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MATHEMATICS COMPETITIONS

Journal of the

WORLD FEDERATION OF NATIONAL MATHEMATICS COMPETITIONS

AUSTRALIAN MATHS TRUST

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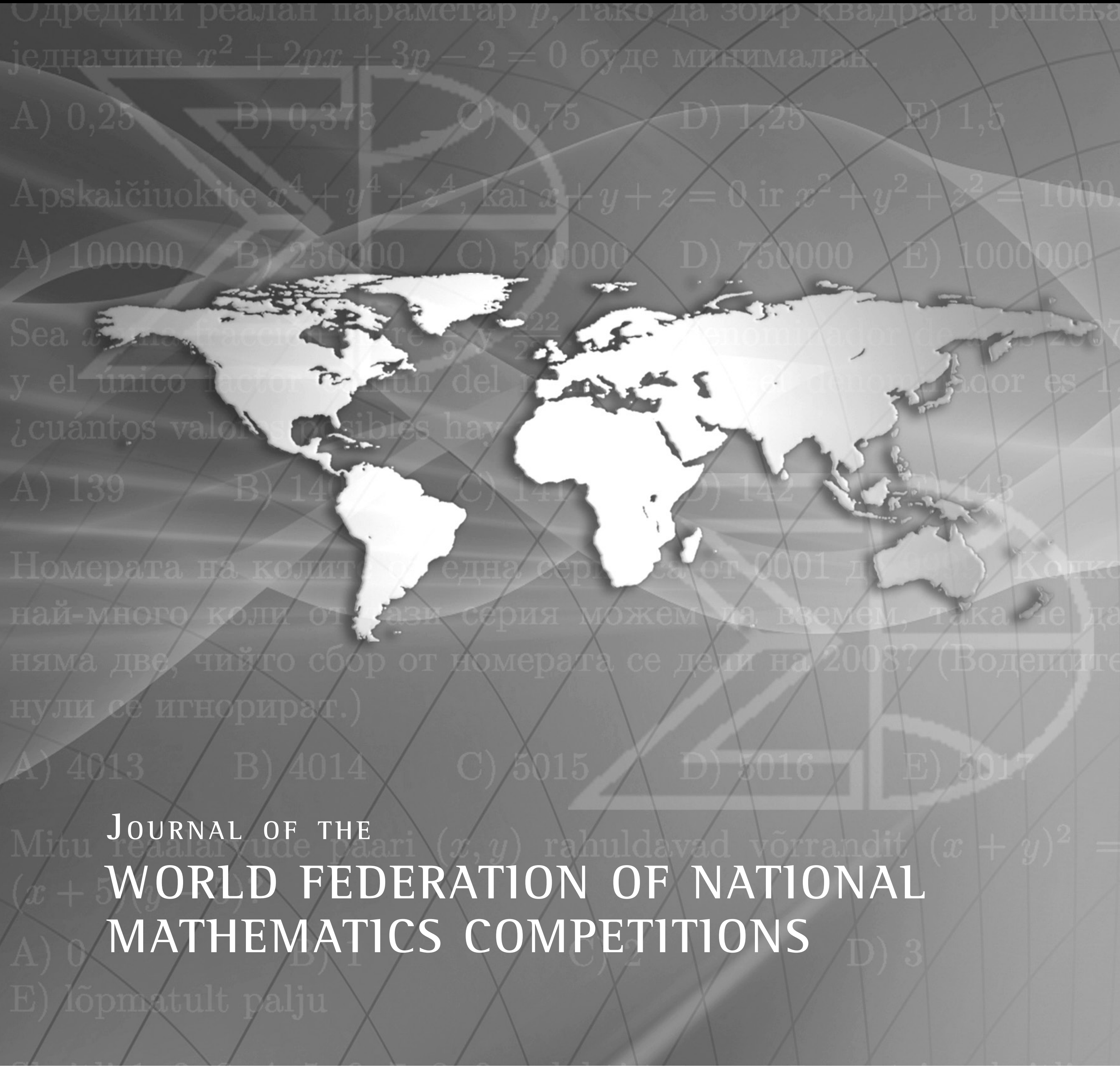

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# MATHEMATICS COMPETITIONS

JOURNAL OF THE WORLD FEDERATION OF NATIONAL MATHEMATICS COMPETITIONS

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Articles (in English) are welcome.
Please send articles to:

Professor Maria Elizabeth Losada
Universidad Antonio Nariño
Bogotá, Colombia
director.olimpiadas@uan.edu.co

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From the President

Dear readers,

By the time this issue of our journal is published, the WFNMC mini-conference and ICME-15 in Sydney will be over. Our conferences are always wonderful opportunities to interact with competition organisers and researchers from all over the world, and this year's meetings will certainly not have dissapointed in that respect.

Of course, there are always many interesting talks given at these conferences, and we plan to include a number of papers in the next edition of this journal, based on the talks given at the mini-conference.

As you can see, this edition already includes my article *Mathematical Competitions Around the World in 2024 - an approximation*, which is an extended version of my talk at the special WFNMC session at ICME. In this article, I attempt to give a broad overview over all the various mathematics competition activities currently being organized all over the world, and I hope that you will find it interesting. I am certain that you will discover some activites mentioned there that you were not previously aware of!

Of course, our work continues after this regularly recurring highlight, and we look forward to upcoming projects. The next WFNMC meeting is planned for Kuala Lumpur in July of 2026, and it would be good to mark the date in your calendars now, just to be safe. And, as always, you are invited to submit papers for publication in this journal. The quality of the journal depends of your submissions, and your collaboration is always appreciated.

Robert Geretschläger

Editor's Page

Dear Competitions enthusiasts, readers of our *Mathematics Competitions* journal!

Mathematics Competitions is the right place for you to publish and read the different activities about competitions in Mathematics from around the world. For those of us who have spent a great part of our life encouraging students to enjoy mathematics and the different challenges surrounding its study and development, the journal can offer a platform to exhibit our results as well as a place to find new inspiration in the ways others have motivated young students to explore and learn mathematics through competitions. In a way, this learning from others is one of the better benefits of the competitions environment.

Following the example of previous editors, I invite you to submit to our journal *Mathematics Competitions* your creative essays on a variety of topics related to creating original problems, working with students and teachers, organizing and running mathematics competitions, historical and philosophical views on mathematics and closely related fields, and even your original literary works related to mathematics.

Just be original, creative, and inspirational. Share your ideas, problems, conjectures, and solutions with all your colleagues by publishing them here. We have formalized the submission format to establish uniformity in our journal.

