue axes. The image $M(A)$ of the set A under the map M forms in the plane a translation of the original 7-point set S_r . If we fix another array of white and blue coordinates, we would get another 7-element set in $\mathbb{R}^{18}$, whose image under M would form in the plane another translation of S_r . Thus, the set W gets partitioned into 72 subsets, each of which maps into a translate of S_r .

Now recall the observation we made after the first solution of problem 2.2 in section

2. It implies here that any translate of the Mosers Spindle S_r contains at most 2 red points out of its 7 points. Since the set W has been partitioned into the translates of S_r , at most $2/7$ of the points of W are red. We can start all over again, and in a similar way show that at most $2/7$ of the points of W are white, and similarly to show that at most $2/7$ of the points of W are blue. But $2/7 + 2/7 + 2/7$ does not add up to 1! This contradiction implies that at least one of the colors realizes all distances, as required. ■

In 2007, I left Princeton and Rutgers Universities after a productive 3-year visit. Two years later, they called me back to create a 3-day international workshop at Princeton-Rutgers-et al Center for Discrete Mathematics and Theoretical Computer Science, DIMACS. I named the workshop “Ramsey Theory Yesterday, Today, and Tomorrow.” 30 brilliant minds attended, from great graduate students of Berkeley and Princeton to the leaders of the field Ronald L. Graham, Joel Spencer, and Jaroslav Nešetřil. I proposed to determine the chromatic number of the plane by democratic means: the Berkeley student chose 4, I voted for 7, and all others equally split between 5 and 6. Thus, I declared the chromatic number to be 5.5. :-)

In April 2018, I received an email from Aubrey de Grey, a gerontologist, Ph.D. in Biology from Cambridge University, with the first breakthrough in the main problem in 68 years:

Aubrey D.N.J. de Grey’s Breakthrough. $\chi(E^2) \geq 5$.

He constructs a unit-distance graph on 20,425 vertices. Can one check whether it is 4-colorable? Normally, one would not even try, especially when you do not have supercomputers, but ordinary “household” means: Mathematica 11 on a standard MacBook Air laptop. De Grey bravely goes for it, and with a clever use of specific properties of his graph succeeds in verifying that this giant graph is indeed not 4-colorable. Hence, we arrive at The de Grey Theorem.

De Grey then dramatically reduces the size of his 5-chromatic unit-distance graph.

De Grey’s Example. There is a 5-chromatic unit-distance graph on 1581 vertices.

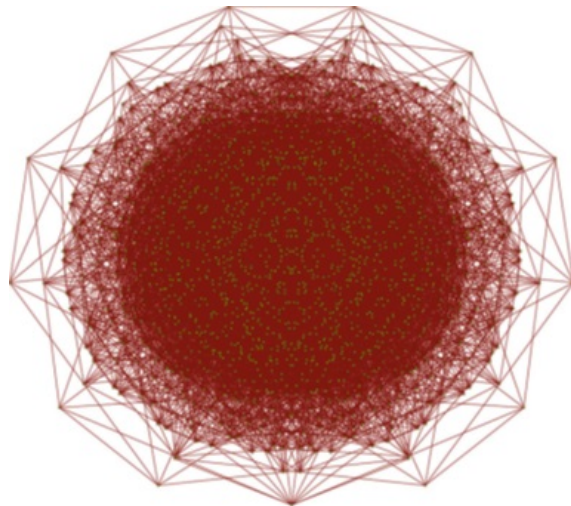


Figure 4: The 1581-vertex 5-chromatic unit-distance De Grey Graph.

With this size, the verification becomes within the reach of his computer program.

Aubrey's remarkable result changed everyone's mind: all started believing that $\chi(E^2) = 7$. However, over a few years, the excitement subsided because it became clear that proving even $\chi(E^2) \geq 6$ is out of reach at the moment.

The present record of the smallest number of vertices in 5-chromatic UDG belongs to Jaan Parts, Kazan, Russia: it is a 5-chromatic unit-distance graph on 509 vertices (Figure 5).

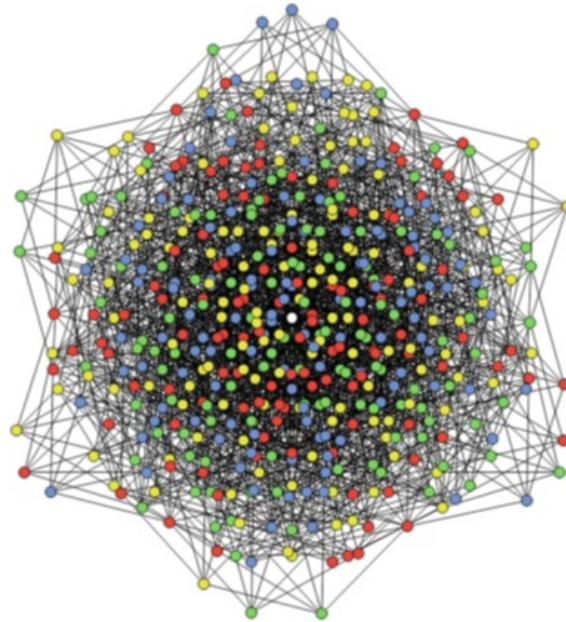


Figure 5: 5-chromatic unit-distance graph on 509 vertices with 2442 edges

Tomorrow

In 2002, I announced the following 3 conjectures, which remain open today, 24 years later.

Chromatic Number of the Plane Conjecture. $\chi(E^2) = 7$.

Chromatic Number of 3-Space Conjecture. $\chi(E^3) = 15$.

General Chromatic Number of Euclidean n-space Conjecture. For any positive $n > 1$, $\chi(E^n) = 2^{n+1} - 1$.

I think it will take circa 300 years to prove the general conjecture, and circa 100 to disprove it. We shall see, as Erdős used to say.

As a possible step toward the chromatic number of the plane, it may make sense to study graphs with two forbidden distances instead of 1. Let us call a graph G a *two-distance graph* and denote it by $G(1, d)$, $1 < d$, if its vertices are a distance 1 or a distance d apart. You understand, of course, that we do not need to use analogously defined $G(c, d)$ because a scaling by c brings the smaller distance to 1. In *coloring a two-distance graph*, we forbid monochromatic pairs of distance 1 and distance d . The *Chromatic number of a two-distance graph* G is the minimum number of colors assigned to the vertices of G that forbids a monochromatic unit and a monochromatic d . Jeffrey Exoo and Dan Ismailescu constructed a few 6-chromatic two-distance graphs. Matthew Huxtable constructed one such graph first. I hope you will enjoy his beautiful, Olympiad-like solution.

Matthew Huddleston Result (2008). There exists a two-distance graph with the vertex set E^2 with the chromatic number at least 6:

$$\chi\left(E^2\left\{1, \frac{1+\sqrt{5}}{2}\right\}\right) \geq 6.$$

Proof. The forbidden two distances should remind you of a unit regular pentagon (Figure 6), in which all pairwise vertex distances are our forbidden 1 and $\frac{1+\sqrt{5}}{2}$. Of course, the 5 vertices of this pentagon must be assigned 5 different colors.

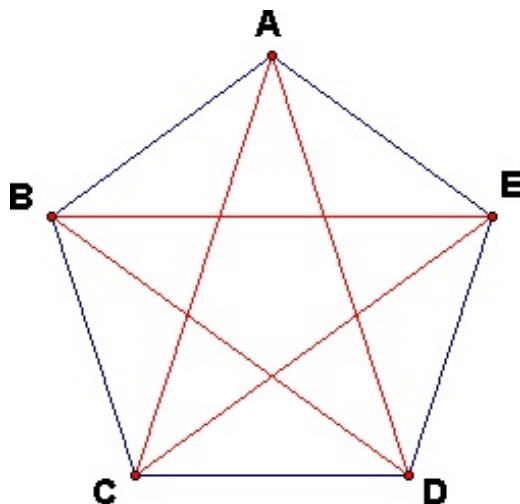


Figure 6

Assume the plane is colored in 5 colors without creating monochromatic pairs of distance 1 and $\frac{1+\sqrt{5}}{2}$. Let S be the set of five points forming the vertices of a regular pentagon of side 1. Let Q be the set of ordered quintuples chosen from S , so that Q has 55 elements. Color Q by assigning to each of its 5-tuples the color of the sum of its five entries in the original coloring of the plane. For any 4-tuple of points in S , note that adding the sum of all its entries to each of the five points of S produces a regular pentagon of side 1, so these five points have different colors. Therefore, each of the five colors is assigned to 54 of the 55 elements of Q . On the other hand, permuting an ordered 5-tuple in Q cannot change its color. The number of permutations of a given quintuple is a multinomial coefficient of the form

$$\binom{5}{a_1, a_2, a_3, a_4, a_5}$$

where a_j is the number of occurrences in the quintuple of the j th element of S ; $a_1 + a_2 + a_3 + a_4 + a_5 = 5$. This multinomial coefficient is a multiple of 5 except for the cases in which one of the a_j is 5 and the rest are 0. In order for the sizes of all the color classes in Q to be multiples of 5, these 5 exceptional cases must all be assigned the same color. Equivalently, the points of $5S$ all have the same color. This shows that in any regular pentagon with side 5, all vertices have the same color, so that any two vertices with distance 5 have same color. Thus, an isosceles triangle with sides of length 5, 5, 1 has vertices of the same color, and we get a monochromatic pair at the distance 1 apart, and that is a contradiction. ■

Two-Distance Conjecture (Soifer, 2002 [1]). There is a two-distance graph of chromatic number 7.

Hard Two-Distance Conjecture (Soifer, 2026). There a two-distance graph of chromatic number 8.

Open Problem (Soifer, 2022). Over all $d > 1$, find in the Euclidean plane a two-distance graph $G(1, d)$ of maximum chromatic number $\chi(G) = \Psi$. What are the values of d that realize Ψ ?

References

- [1] A. Soifer. *The New Mathematical Book: Mathematics of Coloring and the Colorful Life of Its Creators*. Springer 2024.

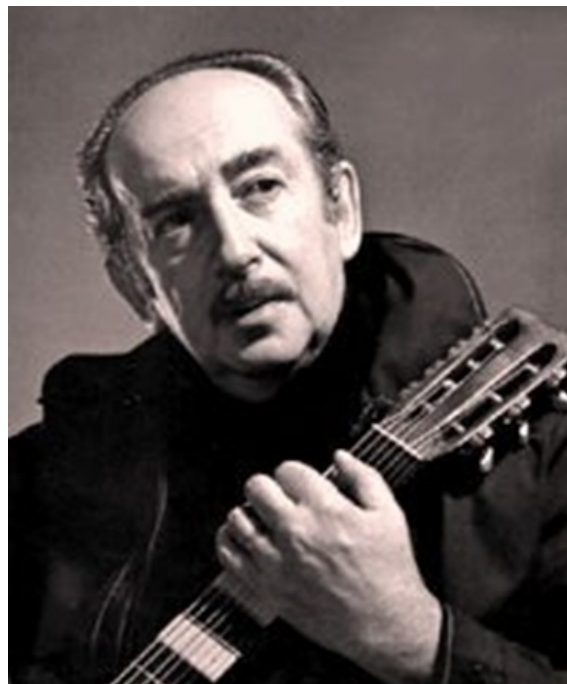
Farewell, or One Popular Belief

A popular belief teaches us, Silence is golden.

In the British Isles, this popular belief is visualized as follows: Closed mouth catches no flies.

This folk wisdom calls for safety, for not speaking truth to power, for power may retaliate. I respectfully disagree with this popular belief. We are responsible not only for what we do, but also for what we don't do; responsible not only for what we say, but also for our silence. Moreover, silence in the face of injustice makes us accomplices.

In my opinion, silence will not conceal truth forever. Truth, like water, will find its way out. Alexander Galich, a dissident poet, songwriter, performer, screenwriter, and playwright writes about futility of covering up the truth in his poem "Kadysh." I am translating a few lines from his Russian original.



“It’s time,” one day said noble prince,
“To overpaint this dirt.”
The painter said: “It’s time, my prince,
Long overdue, my lord.”
And dirt became all dirty-white,
And dirt became all dirty-green,
And dirt became all dirty-blue
Under the painter’s brush.
It all because the dirt is dirt,
Whatever color you insert.

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A Tablet and a problem

Don Barry



Don Barry started his teaching career in 1973 in a dual language school for Turkish students in Tarsus, Turkey. He created a math contest for those schools that were teaching math in English because he believed/believes strongly that math contests really stimulate interest, creativity, and a very satisfying experience when a problem is solved. He taught in Turkey for seven years and then at Phillips Andover in Massachusetts for 34 years. For the last 52 years he has written math contest problems for quite a variety of contests, some of which involved a 35-year commitment. He was the head author of the American Regions Math League (ARML) contest from 1995 to 2008. He is still writing problems, this time for the Math Madness contest.

Background

I love the early history of math, particularly that of Babylon around 1800 BCE. My study of tablets there led to the creation of the tablet in this problem. It was to fill in the 10 days after the Advanced Placement calculus test and it worked beautifully as the students had the divide into groups to discover what the number base was, whether the numbers were written from left to right or right to left, and since it was mathematics, they assumed there had to be a pattern or sequence that would lead them to complete the missing part of the tablet. That proved difficult and led to lots of interesting discussions.

Introduction

In [1], Colin Renfrew argues that the mother tongue of the Indo-European languages originated in the agricultural communities that developed in central Anatolia around 8000 BCE. The spread of language and its differentiation were concomitant with the spread of agriculture since families, not just agricultural techniques, were moving into virgin lands. As luck would have it², not far from the well-known Neolithic village of Çatal Hüyük in south central Turkey, near the village of Olmazköy, by the banks of Imkânsız dere, a shepherd recently discovered a cave filled with ancient clay tablets. The writing on the tablets represents neither Hurrian, Hattic, Phrygian, Minoan Linear A, nor any other known ancient language. It may be that it represents the recently sought mother tongue of all Indo-European languages.

²such a phrase should signify to the discriminating reader that what follows is a flight of fancy

The Tablet and problem

Some of the tablets seem to contain mathematical inscriptions. Shown below is a picture and a grave rubbing of what is, perhaps, the most interesting and exciting tablet. It consists of 3 columns of 7 lines of what must be numerical information. Unfortunately, the tablet is broken off and the other part has not yet been recovered. It is hoped that the missing part may still be found in the cave, but it is equally likely that by now it is a bookend in some swank New York City apartment. If this tablet is of the same general size as the rest, we have about two-thirds of the tablet.



Figure 7: Mathematical tablet

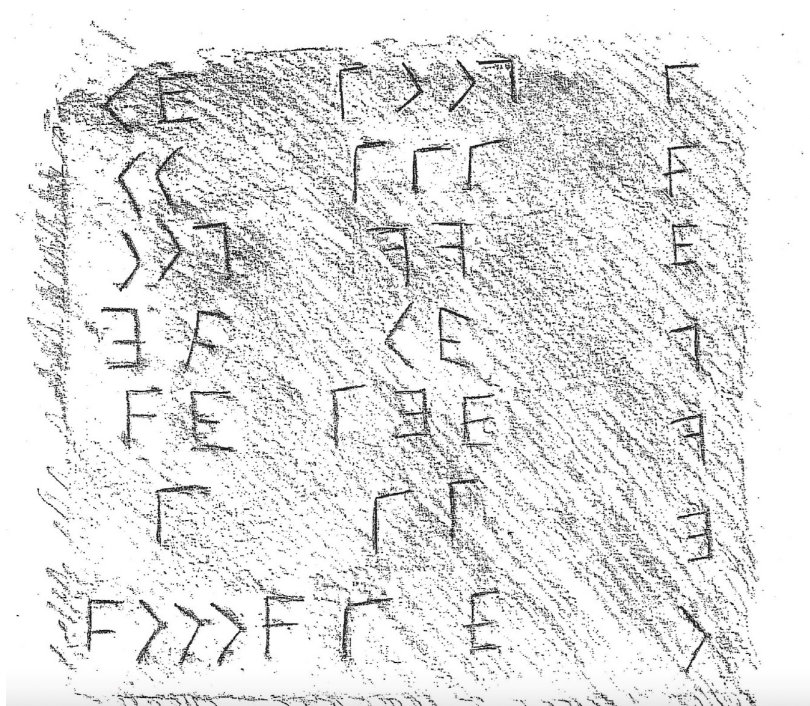


Figure 8: Tablet grave rubbing

Your task, should you choose to accept it, is to decipher the tablet. Determine the numbers in the columns and rows and develop an interpretation of the numerical information. Based

on your interpretation, develop an argument concerning the contents of the remaining rows of the tablet³.

Mathematical Poem

Which of the two following is more beautiful to you?

$$\frac{12 + 144 + 20 + 3\sqrt{4}}{7} + 9 \times 11 = 9^2 + 0$$

which translates to

A dozen, a gross and a score
Plus three times the square root of four
Divided by seven
Plus nine times eleven
Is nine squared and not a bit more

References

- [1] C. Renfrew. *Archaeology and Language: The Puzzle of Indo-European Origins*. Jonathan Cape 1987.

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³sometimes an ancient Babylonian tablet would be a student copy of a master's work and might include mistakes which have to be dealt with

Problems with a Mouse: A Selection from *Delta*

Dominik Burek



Dominik Burek is an assistant professor at Mathematics Institute, Jagiellonian University. He works in algebraic geometry, with a particular interest in Calabi-Yau varieties. He has served as a member of the Main Committee of the Polish Mathematical Olympiad and as the Leader of the Polish team at the International Mathematical Olympiad.

He is also an active popularizer of mathematics and an experienced trainer of mathematical olympiad students. Over the years, he has mentored several dozen finalists, laureates, and medalists of national and international mathematical olympiads. He has authored more than one hundred problems for national Mathematical Olympiad competitions and international mathematical olympiads, including the International Mathematical Olympiad.

Dominik Burek has collaborated with many countries, including the United States, Saudi Arabia, Uzbekistan, and Pakistan, supporting their preparations for national and international olympiad competitions.

Abstract

For more than fifty years, the Polish popular-science monthly *Delta* has published short mathematical problems for young readers. One of its regular columns is known in Polish as *Zadania z myszką*, literally “Problems with a Mouse”. This article presents a small selection of problems from recent years, prepared by the author for the column, together with solutions. The problems are intended to illustrate a particular style of elementary problem solving: short statements, modest technical prerequisites, and solutions usually based on one decisive idea.

History of the Column

Delta first appeared in January 1974 as a monthly magazine devoted to mathematics, physics, astronomy, and later also computer science. Its first editor-in-chief was Marek Kordos, who held this position until 2019. From the beginning it combined expository articles with problem sections addressed to students, teachers, and mathematically curious readers.

One of its long-running mathematical features is the column known in Polish as *Zadania z myszką*. The title is not completely straightforward to translate. Literally, *myszka* means “a little mouse”, but in Polish the phrase also suggests something with a slightly old-fashioned charm or a distinctive flavour. This ambiguity suits the column rather well.

The first editor of the column was Andrzej Mąkowski (1937–2007), a figure of great importance for the Polish mathematical community, in particular for the Polish Mathematical

Olympiad. He served on the Main Committee of the Olympiad for 48 years and is remembered not only for his mathematical taste, but also for his exceptional care for language and clarity of exposition. This aspect of his work is commemorated by the Andrzej Mąkowski Prize, awarded by the Polish Mathematical Olympiad for the best-written correct solution to a problem from the national final. In this sense, the spirit of the column is closely connected with a broader tradition of valuing not only the idea of a solution, but also the way in which that idea is communicated.

The format of *Zadania z myślką* is deliberately compact. Each issue contains three short problems, and the solutions are published in the same issue. The column is therefore not a competition in the formal sense. It is closer to a monthly invitation to problem solving. A typical problem should be easy to state and should not require advanced theory; at the same time, its solution should contain one recognizable idea.

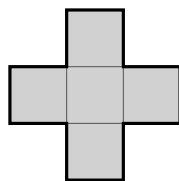
Over the years the column has been edited by several mathematicians, including Waldemar Pompe, Krzysztof Nowiński, Andrzej Mąkowski, Tomasz Tkocz, Rafał Sztencel, Łukasz Wiechecki, Krzysztof Oleszkiewicz, Łukasz Bożyk, Paweł Strzelecki, and Mikołaj Rotkiewicz, among others. Since 2021 I have had the pleasure of continuing this series. In selecting problems for the present article, I have tried to preserve the original character of the column. The examples below are mostly “one-level” problems: they usually do not require a long chain of reductions, but rather one trick, observation, or change of point of view.

