Vol 38 No 2 2025

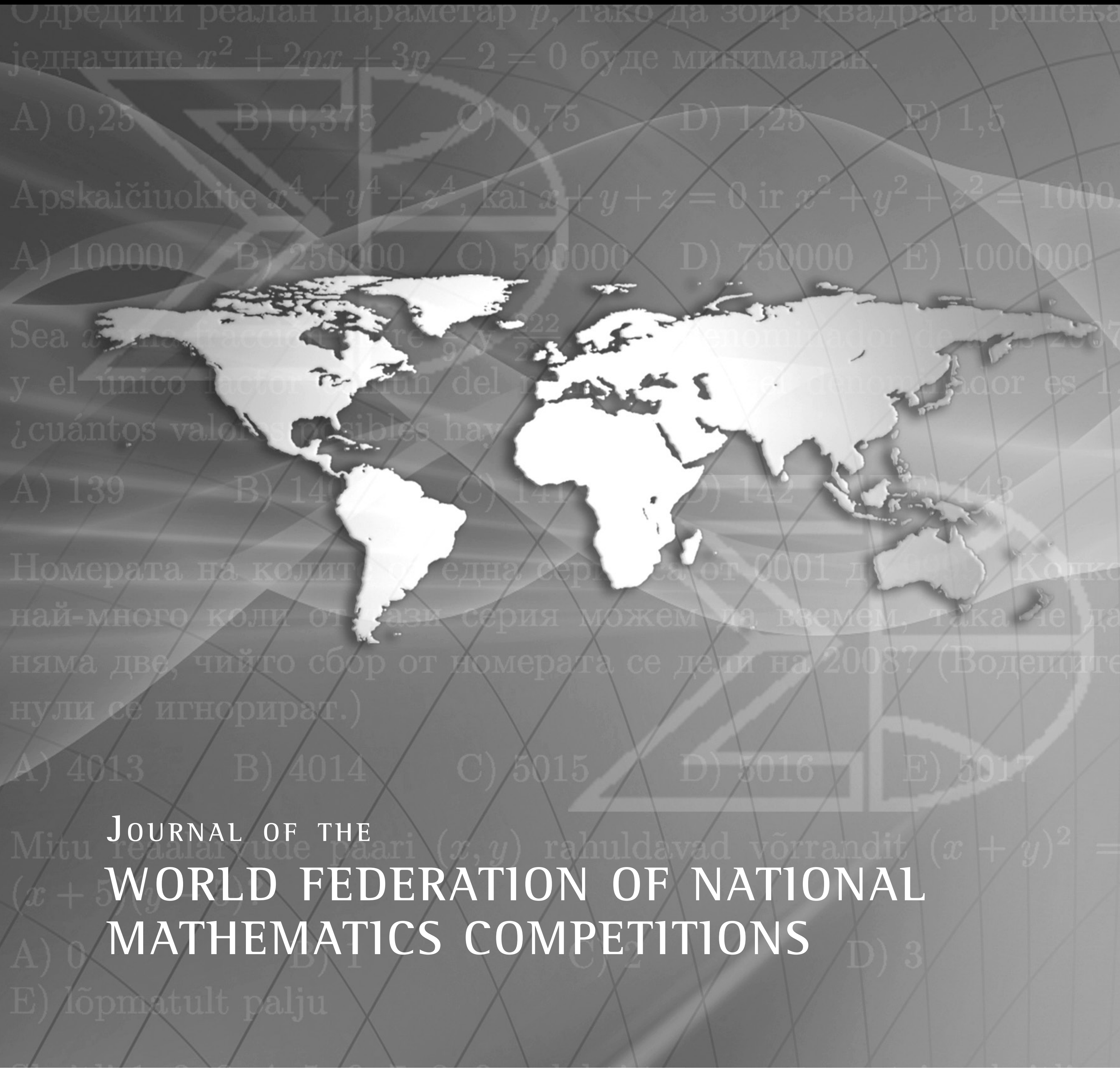
MATHEMATICS COMPETITIONS

Journal of the WORLD FEDERATION OF NATIONAL MATHEMATICS COMPETITIONS

AUSTRALIAN MATHS TRUST

Vol 38 No 2 2025

MATHEMATICS COMPETITIONS



Journal of the

WORLD FEDERATION OF NATIONAL MATHEMATICS COMPETITIONS

AUSTRALIAN MATHS TRUST

MATHEMATICS COMPETITIONS

JOURNAL OF THE WORLD FEDERATION OF NATIONAL MATHEMATICS COMPETITIONS

(ISSN 1031-7503)

Published biannually by

AUSTRALIAN MATHEMATICS TRUST
170 HAYDON DRIVE
BRUCE ACT 2617
AUSTRALIA

Articles (in English) are welcome.
Please send articles to:

Professor Maria Elizabeth Losada
Universidad Antonio Nariño
Bogotá, Colombia
director.olimpiadas@uan.edu.co

Contents

World Federation of National Mathematics Competitions

Executive

President:

Robert Geretschläger
Graz, AUSTRIA
robert@rgeretschlaeger.com

Senior Vice President:

Krzysztof Ciesielski
Jagiellonian University
Krakow, POLAND
Krzysztof.Ciesielski@im.uj.edu.pl

Vice Presidents:

Sergey Dorichenko
School 179
Moscow, RUSSIA
sdorichenko@gmail.com

Meike Akveld
ETH Zürich
Zürich, SWITZERLAND
meike.akveld@math.ethz.ch

Publications Officer:

Maria Elizabeth Losada
Universidad Antonio Nariño
Bogota, COLOMBIA
director.olimpiadas@uan.edu.co

Secretary:

David Crawford
Leicester Grammar School
Leicester, UK
davidmc103@hotmail.com

Immediate Past President:

Kiril Bankov
University of Sofia
Sofia, BULGARIA
kbankov@fmi.uni-sofia.bg

Treasurer:

Peter Taylor
University of Canberra
Canberra, AUSTRALIA
pjt013@gmail.com

Editorial Board

For WFNMC Standing Committees please refer to ABOUT WFNMC section of the WFNMC website http://www.wfnmc.org/.

From the President

Dear readers,

It is my sad duty to inform you of the passing of our dear friend Warren Atkins. He was an important part of the WFNMC right from its beginnings, and was the editor of this journal for many years. He was also a recipient of the Erdős award in 2004. He will be sorely missed by all of us that knew him.

Preparations for our congress (22-28 July, 2026 in Putrajaya, Malaysia) are well under way, and current information is always available at https://www.wfnmc.org/ and https://wfnmc2026.com/. If you have not already done so, you are invited to pre-register now. Also, you are invited to submit a talk for the conference. Submissions are directly to the group Chairs, and information on how to submit can be found at https://www.wfnmc.org/conferences.html. The Topic Groups are

Topic Group A: Building Bridges between Problems of Mathematical Research and Competitions

Chair: Alexander Soifer (asoifer (at) uccs.edu)

Topic Group B: Creating Problems and Problem Solving

Chair: Hayk Sedrakyan (sedrakyan.hayk (at) gmail.com)

Topic Group C: Competitions around the World (with a special focus on developing countries)

Chair: Paul Vaderlind (paul (at) math.su.se)

Topic Group D: Technological Applications in Mathematics Competitions and Didactics of Mathematics Competitions

Chair: Lukas Donner (lukasmarkus.donner (at) gmail.com)

Thank you to Suhaimi Ramly and Krzysztof Ciesielski for all the work they are doing in preparation for the event, and to Alexander, Hayk, Paul and Lukas for agreeing to chair the Topic Groups!

I hope to see you there.

Robert Geretschläger

Editor's Page

Dear Competitions enthusiasts, readers of our *Mathematics Competitions* journal!

Mathematics Competitions is the right place for you to publish and read the different activities about competitions in Mathematics from around the world. For those of us who have spent a great part of our life encouraging students to enjoy mathematics and the different challenges surrounding its study and development, the journal can offer a platform to exhibit our results as well as a place to find new inspiration in the ways others have motivated young students to explore and learn mathematics through competitions. In a way, this learning from others is one of the better benefits of the competitions environment.

Following the example of previous editors, I invite you to submit to our journal *Mathematics Competitions* your creative essays on a variety of topics related to creating original problems, working with students and teachers, organizing and running mathematics competitions, historical and philosophical views on mathematics and closely related fields, and even your original literary works related to mathematics.

Just be original, creative, and inspirational. Share your ideas, problems, conjectures, and solutions with all your colleagues by publishing them here. We have formalized the submission format to establish uniformity in our journal.

Submission Format
FORMAT: should be LaTeX, TeX, or for only text articles in Microsoft Word, accompanied by another copy in pdf. However, the authors are strongly recommended to send article in TeX or LaTeX format. This is because the whole journal will be compiled in LaTex. Thus your Word document will be typeset again. Texts in Word, if sent, should mainly contain non-mathematical text and any images used should be sent separately.
START: with the title centered in Large format (roughly 14 pt), followed on the next line by the author(s)' name(s) in italic 12 pt.
MAIN TEXT: Use a font from the Times New Roman family or 12 pt in LaTex.
END: with your name, address, email and your website (if applicable).
INCLUDE: your high-resolution small photo and a concise professional summary of your works and titles.
ILLUSTRATIONS: must be inserted at about the correct place of the text of your submission in one of the following formats: jpeg, pdf, tiff, eps, or mp. Your illustration will not be redrawn.
The resolution of your illustrations must be at least 300 dpi, or, preferably, done as vector illustrations.
If a text is embedded in illustrations, use a font from the Times New Roman family in 11 pt.
In figures, the letters used in labeling points, distances, etc. must be written in the same font as in the text that refers to these figures. Note that if the manuscript is prepared in Word, then mathematical symbols will be compiled with the use of the math mode in LaTex.
Furthermore, a version of the figure in a LaTex-compatible graphic language (such as TikZ, PSTricks or Asymptote) must be included with the submission to insure compatibility with the text.
REFERENCES: Every reference must be cited in the paper. References should be ordered alphabetically by author. If any webpages are mentioned as webpages, not as sources of articles, they should be in the end of the reference list. Any cited webpage should include a retrieval date.

A recommended citation style is the following:

[1] A. Soifer, *Mathematics, its history, and mathematical olympiads: a golden braid*, Mathematics Competitions 35(2022), No. 2, 8–23.
[2] A. Soifer, *The mathematical coloring book*, Springer 2008.
[3] https://www.imo-official.org/. Retrieved 1 July, 2004.

Please submit your manuscripts to María Elizabeth Losada at
`director.olimpiadas@uan.edu.co`

We are counting on receiving your contributions, informative, inspired and creative. Best wishes,

Maria Elizabeth Losada
EDITOR

Warren Atkins OAM

18 October 1938 – 24 November 2025

Mike Clapper

Warren (left) receiving his Erdős award from Professor Hyman Bass

Warren would often joke about his height, and it is true that, by any metric standard, he was not a man of great stature. However, when it comes to the history of the Australian Maths Trust, he was undoubtedly a giant. Warren started out as a high school mathematics teacher but soon found out that his vocation was to teach teachers and he moved to a position at the Canberra College of Advanced Education where he was in charge of the training of secondary maths teachers for almost thirty years. In 1976, along with colleagues Peter Taylor, Jo Edwards and Peter O'Halloran, Warren, as Chairman of the Problems Committee, was responsible for producing the very first High School Mathematics Competition in 1976, which was so successful that, by 1978, it had morphed into a national competition, the Australian Mathematics Competition.

Warren continued on the Problems Committee for an astonishing 44 years, only stepping down in

2022. For most of that time (1976 – 79 and 1981 – 2012) he was the committee chair, and in 2003 he was instrumental in setting up a second problems committee to develop primary divisions of the competition, which began in 2004.

Warren was a member of the Australian Mathematics Foundation from 1976 – 2012 and was the Chair of that body from 1995 – 2012. He was also deeply involved in the establishment of the World Federation of National Mathematics Competitions (WFNMC) and at the inaugural meeting of this body in 1984, he took on the role of editing the Foundation newsletter and was appointed as editor of the Federation journal *Mathematics Competitions*, a role which he continued until 2017. Along with Peter Taylor, Warren was also responsible, for many years, for the in-house editing of Solutions and Statistics books for the AMC and the publication of a number of books on problem-solving. Warren has also been a member of the AMOC committee. He received the BH Neumann award from the Trust in 1993. The WFNMC honoured Warren with the Erdős award in 2004 and he received an OAM in 2018 for his services to mathematics education.

Impressive though this list of achievements is, it does not capture Warren, the man. An active sportsman, Warren was still swimming regularly and playing tennis until shortly before his final illness. He was a man of great warmth and charm, who, along with his wife Naida, ensured that all problems committee members felt welcome as a part of the AMT community. Warren's leadership style was all about building trust and confidence and valuing all contributions. I had the great privilege of taking over from Warren as Chair of the Problems committees in 2012 and he prepared me for this role over a number of years and was always there to support me. Colleagues have spoken to me about Warren's generosity of spirit and gentle humour and I don't think any committee members thought of the work as a chore, because Warren made it such an enjoyable experience for everybody.

Warren was famous for his problem cards. In the days before everything became computerised, every question submitted would be stuck on an index card and when papers were in preparation, these cards would be shuffled around on the table to produce the best possible paper. Warren always lamented that the giant repository of cards which constituted questions submitted but not used, was a valuable resource and that more should be done with it. I believe these boxes of cards are still sitting in a cupboard somewhere in the AMT office!

It is incredibly sad that in 2025, we have lost two of the giants in the history of the AMT in Bruce Henry and Warren Atkins.