Submission Format
FORMAT: should be LaTeX, TeX, or for only text articles in Microsoft Word, accompanied by another copy in pdf. However, the authors are strongly recommended to send article in TeX or LaTeX format. This is because the whole journal will be compiled in LaTex. Thus your Word document will be typeset again. Texts in Word, if sent, should mainly contain non-mathematical text and any images used should be sent separately.
ILLUSTRATIONS: must be inserted at about the correct place of the text of your submission in one of the following formats: jpeg, pdf, tiff, eps, or mp. Your illustration will not be redrawn. Resolution of your illustrations must be at least 300 dpi, or, preferably, done as vector illustrations. If a text is embedded in illustrations, use a font from the Times New Roman family in 11 pt.
START: with the title centered in Large format (roughly 14 pt), followed on the next line by the author(s)' name(s) in italic 12 pt.
MAIN TEXT: Use a font from the Times New Roman family or 12 pt in LaTex.
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INCLUDE: your high resolution small photo and a concise professional summary of your works and titles.
Please submit your manuscripts to María Elizabeth Losada at
`director.olimpiadas@uan.edu.co`

We are counting on receiving your contributions, informative, inspired and creative. Best wishes,

Maria Elizabeth Losada
EDITOR

Andy Liu

In Memoriam

Mathematics Competitions is sad to report the passing of our frequent contributor and good friend Andrew Chiang-Fung (Andy) Liu. He passed away on March 26, 2024 in Edmonton, Canada at the age of 77 from a combination of medical problems.

Readers will be aware of the many articles he wrote or co-wrote for this journal, as well as his regular contributions with problems and solutions from the Tournament of the Towns. Andy had a special talent for solving problems in unconventional and creative ways. As a book author, he applied this special talent in many volumes of competition problems from all around the world. He had a special knack for popular mathematics, combining his love for puzzles of all kinds with a talent for getting people excited about the joys of problem solving.

Beside his contributions to this journal, he was also an active participant in numerous WFNMC conferences as a lecturer and co-organizer. In 1996, at ICME in Seville, he was honored by the WFNMC with the Hilbert award. In fact, his was the last such award, as the Erdös and Hilbert awards were merged to a single award after that year.

Personal tributes to Andy are also available on our website, http://www.wfnmc.org/.

Andy Liu (right) receiving his David Hilbert Award from WFNMC President Ron Dunkley at the WFNMC meeting at ICME-8 in Sevilla, Spain, in July 1996.

Editor's note: Andy Liu was a friend of decades, he helped people and groups all over the world around Mathematics competitions. He was also always ready with an article with some worthy collaborator, he would send one immediately on request, especially our Tournament of Towns column. He not only wrote the latter articles but would send those who participated in this contest his 'translation' into English of the problems, accompanied with solutions. He was in short a great pen pal and his stories were remarkable. He will be greatly missed!

Mathematical Competitions Around the World in 2024 - an approximation

Robert Geretschläger

Robert Geretschläger is recently retired from active teaching of Mathematics and Descriptive Geometry after 40 years at BRG Kepler in Graz, Austria. He is still as active as ever in the Austrian Mathematical Olympiad and the Austrian Mathematical Kangaroo, as well as continuing on in his roles as treasurer of the Association Kangourou sans Frontières and President of the WFNMC.

Introduction

When we speak of "Mathematics Competitions" in the modern sense, we are generally referring to problem-solving competitions of different types, organized for primary, secondary or tertiary students. The history of such mathematics competitions can be traced back to problem solving competitions in Hungary and Romania at the end of the 19th century. Much has been written on the history of such competitions and the genesis of the different types of such activities, such as [1] or [5], and the intent of this article is not to discuss this history at length, but rather to give as good an overview as possible of the many different competitions currently being run all over the world.

The idea of using competitions as a tool to popularize mathematics has become so popular in recent years that it has become all but impossible to compile a complete list (which would explain the subtitle of this report). We have come a long way from the days when Peter O'Halloran could dream of publishing a complete collection of mathematics competitions in one volume, as he did in 1992 under the title *World Compendium of Mathematics Competitions*, ([6], ISBN 0-646-09564-1).

That book was the result of a three-year project, whose intention was to identify all active mathematics competitions going on around the world at the time. Recalling that this was done in a time before readily accessible internet, the Compendium was remarkably thorough, and remains a fascinating historical document. While the resulting list was not quite complete (several national Olympiads were missing, for instance), O'Halloran was able to identify a total of 231 contests with a (necessarily approximate) total of slightly more than 4 million annual participants. If we compare this to the available data for 2023/2024, we find that the international competition scene appears to have grown by a factor of about 10, with approximately 40 million annual participants taking part in far more contests now than existed in the 1990s.

Some Readily Accessible Lists of Current Mathematics Competitions

Before we embark on our round-the-world expedition, it is worth noting that there is, of course, a wealth of information on various mathematics competitions readily available online. Nowadays, collecting information of this type inevitably begins with an internet search, and it is not difficult to find lists of mathematics competitions online. Note that the websites listed here are all current as of 2024. If you happen to be reading this at some later date, many of these may, of course, no longer be active. If this should be the case, under the assumption that the internet still works in a manner similar to what we are familiar with in 2024, a quick search will almost certainly unearth the appropriate updated information.

If we are looking for a comprehensive list of any kind, an obvious place to start is always Wikipedia. For mathematics competitions, this popular resource gives us

https://en.wikipedia.org/wiki/List_of_mathematics_competitions.

Anyone interested in the topic can certainly discover much of interest at this source, although the list given here is far from complete. Another possible first resource is available at

https://artofproblemsolving.com/wiki/index.php/List%20of%20international%20mathematics%20competitions.

In fact, Art of Problem Solving attempts to compile as complete a list as possible, and this does appear to be the most encompassing such list currently available online. As is the case for the wikipedia list, the information currently available here does not, however, by any means include all active competitions, and we must take a deeper dive into slightly more obscure corners of the internet if we wish to discover more.

One option is to take a close look at some more specialized lists. There is much to be discovered at certain national websites devoted to those competitions offered locally, for instance.

A wonderful example is the compendium of competitions in Russia, which can be found at the website

https://olimpiada.ru/.

Here, we find an incredible list of dozens of competitions, more than half of which are somehow mathematical in nature (if we include things like informatics and linguistics, which require mathematical reasoning at quite a high level).