The problems are presented first, followed by their solutions. They were chosen not as the most difficult examples, but as a representative sample of the column’s style. The label M followed by a number, such as M1822, is the sequential identifier of a mathematical problem published in *Delta*; more than 1,800 such problems have appeared to date!

Problems

Below we present a few selected problems from *Zadania z myślką*. We encourage the reader to spend some time on them independently before turning to the solutions. In most cases the intended solution is quite short, but it may depend on noticing the right point of view.

Problem 1. (*Delta*, July 2025, M1822.) Determine the greatest possible number of pairwise disjoint plus-shaped pentominoes that can be placed on an 8×8 chessboard, with the sides of the pentominoes parallel to the sides of the board. Here a plus-shaped pentomino consists of one unit square together with its four side-neighbours.



Problem 2. (*Delta*, December 2025, M1839.) Let $\omega(n)$ denote the number of distinct prime divisors of n . Positive integers a, b, c are given. Prove that there exists a positive integer n such that

$$\omega(an + c) \geq \omega(bn + c).$$

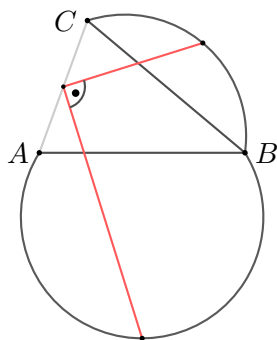
Problem 3. (*Delta*, November 2021, M1692.) Let a, b, c be distinct positive integers such that the numbers

$$\frac{1 + bc}{b - c}, \quad \frac{1 + ca}{c - a}, \quad \frac{1 + ab}{a - b}$$

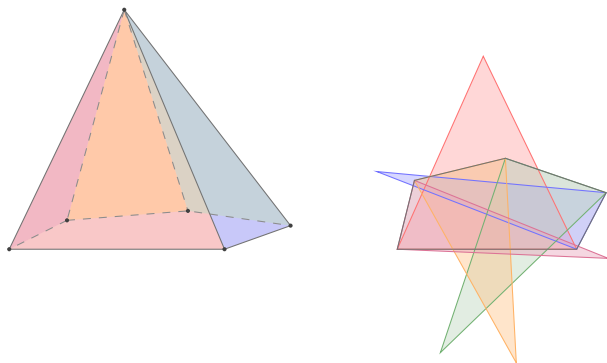
are integers. Prove that their greatest common divisor is equal to 1.

Problem 4. (*Delta*, January 2023, M1733.) Let p_n denote the n -th prime number. The primes $2, 3, 5, \dots, p_n$ are divided into two sets, and the products of the numbers in the two sets are denoted by a and b ; if one of the sets is empty, its product is taken to be 1. Prove that if $a + b < p_{n+1}^2$, then $a + b$ is prime.

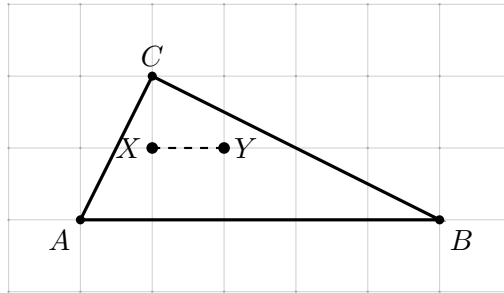
Problem 5. (*Delta*, August 2022, M1717.) A chord AB divides a circle into two circular segments. One of them is rotated about the point B , and under this rotation the image of A is C . Prove that the segments joining the midpoints of the two resulting arcs with the midpoint of AC are perpendicular.



Problem 6. (*Delta*, February 2024, M1773.) Let $SA_1A_2 \dots A_n$ be a pyramid whose base is the convex polygon $A_1A_2 \dots A_n$. For each $i = 1, 2, \dots, n$, in the plane of the base we construct a triangle $X_iA_iA_{i+1}$ congruent to the triangle SA_iA_{i+1} and lying on the same side of the line A_iA_{i+1} as the base; we put $A_{n+1} = A_1$. Prove that the constructed triangles cover the whole base.



Problem 7. (*Delta*, November 2025, M1835.) A point in the plane is called a lattice point if both its coordinates are integers. Consider a triangle with lattice-point vertices that contains exactly two lattice points in its interior. Prove that the line joining these two interior lattice points passes through a vertex of the triangle or is parallel to one of its sides.



Problem 8. (*Delta*, December 2021, M1694.) Let $a_1, a_2, \dots, a_n$ be positive real numbers such that

$$a_1^2 + 2a_2^3 + \dots + na_n^{n+1} < 1.$$

Prove that

$$2a_1 + 3a_2^2 + \dots + (n+1)a_n^n < 3.$$

Problem 9. (*Delta*, March 2024, M1776.) Real numbers a, b, c satisfy

$$\left| \frac{a^2 + b^2 - c^2}{ab} \right| < 2.$$

Prove that

$$\left| \frac{b^2 + c^2 - a^2}{bc} \right| < 2 \quad \text{and} \quad \left| \frac{c^2 + a^2 - b^2}{ca} \right| < 2.$$

Solutions

Problem 1. — *Solution.*

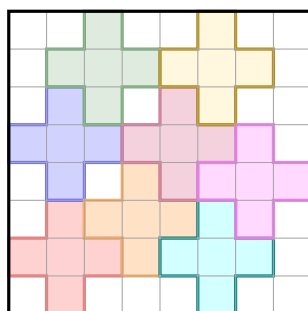
The answer is 8.

First note that the pentominoes can occupy at most two squares in the bottom row. Indeed, each plus-shaped pentomino can occupy at most one square in this row; if three squares in the bottom row were occupied, then the row above it would contain at least nine occupied squares, which is impossible.

The same argument applies to each side of the board. Hence at most

$$6^2 + 4 \cdot 2 = 44$$

squares can be occupied: 36 interior squares and at most two squares on each side. Since each pentomino has area 5, there can be at most $\left\lfloor \frac{44}{5} \right\rfloor = 8$ pentominoes. Conversely, an arrangement of eight such pentominoes is easily drawn, so the maximum is 8.



Problem 2. — *Solution.*

Suppose to the contrary that

$$\omega(an + c) < \omega(bn + c)$$

for every positive integer n .

Fix a positive integer k . Substituting successively

$$n = a^{k-1}, a^{k-2}b, a^{k-3}b^2, \dots, ab^{k-2}, b^{k-1}$$

gives the chain of inequalities

$$\omega(a^k + c) < \omega(a^{k-1}b + c) < \dots < \omega(b^k + c).$$

Therefore $\omega(b^k + c) \geq k + 1$. Thus $b^k + c$ has at least $k + 1$ distinct prime divisors, so it is greater than the product of the first $k + 1$ primes, and in particular greater than $k!$.

On the other hand, for sufficiently large k we have

$$b^k + c < b^k + c^k < (b + c)^k,$$

whereas $\sqrt[k]{k!} \rightarrow \infty$. Hence, $k! > (b + c)^k$ for large k , a contradiction.

Problem 3. — *Solution.*

Set

$$x = \frac{1+bc}{b-c}, \quad y = \frac{1+ca}{c-a}, \quad z = \frac{1+ab}{a-b}.$$

A direct calculation gives

$$xy + yz + zx = 1.$$

Since x, y, z are integers, any common divisor of them divides the left-hand side, and hence divides 1. Their greatest common divisor is therefore 1.

One way to see the displayed identity is to write $a = \tan \alpha$, $b = \tan \beta$, $c = \tan \gamma$, with $\alpha, \beta, \gamma \in (0, \pi/2)$. Then

$$\frac{a-b}{1+ab} = \tan(\alpha - \beta),$$

and similarly for the other two expressions. Since $(\alpha - \beta) + (\beta - \gamma) + (\gamma - \alpha) = 0$, the tangent addition formula gives the desired identity after taking reciprocals.

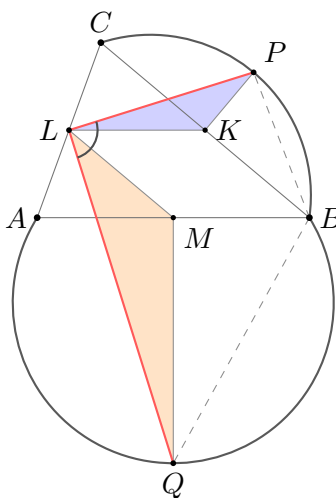
Problem 4. — *Solution.*

None of the primes $2, 3, \dots, p_n$ divides $a + b$, because each of them divides exactly one of the numbers a and b . Therefore every prime divisor of $a + b$ is at least p_{n+1} .

If $a + b$ had at least two prime divisors, counted with multiplicity, then $a + b \geq p_{n+1}^2$, contrary to the assumption. Hence $a + b$ has only one prime divisor and is itself prime.

Problem 5. — *Solution.*

Let P and Q be the midpoints of the two resulting arcs, and let K, L, M be the midpoints of BC, CA, AB , respectively.



By the midpoint theorem in the relevant triangles,

$$\angle PKL = 90^\circ + \angle CBA = 90^\circ + \angle LMA = \angle LMQ,$$

and also

$$LK = \frac{1}{2}AB = \frac{1}{2}BC = LM.$$

Moreover,

$$\angle KPB = \frac{1}{2}\angle CPB = \frac{1}{2}(180^\circ - \angle BQA) = \angle MBQ,$$

so the right triangles BKP and QMB are similar. Hence

$$\frac{KP}{KL} = \frac{KB}{MB} = \frac{KB}{MQ} = \frac{ML}{MQ}.$$

Together with $\angle PKL = \angle LMQ$, this shows that the triangles PKL and LMQ are similar. In particular $\angle QLM = \angle LPK$, and therefore

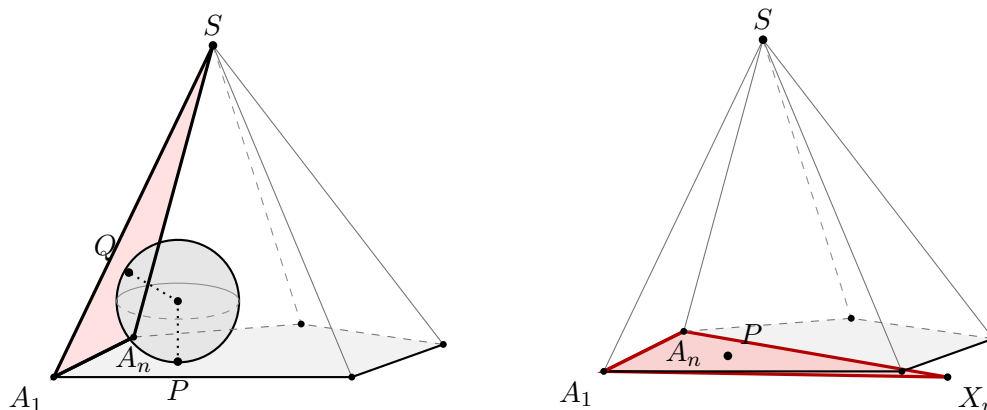
$$\angle QLP = \angle QLM + \angle MLK + \angle KLP = \angle QLM + \angle LMA + \angle MQL = 90^\circ.$$

Thus the two required segments are perpendicular.

Problem 6. — *Solution.*

Take any point P in the base. We prove that it is covered by one of the constructed triangles.

Assume first that P lies in the interior of the base. Consider a sphere inside the pyramid, tangent to the base at P . Increase its radius, keeping the point of tangency with the base fixed, until the sphere first becomes tangent to one of the lateral faces; suppose this face is SA_nA_1 , and let Q be the point of tangency.



The tangent segments from A_1 to the sphere are equal, and so are the tangent segments from A_n . Therefore

$$PA_1 = QA_1, \quad PA_n = QA_n.$$

Thus the triangles PA_nA_1 and QA_nA_1 are congruent. When the face SA_nA_1 is rotated about A_nA_1 into the plane of the base so that it coincides with the constructed triangle $X_nA_nA_1$, the point Q goes to P . Hence P lies in $X_nA_nA_1$. Since P was arbitrary, the constructed triangles cover the interior of the base, and therefore, by taking closures, they cover the whole base.

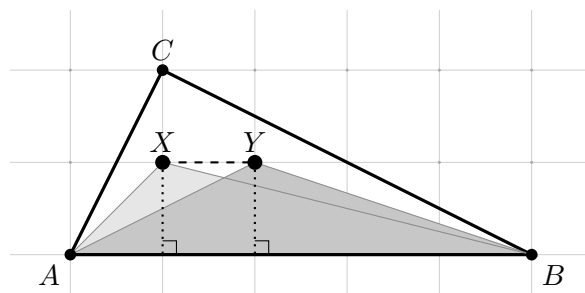
Problem 7. — *Solution.*

We use Pick's theorem: a lattice polygon has area

$$I + \frac{B}{2} - 1,$$

where I and B are the numbers of lattice points in its interior and on its boundary, respectively.

Let the triangle be ABC , and let X, Y be the two lattice points in its interior. Suppose that the line XY does not pass through any vertex of ABC . Then there is a side, say AB , which is not crossed by the line XY .



The triangles XAB and YAB have the same number of lattice points on their boundaries and no lattice points in their interiors. Indeed, the open segments AX , BX , AY , and BY contain no lattice points, since any such point would be an interior lattice point of ABC different from X and Y ; moreover, as the line XY does not cross the side AB , neither of the points X, Y lies in the interior of the triangle formed by the other point and the side AB . By Pick's theorem, their areas are equal. Since they have the same base AB , their altitudes to the line AB are equal. Hence XY is parallel to AB .

Problem 8. — *Solution.*

For each $k = 1, 2, \dots, n$, the AM-GM inequality applied to k copies of a_k^{k+1} and one copy of $\frac{1}{2^{k+1}}$ gives

$$\frac{1}{2}a_k^k \leq \frac{ka_k^{k+1} + \frac{1}{2^{k+1}}}{k+1}.$$

Equivalently,

$$(k+1)a_k^k \leq 2ka_k^{k+1} + \frac{1}{2^k}.$$

Summing these inequalities for $k = 1, \dots, n$ and using the assumption, we get

$$\sum_{k=1}^n (k+1)a_k^k \leq 2 \sum_{k=1}^n ka_k^{k+1} + \sum_{k=1}^n \frac{1}{2^k} < 2 \cdot 1 + 1 = 3.$$

Problem 9. — *Solution.*

The change of the sign of any of the numbers a, b, c does not change the desired inequalities. Also, from the assumption it follows that $ab \neq 0$ and $c \neq 0$. Thus, without loss of generality, we may assume that $a > 0$, $b > 0$, and $c > 0$.

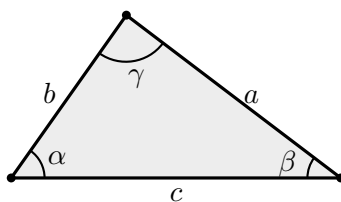
The first inequality can be written as

$$\left| \frac{a^2 + b^2 - c^2}{2ab} \right| < 1.$$

Hence there exists an angle $\gamma \in (0, \pi)$ such that

$$c^2 = a^2 + b^2 - 2ab \cos \gamma.$$

Construct a triangle with two sides a, b and included angle γ .



By the law of cosines its third side is c . If α, β are the angles opposite the sides a, b , respectively, then another application of the law of cosines gives

$$\left| \frac{b^2 + c^2 - a^2}{bc} \right| = 2|\cos \alpha| < 2$$

and

$$\left| \frac{c^2 + a^2 - b^2}{ca} \right| = 2|\cos \beta| < 2.$$

Acknowledgement

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Teamwork and Task Design in Mathematics Team Competitions: Students' Experiences and Organisers' Reflections

Michael Fischer, Stefan Ebner, David Präsent, Yasmin Theile, Robin Gludovatz, Leanka Gmoser & Alexander Slávik



Michael Fischer is a postdoctoral researcher in mathematics education at the University of Graz. Since 2023, he has been involved in organising team mathematics competitions in cooperation with Mathematikmacht Freude (MmF), with a particular focus on establishing and expanding these formats in Styria.



Stefan Ebner is a mathematics teacher at a middle school in Pöllau bei Hartberg in Styria. He conducted the questionnaire study at the three team competitions in 2024 and analysed parts in his master's thesis, which he wrote at the University of Graz.



David Präsent is a mathematics teacher at BRG Kepler in Graz and is involved in mathematics enrichment and competition activities. He has played a central role in organising and further developing numerous MmF team competitions in Styria.



Yasmin Theile is a doctoral researcher and research associate in mathematics education at the University of Cologne. Her research focuses on mathematical problem solving and teachers' diagnostic competencies, particularly the feedback they provide to students during problem-solving processes.



Robin Gludovatz is a mathematics teacher at the Albertus Magnus high school and is active in mathematics competition activities in Vienna. He is involved in MmF Vienna, including the organisation of Náboj, as well as in the Austrian *Kangaroo of Mathematics* competition.



Leanka Gmoser is a student at the University of Graz and an active member of the local mathematics outreach community. She plays an important role in the on-site organisation and delivery of mathematics competitions in Graz.



Alexander Slávik is a mathematician at Charles University in Prague, a mathematics teacher at Nový PORĚ High School, and a long-time organiser and board member of the Náboj competition. He has authored many of the competition's mathematical problems and led the preparation of its problem sets, as well as contributing to the organisation and development of the competition.

Abstract

This paper examines students' experiences and task-design implications in three mathematics team competitions with teams of two, three, and five students. Short post-competition questionnaires from 564 participants included open questions and number-line ratings of overall enjoyment and comfort in teamwork. Explicit mentions of teamwork increased across the three formats. Task-related responses showed that enjoyment was not simply associated with high solution rates. Immediate correctness feedback was frequently valued in Náboj and also affected how students could evaluate individual problems. Analyses of selected tasks illustrate combined cognitive demands, productive designed obstacles, attractive but misleading short-cuts, and deliberately open task formats.

Introduction

Mathematics competitions are a well-established part of mathematical enrichment. Over more than a century, they have developed from local and national events into a broad international network involving students, teachers, organisers, universities, mathematical societies, and international associations [8, 9]. Traditionally, many prominent formats, such as Olympiads or the Mathematical Kangaroo, are organised as individual competitions. At the same time, the landscape of competitions has become increasingly diverse, including popular mass competitions, online formats [6], project-oriented competitions, and team-based formats [4].

Team competitions are particularly interesting because they shift mathematical problem solving from an individual to a collaborative setting [4]. In such formats, success does not depend only on individual mathematical knowledge, but also on communication, coordination, shared decision-making, and the ability to use the strengths of different team members. This is in line with Kenderov [8], who argues that teamwork has become increasingly important in modern mathematics and that mathematics team competitions may contribute to cultivating this ability early on.

At the same time, the quality of such competitions depends on the format, the setting, and the tasks themselves. Competition problems are not simply school exercises in a different environment. They need to be mathematically meaningful, accessible enough to invite participation, and challenging enough to create curiosity, surprise, and satisfaction when solved. Experts from the competition movement have described competition problems as serving several pedagogical goals: they should provide opportunities to engage with meaningful mathematics, strengthen positive attitudes towards mathematics, create cognitive challenges, and surprise students [5, 10]. The specific competition format and venue are also consequential for problem selection, the organisation of feedback, and the preparation of the team environment.

For organisers, this creates a design challenge. Team competitions must provide a positive collaborative experience, but they must also offer a sequence of tasks that sustains mathematical engagement under the constraints of the format, the available setting, and varying student compositions. This paper addresses both aspects. First, we investigate students' experiences in three team-based mathematics competitions with different team sizes. We focus on open responses and number-line ratings concerning enjoyment and teamwork. Second, we combine further open-question responses, task-level solution data, and exploratory analyses of submitted answers to consider feedback, anticipated solution paths, false trails, and designed obstacles in mathematics team competitions.

Theoretical background

Mathematics competitions can be understood as part of a broader ecosystem of mathematical enrichment. Different formats serve different purposes. Some competitions focus on identifying and developing highly talented students, while others aim to reach broader groups and strengthen positive experiences with mathematics [8, 9]. The present contribution examines team competitions primarily as settings in which students can experience mathematics as a collaborative and motivating activity.

Popular and inclusive competitions differ from advanced Olympiad-style competitions in important ways. Their problems need to be interesting and challenging, but they also need

to remain accessible to students who may not have received special preparation. Their primary function is not only to practise a known method or assess a learned concept, but to invite engagement, create cognitive challenge, and offer moments of surprise and satisfaction [5, 10]. In team competitions, this motivational function is embedded in a collaborative setting. Students do not merely solve problems alongside one another; they can discuss ideas, distribute work, compare approaches, and support each other in moments of uncertainty. Students' positive experiences may therefore emerge from at least two intertwined sources: the mathematical quality of the problems and the social process of solving them together.