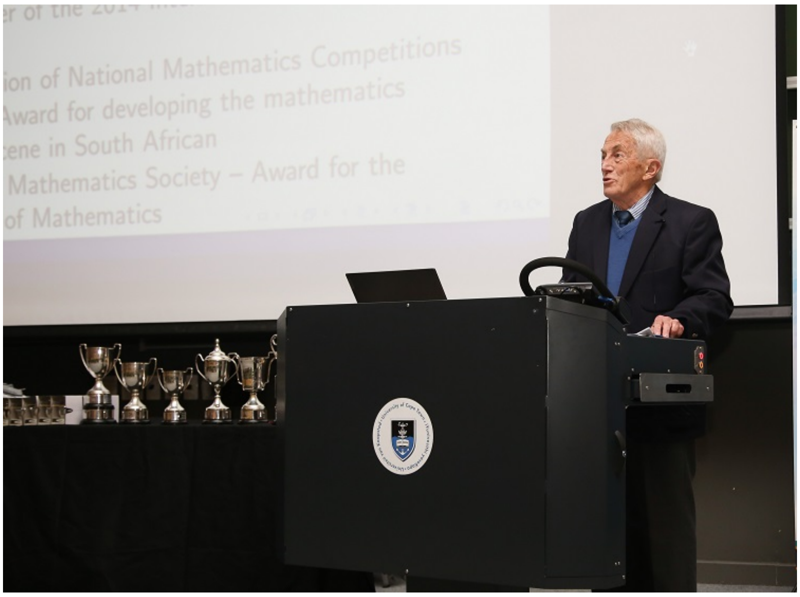
Mike Clapper

Mike.Clapper@amt.edu.au

In Memoriam John Webb

18 April 1942 - 14 May 2025

Mark Saul



John Webb at Awards ceremony of National Mathematics Competitions in South Africa in 2022

Friendships develop in myriad ways. You may meet someone with whom you have a single common interest in, say, railroads. And all you talk about when you have lunch is railroads. Slowly, the friendship expands to other intellectual domains, then overflows into the personal and the emotional. All from railroads.

I first met John Webb in Zhongshan, China, at a meeting of the WFNMC. We found each other in a working group about contest problems for younger students. We soon discovered a deep interest in problem solving. In particular, we both enjoyed finding interesting elementary ways to solve difficult problems and ways of posing difficult problems in elementary ways.

But somehow we sensed a deeper connection than the intellectual. After a few days, we were talking about how we might work together, and John ended up inviting me to South Africa to work on a project with rural teachers there. I stayed with him and his wife Anthea during the visit, and our relationship quickly deepened. We never completely abandoned shop talk: contests, teaching, the joys and sorrows of working with children. And soon our relationship reached a personal and emotional level. I met his children, and we talked about his grandchildren, who just then were not nearby.

“Someday you will meet my grandchildren,” said John. “And someday” he quipped, “I will meet yours.” In fact I didn’t have any at the time.

John and I worked together on various projects after that, and I made several visits to his country.

We worked with teachers, with Olympiad teams, and with written materials. One of my last communications with John was his inquiry into a set of problems I had written last spring, all of which turn out to be bogus. April fool! And John loved it.

John and I met all around the world. In Cambridge (UK), he took me on a tour of his alma mater, Caius College, and recounted an incident when Stephen Hawkings, just then falling ill, stumbled on a sidewalk and John helped him. I eventually did meet his grandchildren (although he missed mine...) and spent an evening at the opera in New York with his son Jonathan.. On our visits, we touched on the history of South Africa, on the mechanisms which legalized apartheid—and on how he evaded its repression and kept his contests and materials open to everyone in the country. He was particularly proud at one point that none of the members of one of his Olympiad teams was a male of European descent.

John pioneered the inclusion of African nations in the International Mathematical Olympiad. In 2002 he was elected Secretary of that organization, a post he held until 2013. His influence thus extended far beyond his own country. His leadership of the IMO never faltered, despite political vicissitudes. He was a dignified and eloquent representative of the organization, to both the mathematical community and the wider public.

The IMO community is strange and wonderful. Like the legend of the Scottish village Brigadoon, it comes to life for ten days each year. And yet it is enough time, and enough continuity, for one to form friendships that abide. John and I knew each other outside of the IMO, and during the event his was a busy schedule. Yet he always found time for lunch or dinner with friends, and so we stayed in contact at least once a year.

Through John I met several of his colleagues, whose work he inspired and whose efforts continue to this day. John's legacy will last longer than any single life, longer than memory of this article or even this journal. It will reside deep in the cultural unconscious of many. He will be missed, even as his legacy endures.

Mark Saul
marksaul@earthlink.net

A new competition: *The Banach Binge for the Brainy*

Krzysztof Ciesielski

Krzysztof Ciesielski is a professor at Jagiellonian University in Kraków and the Chair of Centre of History of Mathematics at this university. For nine years he served as Vice-Head (responsible for teaching) of the Mathematics Institute of this University. His mathematical specialty is topological dynamical systems. He is an author and co-author (mainly with Zdzisław Pogoda) of some books popularizing mathematics that became bestsellers in Poland and were awarded prestigious prizes, including the Steinhaus Prize and "The Golden Rose Prize" for the best books popularizing science published in Poland. In 1980–2025 he was a member of the Kraków Committee of the Polish Mathematical Olympiad, being its vice-Chair in 1991–2008 and its Chair in 2008–2025. Since 2001 he has been vice-Chair of Kraków Committee of the Mathematical Kangaroo. He is currently the Senior Vice President of the WFNMC and a member of the Council of the European Mathematical Society.

Abstract
In 2025, a brand new competition combining mathematics and history was organised in Ostrowsko, a small village in Poland. The central motivation of this competition was the figure of one of the most eminent mathematician of the XXth century, Stefan Banach. In the article, this competition and the problems from its first edition are described.

Stefan Banach and the idea of a new competition

Stefan Banach (1892–1945) is widely regarded as the best Polish mathematician in history. He lived in two cities: Lvov, where he obtained the most of his great mathematical achievements, and Kraków, where he spend his youth. However, there is another place somehow connected with Banach. This is Ostrowsko, a small village (about 1300 citizens) in Nowy Targ County, about 70 kilometers from Kraków. It is located on the road from Nowy Targ to Czorsztyn, by the Dunajec River, and was founded in the XIVth century. The family of Stefan Greczek, Banach's father came from Ostrowsko. Greczek's house is still there and is owned by the family.

In Poland, there is a habit that schools are named after famous persons. The primary school in Ostrowsko is named after Professor Andrzej Waksmundzki, a chemist, the husband of Antonina Waksmundzka, Banach's half-sister. However, in Ostrowsko and in this school also Banach is remembered. The head of this school is Paweł Nowotarski, a mathematician. In this school, with his significant contribution, the new mathematical competition was created. The competition (for primary schools, where pupils in the age 7–15 years old are being educated) was a completely unusual unique event. The participants had to prove their knowledge and skills in mathematics (in the suitable level) and history (concerning Stefan Banach). The competition was intended for pupils from schools in the Nowy Targ area. Each school could delegate a two-person team, working together during the competition.

The competition was called "Banachalia bystrzaków". This phrase is very difficult to translate into English. It may be translated as "The Banach Binge for the Brainy", This translation is due to Katarzyna and Krzysztof Kwaśniewicz[1]. The Polish name requires explanation. The Polish name "Bachanalia" means "the Bacchanalia" which were popular Roman festivals of Bacchus, Roman's god of wine-making, orchards, festivity, insanity and ritual madness, that were organised in ancient Rome in the second century BC. One can see here an ingenious word play, a spoonerism "Bachanalia–Banachalia". The word "bystrzak", in fact, does not exist in the Polish language, but its meaning is obvious (sometimes this word is used in slang). The word "bystry" means "very clever, brainy, quick-witted". An inflection "ak" added to an adjective gives a noun of a person possessing the described property.

The competition

The preparation of problems was a very difficult task, especially because of historical matters. A great deal was written and published about Banach, but much of it is false information (see [4], [5]). It would not be the students' fault if their answers to the question were wrong because they were based on incorrect information. On the other hand, the questions had to be precise and only true answers would be accepted. It was decided that some articles and books that presented Banach correctly should be chosen (in particular, book [11]) and it was stated that these are the sources of information for students preparing for the competition (all these sources were available on the Internet). The problems were prepared by the group of mathematics teachers of Waksmundzki Primary School in Ostrowsko and the author of this article.

The competition took place on March 31, 2025, on the occasion of the anniversary of Banach (Banach's birthday was on March 30, but in 2025 this day was a Sunday).

During the competition, each team had to answer 25 questions in 75 minutes. The papers were corrected immediately, during this time the participants listened to special lectures delivered by Monika Waksmundzka-Hajnos (a daughter of Andrzej Waksmundzki, a half-nephew of Banach), Anna Mlekodaj and Krzysztof Ciesielski. Then, prizes were presented. Each team was also given a special wooden memorial plaque (see Fig.1). In the hall, where the competition took place, participants and guests could see a bust of Banach, made by an artist Marek Szala (see Fig.2).

Figure 1. The memorable wooden plague

[1]The author is greatly indebted to Katarzyna and Krzysztof Kwaśniewicz for this translation.

Figure 2. The bust of Banach

Some participants proved to have very high knowledge and skills. The number of points possible to obtain was $37\frac{1}{4}$. The winning team obtained 89% of the points, the second prize winners – 85%, the third prize winners – 83%. The questions (with the points awarded for solving problems) are presented below. The readers are encouraged to solve problems on their own and compare their results with the results of the recipients of prizes. The answers are given afterward, as well as a more precise description of some historical events concerning the questions.

It is planned to organise a competition like this in subsequent years.

Problems

1. True or false? Select the correct answer ($\frac{1}{4}$ point for each correct answer).

A. Stefan Banach earned money by teaching private lessons.

B. He graduated from university and obtained an MSc diploma with special distinctions.

C. Banach's studies at the Vilnius Technical University were interrupted by the outbreak of World War I.

D. For the German mathematician Steinhaus, his greatest discovery was the result of an accidental meeting with Stefan Banach in Planty Park in Kraków.

E. Stefan Banach became a professor in Lvov.

F. The only results of Lvov mathematicians known to the world between WWI and WWII were those related to functional analysis.

G. Stefan Banach was the author of school textbooks as well as the outstanding mathematical monograph *Théorie des opérations linéaires*.

H. During the German occupation, he protected himself by working for Prof. Wishof's institute as a lice feeder. In this way he also was earning money for a living for his family.

I. Stefan Banach preferred to work on mathematical problems in peace and quiet.

J. When Stefan Banach graduated secondary school, he believed that there was much to be discovered in mathematics and took up mathematical studies.

K. During the discussions in the “Swedish Café” mathematicians used to write mathematical formulas on marble tables in the café,

L. Stefan Banach, although he was left-handed, was writing with his right hand.

2. Select the set of words that includes a word **unrelated** to Stefan Banach’s life. (1 point)

A. Maria Płowa, goose, Steinhaus, Kraków

B. Washerwoman, Prof. Mazur, Lvov, billiards, basketball

C. Ludwik Mien, lice, private lessons, photographer

D. left-handed, tennis, football, walking stick

3. Stefan Banach was raised in a foster family (1 point)

A. Stefan Greczek and Albina Greczek in Kraków

B. Józef Greczek and Antonina Greczek in Ostrowsko

C. Franciszka Płowa and Paweł Płowy in Kraków

D. Franciszka Banach and Paweł Banach in Lipnica Murowana

4. After completing primary school, 10-year-old Stefan became a student at (1 point)

A. IV Gymnasium in Kraków (formerly a branch of I Gymnasium in Kraków)

B. Seweryn Goszczyński Gymnasium in Nowy Targ

C. III Gymnasium in Kraków

D. III Franz Josef Gymnasium in Lvov

5. Select the incorrect answer. (1 point)

A. Stefan Banach was a friend of Witold Wilkosz, with whom he had numerous scientific discussions.

B. Stefan Banach studied mathematics as a self-taught man.

C. After obtaining his doctorate at the University of Lvov, Stefan Banach worked as an assistant at the Lvov Polytechnic.

D. Stefan Banach became a mathematics teacher at the 4th Gymnasium in Kraków.

6. Match the pairs by entering the appropriate letters in brackets ($\frac{1}{4}$ point for each correct answer).

I. [.....] Franciszka Płowa	A. mother
II. [.....] Katarzyna Banach	B. father
III. [.....] Ludwik Mien	C. laundry owner
IV. [.....] Stefan Greczek	D. photographer
V. [.....] Steinhaus	E. bench

7. Choose the correct ending for the sentence. Mark A or B and later C or D. ($2 \times \frac{1}{2}$ point)

Much of the evidence gathered in the café among scientists was lost because:

A. no one wrote it down,

B. it was written on the marble tabletop,

C. the tables in the café were wiped clean.

D. the attempts to prove it went on forever.

8. Stefan Banach was one of the founders of the Polish Mathematical Society in 1919 (1 point)

A. in Vilnius

B. in Kraków

C. in Lvov

D. in Warsaw

9. When Hugo Steinhaus heard a conversation between Banach and Nikodym, his attention was drawn to the word "integral" mentioned in their conversation in connection with the name of a certain mathematician. Indicate the name of this mathematician. (1 point)

A. Darboux

B. Lebesgue

C. Newton

D. Riemann

10. Banach spent the academic year 1924/25 abroad as a Rockefeller Foundation scholar. Which country did he visit? (1 point)

A. Austria

B. France

C. Germany

D. Italy

11. Stefan Banach's wife was (1 point)

A. Łucja Braus, a cousin of H. Steinhaus

B. Helena Alfus from Kraków

C. Maria Puchalska from Kraków

D. Antonina Greczek from Ostrowsko

12. Construct a true sentence by selecting in sequence: A or B, then a word from the second column, then 1, 2 or 3. (1 point)

<table>
<tr><td rowspan="2">A. Banach was quickly solving down mathematical problems himself,</td><td rowspan="4">because/but</td><td>1. he used to note diligently what he had proven.</td></tr>
<tr><td rowspan="2">2. not all of them were published, as he would have needed two or three secretaries to note his ideas.</td></tr>
<tr><td rowspan="2">B. Banach published most of his mathematical proofs,</td></tr>
<tr><td>3. his students, assistants and colleagues were noting them down on the fly.</td></tr>
</table>

13. Write who, why and in which city gave such a peculiar gift to the Swedish mathematician Per Enflo (see the picture)? (2 points)

14. Stefan Banach (1 point)

A. was employed at the University of Lvov in 1920 as a result of a recommendation by Prof. Steinhaus.

B. was employed at the University of Lvov in 1920 in the department headed by Prof. Łomnicki.

C. a few days after his first meeting with Steinhaus, he presented a solution of a problem that Steinhaus had previously been working on for some time.