Some other interesting national lists, that are unfortunately a bit limited in their scope, inasmuch as they tend to name only competitions offered by a single institution, are the following:

for Canada: https://www.cemc.uwaterloo.ca/contests/contests.html

for the USA: https://maa.org/math-competitions

for Australia: https://www.amt.edu.au/competitions

for the UK: https://ukmt.org.uk/

for South Africa: https://math.sun.ac.za/olympiad/ or https://saolympiads.co.za/

The last of these, like the Russian site, lists competitions in various disciplines, and not only in mathematics.

A quite interesting list is also available at the Malaysian website https://smo-testing.com/. Here, among other noteworthy information, we can also find a great wealth of information on Chinese competitions under

https://smo-testing.com/hxc/

An interesting resource for French-language competitions is available at https://www.vinc17.org/cijml/ and a similar resource for Italian-language competitions can be found at https://www.mateinitaly.it/. Finally, a useful list worth mentioning in this context is https://www.math-labyrinth.eu/o1/competitions-and-contests/.

Of course, some time spent on the internet will unearth a number of competitions not listed in any of these sources, but anyone searching for links will find a wealth of them in the websites mentioned here.

Mathematical Olympiads

We are now ready to begin our tour of the mathematics competition universe in earnest with a look at the current state of the Mathematical Olympiads. A case can easily be made for the Olympiad-style of competitions being the type with the longest history. In this context, the expression “Olympiad-style” refers to individual competitions with problems in pure mathematics, requiring solutions with full derivations. In general, such competitions are written in silence under exam conditions with strict time limits. Particpants are not allowed access to literature or electronic tools like calculators or computers, and are expected to write solutions that can be readily understood step-by-step, by any knowledgable reader, for full marks.

Typically, national Olympiads are run either by universities, ministries of education, or institutions funded by some combination of these, with the specific purpose of organising high-level mathematics competitions.

Olympiad-Style Problems

As already mentioned, much has been written on the history of competitions, with [1] being an excellent starting point for anyone interested in reading up on the subject. As noted in this important work, there appears to be a wide consensus that the Hungarian *Eötvös* Competition of 1894 was an important starting point for the modern age of competitions. The following question was posed there as Problem 1.

Problem 1 (Problem 1 of the 1894 Eötvös competition, from [7]):

Prove that the expressions

$$2x+3y \quad \text{and} \quad 9x+5y$$

are divisible by 17 for the same set of integral values of x and y.

Solution: We note that $2x+3y$ is divisible by 17 if and only if

$$30\cdot(2x+3y) = 60x+90y = (9x+5y)+(52x+85y) = (9x+5y)+17\cdot(4x+5y)$$

is divisible by 17, and this is the case, if and only if $9x+5y$ is divisible by 17. □

The general style of problem posing in Olympiad-style competitions is already apparent in this early example. All problems set at such competitions require full proofs for contestants to receive full points, with partial credit awarded for partial solutions. (For readers not so familiar with this competition style, it is worth noting that the full proof can be quite succinct, as long as it is logically complete. In this case, three lines of argument clearly suffice. This does not necessarily mean that the problem in question is easy. Brevity of a possible solution does not always make the solution easy to find, or to formulate in a logically complete way.)

There were probably even earlier competitions of this type (an 1885 contest in Bucharest, Romania is often mentioned in this context in the literature, although details of the specific competition appear to be lost to history), but the problems set there were almost certainly of a similar flavor.

This style of question is the basis of all modern Mathematical Olympiads. Nowadays, the International Mathematical Olympiad (IMO, [8]) is, in many ways, the benchmark against which such competitions are measured. In fact, the tradition of calling such competitions "Olympiads" can be traced to the first International Mathematical Olympiad, held in Brașov, Romania in 1959. (As an interesting aside, it is worth noting that this usage of the term has been contentious ever since, even though the term has become so commonplace in the math competition world.)

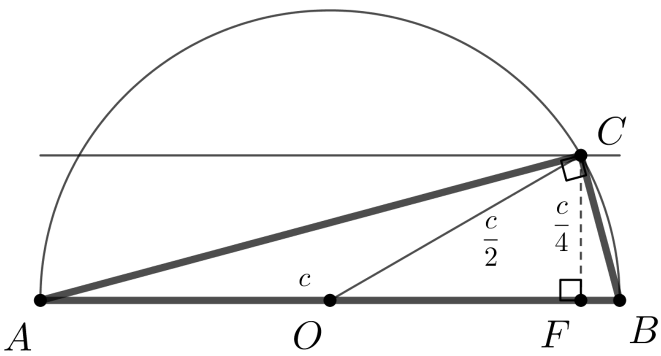
The IMO is probably the best-known of all international mathematics competitions, and arguably the competition held in the highest esteem. IMO gold medallists are well-respected for their mathematical prowess, and are highly sought after as students at universities all over the world after their graduation from secondary school. Many successful IMO participants, like Terence Tao, Maryam Mirzakhani and Grigori Perelman, to name just a few, have gone on to high-profile careers in research, with some going on to win Fields medals and countless other awards [9].

In 2023, 618 students from 112 countries participated in the IMO (as compared to 52 students from 7 countries at the inaugural IMO in 1959). The cachet of the IMO is such that many countries that have not so far been able to participate are hoping to do so in the future, and further growth is certainly to be expected.

The following is an example of a problem from the first IMO in Romania.

Problem 2 (Problem 4 of the 1st IMO, 1959): Construct a right triangle with given hypotenuse c, such that the median drawn to the hypotenuse is the geometric mean of the two legs of the triangle.

Solution: Construct a segment AB of length c with midpoint O and the semicircle on AB. An intersection of this semicircle with the line parallel to AB, and with distance $\frac{c}{4}$ from AB, is point C. The triangle ABC then has the required property.



We need to show that the square of the median OC is equal to the product of the legs CA and CB. We obviously have $OC^2 = \left(\frac{c}{2}\right)^2 = \frac{c^2}{4}$. Let F denote the foot of point C on AB. Since

$$CA \cdot CB = 2 \cdot \text{area}[CAB] = 2 \cdot \frac{1}{2} \cdot AB \cdot CF = 2 \cdot \frac{1}{2} \cdot c \cdot \frac{c}{4} = \frac{c^2}{4},$$

and we are done. □

To compare, consider the following example of a problem from a recent IMO.