Interest research offers an additional lens for this distinction. Interest can be directed not only towards a broad domain, but also towards a particular topic, context, or activity [1]. A student may therefore value not only mathematics in general, but also the specific activity of solving non-routine problems collaboratively in a competition. The present questionnaire measured enjoyment and comfort rather than interest in a strict theoretical sense; nevertheless, the distinction is useful when interpreting what exactly students experience positively in team competitions.

The design of tasks is central to both the mathematical and collaborative dimensions. Tasks can be analysed in terms of their mathematical content, required solution paths, cognitive demand, openness, linguistic complexity, and possible obstacles [15]. In competition settings, these features become especially important because time pressure, answer formats, task distribution, and scoring rules influence how students engage with a task. A problem may be mathematically elegant but still unsuitable for a particular competition if its expected solution path is too formal, too long, too dependent on hidden prerequisites, or too vulnerable to misleading interpretations.

This is particularly relevant for the analysis of students' answers and wrong answers. Research on multiple-choice tasks has shown that students may be attracted by superficial strategies or short-cuts, especially when task formulations or answer options make certain responses appear immediately plausible [2, 11]. Although the competitions analysed here used predominantly open numerical answer formats, similar phenomena can occur: students may follow attractive false trails, rely on salient but misleading cues, or settle on an immediate answer without sufficient verification [14].

Against this background, the contribution combines two perspectives. The empirical part examines whether and how students explicitly mention teamwork as part of what they enjoyed and how they rate their overall enjoyment and comfort in the team. The design-oriented part considers what students' feedback, solution patterns, and organisers' observations suggest about the selection and design of tasks for mathematical team competitions.

Methods

Competition formats

Data were collected using short paper-and-pencil questionnaires completed directly after one of three team-based mathematics competitions: Mathe-Mix-Doppel, m3, and Náboj. The formats differed not only in team size but also in age groups, team composition, duration, task distribution, and feedback structure (Table 1). Mathe-Mix-Doppel and m3 were conducted for the first time in the 2024 year of data collection, whereas Náboj was an

established competition with extensive experience in problem selection and testing. The m3 format was substantially revised after 2024; only the 2024 version is analysed here.

The data set comprised five event groups: Mathe-Mix-Doppel in Graz, m3 in Graz and Vienna, and Náboj in Graz and Vienna. Overall, 564 student questionnaires were included, with 172 responses from Mathe-Mix-Doppel, 152 from m3, and 240 from Náboj.

Table 1: Overview of the competition formats in the 2024 data collection. Immediate feedback means that a jury checked the correctness of an answer during the competition.

Competition	Grade	Team	Time	Tasks, answers, and feedback
Mathe-Mix-Doppel	5–6	2	45 min	Mandatory mixed-gender pairs; complete task set provided at the beginning; predominantly numerical answers; complete answer sheet submitted at the end; no immediate jury feedback.
m3 (2024)	5–8	3	60 min	Freely composed teams; complete task set provided at the beginning; predominantly numerical answers; complete answer sheet submitted at the end; no immediate jury feedback.
Náboj	9–13	5	120 min	Freely composed teams; six single problem sheets available concurrently, normally one copy per team; almost exclusively numerical answers submitted individually; immediate jury feedback.

Questionnaire and analysis

The questionnaire contained six open questions and two number-line items. Students rated how much they enjoyed the competition (Question 1) and how comfortable they felt when collaborating in their team (Question 7, the penultimate question). The open questions asked what they particularly enjoyed, which problem they particularly liked, which problem was too difficult, which problem was too easy, what disturbed them, and whether they wished to add anything else. Figure 9 shows the first rating item and the following open question.

The two number lines were intended to range from 0 to 10. The positions marked by students were measured and entered numerically. A small number of marks lay beyond the printed right endpoint; these were included using their measured positions, resulting in values slightly above 10 rather than being capped at 10.

The open responses were transferred to a spreadsheet and coded into inductively developed categories. For the first part of the present analysis, we focused on whether students explicitly mentioned teamwork as something they enjoyed. The coded categories were analysed descriptively; confidence intervals, a chi-square test, and an exploratory logistic regression were used to describe differences across competition formats. Questions about favourite, difficult, and easy problems were linked to the available task-level solution data. Finally, submitted Mathe-Mix-Doppel answer sheets were examined exploratively for recurring patterns in correct, incorrect, and missing responses.

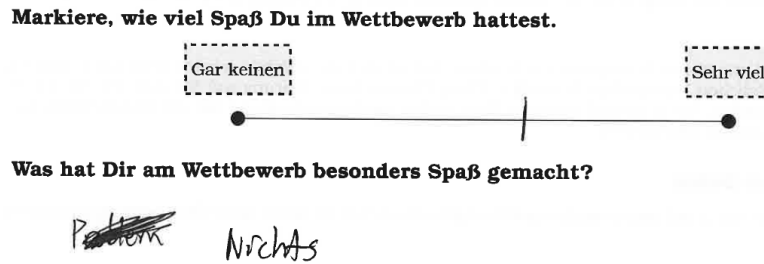


Figure 9: Excerpt from the student questionnaire. The upper item asked students to mark on a number line how much they enjoyed the competition. The lower item asked what they particularly enjoyed.

Results

Team size, teamwork, and enjoyment

In the open-ended question asking what students particularly enjoyed, we coded whether they explicitly mentioned teamwork. The proportion increased across the three settings (Table 2). In Mathe-Mix-Doppel, 32 of 172 students mentioned teamwork (18.6%). In m3, this was the case for 54 of 152 students (35.5%). In Náboj, 114 of 240 students mentioned teamwork (47.5%). Thus, teamwork became increasingly visible in students' open feedback from the two-student format to the three- and five-student formats. A chi-square test indicated that the frequency of mentioning teamwork differed significantly between the three competition formats, $\chi^2(2) = 36.55$, $p < .001$. The corresponding effect size was $V = .255$, indicating a small-to-moderate association between competition format and the explicit mention of teamwork.

Table 2: Team size and explicit mentions of teamwork in the open question on what students particularly enjoyed.

Competition	Team size	<i>n</i>	Teamwork mentioned	% (95% CI)
Mathe-Mix-Doppel	2	172	32	18.6 (13.2–25.4)
m3	3	152	54	35.5 (28.1–43.7)
Náboj	5	240	114	47.5 (41.1–54.0)

To further describe this pattern, we fitted an exploratory logistic regression model with team size as predictor and the mention of teamwork as binary outcome. Larger team formats were associated with higher odds of mentioning teamwork, $OR = 1.50$, 95% CI [1.31, 1.72]. Within this model, each additional team member was associated with approximately 50% higher odds of explicitly referring to teamwork. However, team size and competition format are confounded because each format has a fixed team size. The regression therefore does not isolate a causal effect of team size but visualises a format-related trend across the three competitions (Figure 10).

Students evaluated both the overall competition experience and the teamwork experience positively across all three formats. Mean ratings for comfort in teamwork were consistently higher than the corresponding ratings for enjoyment of the competition. Thus, the collaborative setting was not only frequently mentioned in the open responses, but also very

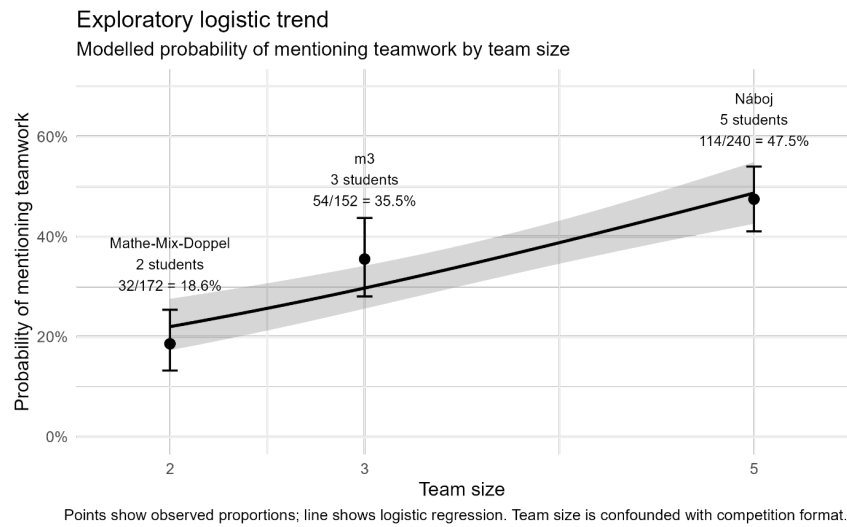


Figure 10: Proportion of students who explicitly mentioned teamwork as an enjoyable aspect, by team size and competition format. Points show observed proportions with 95% confidence intervals; the curve shows an exploratory logistic regression trend.

positively evaluated on the rating-scale item. These converging findings support the overall pattern that the team competitions were experienced positively. However, the rating-scale data alone do not isolate team size as the cause of those evaluations; observed differences should therefore be interpreted descriptively rather than as evidence of format-driven effects.

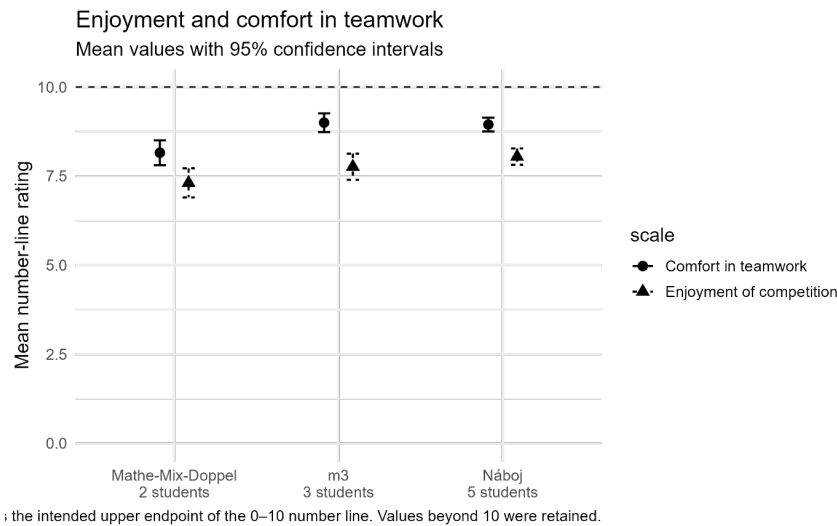


Figure 11: Mean ratings for enjoyment of the competition and comfort in teamwork across the three formats. Error bars represent 95% confidence intervals. Values beyond the intended upper endpoint of 10 were included at their measured positions.

In summary, teamwork was not merely an organisational feature but an important part of students' positive experiences. It was especially salient in the open feedback from the larger team formats, while the rating data indicate highly positive evaluations of collaboration across all three competitions.

Problem-related enjoyment and task feedback

Students' positive experiences were not limited to teamwork. In the open questions, they identified problems that they particularly enjoyed, considered too difficult, or regarded as too easy. The available solution data show that enjoyment was not simply determined by the probability of solving a problem. Some accessible problems were described both as easy and enjoyable, whereas several demanding problems were remembered positively despite comparatively low solution rates. Conversely, a number of problems with high success rates received comparatively few positive mentions. Figure 13 illustrates these patterns for Mathe-Mix-Doppel and Náboj.

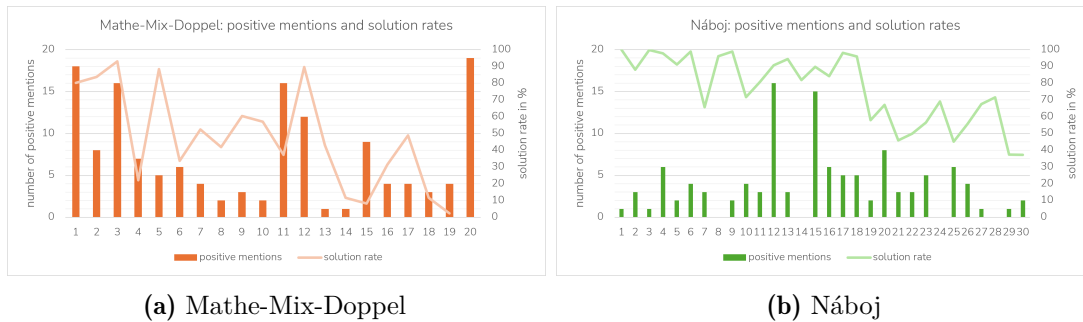


Figure 13: Positive mentions and solution rates for individual problems. Bars show how often each problem was identified as particularly enjoyable; lines show the proportion of teams that solved it successfully.

This was particularly visible in Náboj. Problems 20 and 25 received several positive mentions although their success rates were lower than those of many preceding problems. Only 45% of the teams that received Problem 25 solved it successfully, and successful teams needed approximately 32 minutes on average. Fourteen teams submitted the correct answer during the final two minutes. These data do not establish that greater difficulty necessarily produces greater enjoyment. They do, however, suggest that overcoming a substantial obstacle, especially shortly before the end, can become a memorable positive experience. The late position of these problems may also have increased their salience when the questionnaire was completed. Thus, an accessible early success and a difficult problem that is eventually solved may both contribute to enjoyment in different ways [5].

A second result is that students' evaluations cannot be separated entirely from the feedback structure. In the 2024 versions of Mathe-Mix-Doppel and m3, students submitted their complete answer sheets only at the end. When completing the questionnaire, they therefore did not necessarily know whether a problem they perceived as easy had actually been solved correctly. In Náboj, by contrast, teams submitted problems individually and immediately learned whether an answer was correct. After an incorrect submission, they could return to the problem; after a correct submission, they received a new one.

This feedback cycle was itself an important source of enjoyment. More than 15% of the coded positive mentions in the Náboj responses concerned feedback. Students referred to the feeling associated with a correct submission, appreciated immediately knowing when an answer was wrong, and valued the opportunity to continue working on it. Receiving a new problem after a successful solution was also mentioned positively. Consequently, the format does more than provide information about performance: it structures the emotional rhythm of the competition through repeated cycles of uncertainty, feedback, revision, success, and renewed challenge. Figure 15 gives two examples.

Was hat Dir am Wettbewerb besonders Spaß gemacht? Begeisterung aller Bewerberinnen für Mathe generell: Aufregung und das Gefühl, als die Aufgabe als richtig abgestempelt wurde Translation: "Everyone's enthusiasm for mathematics, the general excitement, and the feeling when a problem was marked as correct."	Was hat Dir am Wettbewerb besonders Spaß gemacht? Wenn man Aufgaben abgegeben hat und es geheißen hat, dass das Abgegebenene richtig war Translation: "When you submitted problems and were told that what you had submitted was correct."	
Was hat Dir am Wettbewerb besonders Spaß gemacht? Der Nervosität, wenn man ein Beispiel gelöst hat und noch keine Lösung ist Translation: "The thrill of solving a problem and going to the front to submit it."	Was hat Dir am Wettbewerb besonders Spaß gemacht? dass man direkt wusste, ob das Ergebnis falsch war Translation: "Knowing immediately whether the result was wrong."	Was hat Dir am Wettbewerb besonders Spaß gemacht? Das sofortige Feedback Translation: "The immediate feedback."
Was hat Dir am Wettbewerb besonders Spaß gemacht? Wenn die Aufgabe richtig war Translation: "When the answer to the problem was correct."	Was hat Dir am Wettbewerb besonders Spaß gemacht? die Freude, wenn etwas richtig ist Translation: "The joy when something is correct."	Was hat Dir am Wettbewerb besonders Spaß gemacht? Den Stempel kriegen Translation: "Getting the stamp."

Figure 15: Examples of students describing immediate feedback and the associated suspense, confirmation, and feelings of success as enjoyable aspects of Náboj. German transcriptions and English translations are provided below each response.

This has a methodological implication for interpreting task feedback. When Náboj students described a problem as enjoyable, they already knew whether they had solved it successfully. For Mathe-Mix-Doppel and m3, a positive evaluation could instead be based on a solution that merely appeared convincing. Comparisons of perceived difficulty and enjoyment must therefore take the availability of immediate correctness feedback into account. From a design perspective, organisers should consider not only which problems are posed, but also when and how teams learn whether their reasoning has succeeded.

Anticipated solution paths, false trails, and designed obstacles

The questionnaire data describe how students evaluated problems, but they do not provide complete records of their problem-solving processes. A complementary exploratory analysis of submitted Mathe-Mix-Doppel answers nevertheless revealed recurring patterns in wrong and missing responses [14]. Problems involving spatial reasoning or systematic counting showed comparatively low success rates for this age group. Formal and multi-step solution expectations also appeared to create obstacles, particularly when students had to translate a verbal situation into an equation, coordinate several dependent calculations, or verify intermediate results. These observations are compatible with previous findings on spatial reasoning and systematic combinatorial strategies [7, 12].

Combined spatial and counting demands. Mathe-Mix-Doppel Problem 6 illustrates how several demands can be hidden within a short and apparently accessible formulation (Figure 16). Students were shown a $4 \times 4 \times 4$ cube composed of smaller cubes. The large cube was painted on the outside, and teams had to determine how many small cubes had exactly two painted faces. Only 29 of the 86 teams answered correctly; 56 submitted an incorrect answer and one team left the problem unanswered.

The intended systematic solution focuses on the edges. Each of the 12 edges contains $4 - 2 = 2$ non-corner cubes with exactly two painted faces, giving $12 \cdot 2 = 24$. Corner cubes must be excluded because they have three painted faces. The task therefore combines spatial visualisation, a distinction between corner and non-corner cases, and systematic

6. Der große Würfel rechts daneben ist aus $4 \times 4 \times 4$ kleinen Würfeln zusammengesetzt.
 Der große Würfel wird außen gefärbt.
 Bei wie vielen kleinen Würfeln sind genau 2 Flächen angemalt?



English transcription:

The large cube shown on the left is composed of $4 \times 4 \times 4$ smaller cubes.

The large cube is painted on the outside.

How many of the small cubes have exactly two painted faces?

Figure 16: Mathe-Mix-Doppel Problem 6 on painted faces of a subdivided cube, shown with an English transcription.

counting. A solver who counts all cubes on the edges without distinguishing the corners obtains a plausible but incorrect result. Even a concise task may thus combine several mathematical activities whose joint difficulty can be underestimated when deciding on its relative placement in a competition.

A productive designed obstacle. An obstacle is not necessarily a weakness of a problem. A well-designed obstacle can interrupt an initially plausible approach and prompt further reasoning. Náboj Problem 12 provides an instructive example (Figure 17). Students had to maximise the number represented by *MATH* in

$$2024 + H A H A = M A T H,$$

where different letters represented different digits. A natural first attempt is to make the thousands digit *M* as large as possible. This initially leads to $H A H A = 7373$ and hence $2024 + 7373 = 9397$. The result is impossible because *M* and *T* would both represent 9. The contradiction forces solvers to reconsider the column-wise addition and recognise that a carry from the units column is required. Testing the remaining possibilities then leads to the largest admissible result, $M A T H = 5963$.

Aufgabe 12. Hier sind *M*, *A*, *T* und *H* verschiedene Ziffern ungleich Null, welche die Gleichung

$$2024 + H A H A = M A T H$$

erfüllen. Bestimme den größtmöglichen Wert der vierstelligen Zahl *MATH*.

Figure 17: Náboj Problem 12, the cryptarithm $2024 + H A H A = M A T H$.

The problem was the most frequently mentioned favourite Náboj problem and had high recognition value: several students referred to it as the “HAHA and MATH problem” rather than by its number. Its attraction may lie in a simple entry point, a meaningful contradiction, and a second idea that uses familiar arithmetic in a non-routine way. The initial route is therefore not merely a trap; it is a productive obstacle whose failure reveals what must be reconsidered.

An attractive short-cut without an internal warning. A contrasting case is the m3 problem “Average Speed” (Figure 18). A journey was made at 20 km/h in one direction and 30 km/h in the other, and students were asked for the average speed. The arithmetic mean,

$$\frac{20 + 30}{2} = 25,$$

is immediately available but incorrect because the slower part of the equal-distance journey takes more time. The correct average is the harmonic mean,

$$\frac{2 \cdot 20 \cdot 30}{20 + 30} = 24 \text{ km/h.}$$

1.16

Fritz fährt mit dem Rad zu seiner Schule und wieder heim. Am Hinweg fuhr er mit konstant 20 km/h, am Rückweg mit konstant 30 km/h. Wie hoch war seine Durchschnittsgeschwindigkeit über die gesamte zurückgelegte Strecke?