D. All of the above statements are true.

15. What **cannot** be attributed to Banach? (1 point)

A. He was interested in a very wide range of mathematical disciplines.

B. His published works did not fully reflect his mathematical passions.

C. He worked on mathematics alone, avoiding discussions with others.

D. He may be regarded as the best of all Polish mathematicians.

16. After his death, Stefan Banach was buried (1 point)

A. at the Lychakov Cemetery in Lvov.

B. at the Meritorious Cemetery in Pęksowy Brzyzek.

C. at the Powązki Cemetery in Warsaw.

D. at the Rakowicki Cemetery in Kraków.

17. Stefan Banach, Poland's greatest mathematician, is commemorated, among other things, by (1 point)

A. coins issued by the National Bank of Poland on the 120th anniversary of his birth.

B. a monument – a bench in the Planty Park in Kraków.

C. the Stefan Banach Medal awarded to outstanding mathematicians.

D. all of the above statements are true.

18. List all divisors of the number that is the sum of the digits of Banach's year of birth. (2 points)

19. Banach's birthday has an interesting property: if we write all days and months using two digits, i.e. in the case of a single-digit number with a zero at the beginning (e.g. 7 January is 07.01), then the number denoting the month is the number denoting the day written backwards (in reverse order). Give all dates in the year with this property. (2 points)

20. Some time ago, we celebrated the 100th anniversary of Banach's "discovery" by Steinhaus in Planty Park in Kraków. Please give the highest power of 2 by which the number of the year of this anniversary is divisible. (2 points)

21. Two single-digit numbers determined by the last two digits of the year of Banach's death are the lengths of the leg and hypotenuse of a certain right triangle. Give the length of the other leg. (1 point)

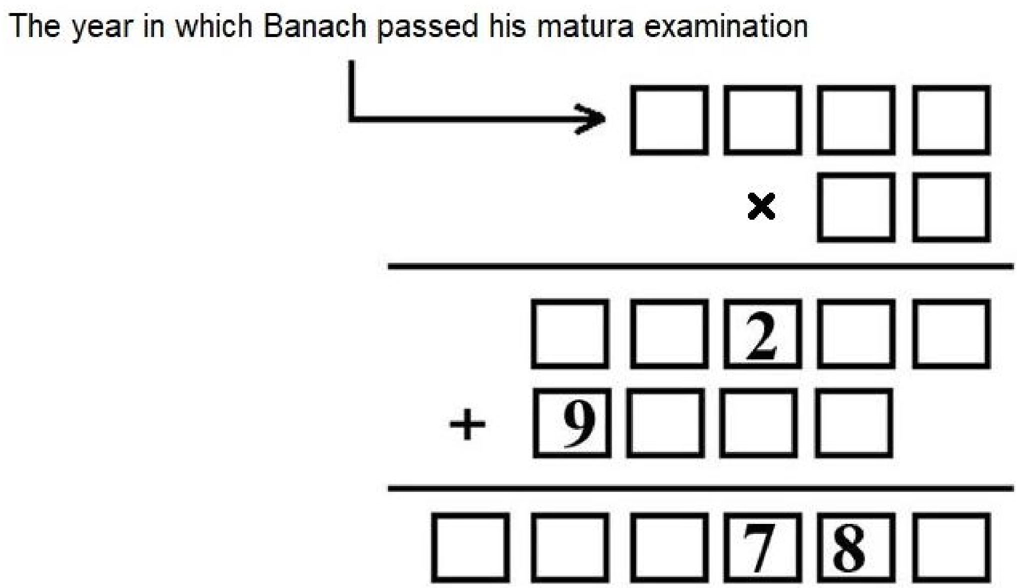
22. Stefan Banach was born on a Wednesday in March in the 19th century. On what day of the week did he celebrate his 10th birthday? Match the correct answer. (1 point)

1. Sunday		A. add 10 days to Wednesday, because 10 years have passed
2. Monday	To calculate the day of the week	B. add 12 days to Wednesday, because there are two leap years
3. Tuesday		C. add 11 days to Wednesday, because there was one leap year

23. Prove that the sum of numbers of years divisible by 12 from Stefan Banach's birth to his death is divisible by 100. (2 points)

24. A bookshop announced a sale of Stefan Banach's book entitled *Linear Operations*. On the first day of the sale, 80% of his books were sold, and on the second day, $\frac{12}{15}$ of the remaining books were sold. There are still 12 copies left to sell. How many books were sold during the two days of the sale? (3 points)

25. Fill in the missing digits in the puzzle in the picture[2]:(4 points)



Hints and answers

In problems where it was possible to get partial points, the rules of assigning points are described.

[2]Matura examination is an exam taken in several countries (including Poland) by young adults at the end of their secondary education.

1. A–T, B–F, C–F, D–F, E–T, F–F, G–T, H–F, I–F, J–F, K–F, L–T

2. B

3. C

4. A

5. D

6. I–C, II–A, III–D, IV–B, V–E

7. BC ($\frac{1}{4}$ point for each correct answer of those two).

8. B

9. B

10. B

11. A

12. A but B (1 point is given for the whole correct answer, a partial answer 0 points).

13. Stanisław Mazur, Warsaw, the prize for the solution of a problem from the Scottish Book. For three correct answers 2 points are given, for two correct answers 1 point.

14. C

15. C

16. A

17. D

18. For listing all six divisors 2 points, for giving a correct sum of the digits without listing all divisors 1 point.

19. 10.01, 20.02, 30.03, 01.10, 11.11, 21.12. For listing all six dates: 2 points, for listing at least three dates 1 point.

20. 5. Hint: the "discovery" took place in 1916. In the case of the lack of the correct answer, but writing the correct date of the 100th anniversary, 1 point was awarded.

21. 3

22. 1C.

23. In the case of the lack of the correct answer, but listing two last digits of each of five proper years (or these five years) 1 point.

24. 288. For partial answers: if somebody found that it was 300 books in the shop – 2 points, for obtaining only 60 (or 240) as the result of the trade in the first day – 1 point.

25. Hint: Banach passed his matura examination in 1910. In the final answer, the second factor equals 58. Marking: For the whole correct digits: 4 points, for at least 9 missing digits – 3 points, for at least 5 missing digits – 2 points, for giving the correct year – 1 point.

Some additional comments

In this section more precise explanations of some answers as well as some details concerning Banach are presented. Fore more information of Banach's life and his results, see [1], [2], [3], [4], [6], [5], [7], [8], [10], [11].

First, let us briefly present some important facts from Banach's life. Banach was born in 1892 in Kraków (a Polish city, then in Austria-Hungary because of the partitions of Poland). He was the son of Katarzyna Banach and Stefan Greczek. His parents were not married. Then, it was not unusual; at the turn of the XIXth century in this area many children were born out of wedlock. Greczek served in the Austro-Hungarian army and he would have not been allowed to marry without the permission from the military authorities. Such permission was denied to him. At the age of several months, the child was handed over to a foster mother, a laundry owner named Franciszka Płowa[3]. She took care of young Stefan (until 1910) together with her daughter Maria.

On completion of his school education in 1910, Banach decided to study engineering in Lvov. He was interested in mathematics, but considered it to be a nearly complete science to which very little was left to be added. Upon the outbreak of World War I, Banach returned to Kraków. Here, he enriched his mathematical knowledge by independent study. He also frequently discussed mathematics with his friends Witold Wilkosz and Otton Nikodym.

In 1916, in the middle of WWI, Hugo Steinhaus was spending some months in Kraków. Steinhaus studied in Göttingen and in 1911 obtained a PhD there. Later he became a notorious mathematician, but then he was just a young, well-educated mathematician. Once, in the summer of 1916, Steinhaus, during an evening walk in the Planty Park in the city centre of Kraków, heard the words "Lebesgue integral". At that time it was a recent idea known almost exclusively to specialists. Steinhaus was intrigued. He joined the conversation between two young people sitting on the bench, who turned out to be Banach and Nikodym. During the conversation, Steinhaus presented them a problem he was currently working on. A few days later, Banach turned up with a solution. Then Steinhaus realized that Banach had a superb mathematical talent and became his mentor. Although Steinhaus could boast many outstanding results, he would often say later that his greatest mathematical discovery was the "discovery" of Stefan Banach. For a detailed description of this event, see [1] and [13].

After WWI, in 1918 Poland regained independence and Kraków and Lvov were again in Poland, as before 1795. In 1920 Steinhaus got a Chair at Jan Kazimierz University in Lvov and he arranged the employment of Banach in the Technical University in Lvov, in the department headed by Antoni Łomnicki. At the turn of 1920 and 1921 Banach earned a doctorate in Jan Kazimierz University, where he got a Chair in 1922. He was a central person of the so-called Lvov School of Mathematics; his main mathematical interest was functional analysis. The notion of Banach space is now one of the most important in higher mathematics. In 1939 WWII broke out and Lvov was gained by the Soviet Union. After WWII it was decided that Lvov would be in the Soviet Union. Banach planned to go back to Kraków, where he would have taken a Chair of Mathematics at the Jagiellonian University, but he died in Lvov in 1945.

Now, let us turn to some answers that perhaps require more detailed explanation.

1. B is false, as Banach did not obtain an MSc diploma anywhere; he passed only the half-diploma (like a BSc) examination in the Engineering Faculty. C is false, as Banach studied in Lvov, not in Vilnius. D is false, as Steinhaus was not a German mathematician; the rest of the

[3] In the Polish language, some surnames have different male and female forms, which is applicable also to this surname: for a man – Płowy, for a woman – Płowa. See Problem 3.

sentence is true. H is false, as the scientist who led and directed the Institute for Typhus and Virus Research in Lvov was Rudolf Weigl, not Wishof; the rest of the sentence is true[4]. The café mentioned in K is the "Scottish Café" in Lvov, not "Swedish Café"[5]; the rest of the sentence is true.

2. The odd one out is basketball. Ludwik Mien was a Frenchman and a photographer, Franciszka Płowa's friend; he taught Banach French and made some photographs presenting Banach in his childhood. Banach used to play billiard when he was a student. He played tennis and it was difficult to play against him, as he was left-handed. He did not behave as a "classical" university professor of these times, he did not follow the rules regarding clothing. Once he went out onto the street wearing a short-sleeved shirt, which was unfashionable at the time, and carrying a walking stick. All these information were given in the interview with Banach's son [9]. Concerning Stanisław Mazur and goose, see Problem 13.

5. In the beginning of the XXth century it was quite common that after graduating university studies the graduates started teaching at schools; some of them were doing scientific research simultaneously and a few were later employed at universities. Banach's friend, Otton Nikodym for some years was teaching at IV Gymnasium in Kraków, the same that Banach and his friend Witold Wilkosz had attended earlier, but Banach never taught at school. Banach, Wilkosz and Nikodym had a habit of taking evening walks in Kraków and discussing mathematics, but Wilkosz was not with them on the day when Steinhaus overheard the talk.

7. The Scottish Café in Lvov was the venue of regular meetings, where mathematicians spent hours eating, drinking, formulating problems, and solving them. They had a habit of writing solutions on marble tables in the café, but these tables were carefully cleaned by the staff each morning. Later Banach's wife, Łucja, bought a special book in which the problems were written down. This book was called "The Scottish Book".

8. The Polish Mathematical Society was founded in April 1919 in Kraków. There were 16 founding members and Banach, who was then still in Kraków was among them.

12. People who knew Banach said that he published only a part of his results. He was introducing new ideas, solving problems all the time. It was said that if two mathematicians had followed him and had been writing down everything he said, then the most of what he invented would have been written down.

13. Sometimes mathematicians, who posed problems in the Scottish Café, offered rewards for solving them. In 1936 Banach's pupil and friend, Stanisław Mazur, offered a live goose for solving Problem no. 153 from the Scottish Book – about the existence of a Schauder basis in a separable Banach space. This problem was solved about forty years later by Per Enflo, a 28-year-old Swede. In 1972 Enflo came to Warsaw and got his reward from Mazur.

15. Banach liked discussing and solving problems in the hustle and bustle of the café, and even frequently sat down at a table near the orchestra.

17. In 2012 the National Bank of Poland issued three coins: a gold 200 PLN, a silver 10 PLN and a Nordic gold 2 PLN on the occasion of 120th anniversary of Banach's birth and 6th European Congress of Mathematics in Kraków (Fig.3). In 2016, on the occasion of the 100th anniversary of the memorable meeting in Planty Park in Kraków, the bench with figures of Banach and Nikodym

[4]It was estimated that Weigl, because of employing people in his Institute, saved probably about 5000 lives during the occupation of Poland in World War II. Weigl developed a technique to produce a typhus vaccine. He was several times nominated for the Nobel Prize in Medicine before WWII.

[5]see [12].

(designed by Stefan Dousa) was unveiled there (Fig.4, see [1]). In 1992, the Stefan Banach Medal (Fig.5) was established, on the centenary of his birth. It is awarded to individuals by the Presidium of the Polish Academy of Sciences in recognition of outstanding contributions to the development of mathematical sciences.