Problem 3 (Problem 1 of the 64th IMO, 2023): Determine all composite integers $n > 1$ that satisfy the following property: if $d_1, d_2, \ldots, d_k$ are all the positive divisors of n with $1 = d_1 < d_2 < \ldots < d_k = n$, then d_i divides $d_{i+1} + d_{i+2}$ for every $1 \leq i \leq k-2$.

Answer: The integers satisfying this property are the prime powers $n = p^r$ with $r \geq 2$. Full solutions are readily available online, for instance at [10]. □

It is clear that this problem, posed as number 1, which is the slot usually reserved for the easiest problem of the competition, is quite a bit more difficult than the problem from the first IMO. However, while the level of difficulty of problems posed at such competitions may have developed (mostly because the amount of special training competitors at the highest levels receive has greatly increased over the years), the way a problem is posed and the expectation for the type of solution a successful particpant will create has remained quite constant.

Nearly all of the approximately 110 participating countries in the IMO hold their own national olympiads. In some countries, the particpants in the national olympiads may only be specially trained students, hoping for a place in their national IMO teams, but in many countries, participation in the various rounds and levels of the national olympiads can be much wider than that, with hundreds or even thousands of participants (or even more, as in China or Brazil). In many places, there are regular training courses held for anyone interested in such topics, and for most students, these courses can be much more important than the competitions themselves. While the prospect of successful event participation certainly offers a strong motivation to attend the courses, the enjoyment of the content, along with the social aspect of working together with others with similar interests, is the more important factor for many.

A Brief Interlude - The Pros and Cons of Olympiad-Style Competitions

The fact that training for a competition can be more important then the competition itself is quite noteworthy. In fact, the motivation for holding olympiad-style competitions is multi-levelled, and it is a common topic of discussion as to whether the stated goals are actually achieved. These can be summed up as follows:

- popularisation - making mathematics fun
- socialisation - bringing students with a common love of mathematics together
- talent identification - finding students with innate mathematical talent and helping to lead them to special training
- research preparation - leading the most able students to mathematical research

There can be little doubt that the first two of these goals are successfully met for many participants. The other two points are, perhaps, a bit more contentious. It is certainly true that the specific type of talents useful in olympiad-style competitions are not common to everyone involved in mathematical research, and some mathematicians, often with a personal preference for applied mathematics, argue that the olympiad structure is too lopsided, and does not adequately cover the breadth of topics and methods useful in mathematical research. The argument is made that only students whose interests align with the specific mathematical content commonly appearing in the olympiads are actually identified. Also, the individualistic style of the competitions themselves is sometimes cited as an antithesis to the collaberative nature of a majority of modern research work. There is certainly some validity to these arguments, but this would not seem to reduce the value of the olympiads for students whose interests and personalities do mesh with this style.

Another counter-argument to the attainment of these goals that is sometimes heard states that success at competitions, even at an international level, does not guarantee success in research in later life. This is due to the fact that the skills required for the one area do not necessarily translate to the other. Being able to deftly apply memorized theorems and methods of proof to problems with a limited scope is something quite different from tackling open ended problems with no obviously clear answer. This point is also well taken. It is certainly true that successful olympiad participation does not prepare everyone for a future career in mathematical research. Nevertheless, a great number of research mathematicians now active all over the world received their first initiation into the world of higher mathematics through their participation in competitions, and are are happy to have done so. More on this aspect can be found at [3].

Regional Olympiads and Olympiad-Style Competitions

In addition to the various national olympiads that can be found all over the world, there are also many popular competitions with participants from many different countries. These are often valuable stepping stones for students eager to qualify for their nations' IMO teams, but they also offer international competition experience to far more students than could possibly take part in the IMO.

Some noteworthy regional olympiads are the following (in alphabetical order; all numbers and web-links are the most recent available):

- **Asian-Pacific Mathematics Olympiad**

  — since 1989

  — 38 countries, 345 participants in 2023

  — website: https://apmo-official.org/

- **Balkan Mathematical Olympiad**

  — since 1984

  — 23 countries, 131 participants in 2023

  — website: https://bmo2023.tubitak.gov.tr/index

  — since 1997 there has also been an annual Junior Balkan Mathematical Olympiad for students under 16; website: https://jbmo2023.al/

- **Baltic Way**

  — since 1990

— 10 countries, 50 participants in 2023

— website: https://balticway2023.de/

- **Ibero-American Mathematical Olympiad**

  — since 1985

  — ∼25 countries, ∼100 participants

  — website: https://www.oma.org.ar/ibero/index.php/pt/

- **Middle European Mathematical Olympiad**

  — since 2007 (formerly, since 1978: Austrian-Polish Mathematics Competition)

  — 10 countries, 60 participants

  — website: https://memo-official.org/ [13]

- **Pan-African Mathematical Olympiad**

  — since 1987

  — 11 countries, 62 participants in 2022, 33 countries in 2023

  — website: https://www.africamathunion.org/AMU-pamo-official.php

- **Tournament of the Towns**

  — since 1980

  — all continents, two levels in four annual rounds

  — website: https://www.turgor.ru/en/

Specialized Competitions

It was perhaps inevitable that the proliferation of international olympiads would lead to the creation of competitions organized with very specific groups of participants in mind. The motivation for the different types of specialisation can be quite diverse, as we see in the following examples.

- Olympiads for female participants: **China Girls Mathematical Olympiad (CGMO)** and **European Girls' Mathematical Olympiad (EGMO)**

  — CGMO since 2002 (∼160 particpants in 40 teams from up to 10 countries, [4]), EGMO since 2012 (213 participants from 55 countries in 2023; https://www.egmo.org/)

  The girls' olympiads were founded with the explicit intent of promoting female participation in high-level competitions. This includes the hope that increased participation at this level will also translate to a higher percentage of females at the IMO and among mathematical researchers. It has often been noted that the percentage of females among the participants in high-level competitions is consistently low (approximately 10 to 15 percent at IMOs, as can readily be checked at [8]), and the girls' olympiads are meant to counterbalance these factors and to motivate more girls to participate actively in mathematics competitions at a high level.

- A superhigh-level olympiad: **The Romanian Masters**

  — since 2009; https://rmms.lbi.ro/rmm2024/index.php?id=home

  This competition features problems that are even harder to solve than those at the IMO. The top 20 countries at the IMO are invited to take part at this elite-level event.

- Olympiads with a restricted topic range: **Iranian Geometry Olympiad (IGO)**

  — IGO since 2014, more than 8300 contestants from 60 countries in four levels in 2022; www.igo-official.com [11])

  The IGO was introduced for olympiad competitors with a special love of geometry. A similar contest for combinatorics, the Iranian Combinatorics Olympiad (ICO) has also been held, but it remains to be seen if this will continue. The IGO meanwhile, appears to be enjoying quite a healthy existence.