English transcription:
 Fritz cycles to school and back home. On the way to school, he travels at a constant speed of 20 km/h, and on the way back, he travels at a constant speed of 30 km/h.
 What was his average speed over the entire distance travelled?

Figure 18: The m3 problem “Average Speed”, shown with an English transcription.

The problem was mentioned as particularly enjoyable by several students and was also the m3 problem most frequently described as too easy. Unfortunately, the students’ written solutions and incorrect answers were not available, so we cannot determine how frequently 25 km/h was submitted. Results from structurally similar tasks nevertheless illustrate how attractive the arithmetic-mean short-cut can be. In a nearly identical task used in the 2026 Styrian Schul5Kampf, a local competition involving students from gifted-education courses, 29 of 59 participants (49%) gave the arithmetic-mean answer, whereas 26 (44%) found the correct solution. In Problem 24 of the 2023 Austrian Mathematical Kangaroo competition in the Cadet category, 72% of the participants who attempted the task selected the arithmetic-mean distractor, while only 4.6% chose the correct average speed [3]. These comparisons do not establish the answer distribution in the m3 sample, but they make a consequential hypothesis plausible: some students’ perception of the problem as “too easy” may have been based on the positive feeling of having an immediate answer rather than on the successful *resolution* of the mathematical problem. Such short-cuts are not restricted to multiple-choice formats because students can generate them independently in open numerical-response tasks [2, 16].

A covert trap in an open answer format. Problem 15 of the 2024 Mathe-Mix-Doppel final offers a further perspective (Figure 19). In “Inverted Land”, integers are written with their tens and units digits exchanged, while the symbols +, ·, −, and : denote multiplication, addition, division, and subtraction, respectively. Students had to write the result of $27 : 45$ under these conventions. The algorithm – switch the operator, calculate, and switch the digits of the result – is not complex, but the final step was overlooked by most teams. In total, 78 of 86 teams submitted an incorrect answer, and almost all of these responses failed to reverse the digits of the computational result.

15. In Verkehrtland schreiben die Leute + wenn sie multiplizieren und · wenn sie addieren. Sie schreiben − wenn sie dividieren und : wenn sie subtrahieren. Deshalb ist dort zum Beispiel $4 + 2 = 8$. Zusätzlich vertauschen sie immer auch noch die Einerziffer mit der Zehnerziffer. Ihr findet in einem Schulbuch im Verkehrtland folgende Rechnung: $27 : 45 = ?$
Wie lautet das Ergebnis dieser Rechnung im Verkehrtland?

English transcription:
 In Inverted Land, people write + when they multiply and · when they add. They write − when they divide and : when they subtract. For example, in Inverted Land, $4 + 2 = 8$. In addition, they always interchange the units digit and the tens digit. In a schoolbook from Inverted Land, you find the following calculation: $27 : 45 = ?$
What is the result of this calculation in Inverted Land?

Figure 19: Mathe-Mix-Doppel Problem 15, set in “Inverted Land”, shown with an English transcription.

In the questionnaire, seven students identified the problem as particularly enjoyable, three as too difficult, and two as too easy. The memorable context itself may have contributed to these mentions. Nevertheless, the response pattern shows how effectively a final, easily

overlooked instruction can function as a covert trap. In a multiple-choice format, the co-appearance of the correct answer 81 and the tempting wrong answer 18 might prompt some students to reconsider their immediate result. Immediate feedback in an open format could similarly provide an opportunity for revision. These are hypotheses for comparative task-format research rather than conclusions supported by the present data.

Deliberate openness as a contrasting design choice. The final Mathe-Mix-Doppel task differs substantially from the preceding examples (Figure 20). It asked students to estimate the number of dimples on a golf ball without access to the physical object and served as a tie-breaker. Although there is a well-defined reference value, participants cannot derive it exactly from the information and resources available. In Yeo [17]’ framework, the problem can therefore be characterised as open primarily with respect to method: students may use different strategies, prior knowledge, or assumptions, while the goal and reference value remain well defined.

20. Schätzaufgabe:

Golfbälle haben auf der Oberfläche kleine Einkerbungen, damit die Flugbahn des Balles stabiler ist.

Wie viele solche Einkerbungen weist der Golfball der Marke „Wilson“ auf?



English transcription:

Estimation problem:

Golf balls have small dimples on their surface so that the ball’s flight path is more stable.

How many such dimples does the Wilson golf ball have?

Figure 20: The final Mathe-Mix-Doppel estimation task on the number of dimples on a golf ball, shown with an English transcription.

This task was mentioned more frequently than any other Mathe-Mix-Doppel problem as one that students particularly enjoyed. It departs from a common expectation of school mathematics that sufficient information and a prescribed toolkit lead to one exact answer. Here, exact determination is deliberately beyond the scope of the task, making estimation an essential part of the problem-solving process. The departure from conventional task structures may have contributed to its appeal.

Together, the examples broaden the design questions for competition problems. Some tasks combine demands that may be underestimated by designers, as in the painted-cube problem. Productive obstacles make an initial idea fail in an interpretable way and support a new insight, as in the cryptarithm. Unproductive false trails or covert traps may produce plausible answers without an internal reason to question them, as in the average-speed and Inverted-Land problems. Deliberately open tasks may, in turn, be appealing precisely because they change the expected mode of mathematical work. Problem setters should therefore ask not only whether a task has an elegant intended solution, but also which response is likely to occur first, what could make students question it, how feedback may alter the process, and whether the task sequence offers varied forms of mathematical activity.

Discussion and practical implications

The more, the merrier?

The results suggest that teamwork becomes more visible in students' feedback as team size increases. In the two-person format, fewer students explicitly mentioned teamwork as enjoyable, whereas in the three- and five-person formats it was mentioned much more often. Together with the high ratings for comfort in teamwork, this indicates that team competitions can offer more than individual problem solving in a shared room. For many students, discussing ideas, distributing work, comparing approaches, and celebrating correct solutions together were part of the positive experience.

However, larger teams also create additional didactical and organisational demands. For teams of three or more, the preparation and choice of the venue require particular care. Students must be able to sit around a common table, see and exchange task sheets, communicate without unduly disturbing neighbouring teams, and retain an overview of distributed work. In feedback-based formats such as Náboj, the placement of jury tables and the space required for movement become primary requirements that many rooms with fixed rows of seats cannot satisfy. A format designed for larger teams cannot simply be placed in an environment intended for individual or pair work.

The tasks and their sequence must also allow a productive balance between distribution and collaboration. Some problems clearly do not exhaust the potential of a larger team, such as a lengthy symbolic rearrangement best completed by one person without meaningful intermediate results. Yet individual work is not necessarily a weakness of the team format. Especially early in a competition, efficient teams may distribute relatively accessible problems according to members' strengths. Later or more complex problems may invite subgroups or whole-team collaboration, joint checking, drawing, estimating, and comparison of approaches. Thus, the practical question is not simply how large a team should be, but which team size, venue, task sequence, material distribution, and feedback structure fit together.

Team size and answer format

The descriptive trend in Figure 10 should be interpreted with caution. The three competitions differed not only in team size but also substantially in students' ages, team-composition rules, duration, stage of format development, task distribution, answer format, and feedback. Mathe-Mix-Doppel additionally required mixed-gender pairs, whereas team composition was voluntary in m3 and Náboj. The observed increase in explicit teamwork mentions therefore cannot be interpreted as a causal effect of team size alone; it is more appropriately understood as a format-related pattern across three real competitions.

The answer and material formats are particularly important. In Mathe-Mix-Doppel and m3, complete task sets were available from the beginning and answers were submitted together at the end. In Náboj, only six single problem sheets were available at a time, normally in one copy per team, and each answer was checked immediately. These differences may affect whether students work individually, form subgroups, or discuss a problem jointly. Immediate feedback creates repeated opportunities for revision and celebration. Additionally, the decision whether to continue working on a problem after an incorrect answer or to redirect team resources is an integral part of the team dynamic. In competitions without feedback, uncertainty may instead increase the importance of jointly validating the final answer

before submission.

Team size and answer format should consequently not be treated as independent features. A wrong answer in a feedback-based five-person format may lead to a redistribution of responsibilities and the generation of new perspectives; a two- or three-person team without feedback may deliberately share the final responsibility for validating an answer. The ways in which team momentum responds to an answer format are varied and cannot be deduced from the present comparison. While this is an important limitation, it is also a productive avenue for further research. Ongoing work examines mathematical problem solving in three-person teams, especially the communicative processes of planning attempts, distributing responsibilities, and validating intermediate and final results [13].

For organisers, the practical implication is that choosing a team size also means choosing a working ecology. Larger teams can make teamwork more salient and enjoyable, but they require suitable rooms, materials, task sequences, answer formats, and feedback routines. Well-designed team competitions therefore require attention not only to the mathematical quality of individual problems, but also to the social and material conditions under which students solve them together.

Summary and perspectives

This study combined post-competition questionnaires from 564 students with analyses of task-related feedback and selected solution patterns in three mathematics team competitions. We reported descriptive results on teamwork and enjoyment, the role of immediate correctness feedback, and task features that may create combined demands, productive obstacles, misleading short-cuts, or deliberately open mathematical activity. Across these strands, the findings suggest that task design cannot be separated from the wider competition ecology: team size, venue, task sequence, material and answer formats, and feedback jointly shape students' opportunities to collaborate and their experience of individual problems.

A promising direction for future research is to examine these experiences from the perspective of interest development. The present questionnaire measured enjoyment and comfort in teamwork, but not interest in a narrower theoretical sense. Nevertheless, the frequent positive references to teamwork and immediate feedback, as well as the positive evaluation of demanding problems after they had been successfully solved, point to activity- and format-specific sources of situational interest. Since interest may be directed not only towards mathematics as a domain, but also towards particular activities, contexts, and forms of participation [1], future studies should distinguish between interest in mathematics, interest in solving non-routine problems, and interest in the collaborative competition format itself. Repeated-event or longitudinal designs could further investigate whether teamwork, immediate feedback, and successfully overcoming challenging problems only trigger situational interest or also contribute to its maintenance and longer-term development.

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Mathematik macht Freude (MmF) is an Austrian mathematics outreach initiative originating at the University of Vienna. In the last years team competitions became an increasingly important part of its activities, including Náboj, Náboj Junior and the newly developed m3 competitions. In 2026, MmF organised m3 at several Austrian locations and the Vienna Náboj venue alone hosted 152 teams, being the largest in Europe.

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A proof-based generalization of Napoleon's Theorem

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Abstract

This paper explores generalizations of Napoleon's Theorem to higher-order polygons. We first prove that constructing equilateral triangles, either outwardly or inwardly, on any arbitrary hexagon yields opposite segment midpoints that form an equilateral triangle. We then generalize this mechanism to any arbitrary $3n$ -sided polygon ($n \geq 1$) using vector geometry, demonstrating that dividing the $3n$ centroids into three modulo-3 groups produces final group centroids that consistently preserve the equilateral configuration regardless of base polygon irregularity.

Introduction

Napoleon's Theorem demonstrates that constructing equilateral triangles on the sides of any triangle produces centroids that form another equilateral triangle ([1],[2]). Extending this symmetric behavior to higher-order polygons introduces significant algebraic challenges because the base symmetries are lost. This paper resolves these challenges by establishing a unified vector framework that operates independently of polygon convexity or coordinate orientation.

First, we address arbitrary hexagons. We prove that constructing outward (Theorem 1) or inward (Theorem 2) equilateral triangles on its sides and connecting opposite centroids yields three segments whose midpoints form an equilateral triangle. We analyze a regular hexagon (Case 1) before proving the unconstrained irregular case (Case 2) using linear vector transformations.

Second, we generalize this system to any arbitrary $3n$ -sided polygon. By sorting the $3n$ centroids into three symmetric groups of n points using a modulo-3 grouping method, we prove that the centroids of these three groups always form a perfect equilateral triangle. This generalizes the classical triangle ($n = 1$) and hexagon ($n = 2$) configurations to any higher-order polygon, such as the nonagon ($n = 3$).

Main results and proofs

Theorem 1. Let $ABCDEF$ be any arbitrary hexagon (convex, concave, or self-intersecting). Build outward equilateral triangles on each side of the hexagon. Let $A_1, B_1, C_1, D_1, E_1, F_1$ denote the centroids of these respective triangles. Connect the centroids of opposite equilateral triangles to form the segments A_1D_1, B_1E_1 , and C_1F_1 . Let points A_2, E_2, C_2 be the midpoints of line segments A_1D_1, E_1B_1 and C_1F_1 . Prove that either points A_2, E_2, C_2 coincide, or triangle $A_2E_2C_2$ is an equilateral triangle.

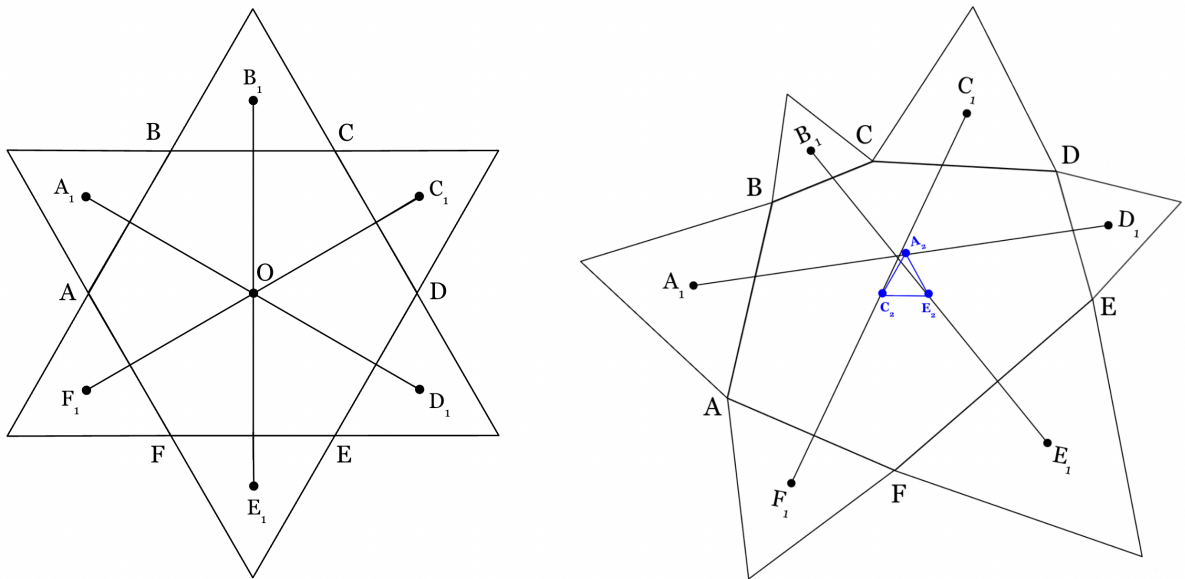


Figure 21: Geometric configurations for the regular hexagon (left) and irregular hexagon (right).

Proof of Theorem 1. Figure 21 shows a convex hexagon, but the proof works for any hexagon.

Case 1: Prove that A_1D_1 , B_1E_1 , and C_1F_1 intersect at a single point O . Moreover, O is the centroid of the regular hexagon and the midpoint of each segment.

Proof of Case 1. Let O be the centroid of $ABCDEF$. Note that $AF \parallel CD$ and $AF = CD$. $\triangle OCD$ and $\triangle OAF$ are congruent equilateral triangles, so their altitudes are congruent too. Thus $OC_1 = OF_1$. This means that O is the midpoint of C_1F_1 . In a similar way, O is the midpoint of A_1D_1 and B_1E_1 . So, A_1D_1 , C_1F_1 , B_1E_1 intersect at point O .

Case 2: Prove that triangle $A_2E_2C_2$ is an equilateral triangle.

Proof of Case 2. Let us consider the following figure.

To analyze the properties of the central triangle $A_2E_2C_2$, we determine the relationship between the directed segments connecting its vertices. By definition, A_2 is the midpoint of A_1D_1 and E_2 is the midpoint of B_1E_1 .

At first, let us prove that:

$$\overrightarrow{A_2E_2} = \frac{1}{2}(\overrightarrow{A_1B_1} + \overrightarrow{D_1E_1}).$$

Let $\mathbf{a}_1, \mathbf{b}_1, \mathbf{d}_1$, and $\mathbf{e}_1$ denote the position vectors of their respective centroids.

We can define the position vectors for midpoints A_2 and E_2 as:

$$\mathbf{a}_2 = \frac{\mathbf{a}_1 + \mathbf{d}_1}{2}, \quad \mathbf{e}_2 = \frac{\mathbf{b}_1 + \mathbf{e}_1}{2}$$

The displacement vector $\overrightarrow{A_2E_2}$ is found by subtracting their positions:

$$\overrightarrow{A_2E_2} = \mathbf{e}_2 - \mathbf{a}_2 = \left(\frac{\mathbf{b}_1 + \mathbf{e}_1}{2} \right) - \left(\frac{\mathbf{a}_1 + \mathbf{d}_1}{2} \right)$$

Combining terms under a common denominator and grouping into known differences yields:

$$\overrightarrow{A_2E_2} = \frac{1}{2}(\mathbf{b}_1 - \mathbf{a}_1 + \mathbf{e}_1 - \mathbf{d}_1) = \frac{1}{2}((\mathbf{b}_1 - \mathbf{a}_1) + (\mathbf{e}_1 - \mathbf{d}_1))$$

Since $\mathbf{b}_1 - \mathbf{a}_1 = \overrightarrow{A_1B_1}$ and $\mathbf{e}_1 - \mathbf{d}_1 = \overrightarrow{D_1E_1}$, we obtain the vector identity we wanted. Extending this process to the other sets of midpoints, we have:

$$\overrightarrow{A_2E_2} = \frac{1}{2}(\overrightarrow{A_1B_1} + \overrightarrow{D_1E_1}),$$

$$\overrightarrow{E_2C_2} = \frac{1}{2}(\overrightarrow{B_1C_1} + \overrightarrow{E_1F_1}),$$

$$\overrightarrow{C_2A_2} = \frac{1}{2}(\overrightarrow{C_1D_1} + \overrightarrow{F_1A_1}).$$

So, in order to prove that $A_2E_2C_2$ is an equilateral triangle, we need to prove

$$|\overrightarrow{A_1B_1} + \overrightarrow{D_1E_1}| = |\overrightarrow{B_1C_1} + \overrightarrow{E_1F_1}| = |\overrightarrow{C_1D_1} + \overrightarrow{F_1A_1}|. \quad (1)$$

According to the property of adding two vectors, we have

$$\overrightarrow{A_1B_1} = \overrightarrow{A_1B} + \overrightarrow{BB_1} = \frac{1}{\sqrt{3}}R^{-30^\circ}\overrightarrow{AB} + \frac{1}{\sqrt{3}}R^{30^\circ}\overrightarrow{BC}.$$

Here, the operator R^θ denotes a counterclockwise rotation of a vector by an angle θ around its initial point. In an outward equilateral triangle constructed on a segment, the angle between a side vector and the segment pointing to its centroid is exactly 30° . Trigonometrically, the distance from any vertex to the centroid of an equilateral triangle is scaled by a factor of $\frac{1}{\sqrt{3}}$ relative to its side length.

Therefore, the term $\frac{1}{\sqrt{3}}R^{-30^\circ}\overrightarrow{AB}$ represents a clockwise rotation (indicated by the negative angle) of side vector $\overrightarrow{AB}$ scaled down to reach the centroid A_1 from vertex B . Conversely, $\frac{1}{\sqrt{3}}R^{30^\circ}\overrightarrow{BC}$ represents a counterclockwise rotation of side vector $\overrightarrow{BC}$ scaled down to locate the centroid B_1 relative to vertex B .