Figure 3. Polish coins commemorating Banach

Figure 4. A bench in Kraków Planty Park with figures of Banach (right) and Nikodym

Figure 5. The Banach Medal

References

[1] D. Ciesielska, K. Ciesielski, *Banach and Nikodym on the bench in Kraków again*, European Mathematical Society Newsletter 104(2017), 25–29.

[2] D. Ciesielska, K. Ciesielski, *Banach in Kraków: a case reopened*, Math. Intelligencer 35 (2013), no.3, 64–68.

[3] D. Ciesielska, K. Ciesielski, *Banach's doctorate: a case of mistaken identity*, Math. Intelligencer 43 (2021), no.3, 1–7.

[4] D. Ciesielska, K. Ciesielski, *Stefan Banach remembered in Kraków*, Math. Intelligencer 30 (2008), no.4, 31–35.

[5] K. Ciesielski, *On some details of Stefan Banach's life*, Opuscula Math. 13(1993), 71–74.

[6] K. Ciesielski, *On Stefan Banach and some of his results*, Banach J. Math. Anal. 1(2007), no.1, 1–10.

[7] K. Ciesielski, Z. Pogoda, *Conversation with Andrzej Turowicz*, Math. Intelligencer 10(1988), no.4, 13–20.

[8] R. Duda, *Pearls from a lost city: the Lvov School of Mathematics*, American Mathematical Society, 2014.

[9] P. Hajłasz, *Stefan Banach o swoim ojcu*, Delta no.10/1992, 1–4.

[10] E. Jakimowicz, A. Miranowicz (eds.), *Stefan Banach. Remarkable life, brilliant mathematics*, Gdańsk University Press, Gdańsk 2010.

[11] R. Kałuża, *Through a reporter's eyes. The life of Stefan Banach*, Birkhäuser, Boston 1996.

[12] R.D. Mauldin (ed.), *The Scottish Book. Mathematics from the Scottish Café*, Birkhäuser, Boston 1981.

[13] H. Steinhaus, *Mathematician for all seasons: recollections and notes,* vol. 1 (1887–1945), Birkhäuser 2015.

Krzysztof Ciesielski
Jagiellonian University, Mathematics Institute
Łojasiewicza 6, 30-059 Kraków, Poland
krzysztof.ciesielski@im.uj.edu.pl

Strengthening Logical Thinking Through Cooperative Strategies For Primary Level Students With Mathematical Talent

Natalia A. Alcalá López, Luis Cáceres

Natalia Alcalá earned a bachelor's degree in mathematics from the University of Tolima and a specialization in pedagogy from the same university. She is currently pursuing a master's degree in mathematics education at the University of Puerto Rico at Mayagüez. Her main academic interests lie in the field of mathematics education, specifically in the development and implementation of educational strategies to improve mathematics teaching at various levels.

Luis Cáceres is a mathematician from Javeriana University in Bogotá, holds a master's degree in applied mathematics from the University of Puerto Rico at Mayagüez, and a doctorate in logic from the University of Iowa in the United States. He is currently a professor at the University of Puerto Rico at Mayagüez, director of the graduate program in the mathematics department, and founding director of the Puerto Rico Mathematical Olympiad. Much of his academic career has been dedicated to leading professional development projects for mathematics teachers, projects for gifted mathematics students, and projects for first-year university students.

Abstract
Children with mathematical talent often excel at solving problems individually; however, when teamwork is properly structured, it can enhance their abilities and significantly improve their performance in competitions such as mathematics Olympiads. This study presents the application of specific cooperative learning strategies in a group of mathematically talented children at the elementary school level, with the aim of improving their performance on questions related to logical thinking. To achieve this, a methodology based on group cohesion activities, the cooperative structure "Think, Share, and Solve," and the "Rotating Folder" technique was implemented. The group's scores in Phases I and II of the Puerto Rico Mathematics Olympiad were analyzed and compared with the results obtained in Phase III of the same competition, after implementing the strategy. The findings show significant improvements in student performance, supporting the effectiveness of cooperative learning in strengthening logical thinking in high-performing mathematics students.
Keywords: Cooperative learning, logical thinking, math Olympiads.

Introduction

Logical thinking is a fundamental skill in mathematical development, especially for students who possess outstanding talent in this discipline. While mathematically gifted children have advanced abilities in abstract reasoning (Krutetskii, 1976), [4] it is essential to enhance their capacity to structure solutions efficiently and collaboratively, as this can directly impact their academic performance and success in competitive contexts. Mathematics Olympiads represent a challenging environment where the ability to analyze patterns, establish relationships, and structure efficient solutions is key to success. However, although many talented children excel in individual problem-solving, they may face difficulties when working in teams or organizing their ideas optimally.

Cooperative learning has proven to be an effective methodology for strengthening logical thinking, as it promotes the exchange of ideas, mathematical argumentation, and critical evaluation of solutions. According to Johnson and Johnson (1999), [1] peer interaction in a structured environment can significantly enhance the understanding and application of mathematical knowledge. In this context, the present study explores the implementation of cooperative strategies in the education of mathematically gifted children, aiming to strengthen their logical reasoning through collaboration and joint analysis of mathematical problems.

Howard Gardner (1983), [3] through his theory of multiple intelligences, highlights logical mathematical intelligence as one of the fundamental human capacities. This intelligence involves the ability to reason logically, solve mathematical problems, and understand patterns and relationships. According to Gardner, the development of this intelligence is enhanced when students are given opportunities to interact in environments that foster critical thinking and shared problem-solving.

The central purpose of this study is to analyze how the application of cooperative strategies further enhances logical thinking in mathematically gifted children. Group cohesion—a key factor in cooperative learning—improves communication, increases trust, and strengthens commitment (Lewin, 1951; Festinger, 1950). [5] [2]. According to J.C Torrego (2013), [6] techniques such as "Think, Share, and Solve" and "Rotating Folder" have proven to be effective methodologies for promoting mathematical reflection and the joint construction of knowledge.

The relevance of this study lies in its potential to optimize teaching practices in mathematics. The implementation of cooperative strategies not only fosters more effective problem-solving but also contributes to the development of metacognitive skills essential for academic success. By studying the impact of these methodologies on mathematically gifted children, evidence-based recommendations can be generated to improve pedagogical practices and strengthen the teaching of logical thinking at the elementary level.

Methodology

The Puerto Rico Mathematical Olympiad, held at the University of Puerto Rico, Mayagüez Campus (UPRM), follows an annual cyclic structure composed of three phases.

In the First Phase, all elementary, middle, and High School students from public and private schools in Puerto Rico are eligible to participate. This phase is conducted virtually and individually through the website https://ompr.weebly.com, where students complete an exam consisting of twenty (20) multiple choice questions and submit their answers within a specific time, which in 2024 was from April 22nd to May 20th . The top 40% of students with the highest scores at each of the three levels are selected to advance to the next phase. In this stage, one thousand one hundred thirty six (1,136) elementary-level students qualified for the Second Phase by scoring seven (7) points or more out of a total of twenty (20).

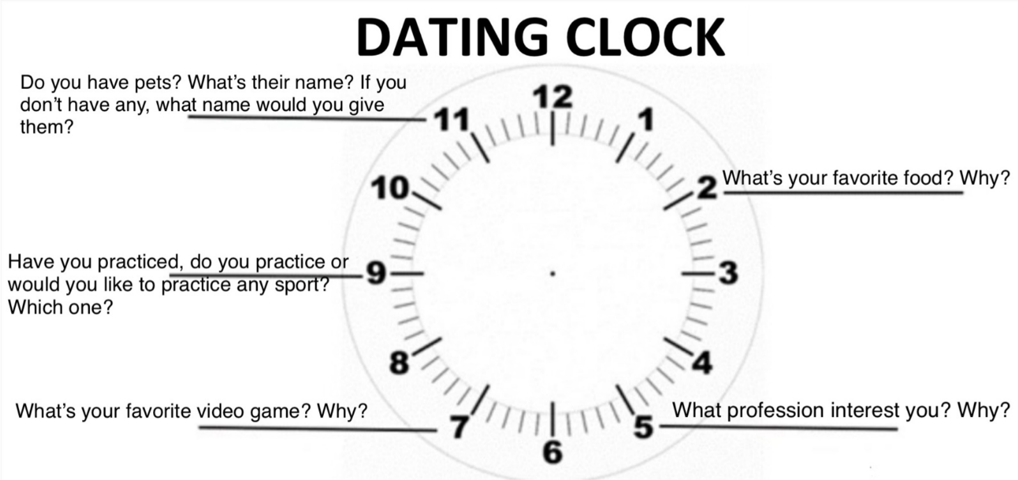
In the Second Phase, selected students take an in-person individual exam. This consists of fifteen (15) questions divided into two parts: Part A, with multiple choice questions, and Part B, with open-ended questions. This phase was held on Saturday, November 16th, 2024. Students with the highest scores are given the opportunity to participate in the Saturday Academies for Mathematically Talented Students, where they receive specialized training in Olympiad problem solving, as well as supplementary study materials. In this phase, sixty six (66) elementary-level students were selected to continue, having scored nine (9) points or more out of a total of fifteen (15).

This study was conducted with the participation of twenty three (23) of the aforementioned sixty six (66) students, who voluntarily attended one of the Saturday academies, held on Saturday, March 1st, 2025. The participants were identified as mathematically talented students and participated in a three hour class session. During this session, a cooperative strategy specifically designed to strengthen logical thinking through group interaction and self-assessment was implemented.

The strategy focused on improving performance in questions related to logical thinking, as prior analysis of results revealed that, among the three types of questions evaluated throughout the phases of the competition-logical, spatial and metric, and numerical thinking-logical thinking was the one with the highest number of incorrect responses. The study methodology was structured in several phases, each with clearly defined objectives and activities.

Phase 1: Group Cohesion and Formation

The session began with a cohesion activity called the "Appointment Clock," which lasted 15 minutes. The main objective was to promote socialization and communication among students, allowing them to interact with various classmates through questions that focused on their interests and personal experiences. Figure 1 shows the template used in the activity.



Template used for the group cohesion activity called "Dating clock"

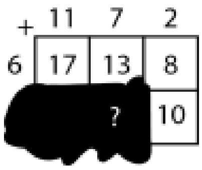
Subsequently, heterogeneous groups of four students were formed, with specific roles assigned to each member: coordinator or moderator, who ensures that all group members participate; secretary or recorder, who writes the answers agreed upon by the group; spokesperson, who is responsible for sharing the group's explanations with the other groups; and timekeeper, who manages the time to complete the task. This role assignment aimed to promote responsibility and equitable participation within each group.

Phase 2: Implementation of "Think, Share, and Solve"

The central phase of the methodology consisted of implementing the "Think, Share, and Solve" strategy, which was carried out over 60 minutes, divided into 3 cycles. Each cycle was structured as follows: 10 minutes for individual problem-solving ("Think"), followed by 5 minutes for comparing and analyzing strategies within the group ("Share"), and finally, 5 minutes to develop a consensual group response ("Solve"). Active breaks were taken between cycles to maintain students' focus and engaged.

Below are three examples of questions used during the problem-solving cycles.

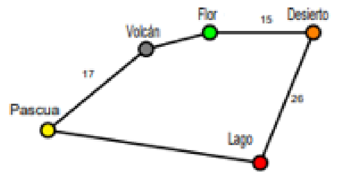
1. If the table in the figure, which has been partially covered by an ink stain, shows correct sums, what number should go in the box with the question mark?



(A) 10 (B) 11 (C) 12 (D) 13 (E) 15

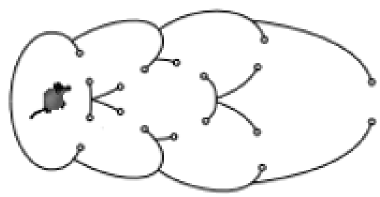
2. Captain Kook wants to sail from the island called Easter to each island on the map and return to Easter. The total journey covers a distance of 100 km. The distance from Desert to Lake is the same as the distance from Easter to Flower passing through Volcano. What is the direct distance

between Easter and Lake?



(A) 17 km (B) 23 km (C) 26 km (D) 33 km (E) 35 km

3. The mouse wants to escape the maze. How many different paths can the mouse take without going through the same door more than once?



(A) 2 (B) 4 (C) 5 (D) 6 (E) 7

Phase 3: "Rotating Folder" Technique and Evaluation

To encourage review and continuous improvement, the “Rotating Folder” technique was implemented for 15 minutes. The goal of the activity is to improve critical thinking through peer review. Folders numbered from one to fourteen were used to collect each group’s solution to the corresponding problem. Then each group was assigned a folder with all the submitted solutions to one of the problems, each group reviewed and improved their own solutions by comparing their work with the solutions submitted by the other groups. Finally, each spokesperson with their group presented and argued their improved solutions to the class.

Finally, 30 minutes were dedicated to sharing and evaluation. The groups presented and defended their answers to the class, and an individual and group evaluation rubric was used to assess participation.

Evaluation Rubric

Figure 2 shows the rubric used to evaluate individual and group participation, time management, and perceived performance. This rubric also allowed measure the perceived impact of the cooperative learning strategy.

OM.PR

Full name: ______________________________ Grade: _________ Date: __________________

Rate your performance within the group work. From 1 to 5, where 1 is the lowest grade and 5 is the highest.

SELF EVALUATION CRITERIA	1	2	3	4	5
Did I participate and contribute in all activities within the established time?					
Did I respect my classmates' opinions even if they were different from mine?					
Did I work responsibly in the team I was assigned to					
Did I develop the activities proposed by the teacher according to the instructions?					
Did I learn and achieve the purpose of the proposed activity?					
GROUP EVALUATION CRITERIA	**1**	**2**	**3**	**4**	**5**
Did we complete the activities within the planned time?					
Did we use time effectively?					
Did we make an effort?					
Did everyone in the group participate in the activity?					
Did we have a good communication within the group?					