University Competitions

Participants in mathematical olympiads in high school often have such wonderful memories of their experiences that they want to continue with this type of activity after they have graduated from school and entered university. This has led to a proliferation of olympiad-style competitions being offered at university level in various places all around the world.

The university level competition with the longest tradition is the prestigious William Lowell Putnam Mathematical Competition, which was first held in 1938. This competition is for undergraduates only, and attracts over 4 000 particpants annually in the US and Canada. More information about this event is available at https://maa.org/math-competitions/william-lowell-putnam-mathematical-competition.

Some other noteworthy competitions of this type are the following:

- **Vojtěch Jarník International Mathematical Competition**

  — since 1991

  — 2 categories, for students under 25 that have not completed their degrees

  — organized in Ostrava / Czech Republic

  — website: https://vjimc.osu.cz/

- **International Mathematics Competition for University Students (IMC)**

  — since 1994

  — for students in year 1-4 only, $\sim$ 650 participants

  — organized in the United Kingdom

  — website: https://www.imc-math.org.uk/

- **Olimpiada Iberoamericana de Matemática Universitaria (OIMU)**

  — since 1997

  — for undergraduate students from iberoamerican universities

  — organized in South America and Central America

  — website: https://paginas.cimpa.ucr.ac.cr/Olimpiadas/index.php/es/inicio/2022

- **Competencia Iberoamericana Interuniversitaria de Matemática (CIIM)**

  — since 2009

  — for undergraduate students from iberoamerican universities

  — organized in South America and Central America

  — website: https://ciim.uan.edu.co/

- **Alibaba Global Mathematics Competition**

  — since 2018

  — held in two rounds for all university students; in second round participants choose specialized topic areas

  — organized internationally in China, US and Singapore

  — website: https://damo.alibaba.com/alibaba-global-mathematics-competition?language=en

- **International Student Team Competition in Mathematics**

  — since 2023

  — for all students up to third year post-graduate

  — organized in Poland

  — website: http://istcim.math.us.edu.pl

While each of these competitions has its own unique style, the problems posed at each are generally of much the same type as would typically be encountered in olympiad competitions at the high-school level. The main difference lies in the mathematical content, which is generally of a higher level at the university competitions. An example of this can be seen in the following algebra problem.

Problem 4 (Problem 2 of the 30th IMC, 2022 [12]):

Let $n \geq 1$. Assume that A is a real $n \times n$ matrix which satisfies the equality

$$A^7 + A^5 + A^3 + A - I = 0.$$

Show that $\det(A) > 0$.

We note that questions pertaining to matrices and determinants are beyond the scope of high school olympiads, but well within the range of topics at most tertiary competitions. The solution to this particular problem can be found at
https://vjimc.osu.cz/storage/uploads/j30solutions1.pdf.

Popular Competitions

We recall that the motivational reasons for olympiad-style competitions include the popularisation of mathematics as a subject, bringing the school subject closer to recreational mathematics and puzzle solving. This is certainly a way to raise the interest for students with an innate interest in the abstractions of pure mathematical reasoning, but only a limited number of pre-disposed students can be reached in this way.

On the other hand, the main thrust of popular, open competitions is the engagement of as many students as possible, bringing this aspect to the forefront. While many regions and countries also make use of the results of such popular competitions for talent recruitment for their olympiads, this aspect is generally considered as secondary. First and foremost, this type of activity is meant to reach as many people as possible, showing students that may not normally be all too interested in abstract problems that there is great joy to be found in solving mathematical puzzles. Of course, there is value in this for its own sake, but it also includes the hope that many participants will transfer the positive feelings generated from solving puzzles to wider areas of rational, logical and scientific reasoning.

Most popular competitions are held in formats that make both answering a question and grading a paper as simple as possible. This almost inevitably means that such competitions are held in some variant of a multiple choice format. There are many advantages to such a format. Not only does it make grading easy, such a contest is easily adapted to an electronic environment, and many such competitions have either been moved online in recent years (this was especially true during the worst days of Covid), or were specifically designed with online participation in mind. Also, since a multiple-choice format does not require a participant to formulate logical arguments, it is well suited to students of all ages and skill levels. All that is required (as if this were so simple...) is for the problems to be chosen at an appropriate level of difficulty for the intended audience. Of course, this means taking many things into consideration, like the ages of potential participants, their likely skill levels, or even the cultures of learning they can be assumed to have experienced.

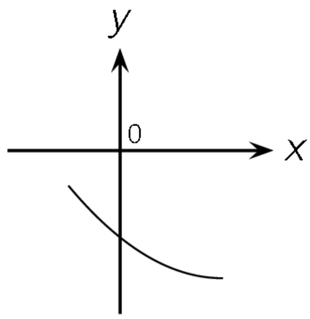
There are many such popular competitions taking place all over the world, and the following problems are typical for the style of question posed there. We note a few of the better known competitions and give an example of a competition problem from each, along with its expected solution.

We begin with the Mathematical Kangaroo. This is the most wide-spread of all competitions, with approximately 6 million participants in over 100 countries taking part in 2024. This competition is offered at six age levels, for all levels of primary and secondary school, from grades 1 through 12/13. Information about the Mathematical Kangaroo is available at the homepage of the Association Kangourou sans Frontières (AKSF), www.aksf.org, where links to the websites of the various national versions of the competition can also be found.

The following was posed as a problem for the oldest group.

Problem 5 (Student 21 of the Mathematical Kangaroo, 2020): The figure shows a section of the parabola with equation $y = ax^2 + bx + c$. Which of the following numbers is positive?

(A) c (B) $b+c$ (C) ac (D) bc (E) ab



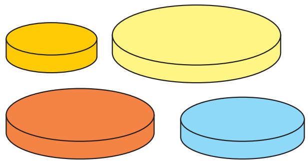
Solution: The answer is (D) bc.

Since $x = 0$ implies $y = c$, and the parabola intersects the y-axis underneath the origin, we have $c < 0$. Since the parabola is open upwards, we have $a > 0$. Finally, since the vertex of the parabola is on the right of the y-axis, we have $b < 0$. We see that c, $b+c$, ac and ab are all negative, and bc is positive. □

To compare, we also take a look at the following, much more elementary problem, posed for students aged 8 to 9:

Problem 6 (Écolier 7 of the Mathematical Kangaroo, 2023): Anna has 4 discs of different sizes. She wants to build a tower of 3 discs so that every disc is smaller than the disc below it.