In a similar way, we obtain

$$\overrightarrow{D_1E_1} = \frac{1}{\sqrt{3}}R^{-30^\circ}\overrightarrow{DE} + \frac{1}{\sqrt{3}}R^{30^\circ}\overrightarrow{EF}.$$

Thus, it follows that

$$\overrightarrow{A_1B_1} + \overrightarrow{D_1E_1} = \frac{1}{\sqrt{3}}\left(R^{-30^\circ}(\overrightarrow{AB} + \overrightarrow{DE}) + R^{30^\circ}(\overrightarrow{BC} + \overrightarrow{EF})\right) \quad (2)$$

In a similar way, we get

$$\overrightarrow{B_1C_1} + \overrightarrow{E_1F_1} = \frac{1}{\sqrt{3}}\left(R^{-30^\circ}(\overrightarrow{BC} + \overrightarrow{EF}) + R^{30^\circ}(\overrightarrow{CD} + \overrightarrow{FA})\right). \quad (3)$$

To simplify Equation (3) and relate it back to Equation (2), we utilize a fundamental property of closed geometric polygons. Because the perimeter of the irregular hexagon $ABCDEF$ forms a continuous, closed boundary loop, the sum of all its sequential side vectors must result in a total net displacement of zero ($\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DE} + \overrightarrow{EF} + \overrightarrow{FA} = \vec{0}$). By applying the associative and commutative properties of vector addition, we can rearrange these six perimeter paths into arbitrary pairs without altering the zero total result:

We have

$$(\overrightarrow{AB} + \overrightarrow{DE}) + (\overrightarrow{BC} + \overrightarrow{EF}) + (\overrightarrow{CD} + \overrightarrow{FA}) = \vec{0}.$$

Thus, from equation (3) it follows that

$$\overrightarrow{B_1C_1} + \overrightarrow{E_1F_1} = \frac{1}{\sqrt{3}}(R^{-30^\circ}(\overrightarrow{BC} + \overrightarrow{EF}) - R^{30^\circ}(\overrightarrow{BC} + \overrightarrow{EF}) - R^{30^\circ}(\overrightarrow{AB} + \overrightarrow{DE})).$$

To evaluate this combination of rotation terms, we apply the algebraic properties of the rotation operator. For the first two segments, we factor out the common vector to evaluate the operator difference $R^{-30^\circ} - R^{30^\circ}$. Expressing these rotations as complex numbers on the unit circle yields $(\cos(30^\circ) - i\sin(30^\circ)) - (\cos(30^\circ) + i\sin(30^\circ)) = -2i\sin(30^\circ)$. Since $\sin(30^\circ) = \frac{1}{2}$, this simplifies directly to $-i$, which represents a pure downward geometric rotation of R^{-90° . For the third segment, the negative coefficient dynamically alters the orientation of the vector by 180° , converting $-R^{30^\circ}$ directly into R^{-150° (since

$30^\circ - 180^\circ = -150^\circ$). This isolates the component rotations cleanly: Thus, the equation above is equivalent to:

$$\frac{1}{\sqrt{3}}(R^{-90^\circ}(\overrightarrow{BC} + \overrightarrow{EF}) + R^{-150^\circ}(\overrightarrow{AB} + \overrightarrow{DE})) = R^{-120^\circ}(\overrightarrow{A_1B_1} + \overrightarrow{D_1E_1}).$$

We get that

$$\overrightarrow{B_1C_1} + \overrightarrow{E_1F_1} = R^{-120^\circ}(\overrightarrow{A_1B_1} + \overrightarrow{D_1E_1}).$$

Thus, it follows that

$$|\overrightarrow{A_1B_1} + \overrightarrow{D_1E_1}| = |\overrightarrow{B_1C_1} + \overrightarrow{E_1F_1}|.$$

In a similar way, we get that

$$|\overrightarrow{B_1C_1} + \overrightarrow{E_1F_1}| = |\overrightarrow{C_1D_1} + \overrightarrow{F_1A_1}|.$$

This ends the proof of (1).

Since the side lengths of the central triangle $A_2E_2C_2$ are given by $A_2E_2 = \frac{1}{2}|\overrightarrow{A_1B_1} + \overrightarrow{D_1E_1}|$, $E_2C_2 = \frac{1}{2}|\overrightarrow{B_1C_1} + \overrightarrow{E_1F_1}|$, and $C_2A_2 = \frac{1}{2}|\overrightarrow{C_1D_1} + \overrightarrow{F_1A_1}|$, relation (1) directly establishes that $A_2E_2 = E_2C_2 = C_2A_2$. Thus, all three sides of the inner configuration are of equal length. If this mutual length is non-zero, the vertices form a perfect equilateral triangle $\Delta A_2E_2C_2$; otherwise, if the vector magnitudes are zero, the three points A_2, E_2, C_2 collapse onto a single coordinate and coincide. Thus, Case 2 is proven. This completes the proof of this theorem. Q.E.D.

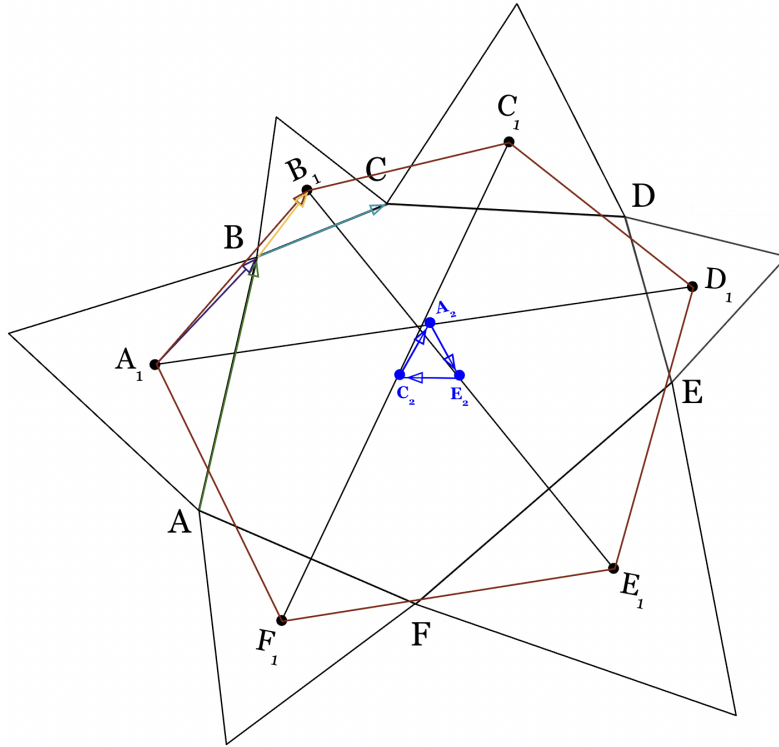


Figure 22: Vector layout mapping the outer triangle centroids to the coordinates of the central midpoints A_2, E_2 , and C_2 .

Theorem 2. Let $ABCDEF$ be any arbitrary hexagon (convex, concave, or self-intersecting). Build *inward* equilateral triangles on each side of the hexagon. Let $A_1, B_1, C_1, D_1, E_1, F_1$

denote the centroids of these respective internal triangles. Connect the centroids of opposite equilateral triangles to form the line segments A_1D_1 , B_1E_1 , and C_1F_1 . Let points A_2 , E_2 , and C_2 be the midpoints of segments A_1D_1 , E_1B_1 , and C_1F_1 , respectively. Prove that either points A_2 , E_2 , and C_2 coincide, or triangle $A_2E_2C_2$ forms a perfect equilateral triangle.

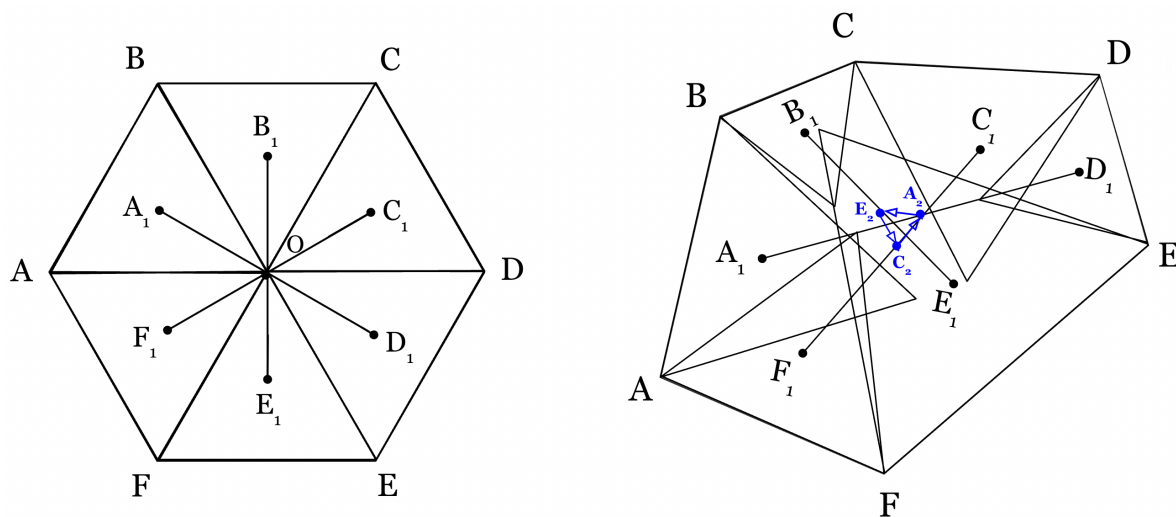


Figure 23: Geometric configurations for the regular hexagon (left) and irregular hexagon (right) with inwardly constructed triangles.

Proof of Theorem 2. The algebraic and vector steps required to prove this theorem are identical to those detailed in the proof of Theorem 1. Transitioning the equilateral triangles to an internal orientation merely reverses the signs of the fixed 30° rotations, leaving the linear combination and the zero-sum perimeter cancellation entirely unchanged. Q.E.D.

Theorem (Generalization). Let P be an arbitrary polygon with $3n$ sides ($n \geq 1$). Construct outward equilateral triangles on each side of the polygon, and let $C_1, C_2, \dots, C_{3n}$ be the centroids of these triangles around the perimeter. Divide these $3n$ centroids into three symmetrical groups of n points each.

- Group A consists of $\{C_1, C_4, C_7, \dots, C_{3n-2}\}$
- Group B consists of $\{C_2, C_5, C_8, \dots, C_{3n-1}\}$
- Group C consists of $\{C_3, C_6, C_9, \dots, C_{3n}\}$

Let A_* , B_* , and C_* be the centroids of the shapes corresponding to Group A, Group B, and Group C, respectively. Prove that either the points A_* , B_* , and C_* coincide, or triangle $A_*B_*C_*$ is an equilateral triangle.

Visual representation of the theorem (generalization). Table 3 illustrates the statement of Theorem (Generalization).

Case	Sides	Polygon Type	Centroid Groupings (mod 3)	Final Balance Points
$n = 1$	3	Triangle	Group A: $\{C_1\}$ Group B: $\{C_2\}$ Group C: $\{C_3\}$	A_*, B_*, C_* coincide with C_1, C_2, C_3
$n = 2$	6	Hexagon	Group A: $\{C_1, C_4\}$ Group B: $\{C_2, C_5\}$ Group C: $\{C_3, C_6\}$	Midpoints of opposite pairs
$n = 3$	9	Nonagon	Group A: $\{C_1, C_4, C_7\}$ Group B: $\{C_2, C_5, C_8\}$ Group C: $\{C_3, C_6, C_9\}$	Centroids of internal triangles
$n = 4$	12	Dodecagon	Group A: $\{C_1, C_4, C_7, C_{10}\}$ Group B: $\{C_2, C_5, C_8, C_{11}\}$ Group C: $\{C_3, C_6, C_9, C_{12}\}$	Centroids of internal quadrilaterals
$n = 5$	15	Pentadecagon	Group A: $\{C_1, C_4, C_7, C_{10}, C_{13}\}$ Group B: $\{C_2, C_5, C_8, C_{11}, C_{14}\}$ Group C: $\{C_3, C_6, C_9, C_{12}, C_{15}\}$	Centroids of internal pentagons
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$n > 5$	$3n$	$3n$ -gon	Group A: $\{C_1, C_4, C_7, \dots, C_{3n-2}\}$ Group B: $\{C_2, C_5, C_8, \dots, C_{3n-1}\}$ Group C: $\{C_3, C_6, C_9, \dots, C_{3n}\}$	Centroids of internal n -sided polygons

Table 3: Systematic classification demonstrating how the skip-counting pattern and center point configurations generalize to any $3n$ -sided polygon.

As we have three groups for each case ($n = 1, n = 2, n = 3, \dots$), that is Group A, Group B, Group C, and as to each group corresponds one centroid, then the theorem states that when we connect these balance points for each case, we get a triangle, which is equilateral triangle $A_*B_*C_*$.

Proof of theorem (generalization). In the case of a hexagon, points A_*, B_*, C_* represent midpoints, whereas for nonagons and higher $3n$ -gons, points A_*, B_*, C_* represent the centroids of the regions formed by the groups listed in the table (see the visual representation of the theorem). To better illustrate this distinction across different $3n$ -gons, we provide the corresponding figures for the nonagon case below.

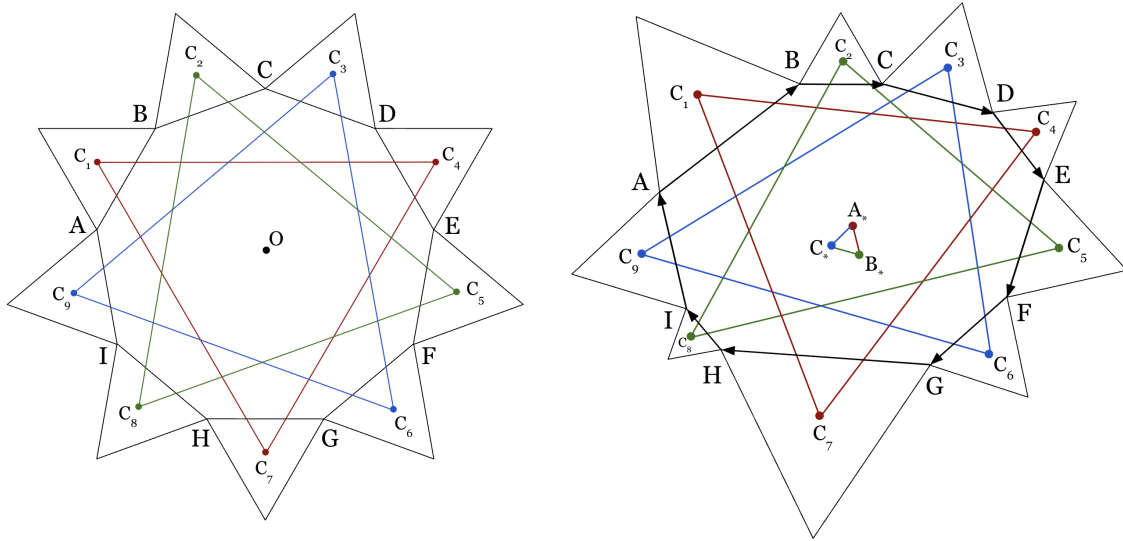


Figure 24: Geometric configurations for the regular nonagon (left) and irregular nonagon (right) showing inner tracking paths and the resulting centroids A_* , B_* , and C_* .

We believe that the proof provided for the hexagon can be extended to the nonagon and, more generally, to any $3n$ -gon. Q.E.D.

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Puzzles, Stories, and Very Large Numbers: Complementary approaches to mathematical engagement

Richard Elwes



Richard Elwes is an associate professor of mathematics at University of Leeds with research interests in graph theory, logic, and random processes. His work in public engagement includes six popular mathematics books, most recently [3]. He is a regular presenter on the YouTube channel *Numberphile* and has written several feature-articles for the New Scientist magazine. For 2019-2025 he served as a Holgate Session leader for the London Mathematical Society and continues to deliver regular lectures and masterclasses at schools around UK. This article is based on a plenary lecture delivered by the author at the conference WFNMC2026 in Kuala Lumpur (pictured).

Abstract

There are two main approaches to engage people in mathematics. The first is through puzzles, problems, games, and tricks. The other is through stories, surveys, and histories. The first is narrow but deep, the second broad but shallow. This is reminiscent of Freeman Dyson's division of research mathematicians into "birds" and "frogs". The first part of this paper discusses these two complementary approaches, with a focus on moments where they work well together: where an individual puzzle plays a starring role in a bigger story. In the second part, we look at one such case: a combinatorial puzzle which, via the unexpected appearance of some extremely large numbers, provide an accessible entry point into deep areas of mathematical logic. The technical content in section 3 onwards discusses work of Harvey Friedman, and largely follows [5].

Birds & Frogs, Puzzles & Stories

Today's mathematics, we all know, is a huge subject. This hugeness manifests in at least two dimensions: breadth and depth. Any mathematical journey, whether a diversion lasting just a few minutes, or a lifelong voyage of exploratory research, will cover different distances along these two dimensions. Likewise any mathematical activity (whether research, teaching, or public engagement) must strike its own balance.

In the context of mathematical research, the contrasting approaches taken by different mathematicians has been written about several times. In [2] Freeman Dyson contrasts mathematical 'birds' with mathematical 'frogs'. The whole essay is recommended reading, but I will quote the opening section:

Some mathematicians are birds, others are frogs. Birds fly high in the air and survey broad vistas of mathematics out to the far horizon. They delight

in concepts that unify our thinking and bring together diverse problems from different parts of the landscape. Frogs live in the mud below and see only the flowers that grow nearby. They delight in the details of particular objects, and they solve problems one at a time. . . It is stupid to claim that birds are better than frogs because they see farther, or that frogs are better than birds because they see deeper. The world of mathematics is both broad and deep, and we need birds and frogs working together to explore it.

Another very readable exploration of a similar theme is [8] where Timothy Gowers discusses ‘the two cultures of mathematics’ dividing mathematical researchers into ‘theory builders’ and ‘problem solvers’:

If you are unsure to which class you belong, then consider the following two statements.

- *The point of solving problems is to understand mathematics better.*
- *The point of understanding mathematics is to become better able to solve problems*

. . . when I say that mathematicians can be classified into theory-builders and problem-solvers, I am talking about their priorities, rather than making the ridiculous claim that they are exclusively devoted to only one sort of mathematical activity.

We might take Alexander Grothendieck as the archetypal theory builder and Paul Erdős as the quintessential problem solver (though to reinforce Gowers’ caveat, Grothendieck also solved problems while Erdős also developed theory).

In any event, the current article does not concern mathematical research but rather different forms of public engagement with mathematics. Here too, questions of breadth versus depth arise.

Very roughly, certain forms of mathematical engagement require the participant to actively do something: to solve a problem, to win a game, to answer a question. I will somewhat inaccurately use the word “puzzle” for all activities of this kind⁴.

Others are more passive, with the participant’s involvement limited to reading, listening, or watching. Often, though not always, engagement of this form can embrace greater breadth than is possible via puzzles. In such cases, I will again inaccurately use the term “story” to describe the activity: elements that may feature include history, biography, scientific survey, discussions of applications.

Thus we can divide instances of mathematical engagement into ‘puzzle-setting’ and ‘story-telling’. Needless to say this is a simplification, and perhaps a caricature, but I hope one containing enough truth to be useful basis for further discussion.

Mathematics it is often said is something one gets better at by *doing*, and as such it might be argued that puzzle-setting is the superior form. I think this is correct, in that if we had to choose only one form of mathematical engagement to pursue, then puzzles would indeed

⁴I wondered whether “problem” or “question” might be better terms, as these reflect the terminology used in competitions, coursework, and exams. However “problem” has an unfortunate second meaning of difficulty or obstacle, and “question” is vaguer still. So I have settled on “puzzle” although this suggests a level of informality which is not always intended.

be the better choice. Happily we are not forced into such a choice, and as someone who specialises in story-telling I hope it is not too controversial to claim that this also has its place.

Indeed, breadth of exploration is exactly what story-telling offers. A certain type puzzle may stimulate thought and help the participant develop skills, but it requires an accompanying story to explain whether, how, and where its ideas fits into the broader mathematical landscape, and where the skills acquired might profitably be applied.

If people get better at mathematics by doing it, it is also true that they will only do it, at least in a sustainable way, if they are motivated and interested to do so. Story-telling can build and reinforce that interest.

Flying frogs

There are some puzzles which are striking for the role they play within a bigger story. Let us give some examples. These are all classic puzzles, thoroughly recounted in many other places, so I will withhold details.