Figure 2. Individual and Group Evaluation Rubric

After this, the students participated in the third phase on Saturday, March 29th, 2025, during which they took an individual, in-person exam. The exam consisted of ten (10) open-ended questions, each worth three (3) points. A total of sixty six (66) students who had qualified from the second phase took part in this stage.

Results

Below is the analysis of the results from Phases II and III of the students who participated in the Puerto Rico Mathematics Olympiad, taking into account their overall performance and those twenty three (23) students who participated in the Saturday academy on March 1st and qualified for these phases.

Statistical Analysis

The students who participated in phases I and II, later voluntarily attended the Saturday academy, and eventually took part in Phase III, were analyzed. To evaluate the impact of the cooperative strategies implemented during the Saturday academy on March 1st in strengthening logical thinking, the non-parametric Wilcoxon signed-rank test for related samples was applied. This test compared the results obtained by the twenty three (23) students in phases II and III, considering only the questions related to logical thinking.

This test was chosen because the data did not meet the normality assumption required by parametric tests. The results of the analysis indicated that there is sufficient evidence, with a significance level of $\alpha = 0.05$, to conclude that the scores obtained by the twenty three (23) students in phase III were significantly higher than those obtained in phase II. This suggests that, after attending the Saturday academy and participating in cooperative learning activities, the students experienced a strengthening of their logical thinking skills.

Table 1. Average performance scores (on a scale from 0 to 1) on logical thinking questions, based on the results obtained by all students participating in phases I, II, and III

	Phase I	Phase II	Phase III
Average	0.47	0.77	0.80

It is important to note that the progressive increase in average scores throughout the phases was an expected trend, given that only the students with the highest scores in the previous stage advanced to the next phase. The level of difficulty in phase I and phase II is similar; however, phase III has a higher degree of difficulty.

Table 2. Average performance scores (on a scale from 0 to 1) on logical thinking questions, based on the results obtained by students from the Saturday Academy in phases I, II, and III

	Phase I	Phase II	Phase III
Average	0.73	0.77	0.87

The data show that the Saturday Academy participants had higher average scores across all phases compared to the overall group, which may indicate that these students possessed stronger logical thinking skills from the outset. Furthermore, the greater improvement from phase II to phase III in this subgroup suggests that the cooperative learning strategies implemented during the Saturday academy had a particularly positive impact on their performance.

Evaluation Rubric

A total of 23 students (attendees of the Saturday academy on March 1st), out of the 66 students who qualified for phase II, participated in the survey. The results show that over 80% of the students perceived a high level of participation and responsibility within their work teams. A strong performance was observed in terms of achieving objectives, with 96% of students expressing satisfaction with the accomplishments. Additionally, time management and its effective use were well rated, with average scores of 4.60 and 4.88, respectively. Effort and commitment were also highly valued, with an average score of 4.84. However, group communication received the lowest rating, suggesting a need to strengthen this aspect in future activities.

Conclusions

The results obtained demonstrate that the implementation of cooperative strategies within the Saturday academy had a positive impact on strengthening the logical thinking skills of mathematically talented students participating in the Puerto Rico Mathematics Olympiad at the University of Puerto Rico at Mayagüez campus. The statistical analysis using the Wilcoxon test showed significant differences between the results of Phases II and III, indicating an improvement in student performance after participating in cooperative activities, with a significance level of $\alpha = 0.05$

Furthermore, the averages observed across the phases suggest a trend of improvement due to the selection process, where only students with the highest scores advance to the next stage. However, the increase observed between phases II and III among the group of students who attended the Saturday Academy suggests that the cooperative strategies applied contributed additionally to this progress.

From a qualitative perspective, the results of the evaluation rubric also support the effectiveness of cooperative learning. Most students reported high levels of participation, responsibility, and satisfaction with their achievements. Notably, skills related to time management, effort, and commitment were especially strong. However, group communication was identified as an area for improvement, offering an opportunity to adjust future implementations. Techniques such as "Think, Share, and Solve" and the "Rotating Folder" promoted not only individual and collective understanding of mathematical problems but also the development of metacognitive skills such as reasoning, critical evaluation, and error review.

In conclusion, when cooperative work is clearly and strategically structured, it not only enhances academic performance in logical thinking skills but also strengthens essential social-emotional and cognitive abilities needed for success in high-level mathematical environments such as Olympiads. Continuing to refine and implement these methodologies will significantly contribute to the comprehensive development of students with high mathematical abilities.

References

[1] R. J. D.W. Johnson. *Learning Together and Alone: Cooperative, Competitive, and Individualistic Learning.* Boston, MA: Allyn & Bacon, 1999.

[2] L. Festinger. Informal social communication. *Psychological Journal*, 57(5):271, 1950.

[3] H. Gardner. *Frames of Mind: The Theory of Multiple Intelligences*. New York, NY: Basic Books, 1983.

[4] V. Krutetskii. *The Psychology of Mathematical Abilities in Schoolchildren*. University of Chicago Press, Chicago, IL, 1976.

[5] K. Lewin. *Field Theory in Social Science: Selected Theoretical Papers*. Harper, New York, NY, 1976.

[6] J. Torrego. *Técnicas formales e informales de aprendizaje cooperativo,*. 2013.

Natalia A. Alcalá López
University of Puerto Rico at Mayagüez
natalia.alcala@upr.edu

Luis Cáceres
University of Puerto Rico at Mayagüez
luis.caceres1@upr.edu

On the Use of Invariants in Solving Olympiad-Level Problems

Kylan Huang, Patrick Galarza

Kylan Huang is a student at Avenues: New York. He has won Honorable Mention in the United States of America Junior Mathematical Olympiad twice and was part of a top team for American Regions Mathematics League in both 2024 and 2025. He has also founded the Highline Mathematics Competition in New York City, bringing the joys of mathematics to his school community.

Patrick Galarza is the Director of the Institute for Advanced Mathematics at Avenues: New York. He received his PhD from Columbia University and often instructs through Columbia's Pre-College program at the School of Professional Studies. An expert in unconventional mathematics learning, teaching, and curriculum design, Galarza has published several texts on the uses of mathematics game play for both secondary-level and university-level students.

Abstract
This paper demonstrates the utility of exploiting invariants in olympiad problems. We provide a compilation of secondary school olympiad problems that may be solved by finding an appropriate mathematical invariant from both regional competitions to national/international competitions, and also submit original problems; collectively, we aim to demonstrate the diversity of applications for invariants and broadly and deeply address their usage in the context of mathematics olympiads.

Introduction

What are Invariants?

In common usage, "invariants" are quantities or expressions that stay constant after specific operations or transformations. Likewise, "monovariants" are quantities or expressions that change in only one direction or fashion (say, increases by 2) after specific operations or transformations.

Common invariant types include the following examples:

- Parity and modular invariants, for which a quantity is unchanged modulo n after applying the given set of operations

- Steadily increasing monovariants, for which a quantity changes by a set amount after applying the given set of operations
- Graphical coloring invariants, for which dividing a chessboard into groups provides insight about properties of a tiling
- Symmetries, for which one of two players in a mathematical game has a method of “mirroring" the second player’s strategy

However, not all invariants can be categorized neatly. This paper mainly focuses on the classical canon of invariant problems, though it also contains problems found outside the canon.

A strategy that leverages the use of a particular invariant is helpful in solving problems of, for example, the following types, all of which are featured in this paper:

- Terminating process problem: “Alex writes the positive integers 1 through n on a whiteboard. Every minute, he replaces the integers a, b, c, and $a+b+c$ with $a+b$, $b+c$, and $c+a$. Prove that this process must end after at most $\frac{n}{4}$ moves. "
- Minimal n problem: “On an n by n board, Bob shades $n-1$ squares. Every second, Bob shades in all squares adjacent to two shaded squares. Will the whole board ever be shaded?"
- Irreversible state problem: “Suppose n indistinguishable stones are scattered across a number line. Each minute, Steven takes two stones currently at the same location and shifts one stone 1 unit to the left and the other 1 unit to the right. Prove that once he moves the first pair of stones, the original setup can never occur again."

Invariants appear in all sorts of contests, from the Harvard-MIT Math Tournament to the International Mathematical Olympiad. Often, these problems are written with the intent for participants to solve the problem by finding the invariant, but on occasion, even the problem writer will miss an invariant-based solution! For example, Vieta’s formulas led to a famous unintentional solution to the final problem of IMO 1988, officially introducing “Vieta Jumping" to the competitive mathematics scene.

The usefulness of invariants goes beyond learning a mathematical technique; while barycentric coordinates provides you with a set of formulas to bash through most area-based or intersection-based geometry problems, finding an invariant often leads to more generalized discoveries. 3Blue1Brown has an excellent lesson titled “Alice, Bob, and the average shadow of a cube," which highlights the usefulness of invariant-based thinking. This is included as the first reference in the Ref

Typically, the best way to develop a feel or intuition for a problem that can be solved by invariants is to play around with the problem. Since invariant problems usually involve a given set of operations, testing what happens after x moves is a great way to gain an understanding of the game state then. We’ll begin by examining problems that are solved using a parity/mods invariant.

Parity/Mods Invariants

Problems under the “Parity/Mods Invariant" category can usually be solved by finding some quantity, whether explicitly stated or elsewise constructed, that stays constant modulo n. Consider the following problems:

1. A game is played as follows: 8 is written on the board. Every minute, you can replace n with $3n$, $5n$, or $8n$, or you can replace it with the number that results from subtracting twice the unit digit from the number resulting from removing the units digit (e.g., 112 would become $11-2\cdot 2=7$), assuming the result is nonnegative. Following the rules, which of the following numbers can you write on the board: a) 1, b) 2, c) 3, d) 4, e) 5, f) 6, and g) 7?

   Solution: The last operation is linked to the divisibility rule of 7, which suggests that we should look at an invariant related to the number 7. The other moves all preserve whether the number is a multiple of 7, so, we've found our invariant: divisibility by 7. Since 8 is not a multiple of 7, we now know that we cannot write 7 on the board at any point. 1, 2, 3, 4, 5, and 6 can be constructed as follows:

   - $8 \to 8\cdot 5^3 = 1000 \to 100-2\cdot 0 = 100 \to 10-2\cdot 0 = 10 \to 1-2\cdot 0 = \boxed{1}$
   - $8 \to 8\cdot 5^2 = 200 \to 20-2\cdot 0 = 20 \to 2-2\cdot 0 = \boxed{2}$
   - $8 \to 8\cdot 5^3 = 1000 \to 100-2\cdot 0 = 100 \to 10-2\cdot 0 = 10 \to 1-2\cdot 0 = 1 \to 1\cdot 3 = \boxed{3}$
   - $8 \to 8\cdot 5 = 40 \to 4-4\cdot 0 = \boxed{4}$
   - $8 \to 8\cdot 5^3 = 1000 \to 100-2\cdot 0 = 100 \to 10-2\cdot 0 = 10 \to 1-2\cdot 0 = 1 \to 1\cdot 5 = \boxed{5}$
   - $8 \to 8\cdot (3\cdot 5) = 120 \to 12-2\cdot 0 = 12 \to 12\cdot 5 = 60 \to 6-2\cdot 0 = \boxed{6}$

2. Milan has a bag of 2020 red balls and 2021 green balls. He repeatedly draws 2 balls out of the bag uniformly at random. If they are the same color, he changes them both to the opposite color and returns them to the bag. If they are different colors, he discards them. Eventually the bag has 1 ball left. Let p be the probability that it is green. Compute $\lfloor 2021p \rfloor$. (2021 HMMT Guts Problem 7)

   Solution: This problem may initially seem impossible, but one nifty observation solves the problem: the difference between the number of green balls and the number of red balls remains invariant mod 4 under the given operations (either 1 color decreases by 2 and the other increases by 2, or both decrease by 1). Since the starting state is 1 mod 4, the final state must also be 1 mod 4. So, if the bag has 1 ball left, then it must be green, so $p = 1$ and $\lfloor 2021p \rfloor = \boxed{2021}$.

We'll now look at steadily increasing monovariants.

Steadily Increasing Monovariants

Problems solved using a steadily increasing monovariant typically involve finding some quantity that changes by a fixed amount after applying any of the given operations. Consider the problems below:

1. Suppose Mr. Inaltong has a 10 by 8 chocolate he wishes to split between his students. He can break any piece of chocolate into two pieces. If he wants to have 80 1 by 1 squares of chocolate, what is the least number of breaks he needs to make? (Note: Mr. Inaltong can only break one piece of chocolate at a time.)

   Solution: Since there are almost infinitely many ways for Mr. Inaltong to break the pieces of chocolate, finding an invariant that holds across all possible arrangements is a natural way to solve the problem. In this case, the total number of pieces increases by 1 after each break, so Mr. Inaltong needs exactly 79 to end up with 80 pieces of chocolate.

2. Suppose n indistinguishable stones are scattered across a number line. Each minute, Steven takes two stones currently at the same location and shifts one stone 1 unit to the left and the other 1 unit to the right. Prove that once he moves the first pair of stones, the original setup can never occur again.

   Solution: The last sentence about the “original setup" is a great hint that a monovariant may solve the problem. Intuitively, stones become more “spread out" as time passes, but how does one rigorize that idea? One such method is to consider the sum of the squares of the locations of the stones. (Any convex function or any concave function will do; x^2 is the simplest such example.) Suppose the two stones Steven moves were originally at x. Now, one is at $x-1$ and the other is at $x+1$. So, the sum of the squares of the locations of the stones has increased by $(x+1)^2+(x-1)^2-2x^2=2$. Since this value increases steadily by 2 with every move, Steven can’t arrive back at the initial board state, and we are done.

The next set of problems involve the use of a coloring invariant.