How many different towers can Anna make?



(A) 1 (B) 2 (C) 4 (D) 5 (E) 6

Solution: The answer is (C) 4.

In any group of three discs, the possible ordering from bottom to top by size in a tower is unique. Since any one of the four discs can be left out, there are four possible towers. □

Next, we consider a problem posed at the competition with the highest number of current annual particpants. This is the Olimpíada Brasileira de Matemática das Escolas Públicas (OBMEP), whose website can be found at https://www.obmep.org.br/. OBMEP offers three levels of competition problems in multiple rounds, with particpants qualifying for successive rounds through their results. This amazing competition has been growing steadily since its inception in 2005, and in 2023, it counted more than 18 million participants.

The following problem was posed for the older students in Round 1.

Problem 7 (OBMEP Brazil 2023, Round 1, Level 3, Nr. 6): Let x be a number such that $x^2-3x+1=0$. What is the value of $x^2+\frac{1}{x^2}$?

(A) 7 (B) 8 (C) 9 (D) 10 (E) 11

Solution: The answer is (A) 7.

We note that

$$x^2-3x+1=0 \iff x+\frac{1}{x}=3.$$

Since

$$x^2+\frac{1}{x^2}=\left(x+\frac{1}{x}\right)^2-2=3^2-2=7,$$

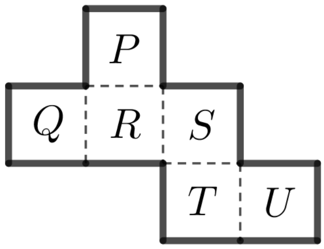
we see that the value of $x^2+\frac{1}{x^2}$ is equal to 7. □

The Kangaroo Competition was named in honour of the Australian Mathematics Competition (AMC), which it was modelled after. The website of the AMC can be found at https://www.amt.edu.au/amc. About a quarter of a million students from more than 30 countries participate in the AMC annually in five levels. The AMC has been held annually since 1978.

The following problem was posed both in the Intermediate and Junior levels.

Problem 8 (Junior 8 and Intermediate 4 of the Australian Mathematics Competition, 1986): When the diagram shown is folded to make a cube then the face marked U is opposite the face marked

(A) P (B) Q (C) R (D) S (E) T



Solution: The answer is (C) R.

If a particpant can picture the folding process mentally, this question can be answered by pure visualization. Another way to find the answer, is to note that face R has a common edge with each of the faces P, Q and S, and will also have a common edge with face T after folding. This means that face R must be opposite face U. □

There are several other similar competitions that reach hundreds of thousands of students every year. Among others, we would be remiss not to mention the following:

in the United States: American Mathematics Competitions (AMC), https://amc-reg.maa.org/

in Canada: The Gauss, Pascal, Cayley and Fermat Mathematics Contests, https://www.cemc.uwaterloo.ca/contests/contests.html

in Europe: Pangea, https://www.pangea-wettbewerb.de/

in the United Kingdom: UKMT Math Challenges, https://ukmt.org.uk/competitions

Other Formats: Correspondence Olympiads, Team Competitions and More

So far, we have been concentrating almost exclusively on competitions organised in a "classic" way. This means that participants are expected to work individually, with strict time limits and without electronic tools, books or notes. Point scores are calculated from their answers, and prizes are awarded to the participants with the highest scores. This is certainly the scenario most likely to come to mind when we talk about a "mathematics competition".

There are, however, many alternative initiatives that deserve to be included here. Some may not count as "competitions" in a strict sense, but it is hard to define an exact boundary of the concept, as we shall see when we consider the following examples.

Non-individual Formats

There are many different ways in which team-work can be integrated into a competition or competition - like activity. One option is to include a team competition in an otherwise classic olympiad-style competition, and an example of this idea is the Middle European Mathematics Competition (MEMO, [13]). This competition is held in two parts, with an individual olympiad-style problem set, followed by a similar problem set the next day, which participants are free to solve together in their (national) teams. This allows for tactical considerations on how best to divide the challenge. A team may decide to hand out the problems to each of the team members to work on individually, or to discuss the problems and develop solutions together. Of course, a combination of these tactics is also possible, and tactical considerations of this type tend to be an element in most team competitions.

Going one step further, some competitions are held purely in a team format. Two noteworthy examples of such competitions are the long-running American Regions Mathematics League (ARML, `https://arml3.com/`) and the more recent Náboj competition (`https://math.naboj.org/at/de/`). A brief overview of the methodology applied in these two competitions gives a good idea on how such things can work.

The American Regions Mathematics League was started in 1976 for teams of 15 students from various US states. It has developed in many ways over the years, including an international offshoot named IRML (for International Regions Mathematics League). The ARML competition is held in multiple rounds, including classic individual and team rounds, but also a "power" round and a "relay" round. The power round includes multiple problems with a related, developing theme, which are solved in teams. The relay round is a mathematical race, similar to the Náboj competition, which we will describe next.

The Náboj was first held internationally in 2005, although its origins date to a local competition in Bratislava/Slovakia in 1998. It has been growing steadily (with a brief online dip during the Covid years), with more than 600 schools from 12 countries taking part in 2024.

The format of the Náboj is typical for relay-type competitions. Its name literally means *Bullet*, and this is quite appropriate, as the action at such a competition is fast and furious, with teams competing against the clock to solve as many problems as possible in the alloted time. (Some relays require teams to solve a certain fixed number of problems, and the first to finish them all is the winning team.) At the Náboj, teams are composed of five members (with Junior and Senior categories offered). At the start of the two-hour competition, each team opens an envelope containing the first six of the dozens of problems that have been prepared in advance. The solution of each problem is a number, and teams are not expected to explain how they derive their answers (in contrast to the olympiad-style, where proofs are of the utmost importance). As with other team competitions, the teams are free to organize their work as they see fit. Whenever a team is convinced that they have found a solution, a team member goes to a judge's table, where the answer is checked. If it is incorrect, the team member is sent back to try again. If the answer is correct, the sheet is stamped, and the team member goes to another table, where he or she hands in the problem and receives a new one. The correct answer is credited to the team, and this credit is entered in the online system. This makes it possible to see how teams are ranked at any given moment, since the competition is started simultaneously in all competition centers. The level of difficulty of the problems rises as the numbers get higher, so more time is generally required to solve the higher-numbered problems. The goal is for the teams to solve as many problems as possible in the alloted time.