1. The Bridges of Königsberg plays an important role in the story of the birth of graph theory, especially vertex-degrees and Eulerian circuits.
2. The Dinner Party Problem (to determine how many guests are needed to guarantee three mutual friends or three mutual strangers at a gathering under the assumption that every pair of people are either friends or strangers) is the first step in the development of Ramsey theory.
3. Three Utilities Puzzle, to determine whether three vertices (the utilities) can be connected to three others (houses) with none of the connections crossing was written about by Henry Dudeney in 1917 ([1]), who observed that it is “as old as the hills...much older than electric lighting, or even gas”. The graph described in this puzzle is $K_{3,3}$ which plays a central role in the theory of planar graphs (via, for example, Kuratowski’s theorem). In turn, the theory of planar graphs is the first chapter in the tale of topological graph theory, and a much longer story about forbidden minor families of graphs.
4. Many simple puzzles ask whether a certain map can be coloured using a specified number of colours (under the condition that neighbouring territories may not have the same colour). Such a puzzle naturally invites generalisation, opening the door to discussion of the four colour theorem, a milestone result not only in discrete geometry, but in the history of mathematics being the first proof which made such substantial use of computational technology that the resulting proof was too large for any individual human to digest unaided. This story, of course, continues to develop and at time of writing is very topical in the fast-moving context of AI and proof formalisation.
5. The puzzle of the blue-eyed islanders (it has many isomorphic variants, such as muddy children, cheating wives, cheating husbands, messy eaters, and so on) is an excellent introduction to the theory of common knowledge, as discussed for example in [11], an important topic in game theory, with highlights including Aumann’s agreement theorem.

A blocky puzzle

In examples 1-5 above, the puzzle plays such a central role in the story that to omit it might even be said to be remiss. (Of course this depends on the story; in example 1, one can tell a perfectly good story about graph theory without mentioning Königsberg, but if the story is the origin of that subject then the point stands.) The author would welcome other examples where puzzles play such central roles in bigger stories.

More numerous, however, are cases where a puzzle is a good, though not indispensable, door to a larger story. For instance, the Rubik's cube is an excellent illustration of certain topics from group theory, though far from an essential prop. Likewise I have used the adjacency matrix of a network and invited students to compute and manipulate a few examples, as a way to introduce elements of linear algebra, but achieve a similar effect in other ways.

For the remainder of this paper we will introduce a puzzle of this kind, a convenient and engaging, but nevertheless optional, door to a much bigger story. I argue that this has particular value, as the field being approached is otherwise highly technical and difficult to access, although its underlying ideas are (I would claim) intuitive. The field in question is that of reverse mathematics, as invented by Harvey Friedman and developed by Stephen Simpson in [13], a technical area of mathematical logic closely related to proof theory. I will develop this story further, but let us first introduce the puzzle, devised by Friedman, whose work this discussion is taken from (any errors are mine). See principally [5].

The blocky puzzle

The set-up. Consider finite strings from a two-letter alphabet, for example $BAABBBBA$. We index the positions starting at 1, so write the above as $x_1x_2x_3x_4x_5x_6x_7$ where $x_1 = B$ and $x_2 = A$ and so on.

A **block** within such a string is a substring running from some index to its double (inclusive): x_1x_2 , or $x_2x_3x_4$, or more generally $x_ix_{i+1} \dots x_{2i}$. For the string $BAABBBBA$ we get just three blocks: BA , AAB , $ABBB$. Strings of length ≥ 8 will have more blocks.

We are interested in the phenomenon of a smaller block s being a *substring* of a larger one t , written $s \leq t$, meaning that s can be obtained from t by deleting elements. Crucially, reordering is not allowed. So $AB \leq BAB$ (deleting the first element) and $AA \leq ABBA$ (deleting the second and third elements). However $AB \not\leq BBAA$.

The problem. The above example $BAABBBBA$ has the property that no block is a substring of any other. Call such a string *blocky*. What is the longest possible blocky string?

Extension. This puzzle, as we shall see shortly, is perfectly soluble with a little trial and error, and on its own does not open the door to the wider story. Now change the size of the alphabet, keeping other details the same. What is the longest blocky string with just one symbol (A)? What if we extend the alphabet to three symbols (A, B, C), or four, or to general $K \in \mathbb{N}$?

Solutions and proof for $K = 1, 2$

For the single-letter alphabet, it is easy to see that the longest blocky string is AAA as any longer string will have $b_1 \leq b_2$.

For the original two-symbol alphabet, the longest blocky string has length 11. There are four such winners, one being

$$ABBBAAAAAA$$

and the others derived from this by exchanging the roles of A and B and/or flipping the final element.

The following proof follows [5] although the presentation is different. It not essential to the subsequent discussion and the reader may choose to skip it (or recreate it independently).

Proof. We use b_i to refer to the i th block, $x_i \dots x_{2i}$. We take $x_1, x_2, \dots$ step by step aiming to generate blocky strings of maximal length.

Step 1. Without loss of generality, we assume $x_1 = A$.

Step 2. First consider strings $s = AAx_3x_4 \dots$. We will show that starting this way we can grow blocky strings of length 7 and no more.

We have $b_1 = AA$ and $b_2 = Ax_3x_4$, since $b_1 \not\leq b_2$ we must have $x_3 = x_4 = B$.

Thus $s = AABBAx_5x_6 \dots$ and $b_1 = AA$, and $b_2 = ABB$, and $b_3 = BBx_5x_6$. Since $b_1 \not\leq b_3$, we cannot have $x_5x_6 = AA$, but the other three possibilities are all legally possible: $x_5x_6 = AB, BA, BB$. With each of these we can append any x_7 without creating any new blocks, creating six blocky strings of length 7 beginning AA :

$s_1 = AABBAABA$, $s_2 = AABBAABB$, $s_3 = AABBBAA$, $s_4 = AABBBAB$, $s_5 = AABBBBA$, and $s_6 = AABBBBB$.

In all six cases, we shall show no x_8 can be appended. In s_3 , appending any x_8 causes $b_1 = AA \leq b_4 = BBAAx_8$. For s_6 , appending any x_8 causes $b_3 = BBBB \leq b_4 = BBBBx_8$.

For s_1, s_2, s_4, s_5 we consider the two possibilities for x_8 in turn. First, if $x_8 = A$, then in all cases s_1, s_2, s_4, s_5 this causes $b_1 = AA \leq b_4$.

Now consider appending $x_8 = B$ to s_1, s_2, s_4, s_5 . In cases s_1, s_2, s_4 , this causes $b_2 = ABB \leq b_4$. For s_5 , it causes $b_3 = BBBB \leq b_4 = BBBAB$.

Thus for blocky strings of length ≥ 8 , we must take $x_2 = B$.

Step 3. We are looking at strings of the form $s = ABx_3x_4 \dots$. If $x_3 = A$ then we must also have $x_4 = A$ as $b_1 \not\leq b_2$, and iteratively since $b_1 \not\leq b_i$ we must have $x_i = A$ for all $i \geq 3$. This again generates blocky strings as far as 7 (namely $ABAAAAA$), but no further as $b_3 = AAAA \leq b_4 = AAAAA$.

So we are reduced to strings of the form $s = ABBx_4x_5x_6 \dots$

Step 4. If $x_4 = A$, then $s = ABBAx_5x_6 \dots$ where $b_1 = AB$ and $b_4 = Ax_5x_6x_7x_8$ forcing $x_5 = x_6 = x_7 = x_8 = A$. Extending by one unit creates no new blocks, yielding two blocky strings of length 9: $s_7 = ABBAAAAA$ and $s_8 = ABBAAAAAB$. Extending s_8 with

any x_{10} causes $b_1 = AB \leq b_5$. Extending s_7 by $x_{10} = B$ likewise causes $b_1 = AB \leq b_5$. Extending s_7 with $x_{10} = A$ causes $b_4 = AAAAA \leq b_5 = AAAAA$.

Hence for blocky strings of length ≥ 10 we can only have $s = AB BB x_5 x_6 \dots$

Step 5. For $s = AB BB x_5 x_6 \dots$ we need $b_2 = BB B \not\leq b_3 = BB x_5 x_6$ forcing $x_5 = x_6 = A$ and $s = AB BB AA x_7 x_8 \dots$. Then as $b_1 = AB \not\leq b_4 = BAA x_7 x_8$ we need $x_7 = x_8 = A$ and likewise since $b_1 = AB \not\leq b_5 = AAAA x_9 x_{10}$ we must have $x_9 = x_{10} = A$.

As before adding any x_{11} creates no new blocks, giving us two blocky strings of length 11: $s_9 = AB BB AAAAAA$ and $s_{10} = AB BB AAAAAAB$.

These are our final answers since neither can be legally extended. Appending any x_{12} to s_{10} violates $b_1 = AB \leq b_6$, as does appending B to s_9 , while appending A to s_9 violates $b_5 = AAAAAA \leq b_6 = AAAAAA$. ■

Proof of solubility for all K

Write $n(1) = 3$ and $n(2) = 11$ to indicate the results above, i.e. $n(K)$ is the length of the longest blocky string in a K -symbol alphabet.

This problem is of broader interest because of what happens for larger alphabets. Firstly, let us sketch a proof that a solution to the puzzle always exists.

Theorem 1 (Friedman, [5]). For all $K \geq 1$, the value $n(K)$ exists uniquely. That is arbitrarily long (or infinitely long) blocky strings do not exist.

Proof. Fix $K \geq 1$. We work with strings from a K -symbol alphabet. We first prove the following:

Claim. Let $s_1, s_2, \dots$ be an infinite sequence of finite strings. Then there are $i < j$ where $s_i \leq s_j$.

Proof of claim. Suppose not. Pick s_1 of minimal length so that there exists an infinite counterexample sequence starting with s_1 .

Then pick s_2 of minimal length so that there exists a counterexample starting with s_1, s_2 . Continuing this way, we build a counterexample $s_1, s_2, s_3, \dots$

Let $s_n = s_{j_0}$ be the first string so that there exist infinitely many strings $s_{j_0}, s_{j_1}, s_{j_2}, \dots$ all starting with the same symbol. (Clearly such must exist.)

Let t_n be $s_n = s_{j_0}$ with the first symbol removed, and similarly t_{n+i} is s_{j_i} with the first symbol removed. Then $s_1, s_2, \dots, s_{n-1}, t_n, t_{n+1}, \dots$ is a counterexample but t_n is shorter than s_n , which contradicts its minimality. **QED Claim.**

An important role in the remainder of the proof is played by König's lemma, first presented in [10], that a finitely branching tree with infinitely many vertices has an infinite branch. We can use this to complete the proof.

Suppose for a contradiction there are arbitrarily long blocky strings. These form a tree, which is finitely branching since the alphabet is finite. Thus by König's lemma, there exists an infinite blocky string. Let $b_1, b_2, \dots$ be its blocks. This sequence contradicts the claim. ■

Finding $n(K)$ for $K \geq 3$ is non-trivial due to their unexpected immensity.

A long sequence for $K = 3$

We will give an example illustrating that $n(3) \geq 5000$, and show how to adapt it to give much larger results. As with most else in this paper, we follow [5], though the details there are technical and the resulting sequence is somewhat buried in the proof of blockiness, which we will not reproduce here. We have done our best here to extract the construction explicitly.

We use the notation A^{10} to mean $AAAAAAAAAA$ and so on. We begin with a seed sequence (found by hand in [5]).

$$\mathbf{S} : AB^2ACA^7C^2ACAC^8AC^5AC^{20}A^2C^{53}AC^{108} \quad \text{length 216.}$$

We build a longer sequence as follows:

$$\mathbf{S} \mathbf{B} \mathbf{B} \mathbf{C}_1 \mathbf{B} \mathbf{C}_2 \mathbf{B} \mathbf{C}_3 \mathbf{B} \dots \mathbf{C}_d$$

where d is to be determined and will be large, $\mathbf{B}$ is a repeating block and the $\mathbf{C}_i$ vary:

$$\mathbf{B} : C^{25} \quad \text{length 25.}$$

$$\mathbf{C}_1 : B^{73}C^8B^2 \quad \text{length 83}$$

$$\mathbf{C}_2 : B^{75}C^8B \quad \text{length 84}$$

So far, $\mathbf{S} \mathbf{B} \mathbf{B} \mathbf{C}_1 \mathbf{B} \mathbf{C}_2 \mathbf{B}$ is a blocky string of length 459. First we will extend this with explicit but non-optimal choices for $\mathbf{C}_3, \dots, \mathbf{C}_9$ and then discuss how this could be taken hugely further.

$$\mathbf{C}_3 : BCB^{169}CB^2CB^2CB^2CB^2CB^2CB \quad \text{length 191}$$

$$\mathbf{C}_4 : BCB^2CB^{170}CB^2CB^2CB^2CB^2CB^2CB^2 \quad \text{length 193}$$

$$\mathbf{C}_5 : BCB^2CB^2CB^{171}CB^2CB^2CB^2CB^2CB^{215} \quad \text{length 407}$$

$$\mathbf{C}_6 : BCB^2CB^2CB^2CB^{172}CB^2CB^2CB^2CB^{218} \quad \text{length 411}$$

$$\mathbf{C}_7 : BCB^2CB^2CB^2CB^{173}CB^2CB^2CB^{645} \quad \text{length 839}$$

$$\mathbf{C}_8 : BCB^2CB^2CB^2CB^2CB^{174}CB^2CB^{652} \quad \text{length 847}$$

$$\mathbf{C}_9 : BCB^2CB^2CB^2CB^2CB^2CB^{175}CB^{1507} \quad \text{length 1703}$$

With these choices, we get a blocky string of length 5200:

$$\mathbf{S} \mathbf{B} \mathbf{B} \mathbf{C}_1 \mathbf{B} \mathbf{C}_2 \mathbf{B} \mathbf{C}_3 \mathbf{B} \mathbf{C}_4 \mathbf{B} \mathbf{C}_5 \mathbf{B} \mathbf{C}_6 \mathbf{B} \mathbf{C}_7 \mathbf{B} \mathbf{C}_8 \mathbf{B} \mathbf{C}_9.$$

To take this further we show how to redefine $\mathbf{C}_3, \mathbf{C}_4, \dots$ up to a high limit. The length of each block $\mathbf{C}_i$ for $i \geq 3$ is predetermined at

$$|\mathbf{C}_i| = F(i+1) - F(i) - 25$$

where F is the recursively defined (and exponentially growing) function:

$$F(1) = 216, \quad F(2) = 324, \quad F(i) = 2 \cdot F(i-2) + 1 \text{ for } i \geq 3.$$

In general $\mathbf{C}_i$ contains exactly 8 C s in the following pattern:

$$\mathbf{C}_i : BCB^{y_1^i}CB^{y_2^i}CB^{y_3^i}CB^{y_4^i}CB^{y_5^i}CB^{y_6^i}CB^{y_7^i}CB^{z_i} \quad \text{length } |\mathbf{C}_i|$$

where for each i we have $\sum_{j=1}^7 y_j^i = 178 + i$ and z_i is the make-up term $z_i = |\mathbf{C}_i| - 187 - i$ (so $z_3 = 1$).

The key idea is that the sequence of septuples y^i must have the property that for no $i < j$ do we have $y^i \leq y^j$, where

$$(y_1^i, \dots, y_7^i) \leq (y_1^j, \dots, y_7^j) \Leftrightarrow \forall \ell \leq 7 \quad y_\ell^i \leq y_\ell^j.$$

Notice that $y^i \leq y^j \Leftrightarrow \mathbf{C}_i \leq \mathbf{C}_j$ in the sense of substrings.

Clearly this holds for our choices above: $y^3 = (169, 2, 2, 2, 2, 2, 2), \dots, y^9 = (2, 2, 2, 2, 2, 2, 175)$. Equally clearly these are not optimal and much longer sequences (y^i) are possible. How long?

Size of solutions for $K = 3, 4$

The function that arose in the last section is $f_7(179)$, the instance $k = 7$ and $p = 179$ of the following: $f_k(p)$ is the longest sequence y^i of tuples of $\mathbf{N}^k$ such that:

- For all i we have $\sum_{j=1}^k y_j^i \leq i + p - 1$.
- For no $i < j$ do we have $y^i \leq y^j$.

Similarly to the proof of Theorem 1, [5] shows that $f_k(p)$ is well-defined for all $k, p \geq 1$ by first eliminating the possibility of infinitely long legal sequences and then appealing to König's Lemma to rule out finite sequences of arbitrary length.

Let us try to get a handle on the size of $f_k(p)$. Consider first $f_2(5)$ and the following sequence of pairs y^i :

$$\begin{aligned} y^1 &= (5, 0) \\ y^2 &= (4, 2), \quad y^3 = (4, 1), \quad y^4 = (4, 0) \\ y^5 &= (3, 6), \quad y^6 = (3, 5), \dots, y^{11} = (3, 0) \\ y^{12} &= (2, 14), \dots, y^{26} = (2, 0) \\ y^{27} &= (1, 30), \dots, y^{57} = (1, 0) \\ y^{58} &= (0, 62), \dots, y^{120} = (0, 0) \end{aligned}$$

where row j runs from $(6 - j, 2^j - 2)$ to $(6 - j, 0)$ with length $2^j - 1$. This example shows $f_2(5) \geq 120$. Generalising, $f_2(p) \geq g_2(p) := 2^{p+2} - p - 3$.

Now consider $f_3(5)$, via the following sequence initially:

$$\begin{aligned} y^1 &= (5, 0, 0) \\ y^2 &= (4, 2, 0), \dots, y^{12} = (4, 0, 0) \end{aligned}$$

where on the second row, the second and third coordinates witness $f_2(2) \geq g_2(2) = 11$ as above:

$$\begin{aligned} y^2 &= (4, 2, 0) \\ y^3 &= (4, 1, 2), \quad y^4 = (4, 1, 1), \quad y^5 = (4, 1, 0) \\ y^6 &= (4, 0, 6), \quad y^7 = (4, 0, 5) \dots, y^{12} = (4, 0, 0) \end{aligned}$$

We continue:

$$y^{13} = (3, 11, 0) \dots, y^{8190} = (3, 0, 0)$$

where this third row witnesses $g_2(g_2(2)) = g_2(11) = 8178$. Notice this is not optimal as we could begin with $(3, 14, 0)$. Likewise the fourth row will have first coordinates 2 and length $g_2(g_2(g_2(2)))$, and so on, until the final line has initial coordinates 0 and length $g_2(g_2(g_2(g_2(g_2(2)))))$. Since $g_2(p) \geq 2^{p+1}$ this gives us

$$f_3(5) > 2^{2^{2^{2^3}}}.$$

Generalising, we see that

$$f_{k+1}(p) > f_k \circ \dots \circ f_k(2)$$

where the number of f_k is p .

This is strongly reminiscent of the famously fast-growing Ackermann function⁵, A be defined recursively by:

- $A_1(n) = 2n$
- $A_{k+1}(n) = A_k A_k \dots A_k(1)$ where there are n many occurrences of A_k .

So $A_2(n) = 2^n$ and $A_3(n) = 2^{2^{\dots^2}}$ where the tower contains n many 2s.

For three letters, the recursive system above allows an adaptation of the sequence $\mathbf{C}_1, \dots, \mathbf{C}_d$ where $d = A_7(184)$ (note the 7 here corresponding the tuples $y^i \in \mathbf{N}^7$, and to the depth of the recursion outlined above). This trivially implies that $n(3) > A_7(184)$. Friedman reports that Randall Dougherty has found a seed sequence $\mathbf{S}$ of length 187196, and has thereby improved this to: $n(3) > A_{7198}(158386)$.

So much for $n(3)$. What of $n(4)$? That is, in the words of Friedman in [4], “a whole ‘nother kettle of fish”, requiring a further extension of the Ackermann function. Writing $A(k) = A_k(k)$, then $n(4) > AA\dots A(1)$, where there are $A(187196)$ many A s.

All of these gigantic numbers are not mere curiosities, but symptoms of the logical strength of the system required to prove Friedman’s result that $n(K)$ is well-defined for all k .

⁵Various versions of the Ackermann function are in circulation. We follow the variant used in [5].

A logical story, in fast forward

Let us end with a very brief discussion of how the results discussed above fit into the story of mathematical logic over the last two centuries. Fuller accounts of these matters can be found in many places, including for a general audience in my recent book [3].

Around 1900, David Hilbert set in motion a project to axiomatise arithmetic (meaning for current purposes the natural numbers equipped just with addition and multiplication). He hoped for a system which would be *consistent* (i.e. it should not be possible to deduce a falsehood from the axioms) and *complete* (any true statement about arithmetic should be deducible from the axioms), as well as being *decidable* (which we will ignore for current purposes).

Hilbert's programme, at least in its maximalist form, was ended by Kurt Gödel's Incompleteness Theorem of 1931. Gödel showed that any axiomatisation we can ever write down (meaning, in modern terminology, any which is computationally enumerable) is necessarily incomplete (assuming it is consistent).