Coloring Invariants

Problems under this category often revolve around tiling an $m \times n$ board with certain dominoes, with the underlying strategy being coloring the board in such a way that any placement of the domino will result in a fixed property. Consider the following problems:

1. Consider an 8×8 chessboard. Is it possible to tile it with 31 non-overlapping dominoes if two adjacent corners are removed? What if two opposite corners are removed? Dominoes must be placed along the gridlines.

   Solution: It is trivial to create a tiling if two adjacent corners are removed, but seemingly impossible if two opposite corners are removed. Why? If we color the 8×8 board sans the opposite corners like a standard chessboard, we see that there are 32 dark squares and 30 light squares. Any domino on the board must cover 1 dark square and 1 light square, so 31 dominoes would cover 31 dark squares and 31 light squares, and thus such a tiling is impossible.

2. Is it possible to tile a 10×10 board with 25 4×1 tetrominoes?

   Solution: Coloring the 10×10 board like a chessboard doesn’t give us the desired result—each tetromino covers 2 dark squares and 2 light squares, and the board has 50 dark squares and 50 light squares. However, if we color the board using 4 colors, then we can prove impossibility: each tetromino must cover 1 square of each color, but the board has 26 squares each of 2 of the colors and 24 squares each of the remaining 2 colors. Thus, such a tiling is impossible.

Next, we’ll look at problems solved using symmetry.

Symmetry Invariants

The problems under this category are often 2- or multi-player games, where one player (typically the second, though on occasion, it can be the first) finds a strategy that mirrors the other in some aspect. Consider the following problems:

1. Arya and Bran are playing a game. They begin with 2008 coins arranged in a circle, and alternate turns, starting with Arya. On his or her turn, a player may remove any one coin, or

if two adjacent coins remain, he or she may instead remove both. The player who removes the last coin wins. Show that Bran has a winning strategy, no matter how Arya plays. (Bernoulli Trials, 1998)

Solution: Bran's guaranteed win suggests that there is some symmetry strategy he can adopt. Indeed, one such strategy exists: after Arya's first move, Bran will take the coin(s) diametrically opposite. The board is now split into two congruent halves. The rules of the game only allow players to make a move that removes coins on one half; therefore, Bran can always mirror Arya's move. Eventually, Arya will run out of moves (since the number of coins is always decreasing), i.e., there are no more coins left to take. Q.E.D.

2. Alice and Bob are playing a game. The positive integer n is currently written on the board. On a turn, a player replaces the number on the board, say, k, with positive integer k' such that $\frac{k}{2} \leq k' < k$. Alice goes first. For what n does Alice win?

   Solution: Bob wins for all $n = 2^a - 1$: trivially, if the number on the board is 1, Alice loses. No matter what Alice replaces $2^a - 1$ with, Bob can always replace Alice's number with $2^{a-1} - 1$, and an induction argument finishes.

   Now, for all $n \neq 2^a - 1$, Alice can replace the number on the board with a number of the form $2^a - 1$, so Alice wins for all $n \neq 2^a - 1$.

Lastly, we'll look at problems that involve other invariants.

Other Invariants

The invariants below don't fit neatly into the above categories, and, as such, are listed below. Consider the following problems:

1. Define a sequence of positive rational numbers x_0, x_1, x_2, x_3, ..., by $x_0 = 2$, $x_1 = 3$, and for all $n \geq 2$,
$$x_n = \frac{x_{n-1}^2 + 5}{x_{n-2}}.$$
   Prove that x_n is an integer for all $n \geq 0$. (USAMTS 5/1/33)

   Solution:
$$\begin{aligned}
x_n &= \frac{x_{n-1}^2 + 5}{x_{n-2}} \\
x_n x_{n-2} &= x_{n-1}^2 + 5 \\
x_n x_{n-2} - x_{n-1}^2 &= 5 \\
x_n x_{n-2} - x_{n-1}^2 &= x_{n+1} x_{n-1} - x_n^2 = 5 \\
x_n x_{n-2} + x_n^2 &= x_{n+1} x_{n-1} + x_{n-1}^2 \\
\frac{x_{n-2} + x_n}{x_{n-1}} &= \frac{x_{n+1} + x_{n-1}}{x_n}
\end{aligned}$$

   Hence, $\frac{x_{n-2}+x_n}{x_{n-1}}$ is invariant for all n. $x_2 = 7$, so $\frac{x_{n-2}+x_n}{x_{n-1}} = \frac{2+7}{3} = 3$ for all n. We can rearrange this to get $x_n = 3x_{n-1} - x_{n-2}$, so x_n is clearly an integer for all $n \geq 0$.

2. Initially, 4 chips are placed at the origin. A move consists of taking a chip at position (x, y) and replacing it with two chips at $(x+1, y)$ and $(x, y+1)$. Prove there will always be a coordinate with two or more chips on it. (Waterloo)

Solution: Assign the coordinate (x,y) the weight $\frac{1}{2^{x+y}}$. Note that the original sum of weights is equal to 4, and after any move, this sum does not change. Also, note the total sum of all coordinates is equal to $(2^0+2^{-1}+...)^2 = 2^2 = 4$, and since after a finite number of moves, it is not possible to have all coordinates have a chip on it. Hence, if there was no coordinate with two chips, then the sum of the weights would be less than 4, a contradiction, Q.E.D.

Problem Set

The problems in this set are roughly ordered by difficulty, but this is only a general guideline—some later problems may be more approachable than earlier ones. You are encouraged to attempt every problem and to spend time thinking deeply about each before reading the solutions. If you find yourself stuck on a particular problem, don't let that discourage you from trying the ones that follow. Enjoy!

1. Is it possible to choose signs in the expression $1 \pm 2 \pm 3 \pm ... \pm 10$ to make it equal to 0?

   Solution: $\boxed{\text{No}}$. The parity of the sum is invariant regardless of the signs. Therefore, $1 \pm 2 \pm 3 \pm ... \pm 10$ is congruent to 1 mod 2, so it cannot be equal to 0; Q.E.D.

2. Prove that it is impossible to tile a 10×10 chessboard with all its corners missing using 32 1×3 triminoes.

   Solution: Color the chessboard using 3 colors in a similar style as a regular two-colored chessboard. Note that any trimino covers 1 square of each color, but the board has 30 of one color and 33 of the other two colors, a contradiction; Q.E.D.

3. Prove that one cannot tile a 9×9 board with 26 3×1 triminoes and 1 2×2 square with a corner missing.

   Solution: Assume the contrary; it must be possible to rotate the board until the L-shaped trimino makes is oriented as an “L". Consider the following coloring:
   A B C A B C A B C
   C A B C A B C A B
   B C A B C A B C A
   A B C A B C A B C
   C A B C A B C A B
   B C A B C A B C A
   A B C A B C A B C
   C A B C A B C A B
   B C A B C A B C A

   Note that the L-shaped trimino will cover 2 squares of one letter and 1 square of another, while the 3x1 triminoes will cover 1 square of each letter. Hence, there cannot be a tiling with 27 squares of each letter; Q.E.D.

4. 1 marble is placed at n equally-spaced points around a circle. Every minute, Catherine moves one marble to the adjacent counterclockwise point and one marble to the adjacent clockwise point. For what n is it possible that at some point, all the marbles are at the same point?

   Solution: n is odd. To prove it is possible for all odd numbers, rotate the circle such that there is a “top" point. Now, for each pair of points that is horizontally level, it is possible to move them to the top point; hence, it is possible for all marbles to end up at the top point.

To prove it is impossible for even n, let $n = 2k$ and assign values 1 through $n = 2k$ to the points along the circle. Let the value of marble i, a_i be equal to the value of the point it is at. What remains is to note that $\sum_{i=1}^{n} a_i$ is invariant mod n. The starting state is $\sum_{i=1}^{2k} i = k(2k+1) \equiv k(\bmod 2k)$, while the final state is congruent to $\sum_{i=1}^{2k} a \equiv 0(\bmod 2k)$, so we can never reach the final state from the starting state; Q.E.D. (Remark: it is possible to rephrase this solution using nth roots of unity.)

5. Suppose there are N lodges in a circle, and $N-1$ travellers who live in the lodges. They're bored, so they play a game. Every minute, some lodge with two or more people inside send one person to each neighboring lodge (they travel in a circle; no "skipping" lodges). Prove that this game must stop at some point.

   Solution: Define there to be a "path" between each neighboring pair of lodges. The key insight is that one of these paths will never be traversed: once this is proved, the problem reduces to the example problem about stones on a line.

   Now, for each time two travellers move, see if the path they traversed has been travelled yet. If it has not been travelled yet, name the path after them; from this move on, let them only travel on this path. If the path has been travelled, let nothing happen. Now, note that since there are $N-1$ travellers, and each traveller can only have at most one path named after them, at most $N-1$ paths have been named, as, if we assume the last path to be travelled, it must be named after someone. Hence, one of these paths is not travelled, so we are done; Q.E.D.

6. A triangle with sides a, b, and c is given. Denote by s the semiperimeter, that is $s = \frac{a+b+c}{2}$. Construct a triangle with sides $s-a$, $s-b$, and $s-c$. This process is repeated until a triangle can no longer be constructed with the side lengths given. For which original triangles can this process be repeated indefinitely? (1992 APMO #1)

   Solution: To keep the perimeter invariant, we'll modify each triangle to have side lengths $2(s-a) = b+c-a$, $2(s-b) = c+a-b$, and $2(s-c) = a+b-c$. WLOG, assume $a \le b \le c$. Note that $2(s-a) \ge 2(s-b) \ge 2(s-c)$.

   Consider the difference between the longest and shortest sides of the triangle. After an operation is applied, this quantity doubles: $2(s-a)-2(s-c) = 2(c-a)$. If $c-a = 0$, then the triangle is equilateral and trivially works. Hence, if $c-a > 0$, then at some point the difference between the longest and shortest length of the triangle will exceed the perimeter $a+b+c$, which isn't possible.

   Hence, only equilateral triangles work. Q.E.D.

7. On the island of Camelot live 13 gray, 15 brown and 17 crimson chameleons. If two chameleons of different colors meet, they both simultaneously change color to the third color (e.g. if a gray and a brown chameleon meet each other, they both change to crimson).

   (a) Is it possible that they will eventually all be the same color?

   (b) Is it possible that there will eventually be the same numbers of gray, brown, and crimson chameleons?

   (1984 Tournament of Towns Fall Round #5)

   Solution: Consider the number of gray, brown, and crimson chameleons mod 3: any meeting keeps the values (in some order). We start with 0, 1, and 2 mod 3, and both (a) and (b) end with 3 congruent to 0 mod 3, so both are not possible; Q.E.D.

8. Bob lost a bet to Alice, and they are playing a game to decide how much money Bob will pay Alice. The numbers 0, 1, 2,..., 1024 are written on a board. Alice erases 512 numbers, Bob erases 256 numbers of his choice, Alice erases 128 numbers, Bob erases 64 numbers, etc. After 5 moves by both players, two numbers will be left. Bob will pay Alice the positive difference of the two numbers. Assuming optimal play, how much money will Bob give Alice?

   Solution: $\boxed{32}$. We will prove this by proving Alice can guarantee a minimum of 32 dollars, and Bob can guarantee a maximum of 32 dollars.

   Alice starts off by removing all odd numbers. Thus, the only remainders left mod 4 are 0 and 2, of which there are 256 each. By the pigeonhole principle, after Bob's move, one of these mods has at most 128 numbers left, which Alice can remove. This guarantees that there are at most 2 different remainders mod 8 left, and that all numbers are congruent mod 4. Continuing in this fashion, Alice can guarantee after her fifth move that all numbers are congruent mod 32, so she can guarantee a minimum payout of 32 dollars.

   In a similar vein, after Alice's first move, at least half of the numbers in at least one of $0-512$ or $512-1024$ are removed, so Bob can shrink the range down to 512. In this fashion, Bob can shrink the range after his fifth move down to 32, so he pays out at most 32 dollars.

9. Alice has a large amount of Halloween candy she collected, which she has sorted into piles. Getting bored, she decides to play a game with one valid operation: If two piles of candy have p and q candies, respectively, with $p \geq q$, then she can move q candy from the first pile into the second pile.

   (a) If Alice started with piles of 7, 24, 37, 88, and 47, can she combine all her candy into one pile?

   (b) Prove that if the number of pieces of candy Alice has is a power of two, she can combine them into one pile.

   Solution: To prove (a) is no, assume the contrary. On the last move, Alice must have moved a pile of k candies to another pile of k candies. This implies that the total number of candies must be even. However, there are initially an odd number of candies, so the answer is $\boxed{\text{No}}$.

   To prove part (b), we proceed by induction. Let our inductive claim be that if Alice has 2^n candies, she can combine them all into one pile. Clearly, the base case $n = 0$ is true.

   Assume the case $n = k$ is true. When $n = k+1$, notice that there must be an even number of odd piles (since the total number of candies is even). Pair them up, and for each pair, apply the operation. If the two piles originally had p and q candies, where p and q are odd, note that the new piles of $p-q$ and $2q$ candies both have an even number of candies. Therefore, we now only have piles with an even number of candies.

   Group the 2^{k+1} candies into 2^k candy-pairs, which can be done since each pile has an even number of candies. Applying our inductive claim on the 2^k candy-pairs finishes; Q.E.D.

10. Let $n \geq 3$ be an integer. Rowan and Colin play a game on an $n \times n$ grid of squares, where each square is colored either red or blue. Rowan is allowed to permute the rows of the grid, and Colin is allowed to permute the columns of the grid. A grid coloring is *orderly* if:

    - no matter how Rowan permutes the rows of the coloring, Colin can then permute the columns to restore the original grid coloring; and
    - no matter how Colin permutes the column of the coloring, Rowan can then permute the rows to restore the original grid coloring;

In terms of n, how many orderly colorings are there? (2024 USAJMO Problem 4)

Solution (Evan Chen): The answer is $2n! + 2$. In fact, we can describe all the orderly colorings as follows:

- The all-blue coloring.
- The all-red coloring.
- Each of the $n!$ colorings where every row/column has exactly one red cell.
- Each of the $n!$ colorings where every row/column has exactly one blue cell.