It is not difficult to understand the popularity of relay competitions. The atmosphere is infectious, even for non-participating observers, with the excitement of finding a solution and immediately moving on to a new challenge alternating with the disappointment of the identification of an incorrect answer.

Of course, there are also completely different types of team competitions. A very interesting example of an event featuring multiple fascinating alternatives is EUROmath (https://euromath.org/). This event has been held annually since 2009 in various European venues and there has also been a EUROmath Asia held in South Korea. (In addition, the event has also been extended to include analogous EUROscience competitions.)

In EUROmath, there are four different competitions. A poster competition asks students to develop a poster presenting some kind of mathematical content. This can be done individually or in teams. Also, competitions called *Mathpresentation* and *Mathfactor* invite students to present some freely chosen mathematical content, either individually or in a group. The rules of the two competitions differ, but in each case the presentation element is considered by a jury, which awards point scores. Finally, and perhaps most unusually, there is a competition named *Matheater*, which is just what it sounds like. Students create a short play with some kind of mathematical content, and their presentations are again awarded points by a jury.

All four EUROmath competitions are unique, in that the mathematical content is freely chosen by the participants. Of course, they can receive help from teachers or trainers in choosing their subject material, but this freedom of choice is quite unlike the content of nearly all other such activities, perhaps with the exception of the SNAP Math Fairs, which we will be encountering in the next sub-section.

Non-competitive Competitions

The heading of this sub-section may seem oxymoronic, but its meaning will soon become clear.

Thinking back to the reason we offer mathematics competitions in the first place, we recall that the main motivation is to give students a reason to engage more deeply with the subject. This is done by organizing a structure that participants find diverting, and possibly even amusing. Competing against each other for points, honors and prizes is one way to do this. Of course, in all such activities, the participants are also competing against themselves and against the subject material. Seen this way, it is not such a stretch to consider an activity a "competition", even if it only offers the latter aspect to participants, and not the former.

Any Math Club would qualify to be included in this category, but there are several exceptional initiatives worth mentioning specifically. Topping the list, the concept of a *Mathematical Circle* can look back on nearly a century of history. Such groups have been meeting in Russia since the 1930's [2], and the concept has since taken root in places all over the world. (Information on Math Circles in the US is avaliable at https://mathcircles.org/, for instance.) What makes a math circle special is the way in which its topics are structured. Typically, a series of questions on a common topic is posed (in a manner quite reminiscent of the classic Socratic method), leading deeper and deeper into the difficulties of the area under consideration. This is quite a close approximation of mathematical research, and it is not unheard of for open-ended research problems to be ultimately posed in such circles, the solutions of which are not (yet) known. Particpants may work individually on the problems, in pairs, or in large groups. In any case, the goal is to get as far as possible into the depths of the topic, and success is defined by the number of insights that can be attained.

A special example of this concept is the Julia Robinson Mathematics Festival (https://jrmf

.org/). The target audience of the JRMF is made up of younger students, and so the “research” topics tend to be more puzzle-oriented than would be the case in math circles intended for a more mature audience. Also, these festivals are generally organised as unique events, unlike most math circles, which are usually organised to meet at regular intervals.

A related idea is the SNAP Math Fair (https://www.mathfair.com/). The acronym SNAP is short for *Student-centered, Non-competitive, All-inclusive, Problem-based*. In such a math fair, participants are invited to choose any mathematical topic that interests them (much as is the case at the EUROmath competitions). Students are then invited to prepare a presentation of their topics, and the various presentations are then displayed in a public venue at a special event. The presentations can include posters, models, toys, computer programs, or anything else the presenters feel is appropriate, with special encouragement given to making the displays as interactive as possible. The projects are then displayed on a special day at an appropriate venue, which can be a school or some more public spot, like a shopping mall. The presenters are on-site at the venue to guide visitors through “their” topic. This structure allows participants to take emotional ownership of their topics, as they first spend some time understanding the details, then dive deeper as they prepare the displays, and finally achieve a deep confidence in their understanding, as they explain the topic to multiple visitors at the public event.

Correspondence Competitions

When we think of mathematical competitions, an image immediately comes to mind of students sitting together in a room, thinking and writing. Nowadays, the common room may be virtual, but some wonderful competitions do not depend on any such gathering of participants at all. In fact, one of the very earliest competitions was the correspondence competition offered by the Hungarian magazine KÖMAL. This competition, still going strong after over a century, is the model on which many similar activities are based.

KÖMAL (Középiskolai Matematikai és Fizikai Lapok, https://www.komal.hu/home.h.shtml) was founded in 1893 as a magazine on mathematics and physics for high school students. Along with informative articles, each issue contains a number of olympiad-style problems, and readers are invited to solve these and submit their solutions. Since the time limits are measured in weeks rather than hours, the highest-level problems offered here can be quite difficult, and it is not surprising that many future research mathematicians of renown were captivated by the opportunity to tackle these challenges in their younger years. Perhaps the most famous of the KÖMAL alumni was Paul Erdös, the namesake of the WFNMC prize awarded to mathematicians whose work in competitions has contributed to mathematics enrichment in an exceptional way.

The idea of a competition offering students the opportunity to think deeply about a problem for a longer time has been taken up in many other places. An especially successful competition of this type is offered by the Russian magazine Kvant (https://kvant.mccme.ru/), and another note-worthy initiative of this type is the USA Mathematical Talent Search (USAMTS, https://usamts.org/).

Alternative Content

In closing this section, it is worth mentioning that there are numerous experiments being made with alternative content. Typically, the mathematical content in competitions starts with logical puzzles, and advances to “advanced elementary mathematics”, which includes Euclidean geometry, combinatorics, number theory, equations and inequalities, and a few other topics. It is difficult to draw a clear border delineating where classic content ends and alternative content begins, and it is probably not something we would want to do anyway, since the stated intentions of popularisation and talent fostering in mathematics are well represented by anything that works.

There are always discussions about expanding the content options of competition problems, with arguments often brought in favor of the inclusion of additional topics such as statistics, solid geometry or calculus. There are also interesting initiatives attempting to extend the limits of "mathematics" competitions in a much stronger way. Some of these are the following:

- Bebras, the "International Challenge on Informatics and Computational Thinking", https://www.bebras.org/

  The problems in "computational thinking" posed at the Bebras competition are quite mathematical, but often algorithmic, and decidedly oriented toward computer science.