That is, there will always be a statement G , where neither G nor $\neg G$ (the negation of G) logically follows from the axioms.

Gödel's original statement was, essentially, "*This statement has no proof*" as translated into the language of arithmetic. His genius was to recognise the highly non-obvious fact that statements like this can be encoded into the language of arithmetic itself: any sufficiently powerful axiomatic system for arithmetic is strong enough to express statements about itself.

Now, Gödel's original unprovable statement, along with most others which are known, have a distinctly metamathematical flavour. Although fascinating on their own terms, the question arises: are there any *concrete* instances of Gödelian incompleteness, that is true statements about arithmetic which are unprovable, but which are at the same time broadly accessible, requiring no knowledge of (or indeed interest in) logic to understand?

According to Theorem 0.7.9 of [6], Theorem 1 is precisely such a instance of concrete incompleteness. However this requires clarification, since unprovability is relative to a given axiomatic system.

In this case, we can specify Elementary Function Arithmetic (EFA) as the axiomatic system. This is a simple but efficient axiomatisation of arithmetic, whose efficacy is captured by Friedman's so-called 'grand conjecture' of 1999: "Every theorem published in the *Annals of Mathematics* whose statement involves only finitary mathematical objects... can be proved in EFA."

The unprovability of the statement that $n(K)$ exists for all K is closely related to the very high rate of growth of the sequence: $n(K)$ eventually dominates all functions $\mathbf{N} \rightarrow \mathbf{N}$ which are provably total in that system. In fact, Theorem 1 remains unprovable in the stronger system $I\Sigma_2$ (see Theorem 0.7.9 of [6]) and dominates all functions $\mathbf{N} \rightarrow \mathbf{N}$ which are provably total there too, which include exotic multiply nested extensions of the Ackermann function A .

Every unprovable theorem, of course, becomes provable under a suitable strengthening of the axioms. In this case, Theorem 1 is provable in, for example, Peano Arithmetic (or more weakly in $I\Sigma_3$).

We know a small number of other examples of concrete incompleteness beyond this, some at higher levels of unprovability: the cases of Goodstein's metasequence and relatively

large Ramsey numbers were the first to be discovered in [9], and are unprovable in Peano Arithmetic. The Robertson-Seymour Graph Minor Theorem, proved in a sequence of twenty papers culminating in [12], is arguably the pinnacle of this line of work. It is a major theorem of broad mathematical interest, involving nothing more exotic than finite graphs. Yet in [7] Friedman, Robertson, and Seymour have nevertheless shown it to be unprovable at a very high level.

This unprovability is accompanied by some exceptionally fast-growing sequences, far outpacing the sequence $n(K)$ discussed here, namely Friedman's notorious TREE and Subcubic graph numbers – exciting flying frogs to be discussed on another occasion.

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From Heuristic to Theorem: Higher Mathematical Ideas in Middle School Competitions

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Abstract

Several ideas conventionally taught at secondary level and above appear regularly, in simplified form, in middle school competition problems. For each, we describe the informal reasoning younger students bring to a representative problem unprompted, then state the formal result and prove it, showing how the proof generalizes that informal reasoning into a technique that transfers beyond the specific numbers of the motivating problem. We argue, drawing on Polya's account of heuristic reasoning and Bruner's spiral-curriculum principle, that these results are not premature content transplanted downward, but formal names and proofs for strategies students already use instinctively.

Introduction

Elegant middle school competition problems are often solvable precisely because an underlying higher-level principle has an intuitive, low-prerequisite special case. The ten results in Sections 1–10 are existing, standard mathematics, but this article's contribution is a reusable instructional framework: an intuitive competition problem leads to heuristic discovery, which leads to a formal theorem, which transfers to broader mathematics beyond the motivating problem. The ten topics were selected because each recurs frequently in middle-school competitions while exemplifying one of a small number of recurring mathematical modes of thought. The descriptions of student reasoning throughout are based on the author's experience in coaching middle-school students in Myanmar; they are offered as heuristics observed in coaching rather than findings from a controlled study. Section 11 supplements this core material with problems drawn directly from Junior I and Junior II papers of the Myanmar Open Mathematics Competition [5], organized by the Mathematical Society of Myanmar, presented on their own terms since several illustrate techniques adjacent to, but distinct from, the main sequence.

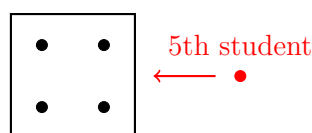
Theoretical Background

Two established lines of work motivate the approach taken here. Polya [7] argued that problem-solving proceeds through heuristics, such as looking for patterns, testing special cases, and guessing before proving, that can be learned independently of formal theory, and that proofs are best introduced once a reasonable conjecture has already been developed. Bruner's spiral-curriculum principle [2] holds that a concept can be grasped meaningfully at an early age in an intuitive form, then revisited later in more formal terms. Together these suggest an instructional sequence: introduce the problem, let the heuristic (Polya) phase run its course, and only then give the formal statement and proof (Bruner). This sits alongside more recent work: Schoenfeld [8] similarly distinguishes the exploratory phase of solving a problem from its formal write-up, and Cuoco, Goldenberg, and Mark's *mathematical habits of mind* [3] argues that curricula should focus on the recurring modes of thought, such as pattern recognition, abstraction, and arguing by extreme cases, rather than only on results, a theme that underlies the organization of this paper. NCTM [6] likewise emphasizes conceptual understanding built through problem solving over rule memorization. More recently, Kapur's *productive failure* research [4] has found that students first exposed to a task in a problem-solving setting, even when they fail to solve it, often outperform students taught the procedure and concept beforehand. Boaler [1] similarly argues that students benefit from encountering mathematical ideas as open problems to explore before they are reduced to procedures.

1. The Pigeonhole Principle

Statement (Generalized Pigeonhole Principle). *If N objects are distributed among n boxes, some box contains at least $\lceil N/n \rceil$ objects. Consequently, $N = n(k - 1) + 1$ objects are sufficient, and $n(k - 1)$ are not sufficient, to guarantee that some box contains at least k objects.*

Problem. *What is the minimum number of students in a class needed to guarantee that at least 5 of them share the same zodiac sign, given 12 possible signs?*



one sign's box, already holding 4

Figure 25: Any single sign can absorb at most 4 students before a 5th forces a repeat. Applying this bound across all 12 signs at once gives $N = 12 \times 4 + 1 = 49$.

Student intuition. A common heuristic observed in olympiad coaching is to reason from the worst case outward: in the worst arrangement, each of the 12 signs could hold exactly 4 students ($12 \times 4 = 48$) before any sign reaches 5; the next student, regardless of sign, forces a fifth match.

Formalization. With $n = 12$ boxes and $k - 1 = 4$, if $N \leq n(k - 1) = 48$ it is possible to place at most 4 objects in each of the 12 boxes with none reaching 5, so 48 does not suffice. If $N = 49 = 12 \cdot 4 + 1$, then $49 > 12 \cdot 4$, so by the averaging bound at least one box must contain more than 4, i.e. at least 5, objects. Hence 49 is the minimum. ■

2. The AM-GM Inequality

Statement. For $a, b \geq 0$, $\frac{a+b}{2} \geq \sqrt{ab}$, with equality if and only if $a = b$.

Problem. Two positive integers sum to 20. What is the largest possible value of their product?

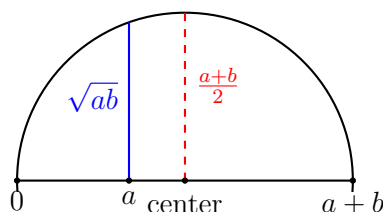


Figure 26: Classical geometric proof: on a semicircle of diameter $a + b$, the radius equals $\frac{a+b}{2}$ and the perpendicular chord at the point splitting the diameter into a and b has length $\sqrt{ab}$ (the geometric-mean relation for a right triangle inscribed in a semicircle). A chord can never exceed the radius, so $\sqrt{ab} \leq \frac{a+b}{2}$ directly, with equality exactly when the chord passes through the center, i.e. $a = b$.

Student intuition. One natural line of reasoning is to test 1×19 , 8×12 , 10×10 , noticing the product grows as the two numbers converge, and to conjecture that the balanced pair $(10, 10)$ is optimal.

Formalization. Since $(a - b)^2 \geq 0$, expanding gives $a^2 - 2ab + b^2 \geq 0$, so $(a + b)^2 \geq 4ab$, and taking square roots, $a + b \geq 2\sqrt{ab}$, which is the claim after dividing by 2. With $a + b = 20$ fixed, $10 \geq \sqrt{ab}$, so $ab \leq 100$, attained at $a = b = 10$. This confirms the conjectured maximum for every real pair summing to 20, not only the integer pairs tested. ■

3. Invariants and Parity Arguments

Statement. A quantity preserved by every allowed operation (an invariant) can be used to prove that a target configuration is unreachable, by showing the invariant's value at the target differs from its value at the start.

Problem. A row of 8 lightbulbs starts with exactly one bulb on. A move toggles any two adjacent bulbs simultaneously. Can all 8 bulbs end up on?

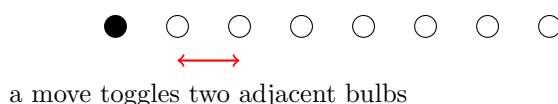


Figure 27: Eight bulbs, one lit. Each move changes the lit-count by $+2$, 0 , or -2 , so its parity is invariant: starting odd (1), the all-on state (even, 8) is unreachable.

Student intuition. A frequent starting point is trial sequences of moves that get stuck, from which one often notices informally that the number of lit bulbs seems to always change by an even amount.

Formalization. Toggling two adjacent bulbs changes the count of lit bulbs by +2 (both were off), −2 (both were on), or 0 (one of each), always an even change. Parity of the lit-bulb count is therefore invariant. The count starts at 1 (odd); the target, 8 (even), has different parity and so is unreachable. ■

Remark. A genuine invariant stays exactly fixed; a weaker but equally useful cousin, a *monovariant*, only moves in one direction (always increasing, or always decreasing) under the allowed operation. A monovariant cannot rule out a target the way an invariant does, but it can prove that a process must terminate: a different kind of impossibility argument, ruling out infinite repetition rather than a particular end state, though built on the same idea of tracking a quantity across each move.

4. Combinatorial Counting via Line Selection

Statement. In a grid formed by m horizontal and n vertical lines, the number of rectangles is $\binom{m}{2}\binom{n}{2}$, since a rectangle is uniquely determined by a choice of 2 horizontal and 2 vertical lines.

Problem. How many rectangles, of any size, are there in an 8×8 grid?

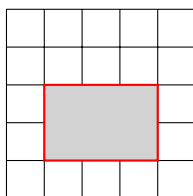


Figure 28: In a smaller 5×5 illustrative grid, choosing 2 of the 6 vertical and 2 of the 6 horizontal lines determines exactly one rectangle (shaded). Applied to the 9 lines of an 8×8 grid, the same bijection gives $\binom{9}{2}^2 = 1296$.

Student intuition. A common approach is to count rectangles formed by 1 unit squares, 2 units squares and so on, summing separate cases which can lead to correct answer, but without noticing that every rectangle is determined by a pair of vertical and a pair of horizontal lines.

Formalization. An 8×8 grid has 9 horizontal and 9 vertical lines. Any 2 of the 9 horizontal lines together with any 2 of the 9 vertical lines determine exactly one rectangle, and every rectangle arises this way exactly once, so the count is $\binom{9}{2}\binom{9}{2} = 36 \times 36 = 1296$. ■

5. Recurrence Relations

Statement. If $f(n)$ counts the ways to reach step n by moves of size 1 or 2, then $f(n) = f(n-1) + f(n-2)$ for $n \geq 2$, with $f(0) = f(1) = 1$.

Problem. A staircase has 10 steps. Each move covers 1 or 2 steps. How many distinct ways are there to reach the top?

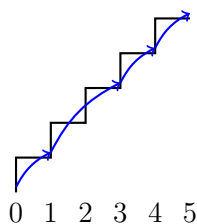


Figure 29: A 5-step staircase with one sample climbing path using steps 1, 2, 1, 1. The recurrence $f(n) = f(n-1) + f(n-2)$ counts all such paths by conditioning on whether the last move is a 1-step or a 2-step.

Student intuition. One natural line of reasoning computes the count for 1, 2, and 3 steps by hand (1, 2, 3 ways) and conjectures that each count is the sum of the two preceding it, without yet justifying why.

Formalization. The final move to reach step n is either a 1-step from step $n-1$, or a 2-step from step $n-2$; every path falls into exactly one of these two disjoint cases, so $f(n) = f(n-1) + f(n-2)$. Computing forward from $f(0) = f(1) = 1$: 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, so $f(10) = 89$. The same conditioning argument extends immediately to k allowed step sizes; e.g. with steps of 1, 2, or 3, the analogous recurrence gives $g(10) = 274$. ■

6. Graph Theory: Eulerian Paths

Statement (Euler, 1736). *A connected graph admits a trail traversing every edge exactly once if and only if it has exactly 0 or exactly 2 vertices of odd degree; in the latter case, every such trail begins at one odd-degree vertex and ends at the other.*

Problem. *Given a diagram of line segments meeting at various points, can it be drawn without lifting the pen and without tracing any segment twice?*

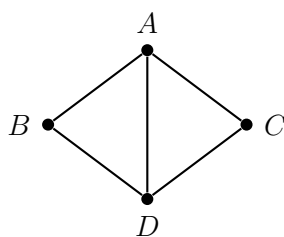


Figure 30: $\deg(A) = 3$, $\deg(D) = 3$, $\deg(B) = \deg(C) = 2$: exactly two odd-degree vertices (A and D), so an Eulerian trail exists from A to D (or D to A), but no Eulerian circuit.

Student intuition. A common heuristic is to trace by trial, backtracking when stuck, noticing that success depends on the starting point, with junctions where an odd number of segments meet causing particular trouble.

Formalization. Necessity. Every visit to a vertex other than the trail's two endpoints uses one edge to enter and a different edge to leave, pairing up its incident edges, so every such vertex must have even degree.

Sufficiency (sketch). If every vertex has even degree, walking from any vertex along unused edges can only get stuck back at the start, producing a closed trail; if edges remain, the

same argument applied to what is left (which is still all even-degree, and shares a vertex with the trail already found, since the graph is connected) produces another closed trail that can be spliced in, and repeating this uses every edge. The 2-odd-vertex case reduces to this by temporarily joining the two odd vertices, finding a closed trail, then removing that edge to open it. ■

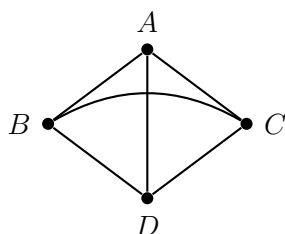


Figure 31: Adding a single edge BC to the diagram above raises every vertex to degree 3: $\deg(A) = \deg(B) = \deg(C) = \deg(D) = 3$, four odd-degree vertices. Since Euler's criterion admits only 0 or 2 odd-degree vertices, this graph (the complete graph K_4) has no Eulerian trail at all: one extra edge is enough to destroy the property.

7. Number Theory: Counting Divisors

Statement. If $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$, the number of positive divisors of n is $(a_1 + 1)(a_2 + 1) \cdots (a_k + 1)$.

Problem. How many positive divisors does 2024 have?

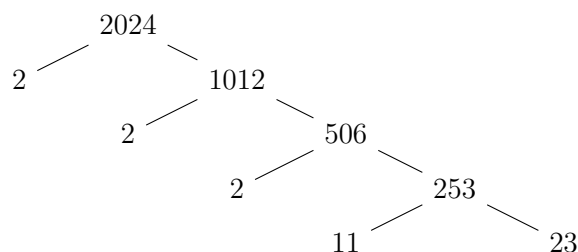


Figure 32: Repeatedly splitting off a prime factor of 2024 gives the leaves 2, 2, 2, 11, 23, so $2024 = 2^3 \times 11 \times 23$. A divisor is exactly a choice of how many copies of each repeated prime to keep: 0 to 3 of the factor 2, and 0 or 1 of each of 11 and 23, giving $4 \times 2 \times 2 = 16$ divisors.

Student intuition. A common approach lists divisors in pairs by trial division (1×2024 , 2×1012 , $4 \times 506, \dots$), which works but becomes unreliable to verify as numbers grow.

Formalization. Any divisor of $n = p_1^{a_1} \cdots p_k^{a_k}$ has the form $p_1^{b_1} \cdots p_k^{b_k}$ with $0 \leq b_i \leq a_i$, and each choice of exponents gives a distinct divisor, so there are $(a_1 + 1) \cdots (a_k + 1)$ divisors in total. Since $2024 = 2^3 \times 11 \times 23$, the count is $(3 + 1)(1 + 1)(1 + 1) = 16$. ■

8. Arithmetic Series

Statement. $1 + 3 + 5 + \cdots + (2n - 1) = n^2$.

Problem. Find $1 + 3 + 5 + \cdots + 99$.

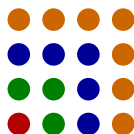


Figure 33: The first four odd numbers (1, 3, 5, 7, where each gnomon gets its own color) tile a square array of dots of size 4×4 , yielding the equality $1 + 3 + 5 + 7 = 4^2$, the same identity proved below using pairing and again using telescoping.

Student intuition. A common heuristic pairs the first and last terms ($1 + 99 = 100$, $3 + 97 = 100, \dots$), or notices that running totals (1, 4, 9, 16, 25, $\dots$) are perfect squares, without yet proving the pattern continues.

Formalization. Pairing the i -th term from each end, $(2i - 1) + (2n - 2i + 1) = 2n$, and there are $n/2$ such pairs, giving total $n \cdot n = n^2$ (a short adjustment handles odd n identically via the standard arithmetic-series formula $\frac{n}{2}(\text{first} + \text{last})$). Since $99 = 2(50) - 1$, this is the sum of the first 50 odd numbers, $50^2 = 2500$. ■

9. Telescoping Sums

Statement. $\sum_{k=1}^n \left(\frac{1}{a_k} - \frac{1}{a_{k+1}} \right) = \frac{1}{a_1} - \frac{1}{a_{n+1}}.$

Problem. Find $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{99 \cdot 100}.$

$$1 \quad -\frac{1}{2} \quad +\frac{1}{2} \quad -\frac{1}{3} \quad +\frac{1}{3} \quad -\frac{1}{4} \quad \cdots$$

Figure 34: Writing $\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$ and laying the terms end to end, each $-\frac{1}{k}$ is immediately canceled by the next term's $+\frac{1}{k}$; only the very first term (1) and the eventual last term ($-\frac{1}{100}$) survive.

Student intuition. One natural line of reasoning computes partial sums ($\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots$) and notices the pattern $\frac{n}{n+1}$ without yet knowing why each new term nearly cancels the previous growth.

Formalization. Since $\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$, writing out the sum, every interior term cancels: $\left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{99} - \frac{1}{100}\right) = 1 - \frac{1}{100} = \frac{99}{100}$. ■

10. Euler's Totient Function

Statement. If $n = p_1^{a_1} \cdots p_k^{a_k}$, the number of integers in $\{1, \dots, n - 1\}$ coprime to n is $\varphi(n) = n \prod_{i=1}^k \left(1 - \frac{1}{p_i}\right).$

Problem. How many positive integers x less than 2023 are there such that $\frac{x}{2023}$ is in reduced (simplest) form?

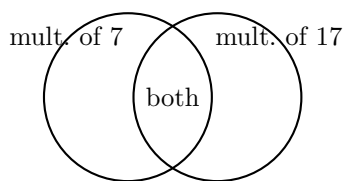


Figure 35: Counting integers in $\{1, \dots, 2023\}$ coprime to $2023 = 7 \times 17^2$ is inclusion-exclusion on the two overlapping regions of multiples of 7 and 17; the surviving (unshaded) count is $\varphi(2023) = 1632$.

Student intuition. A common heuristic crosses off numerators sharing a factor with the denominator one prime at a time, an informal, unlabeled instance of inclusion-exclusion.

Formalization. By inclusion-exclusion on the primes dividing n , the proportion of $\{1, \dots, n\}$ not divisible by any p_i is $\prod_i \left(1 - \frac{1}{p_i}\right)$, giving the stated formula. Since $2023 = 7 \times 17^2$, $\varphi(2023) = 2023 \times \frac{6}{7} \times \frac{16}{17} = 1632$. ■

11. Further Examples from the Myanmar Open Mathematics Competition

The six problems below, quoted verbatim from actual MOMC Junior I and Junior II papers [5], stand as a self-contained set rather than being distributed among Sections 1–10, and are restricted to problems that go beyond what an average Grade 7–8 student would solve without difficulty.