These obviously work; we turn our attention to proving these are the only ones.

For the other direction, fix a orderly coloring $\mathscr{A}$.

Consider any particular column C in $\mathscr{A}$ and let m denote the number of red cells that C has. Any row permutation (say σ) that Rowan chooses will transform C into some column $\sigma(C)$, and our assumption requires $\sigma(C)$ has to appear somewhere in the original assignment $\mathscr{A}$.

11. A group of 100 friends stands in a circle. Initially, one person has 2019 mangos, and no one else has mangos. The friends split the mangos according to the following rules:

    - *sharing*: to share, a friend passes two mangos to the left and one mango to the right.
    - *eating*: the mangos must also be eaten and enjoyed. However, no friend wants to be selfish and eat too many mangos. Every time a person eats a mango, they must also pass another mango to the right.

    A person may only share if they have at least three mangos, and they may only eat if they have at least two mangos. The friends continue sharing and eating, until so many mangos have been eaten that no one is able to share or eat anymore.

    Show that there are exactly eight people stuck with mangos, which can no longer be shared or eaten. (USAMTS 4/1/31)

    Solution: Number the people in the circle 0 through 99, and let person i hold m_i mangos. WLOG, let person 0 have the 2019 mangos. The quantity

    $$\sum_{i=0}^{99} 2^i \cdot m_i$$

    is invariant mod $2^{100} - 1$: if person i shares, then person $i-1$ gains 2 mangoes, person i loses 3 mangos, and person $i+1$ gains 1 mango, where indices are taken cyclically. Note that $2^{i-1} \cdot (m_{i-1} + 2) + 2^i \cdot (m_i - 3) + 2^{i+1} \cdot (m_{i+1} + 1) = 2^{i-1} \cdot m_{i-1} + 2^i \cdot m_i + 2^{i+1} \cdot m_{i+1}$. If person i eats, person i loses 2 mangos and person $i+1$ gains 1 mango. Note that $2^i \cdot (m_i - 2)2^{i+1} \cdot (m_{i+1} + 1) = 2^i \cdot m_i + 2^{i+1} \cdot m_{i+1}$.

    To finish, note that $\sum_{i=0}^{99} 2^i \cdot m_i = 2019$ at the start, and in the final state, m_i is either 0 or 1. Thus, m_i are uniquely determined by the binary expansion of $2019 = 11111100011_2$, so $m_i = 1$ for exactly 8 people; Q.E.D.

12. Alex writes the positive integers 1 through n on a whiteboard. Every minute, he replaces the integers a, b, c, and $a+b+c$ with $a+b$, $b+c$, and $c+a$. Prove that this process must end after at most $\frac{n}{4}$ moves.

    Solution: The key observation is that the sum of the numbers on the board, as well as the sum of the squares of the numbers on the board, are both invariant. These numbers are equal to $1+2+...+n = \frac{n(n+1)}{2}$ and $1^2+2^2+...+n^2 = \frac{n(n+1)(2n+1)}{6}$. Suppose that k moves have been made, and the numbers on the board are a_1, ..., a_{n-k}. By the Cauchy-Schwarz Inequality,

$$(a_1^2+a_2^2+...+a_{n-k}^2)(1^2+1^2+...+1^2) \geq (a_1+a_2+...+a_{n-k}^2)^2.$$

To finish,

$$\begin{aligned}
\frac{n(n+1)(2n+1)}{6}\cdot(n-k) &\geq \left(\frac{n(n+1)}{2}\right)^2 \\
n-k &\geq \frac{3n(n+1)}{2(2n+1)} \\
k &\leq n-\frac{3n(n+1)}{2(2n+1)} \\
&\leq n-\frac{3n(n+1)}{2(2n+2)} \\
&= \frac{n}{4}
\end{aligned}$$

Q.E.D.

13. On a board the following six vectors are written: $(1,0,0)$, $(-1,0,0)$, $(0,1,0)$, $(0,-1,0)$, $(0,0,1)$, $(0,0,-1)$. Given two vectors v and w on the board, a move consists of erasing v and w and replacing them with $\frac{1}{\sqrt{2}}(v+w)$ and $\frac{1}{\sqrt{2}}(v-w)$. After some number of moves, the sum of the six vectors on the board is u. Find, with proof, the maximum possible length of u. (2022 HMMT February Team #10)

Solution: $\boxed{2\sqrt{3}}$. One construction is the following:

$$(1,0,0),(-1,0,0) \to (\sqrt{2},0,0),(0,0,0) \to (1,0,0),(1,0,0)$$

Similarly, convert $(0,-1,0)$ to $(0,1,0)$ and convert $(0,0,-1)$ to $(0,0,1)$. Thus, the magnitude of u is equal to $2\sqrt{3}$.

To prove $2\sqrt{3}$ is the upper bound, let $n=(x,y,z)$ be any unit vector. Then, $\sum_i(n\cdot v_i)^2$ is invariant:

$$\begin{aligned}
\left(n\cdot\frac{1}{\sqrt{2}}(v+w)\right)^2+\left(n\cdot\frac{1}{\sqrt{2}}(v-w)\right)^2 &= \left(\frac{n\cdot v+n\cdot w}{\sqrt{2}}\right)^2+\left(\frac{n\cdot v-n\cdot w}{\sqrt{2}}\right)^2 \\
&= \frac{2\,(n\cdot v)^2+2\,(n\cdot w)^2}{2} \\
&= (n\cdot v)^2+(n\cdot w)^2
\end{aligned}$$

To finish, at the beginning, $\sum_i(n\cdot v_i)^2 = 2(x^2+y^2+z^2) = 2$, so $S=2$. Cauchy-Schwarz gives

$$n\cdot u = \sum_i n\cdot v \leq \sqrt{\sum_i (n\cdot v_i)^2}\cdot\sqrt{6} = \sqrt{2}\cdot\sqrt{6} = 2\sqrt{3}.$$

Q.E.D.

14. On an n by n board, Bob shades $n-1$ squares. Every second, Bob shades in all squares adjacent to two shaded squares. Will the whole board ever be shaded?

    Solution: Look at the total perimeter of the shaded squares. To begin, this perimeter is at most $4(n-1) = 4n-4$, and the perimeter of a shaded board is $4n$. Note that shading any square adjacent to two shaded squares cannot increase the perimeter, so it must be non-decreasing. However, $4n-4 < 4n$, so the answer is $\boxed{\text{No}}$.

15. To each vertex of a regular pentagon an integer is assigned, so that the sum of all five numbers is positive. If three consecutive vertices are assigned the numbers x, y, z respectively, and $y < 0$, then the following operation is allowed: x, y, z are replaced by $x+y, -y, z+y$ respectively. Such an operation is performed repeatedly as long as at least one of the five numbers is negative. Determine whether this procedure necessarily comes to an end after a finite number of steps. (IMO 1986 #3)

    Solution: The procedure will always eventually come to a stop. Let the integers at the vertices be $x_1, x_2, ..., x_5$, in that order. Note that the operation replaces integers with integers, and that $\sum_{i=1}^{5} x_i$ is invariant. Since $\sum_{i=1}^{5} x_i > 0$ at the start, it is always positive. Now, consider the quantity (taking indices mod 5)

$$\sum_{i=1}^{5} (x_{i+2} - x_i)^2 .$$

The quantity is always nonnegative, since it is a sum of squares. Suppose we apply the operation on $x_1, x_2 < 0, x_3$ in $(x_1, x_2, x_3, x_4, x_5)$, getting $(y_1, y_2, y_3, y_4, y_5) = (x_1 + x_2, -x_2, x_2 + x_3, x_4, x_5)$. Then,

$$\begin{aligned}
&\sum_{i=1}^{5} (y_{i+2} - y_i)^2 \\
=&(x_3 - x_1)^2 + (x_4 + x_2)^2 + (x_5 - x_3 - x_2)^2 + (x_4 - x_1 - x_2)^2 + (x_5 + x_2)^2 \\
=&2x_1^2 + 4x_2^2 + 2x_3^2 + 2x_4^2 + 2x_5^2 + 2x_1x_2 - 2x_1x_3 - 2x_1x_4 + 2x_2x_3 - 2x_3x_5 \\
=&\sum_{i=1}^{5} (x_{i+2} - x_i)^2 + 2x_2 \sum_{i=1}^{5} x_i \\
<&\sum_{i=1}^{5} (x_{i+2} - x_i)^2
\end{aligned}$$

Hence, $\sum_{i=1}^{5} (x_{i+2} - x_i)^2$ is a decreasing monovariant. But, it is bounded by 0, so the sequence must eventually terminate; Q.E.D.

Conclusion

As the above paper illustrates, invariants are a powerful problem-solving tool that can be leveraged to solve many mathematical olympiad problems. By reframing problems to be about certain quantities, one can find the fascinating patterns that underlie different systems.

References

[1] 3Blue1Brown. https://www.3blue1brown.com/?v=shadows#video-section. Retrieved September 16, 2025.

[2] Art of Problem Solving. *AoPS Wiki*. https://artofproblemsolving.com/wiki. Retrieved September 16, 2025.

[3] Asian Pacific Math Olympiad. *APMO Problems Archive*. https://apmo-official.org/problems. Retrieved September 16, 2025.

[4] E. Chen. *Public Repository of Olympiad Problems*. https://web.evanchen.cc/problems.html. Retrieved September 16, 2025.

[5] Harvard-MIT Mathematics Tournament. *HMMT Problem Archive*. https://www.hmmt.org/archive/problems. Retrieved September 16, 2025.

[6] Tournament of Towns. *Tournament of Towns Problem Archive*. https://www.turgor.ru/en/problems/. Retrieved September 16, 2025.

[7] USA Mathematical Talent Search. *USAMTS Past Problems*. https://usamts.org/contest/past-problems/. Retrieved September 16, 2025.

[8] University of Waterloo. *Bernoulli Trials*. https://www.math.uwaterloo.ca/~snew/Contests/Bernoulli/index.html. Link deprecated as of September 2025.

[9] University of Waterloo. *Problem Sessions*. https://www.math.uwaterloo.ca/~snew/Contests/ProblemSessions/Problems2016/Lesson0soln.pdf. Link deprecated as of September 2025.

Kylan Huang
Avenues: The World School, Institute for Advanced Mathematics, New York, NY
kylan_huang@avenues-ny.org

Patrick Galarza
Avenues: The World School, Institute for Advanced Mathematics, New York, NY
pdg2112@tc.columbia.edu

THE LEGEND LIVES ON 500 YEARS LATER
From Nessebar to Milan – in Search of Ferrari's Face

Lyubomir Lyubenov

Lyubomir Lyubenov is a Bulgarian mathematics teacher who graduated from Plovdiv University in 1981. He has extensive experience as a mathematics lecturer in both secondary schools and universities, and he has also worked in the state administration of the Bulgarian Ministry of Education. He is the publisher of the newspaper Mathematical Post and an honorary citizen of Stara Zagora. Lyubenov is the organizer of several prestigious international mathematics tournaments: Friendship (1986–1993), Mathematics Without Borders (since 2013), and the Lodovico Ferrari Tournament (since 2021). As a student, he was a member of the university track and field team, competing in the 100 m, 200 m, and 400 m sprints.

Introduction

Italy gave humanity — in the words of Felix Klein — "the great art" of algebraic solutions of cubic equations. In the 16th century, public contests in solving equations, known as mathematical duels, were held there. Among the most prominent "duelists" were Scipione del Ferro, Niccolò Tartaglia, Girolamo Cardano, and Lodovico Ferrari.

The famous duel between Ferrari and Tartaglia on 10 August 1548 in Milan (in the church of Santa Maria del Giardino, demolished in the 19th century) is considered the culmination of this era. Tradition holds that Ferrari was the winner. The crowd applauded, but the true victor was the idea: the techniques for solving equations outlived both places and people.

Today, mathematical "duels" are no longer held in public squares but on the shore of the Black Sea — in Nessebar. The city is one of the oldest in Bulgaria and is inscribed on the UNESCO World Heritage List, which adds a special atmosphere to the tournament. Since 2021, Nessebar has hosted the international tournament "Lodovico Ferrari" for five consecutive years. The first edition, in 2021, was dedicated to the 500th anniversary of Lodovico Ferrari's birth — the man who gave the world the method for solving quartic equations.

The tournament is organized by the Mathematics Without Borders Foundation and the team under my leadership, continuing the tradition of bringing together young talents from all over the world for a true celebration of mathematics.

When I began preparing the article with the problems and results from the fifth tournament (July 2025), I decided to look for an authentic portrait of Ferrari. Instead of an easy answer, a real detective story was born — taking me through archives, libraries, and museums in Bologna, Milan, Florence, the Vatican, and Mantua.

What We Were Looking For

- An authentic portrait or engraving of Lodovico (Ludovico) Ferrari (1522–1565)
- Iconographic evidence of the location of the Ferrari–Tartaglia duel
- Printed and manuscript sources of the "cartelli" (1547–1548)
- Clues to the place of Ferrari's funeral and burial

What We Found

1. There is no universally recognized authentic portrait of Ferrari. This was independently confirmed by:
   - The Historical Archive and Library of the University of Bologna
   - The University of Pavia
   - Museo Galileo (Florence)
   - Biblioteca Apostolica Vaticana

   Historical note: Unlike Cardano and Tartaglia, whose images have reached us, Ferrari remains "faceless" in history.
2. The portrait widely circulated online as Ferrari's is actually of Cardinal Alessandro Farnese.
3. We found three key pieces of evidence regarding the duel's location:
   - An engraving by Domenico Aspari (1790)
   - The plan of the church (Raccolta Bianconi)
   - A photograph of the church's demolition in the late 19th century
4. The "cartelli" (duel announcements) were confirmed through the editions of Masotti (1974) and Giordani (1876), as well as the manuscript frontispiece Contrasto...
5. Ferrari's funeral in Bologna (1565) — the Archdiocesan Archive confirmed that at the time only a few parishes kept a Liber mortuorum, so a record may not exist or may have survived only locally.

Quotes and Notes for Accuracy

- Museo Galileo (Florence): "...the iconographic archive contains no discovered portrait/engraving of Lodovico Ferrari."
- Biblioteca Apostolica Vaticana — Simona De Crescenzo: confirmation that the collections were checked and that no images of Ferrari or the church have been found so far; recommendation to mention only the fact that the research was conducted, since new materials may surface in the future.
- Archivio Generale Arcivescovile di Bologna — Dr. Simone Marchesani: clarified that registers are preserved by parish, and in 1565 only a small part had started keeping a Liber mortuorum.

Sources and Contacts

(listed alphabetically, as required by scholarly standards)

- Archivio Generale Arcivescovile di Bologna — Simone Marchesani
- Archivio Storico Civico e Biblioteca Trivulziana (Milan)
- Archivio Storico Diocesano di Milano
- Biblioteca Apostolica Vaticana — Simona De Crescenzo
- Biblioteca Nazionale Braidense (Milan) — Matteo Vacchini
- Biblioteca Universitaria di Bologna — Stefania Filippi
- Museo del Risorgimento – Bologna — Otello Sangiorgi
- Museo Galileo (Florence) — Sabina Bernacchini
- Servizio Biblioteche, Università di Pavia — Gabriele Rossini
- Archivio storico, Alma Mater Studiorum – Università di Bologna — Pier Paolo Zannoni

Ferrari's story continues to challenge us. Our team will continue searching for traces of his life — including the specific parish where he was buried and any visual evidence that may still exist. Students who wish to take part in the mathematical challenge in Nessebar can write to us at: `mwb_bg@mathematicalmail.com`

REGULATIONS OF THE "LODOVICO FERRARI" TOURNAMENT

The "Lodovico Ferrari" Tournament is an international competition in solving arithmetic puzzles and algebraic equations for students aged 7 to 19. Organizers: Mathematics Without Borders Foundation and the team under the leadership of Lyubomir Lyubenov.

1. Stages
   - Qualification
   - Final Round

   The final round is held in Nessebar — one of the oldest towns in Bulgaria and a UNESCO World Heritage site.
2. Participants Students from Grade 1 to Grade 12 (or aged 8 to 19 in the year of the final).
3. Competition Format
   - 5 problems with a short answer
   - 2 problems requiring full solutions with justification Age Groups:
   - Arithmetic puzzles: 8–9 yrs, 10–11 yrs, 12–13 yrs
   - Algebraic equations: 14–15 yrs, 16–19 yrs

   Time limit: up to 90 minutes.

4. Scoring
    - Short-answer problems:
        - 2 pts – correct answer
        - 1 pt – at least half of the correct answers, or correct + one wrong answer
        - 0 pts – wrong or missing answer
    - Full-solution problems:
        - 4 pts – complete solution with justification and correct answer
        - 2 pts – correct answer without justification
        - 1 pt – partially correct solution
        - 0 pts – all other cases
5. Ranking and Awards Competitors are ranked by their total score (sum of all seven problems).
    - The top scorer in each age group receives a champion's cup and certificate.
    - Second and third places receive silver and bronze medals plus certificates.
    - All finalists receive certificates.
    - In case of a tie, the prize is shared.
6. Organization Travel and accommodation in Nessebar are arranged and financed by the participants, parents, or schools.

On June 29, 2025, at "Lyuben Karavelov" Secondary School in Nessebar, within the framework of the „Mathematics Without Borders" tournament, the final round of the "Lodovico Ferrari" competition in solving arithmetic puzzles and algebraic equations was held for the fifth time. Students from Bulgaria, Estonia, Kazakhstan, Kyrgyzstan, Romania, North Macedonia, Serbia, Turkey, the Philippines, and Uzbekistan took part.

Here are the names of the winners:

- Matei Hanu (**Romania**, Bucharest, International School of Bucharest, grade 3)
- Martin Ivanov Binev (**Bulgaria**, Burgas, SU "Yordan Yovkov", grade 3)
- Mina Dabovic (**Serbia**, Belgrade, Mathematical Grammar School, grade 10)
- Miroslav Dimirtov Srebkov (**Bulgaria**, Burgas, PPMG "Akad.Nikola Obreshkov", grade 5)
- Nikol Maeva Dimitrova (**Bulgaria**, Stara Zagora, PPMG "Geo Milev", grade 6)
- Nikol Zdravkova Nikolova (**Bulgaria**, Plovdiv, MG "Akad. Kiril Popov", grade 8)
- Pavel Vladimirov Lozanov (**Bulgaria**, Ruse, OU Otets Paisii, grade 1)
- Pornillos, Lebron James (**Philippines**, Gen. Trias, Sunny Brooke Elementary School, grade 4)
- Preslav Ganev Ganev (**Bulgaria**, Stara Zagora, OU "Kiril Hristov", grade 7)
- Radoslav Svetoslavov Balkanski (**Bulgaria**, Stara Zagora, PPMG "Geo Milev", grade 11)
- Robin Haamer (**Estonia**, Tallinn, Gustav Adolf Grammar School, grade 12)

- Sava Deyanov Troanski (**Bulgaria**, Pazardzhik, OU "Prof. Ivan Batakliev", grade 5)
- Sigrid Siinmaa (**Estonia**, Tallinn, Gustav Adolf Grammar School, grade 12)
- Valentin Nikolaev Stanchev (**Bulgaria**, Dimitrovgrad, PMG Ivan Vazov, grade 9)
- Velislav Teodorov Dzhelepov (**Bulgaria**, Plovdiv, High School of Mathematics "Acad. Kiril Popov", grade 11)
- Vladimir Valeriev Sabev (**Bulgaria**, Burgas, Luben Karavelov, grade 2)
- Kristiyan Svetlinov Krastanov (**Bulgaria**, Sofia, Sofia High School of Mathematics, grade 6)
- Kovacevic Aleksandra (**Serbia**, Pozega, grammar school "Sveti Sava", grade 12)
- Zoran Dimitrijevic (**Serbia**, Belgrade, Mathematical Grammar School, grade 10)
- Georgi Lenkov Mitev (**Bulgaria**, Stara Zagora, 5OU "Mityo Stanev", grade 3)
- Denis Nikolaev Sirakov (**Bulgaria**, Sliven, PPMG "Dobri Chintulov", grade 11)
- Anja Tesevic (**Serbia**, Belgrade, Mathematical Grammar School, grade 10)
- Georgi Plamenov Dimitrov (**Bulgaria**, Ruse, OU "Ivan Vazov", grade 4)
- Georgi Manov (**Bulgaria**, Sofia, 88 SU, grade 4)
- Daniel Vladimirov Kolovski (**Bulgaria**, Vratsa, PPMG "Akademik Ivan Tsenov", grade 6)

Here are the competition problems:

ARITHMETIC PUZZLES FOR 8- AND 9-YEAR-OLDS

Problem 1. A two-digit number with 2 as its units digit was added to another two-digit number with 8 as its tens digit. The result was a three-digit number with 0 as its tens digit. Find the sum of the digits of the two numbers being added.

Problem 2. The sum of the three-digit numbers $\overline{SUN}$ and $\overline{SEA}$ is 232. What is the maximum possible value of the three-digit number $\overline{USA}$? Identical letters must represent the same digits, and different letters must represent different digits.

Problem 3. If F, O, T, B, A, and L are different one-digit numbers, and the sum: $F+O+O+T+B+A+L+L=56$. calculate the value of $O+LL$, where LL is a two-digit number formed by repeating the digit L.

Problem 4. All numbers in the equations are two-digit numbers. What is the value of the two-digit number CH? $SU+NO=RB+EA=CH$. Identical letters must represent the same digits, and different letters must represent different digits.

Problem 5. From the number 19724953, delete some digits so that the remaining digits form a strictly decreasing number (from left to right). What is the largest number you can obtain?

Problem 6. The difference between the two-digit numbers AB and CD, written using four different digits, is 1. How many and which three-digit numbers can result from the sum of these numbers AB and CD? Problem 7. How many two-digit numbers $\overline{AB}$ satisfy the following conditions:

- $\overline{AB}$ is divisible by the sum of its digits $(A+B)$;
- the product AB is a two-digit number;
- Identical letters must represent the same digits, and different letters must represent different digits.

ARITHMETIC PUZZLES FOR 10- AND 11-YEAR-OLDS

Problem 1. What is the largest even two-digit number that is divisible by its units digit?

Problem 2. Find the three-digit number $\overline{abc}$, if $a+b=10$ and $b+c=18$.

Problem 3. If F, O, T, B, A, and L are different one-digit numbers, and the sum: $F+O+O+T+B+A+L+L=56$. calculate the value of $O+LL$, where LL is a two-digit number formed by repeating the digit L.

Problem 4. What is the maximum value of $S\times\overline{EA}+S\times\overline{UN}$. Identical letters must represent the same digits, and different letters must represent different digits.

Problem 5. What is the smallest three-digit number $\overline{SEA}$, such that it equals the sum of the two-digit numbers $\overline{NE}$, $\overline{SS}$, $\overline{EB}$ and $\overline{AR}$? Identical letters must represent the same digits, and different letters must represent different digits.

Problem 6. Let $a>b>c>d$ be four distinct integers. The different values of the differences between each pair (where the larger number minus the smaller) are 2, 3, 4, 5, and 7. What is the value of $b-c$?

Problem 7. How many three-digit even numbers have a digit sum equal to 22?

ARITHMETIC PUZZLES FOR 12- AND 13-YEAR-OLDS

Problem 1. What is the greatest three-digit number $\overline{SEA}$, if

$$S\times E\times A+S\times U\times N=20\times 25?$$

Identical letters must represent the same digits, and different letters must represent different digits.

Problem 2. Let $a>b>c>d$ be four distinct integers. The different values of the differences between each pair (where the larger number minus the smaller) are 3, 5, 6, 8, and 11. What is the value of $c-b$?

Problem 3. Find the smallest number x such that $x=a+b+c+d=e+f+g+h$, where a, b, c, d, e, f, g and h are distinct positive integers.

Problem 4. How many solutions does the puzzle have? $\dfrac{\overline{NN}}{\overline{SUN}}=0.\overline{16}$.

Identical letters must represent the same digits, and different letters must represent different digits.

Problem 5. How many three-digit even numbers, written with even digits, have a digit sum of 20?

Problem 6. Find the sum of all four-digit numbers whose digits add up to 5.

Problem 7. Arrange two digits 1, two digits 2, two digits 3, two digits 4, and two digits 5 so that:

- there are exactly 2 digits between the two 1s,
- there are exactly 3 digitsi between the two 2s,
- there are exactly 4 digits between the two 3s,
- there are exactly 5 digits between the two 4s,
- there are exactly 6 digits between the two 5s,
- the resulting number is divisible by 5.

ALGEBRAIC EQUATIONS FOR 14- AND 15-YEAR-OLDS

Problem 1. Solve the equation in integers. $xy = x + y + 6$

Problem 2. Calculate the product of the coefficients a, b, and c if the roots of the equation $x^3 + ax^2 + bx + c = 0$ are the numbers 1, 2, and 3.

Problem 3. Write a quadratic equation with rational coefficients and leading coefficient 2, for which $6 - \sqrt{7}$ is a root.

Problem 4. Solve the equation $x^4 - 6x^3 + 13x^2 - 12x + 4 = 0$

Problem 5. Calculate the product of the real roots of the equation.

$$(x-1)^2 + \sqrt{(x^2 - 2x + 1)} - 6 = 0.$$

Problem 6. Find the real roots of the equation

$$(x^3 - 6x^2 + 12x - 8)^4 = (4x^2 - 12x + 9)^6.$$

Problem 7. Find the real roots of the equation $x^2 - \sqrt{(2-x)} = 2$.

ALGEBRAIC EQUATIONS FOR 16- TO 19-YEAR-OLDS

Problem 1. Calculate $A + B + C + D + E$, if

$$\begin{aligned} A + B &= 7 \\ B + C &= 9 \\ C + D &= 11 \\ D + E &= 8 \\ A + C + E &= 10 \end{aligned}$$

Problem 2. How many real roots does the equation have?

$$\frac{(x^9 - 1)}{(x^5 - 1)} = \frac{(x^5 - 1)}{(x^3 - 1)}$$

Problem 3. Calculate the product of the real roots of the equation.

$$(x^2 - x + 1)(2x^2 - x + 2) = 6x^2$$

Problem 4. How many integers are solutions of the equation?

$$\sqrt{(x+23-10\sqrt{(x-2)})}+\sqrt{(x+7-6\sqrt{(x-2)})}=2.$$

Problem 5. What is the value of N if the number of non-negative integer solutions x_1, x_2, x_3, x_4 to the equation $x_1+x_2+x_3+x_4=N$ is 84?

Problem 6. Find the real roots of the equation $x^9+7x^7-8=0$.

Problem 7. If a and b are the roots of the equation $x^2-2x-1=0$ and c and d are the roots of the equation $x^2-4x-3=0$, calculate

$$\frac{(a^2-36)}{(b+c+d)}+\frac{(b^2-36)}{(c+d+a)}+\frac{(c^2-36)}{(d+a+b)}+\frac{(d^2-36)}{(a+b+c)}.$$

Lyubomir Lyubenov
mwb_bg@mathematicalmail.com

Australian Maths Trust
170 Haydon Drive
Bruce ACT 2617 AUSTRALIA
Tel: +61 2 6201 5136