11.1 Pigeonhole on Angles

Problem (MOMC Junior II, October 2023 [5]). “Draw straight lines through a point P on a paper. How many lines do you have to draw at least to be sure that there are two lines that make an angle of less than 15° ?”

Student intuition. A common heuristic tests small numbers of lines directly, sketching them spaced as far apart as possible, before noticing the question is really about how many equal arcs a 180° half-turn splits into.

Formalization. A line’s direction is an angle in $[0^\circ, 180^\circ)$, since a line and its reverse direction coincide. Splitting $[0^\circ, 180^\circ)$ into $180/15 = 12$ arcs of exactly 15° , one line can be placed at each arc boundary ($0^\circ, 15^\circ, \dots, 165^\circ$) with every pairwise angle a multiple of 15° , so 12 lines can avoid the condition. A 13th line must fall inside one of these 12 arcs alongside an existing line, forcing an angle strictly less than 15° between them by the Pigeonhole Principle (Section 1). So 13 lines are needed. ■

11.2 A Permutation Parity Argument

Problem (MOMC Junior II, October 2024 [5]). “Suppose that $x_1, x_2, \dots, x_{2025}$ is an arbitrary arrangement of $1, 2, 3, \dots, 2025$. Prove that $(x_1 - 1)(x_2 - 2) \cdots (x_{2025} - 2025)$ is even.”

Student intuition. One natural line of reasoning tests small cases, such as arrangements of $1, 2, 3$ or $1, \dots, 5$, and notices at least one factor $x_i - i$ always comes out even, without yet seeing why it must be forced.

Formalization. Suppose, for contradiction, that every factor $x_i - i$ is odd. Since $x_1, \dots, x_{2025}$ is a permutation of $1, \dots, 2025$, $\sum_{i=1}^{2025} (x_i - i) = \sum x_i - \sum i = 0$. But a sum of 2025 odd terms (an odd count of odd numbers) is itself odd, while 0 is even, a contradiction. So some factor $x_i - i$ is even, making the product even. This is a parity argument in the spirit of Section 3, though the invariant here is the parity of a sum forced to zero by the permutation structure, rather than a quantity tracked across a sequence of moves. ■

11.3 Subtracting Two Quadratics

Problem (MOMC Junior II, January 2025 [5]). “There is a real number r that is a solution of both equations $x^2 + 2ax + b = 0$ and $x^2 + 2bx + a = 0$. Given that $a \neq b$, find the value of $a + b$.”

Student intuition. A common approach substitutes r into both equations and tries to solve the resulting system for a and b separately, before noticing that subtracting the two equations eliminates the quadratic term entirely.

Formalization. Subtracting the second equation from the first: $2ar + b - 2br - a = 0$, i.e. $(a - b)(2r - 1) = 0$. Since $a \neq b$, $2r - 1 = 0$, so $r = \frac{1}{2}$. Substituting back into the first equation: $\frac{1}{4} + a + b = 0$, so $a + b = -\frac{1}{4}$. ■

11.4 A System from Ratios

Problem (MOMC Junior II, October 2023 [5]). “A box contains red, white and blue balls. The number of red balls is an even number and the total number of balls is less than 100. The number of white and blue balls together is exactly 4 times the number of red balls. The number of red and blue balls together is exactly 6 times the number of white balls. How many balls are in the box?”

Student intuition. A common approach assigns variables to each color and writes down the two given relations, then experiments with small values of the red count before organizing the search algebraically.

Formalization. Let R, W, B be the red, white, blue counts. From $W + B = 4R$, the total is $R + W + B = 5R$. From $R + B = 6W$ and $B = 4R - W$, substitution gives $R + 4R - W = 6W$, so $7W = 5R$, i.e. $W = \frac{5R}{7}$, forcing $7 \mid R$. Combined with R even and $5R < 100$ (so $R < 20$), the only option is $R = 14$, giving $W = 10$, $B = 4(14) - 10 = 46$, and total $= 5(14) = 70$. ■

11.5 The Remainder Theorem for Divisibility

Problem (MOMC Junior I, January 2026 [5]). “Given that $(m - 2)$ is a positive integer and that it is also a factor of $3m^2 - 2m + 7$, find all possible values of m .”

Student intuition. A common approach substitutes small integer values of m directly and checks divisibility by hand, before noticing that polynomial division would settle the question for every m at once.

Formalization. Dividing, $3m^2 - 2m + 7 = (m - 2)(3m + 4) + 15$, so $(m - 2) \mid (3m^2 - 2m + 7)$ exactly when $(m - 2) \mid 15$. The positive divisors of 15 are 1, 3, 5, 15, giving $m - 2 \in \{1, 3, 5, 15\}$, so $m \in \{3, 5, 7, 17\}$. ■

11.6 An AM-GM Minimization

Problem (MOMC Junior II, October 2023 [5]). “For positive numbers x and y , find the minimum value of $\left(2x + \frac{1}{3y}\right)\left(4y + \frac{1}{6x}\right)$.”

Student intuition. A common approach tries specific values of x and y by trial, or attempts to expand and minimize, before noticing that the expansion collapses to a function of the single product xy .

Formalization. Expanding: $\left(2x + \frac{1}{3y}\right)\left(4y + \frac{1}{6x}\right) = 8xy + \frac{2x}{6x} + \frac{4y}{3y} + \frac{1}{18xy} = 8xy + \frac{1}{18xy} + \frac{5}{3}$. Writing $t = xy > 0$ and applying AM-GM (Section 2) to $8t$ and $\frac{1}{18t}$: $8t + \frac{1}{18t} \geq 2\sqrt{8t \cdot \frac{1}{18t}} = 2\sqrt{\frac{4}{9}} = \frac{4}{3}$, with equality at $t = \frac{1}{12}$. The minimum value is therefore $\frac{4}{3} + \frac{5}{3} = 3$. ■

Limitations

This article is an expository and pedagogical reflection rather than an empirical study: the student-reasoning descriptions are a practitioner’s field observations from coaching, not data from a controlled study. A natural next step is a classroom study comparing this sequencing against formalization-first instruction.

Conclusion

Across the results in Sections 1–10, the mathematics is standard, but two things are not: the analysis of the informal reasoning that precedes formal exposure, and the observation that these ten results collapse into a smaller number of recurring deep ideas appearing under different names. Consonant with Polya’s heuristic-then-proof sequence and Bruner’s spiral curriculum, the resulting instructional order poses the accessible instance, lets the informal strategy surface on its own, and only then supplies the formal statement and proof. Section 11’s further examples, drawn directly from MOMC papers, are offered in the same spirit but as standalone problems worth knowing on their own terms. Competition problems, in this view, serve not merely as exercises in ingenuity, but as natural entry points into the formal structures students will later encounter throughout higher mathematics.

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2024 International Mathematical Olympiad

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Angelo was twice a contestant at the International Mathematical Olympiad. He completed a PhD in mathematics at the University of Melbourne studying algebraic curves. He is currently Director of Training for the Australian Mathematical Olympiad Committee (AMOC), and Australian Team Leader at the International Mathematical Olympiad. He enjoys composing Olympiad problems for mathematics contests.

The 65th International Mathematical Olympiad (IMO) was held 11–22 July 2024 in the city of Bath, United Kingdom. This was the fourth time that the United Kingdom has hosted the IMO.

A total of 609 high school students from 108 countries participated. Of these, 81 were female.

As per normal IMO rules, each participating country may enter a team of up to six students, a Team Leader and a Deputy Team Leader.¹

Participating countries also submit problem proposals for the IMO. This year there were 229 problem proposals from 63 countries. The IMO Problem Selection Committee shortlisted 31 of these for potential use on the IMO exams.

At the IMO the Team Leaders, as an international collective, form what is called the *Jury*. The Jury makes the important decisions that shape each year's IMO. Their first task is to set the two IMO competition papers from the aforementioned shortlist and approve marking schemes. During this period the Leaders and their observers are trusted to keep all information about the contest problems completely confidential.

The six problems that ultimately appeared on the IMO exam papers may be described as follows.

1. A very easy algebra problem proposed by Colombia.
2. A medium number theory problem proposed by Indonesia.
3. A very difficult combinatorics problem about sequences proposed by Australia.
4. An easy classical geometry problem proposed by Poland.

¹The IMO regulations also permit countries to enter a small number of additional staff as Observers. These may fulfil various roles such as meeting child safety obligations, assisting with marking and coordination, or learning about how to host an IMO.

5. A medium game-style combinatorics problem proposed by Hong Kong.
6. A very difficult algebra problem about functions defined on the set of rational numbers proposed by Japan.

These six problems were posed in two exams held on Tuesday 16 July and Wednesday 17 July. Each exam paper had three problems. The contestants worked individually. They were allowed four and a half hours per paper to attempt the problems. Each problem was scored out of a maximum of seven points.

After the exams, the Leaders and their Deputies spent about two days assessing the work of the students from their own countries, guided by the marking schemes which had been agreed to earlier. A local team of markers called Coordinators also assessed the papers. They too were guided by the marking schemes but are allowed some flexibility if, for example, a Leader brought something to their attention in a contestant's exam script that is not covered by the marking scheme. The Team Leader and Coordinators must agree on scores for each student of the Leader's country in order to finalise scores. Any disagreements that cannot be resolved in this way are ultimately referred to the Jury. No such referrals occurred this year.

The contestants found Problems 1 and 4 to be the easiest with average scores of 5.57 and 4.85, respectively. Problem 6 was the hardest, with only 5 contestants receiving full marks on it. It averaged just 0.40 overall. The score distributions by problem number were as follows.

Mark	P1	P2	P3	P4	P5	P6
0	29	217	501	129	316	482
1	54	80	40	39	97	76
2	22	129	38	19	3	27
3	35	10	10	10	9	9
4	13	2	3	4	1	2
5	12	9	2	8	18	7
6	31	6	7	7	11	1
7	413	156	8	393	154	5
Mean	5.57	2.54	0.44	4.85	2.25	0.40

The medal cuts were set at 29 points for Gold, 22 for Silver and 16 for Bronze. The medal distributions² were as follows.

	Gold	Silver	Bronze	Total
Number	58	123	145	326
Proportion	9.5%	20.2%	23.8%	53.5%

These awards were presented at the closing ceremony. Of those who did not get a medal, 170 contestants received an Honourable Mention for scoring full marks on at least one problem.

Haojia Shi of the People's Republic of China was the sole contestant who achieved the most excellent feat of a perfect score of 42. He was given a standing ovation during the presentation of medals at the closing ceremony.

²The total number of medals is approved by the Jury and should not normally exceed half the total number of contestants. The numbers of Gold, Silver and Bronze medals should be approximately in the ratio 1:2:3.

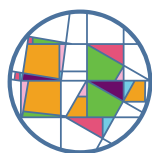
The 2024 IMO was organised by Purposeful Ventures with the support of XTX Markets.

Hosts for future IMOs have been secured as follows.

10-20 July, 2025	Sunshine Coast, Australia
2026	People's Republic of China
2027	Hungary
2028	Saudi Arabia

Much of the statistical information found in this report can also be found on the official website of the IMO.

www.imo-official.org



IMO2024
65th International
Mathematical Olympiad

English (eng), day 1

Tuesday, 16. July 2024

Problem 1. Determine all real numbers α such that, for every positive integer n , the integer

$$\lfloor \alpha \rfloor + \lfloor 2\alpha \rfloor + \cdots + \lfloor n\alpha \rfloor$$

is a multiple of n . (Note that $\lfloor z \rfloor$ denotes the greatest integer less than or equal to z . For example, $\lfloor -\pi \rfloor = -4$ and $\lfloor 2 \rfloor = \lfloor 2.9 \rfloor = 2$.)

Problem 2. Determine all pairs (a, b) of positive integers for which there exist positive integers g and N such that

$$\gcd(a^n + b, b^n + a) = g$$

holds for all integers $n \geq N$. (Note that $\gcd(x, y)$ denotes the greatest common divisor of integers x and y .)

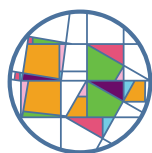
Problem 3. Let $a_1, a_2, a_3, \dots$ be an infinite sequence of positive integers, and let N be a positive integer. Suppose that, for each $n > N$, a_n is equal to the number of times a_{n-1} appears in the list $a_1, a_2, \dots, a_{n-1}$.

Prove that at least one of the sequences $a_1, a_3, a_5, \dots$ and $a_2, a_4, a_6, \dots$ is eventually periodic.

(An infinite sequence $b_1, b_2, b_3, \dots$ is *eventually periodic* if there exist positive integers p and M such that $b_{m+p} = b_m$ for all $m \geq M$.)

Language: English

*Time: 4 hours and 30 minutes.
Each problem is worth 7 points.*



IMO2024
65th International
Mathematical Olympiad

English (eng), day 2

Wednesday, 17. July 2024

Problem 4. Let ABC be a triangle with $AB < AC < BC$. Let the incentre and incircle of triangle ABC be I and ω , respectively. Let X be the point on line BC different from C such that the line through X parallel to AC is tangent to ω . Similarly, let Y be the point on line BC different from B such that the line through Y parallel to AB is tangent to ω . Let AI intersect the circumcircle of triangle ABC again at $P \neq A$. Let K and L be the midpoints of AC and AB , respectively.

Prove that $\angle KIL + \angle YPX = 180^\circ$.

Problem 5. Turbo the snail plays a game on a board with 2024 rows and 2023 columns. There are hidden monsters in 2022 of the cells. Initially, Turbo does not know where any of the monsters are, but he knows that there is exactly one monster in each row except the first row and the last row, and that each column contains at most one monster.

Turbo makes a series of attempts to go from the first row to the last row. On each attempt, he chooses to start on any cell in the first row, then repeatedly moves to an adjacent cell sharing a common side. (He is allowed to return to a previously visited cell.) If he reaches a cell with a monster, his attempt ends and he is transported back to the first row to start a new attempt. The monsters do not move, and Turbo remembers whether or not each cell he has visited contains a monster. If he reaches any cell in the last row, his attempt ends and the game is over.

Determine the minimum value of n for which Turbo has a strategy that guarantees reaching the last row on the n^{th} attempt or earlier, regardless of the locations of the monsters.

Problem 6. Let $\mathbb{Q}$ be the set of rational numbers. A function $f: \mathbb{Q} \rightarrow \mathbb{Q}$ is called *aquaesulian* if the following property holds: for every $x, y \in \mathbb{Q}$,

$$f(x + f(y)) = f(x) + y \quad \text{or} \quad f(f(x) + y) = x + f(y).$$

Show that there exists an integer c such that for any aquaesulian function f there are at most c different rational numbers of the form $f(r) + f(-r)$ for some rational number r , and find the smallest possible value of c .

Language: English

Time: 4 hours and 30 minutes.
Each problem is worth 7 points.

Some IMO Country Totals

Rank	Country	Total
1	United States of America	192
2	People's Republic of China	190
3	Republic of Korea	168
4	India	167
5	Belarus	165
6	Singapore	162
6	United Kingdom	162
8	Hungary	155
9	Poland	151
9	Türkiye	151
11	Taiwan	149
12	Romania	145
13	Bosnia and Herzegovina	144
14	Italy	143
14	Japan	143
16	Israel	142
16	Mongolia	142
18	Hong Kong	140
19	Islamic Republic of Iran	137
20	Brazil	134
21	France	133
22	Serbia	132
23	Canada	131
24	Mexico	129
25	Austria	127
25	Kazakhstan	127
27	Bulgaria	126
27	Greece	126
29	Kyrgyzstan	122
29	Peru	122
31	Germany	120
31	New Zealand	120
33	Malaysia	118
33	Vietnam	118
35	Croatia	116
35	Slovakia	116
35	Thailand	116
38	Armenia	113
38	Australia	113
38	Ukraine	113

Distribution of Awards at the 2024 IMO

Country	Total	Gold	Silver	Bronze	HM
Albania	51	0	0	1	3
Algeria	86	1	0	0	4
Argentina	90	0	1	3	1
Armenia	113	0	0	5	1
Australia	113	1	1	2	2
Austria	127	0	5	0	1
Azerbaijan	64	0	0	1	5
Bangladesh	83	0	0	2	4
Belarus	165	4	0	2	0
Belgium	80	0	1	2	1
Bhutan	8	0	0	0	0
Bolivia	26	0	0	1	1
Bosnia and Herzegovina	144	3	1	2	0
Botswana	15	0	0	0	1
Brazil	134	1	3	2	0
Bulgaria	126	0	3	2	1
Canada	131	0	4	1	1
Chile	17	0	0	0	1
Colombia	87	0	0	2	4
Costa Rica	82	0	0	2	3
Croatia	116	0	3	2	1
Cuba	68	0	0	0	5
Cyprus	88	0	2	1	3
Czech Republic	103	0	2	2	2
Denmark	88	0	0	2	4
Dominican Republic	39	0	0	0	3
Ecuador	46	0	0	1	2
El Salvador	17	0	0	0	1
Estonia	91	0	1	1	4
Finland	85	0	1	1	3
France	133	1	3	1	1
Georgia	108	1	0	3	1
Germany	120	0	2	4	0
Ghana	4	0	0	0	0
Greece	126	1	2	3	0
Honduras	1	0	0	0	0
Hong Kong	140	0	5	1	0
Hungary	155	2	3	1	0
Iceland	35	0	0	0	2
India	167	4	1	0	1

Country	Total	Gold	Silver	Bronze	HM
Indonesia	111	1	0	3	2
Iraq	50	0	0	2	1
Ireland	67	0	0	0	4
Islamic Republic of Iran	137	1	3	1	1
Israel	142	2	2	2	0
Italy	143	1	3	2	0
Ivory Coast	19	0	0	0	0
Japan	143	2	2	1	1
Kazakhstan	127	0	3	2	1
Kenya	7	0	0	0	0
Kosovo	48	0	0	0	4
Kyrgyzstan	122	1	3	0	2
Latvia	87	0	0	2	4
Liechtenstein	14	0	0	0	1
Lithuania	94	1	0	2	2
Luxembourg	60	0	0	1	3
Macau	56	0	0	0	5
Malaysia	118	0	2	4	0
Mexico	129	1	2	2	1
Mongolia	142	1	2	3	0
Myanmar	30	0	0	0	3
Nepal	28	0	0	0	2
Netherlands	109	0	1	4	1
New Zealand	120	0	3	3	0
Nicaragua	16	0	0	1	0
North Macedonia	107	1	0	2	3
Norway	85	0	0	3	3
Oman	4	0	0	0	0
Pakistan	64	0	1	1	1
Panama	10	0	0	0	1
Paraguay	20	0	0	0	1
People's Republic of China	190	5	1	0	0
Peru	122	0	2	3	1
Philippines	102	0	1	3	2
Poland	151	1	4	1	0
Portugal	71	0	0	1	4
Puerto Rico	18	0	0	0	1
Republic of Korea	168	2	4	0	0
Republic of Moldova	84	0	1	2	2
Romania	145	1	4	1	0
Rwanda	39	0	0	0	4
Saudi Arabia	111	0	1	4	1

Country	Total	Gold	Silver	Bronze	HM
Serbia	132	0	4	1	1
Singapore	162	1	5	0	0
Slovakia	116	1	1	3	1
Slovenia	92	0	1	1	3
South Africa	93	0	1	2	1
Spain	80	0	0	2	3
Sri Lanka	71	0	0	2	4
Sweden	102	0	1	3	2
Switzerland	106	1	0	1	4
Syria	64	0	0	1	4
Taiwan	149	2	2	2	0
Tajikistan	80	0	0	2	3
Thailand	116	0	3	1	2
Trinidad and Tobago	37	0	0	0	2
Tunisia	67	0	0	1	3
Türkiye	151	2	2	2	0
Turkmenistan	107	0	2	2	2
Uganda	22	0	0	0	1
Ukraine	113	0	2	3	1
United Arab Emirates	1	0	0	0	0
United Kingdom	162	2	3	1	0
United States of America	192	5	1	0	0
Uruguay	48	0	0	0	3
Uzbekistan	110	0	2	3	0
Venezuela	14	0	0	0	0
Vietnam	118	0	2	3	1
Total (108 teams, 609 contestants)		58	123	145	170

(Not all countries entered a full team of six students.)

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Editor's note: This report was submitted for Volume I 2025 but got lost in the compiling universe. Because of this, the Report of IMO 2025 will be included in Volume II 2026.



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