- Let'sMOD, https://letsmod.com/

  Another competition pointing directly to computer science, Let'sMOD lets participants create "machines". These are simple computer games, which are programmed in a user-friendly programming language. The results are then rated by users, with the goal of the competition being to create the most popular game possible.

- mathematical modelling competitions, https://www.mathmodels.org/contests.html

  The problems posed at mathematical modelling competitions are meant to offer a path to applied mathematics, analogous to the path to pure mathematics offered by the more traditional competition curriculum.

Future Developments

It is interesting to speculate on what the near future holds in store for the mathematics competition universe. Extrapolating the development of recent decades would seem to allow us to expect continued growth, as more and more interested students gain access to such activities.

Specifically, the continent of Africa would appear to offer a good deal of growth potential. There are currently a number of initiatives being put in place to help overcome the infrastructure problems that have thus far made it so difficult to offer regional or national mathematics competitions in large parts of Africa. At the moment, corporate sponsorship would appear to make it possible to finance these projects, and there is every reason to believe that they will quickly bear fruit, bringing the joys of mathematics to many areas that have traditionally been underserved in this respect.

Another promising aspect is the growth of online competitions. A big stumbling block to participation in competitions, even in countries with traditionally high levels of participation, is the difficulty in travelling to far-off venues for contests or training from some more remote areas. Currently, more and more websites offer online training and competition participation, which makes it ever easier for anyone interested in the topic to find some reasonable avenue of access. (Perhaps the most important online source for high-level training is currently Art of Problem Solving, https://artofproblemsolving.com/.)

Some possible developments seem to be peeking over the horizon, with their eventual impact completely unclear at the moment. For instance, some competitions now consciously allow participants to utilize technology. Since traditional competitions usually have explicit prohibitions on the use of calculators or computers, it will be interesting to see what influence this development will have in the future.

Another aspect that may or may not have an impact in the future is the role of for-profit competitions. At the moment, nearly all competitions are either free for participants, or involve a nominal participation fee, which is meant to finance the operating costs of the organisation to some extent. In general, organising entities do not expect to make a profit from their competitions, but rather struggle to find external funding in the form of government support, institutional support (from schools or universities) or corporate sponsorship. There are, however, groups that attempt to profit from the organization of mathematics competitions, and while their role has been minimal so far, it is quite conceivable that this could change at some point. We will simply have to wait and see.

Conclusion

The motivation for this article was to present an overview of all that is going on around the globe in mathematics competitions in 2024. The idea was both to create a portfolio of these activities available for any interested parties, who may not be aware of the things that are going on in this respect (especially in the math education community, but not exclusively so), and to offer a starting point for an internet search for anyone interested in a deep dive into this fascinating world.

I have tried my best to include all types of competitions and to name the largest and most impactful among them. If I have missed any competition of note, I apologize, and would be grateful if you could communicate this to me. As WFNMC president, I would very much like to have a complete picture of all that is going on in the world in this exciting specialty, and I appreciate any information you could pass on to me that I may not have been aware of. I can easily be reached by email at `robert@rgeretschlaeger.com`.

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Robert Geretschläger
robert@rgeretschlaeger.com

Analyzing Proficiency and Growth: A Study of Math Competitions in Elementary School Students in Puerto Rico

Junior Arauz-Aguilar and Luis Cáceres

Luis Caceres is a professor of mathematics at the University of Puerto Rico, Mayaguez Campus. He is the founder and director of the Puerto Rico Mathematics Olympiads Program. He holds a Ph.D. in mathematical logic from the University of Iowa in the United States. He has worked with the Math Olympiads for the past two decades and has collaborated on different educational and math competition projects in various countries. He received the Paul Ërdos Award from the WFNMC in 2016. He is vice president of the Association "Kangourou sans Frontieres", which organizes the largest international mathematical competition in the world.

Junior Arauz obtained his bachelor's degree in Teaching Mathematics with Technological Environments at the Technological Institute of Costa Rica. He is pursuing a master's degree in Teaching Mathematics at the Secondary Level at the University of Puerto Rico, Mayaguez campus. He is part of the Puerto Rico Math Olympiads team and he is interested in continuing with his doctoral studies and researching topics related to math education and statistics.

Abstract
In this article we study the elementary level test proposed for the second eliminatory phase of the 2023-2024 mathematics Olympiads in Puerto Rico. The test consisted of multiple-choice and open-response questions. Each of these questions was classified within an area of mathematics such as geometry, algebra, number theory, counting, among others. A difficulty analysis of the proposed questions was carried out by classification area to determine the level of knowledge of the students and the strengths and weaknesses within each of the areas evaluated.

A statistical study was carried out comparing the results in the different math areas and grades, identifying the areas where students exhibit higher levels of competence and areas where their mastery is comparatively lower.

Introduction

Mathematics Olympiads are academic competitions that consist of solving mathematical problems in a set time. These competitions are carried out by students at different academic levels, as is the case of the Mathematics Olympiads of Puerto Rico, which have been continuously developed by the University of Puerto Rico since 2001. This institution invites both elementary and secundary school students to take a series of tests consisting of several phases to determine the students with the best scores. In particular, these students represent Puerto Rico in mathematics competitions at the international level. Therefore, it is important to design efficient tests that serve as instruments to help select the best students.

On Saturday, November 4, 2023, the second stage of the Puerto Rico mathematics Olympiads was held, in which elementary and secondary school students, who managed to pass the first stage of this competition, participated. The participants of this second test, took an exam according to their classification:

- Students in the fourth, fifth and sixth grades of elementary education took the elementary test.
- Students in the seventh, eighth and ninth grades of secondary education took the intermediate test.
- Participants in the tenth, eleventh and twelfth grades of secondary education took the superior test.

The students with the highest scores move on to the third phase of the annual cycle of Olympiads. The objective of this article focuses on determining areas of difficulty present in the participants of this competition who took the elementary test, according to the results obtained from the classification test for the third phase of the Puerto Rico Mathematics Olympiads.

Methodology

Type of research

The research conducted, utilizes a descriptive quantitative methodology, according to Hernandez et al (2018) [5], a data collection and analysis of this type is performed to determine characteristics and important features by establishing the behavior of a population by relying on statistical numerical measurement.

Population and sample

In this research, the sampled population consisted of students who made it to the second round of the 2023-2024 Puerto Rico Mathematics Olympiads. The sample included all the students who took the elementary level exam, which consisted of 119 fourth grade students, 134 fifth grade students and 87 sixth grade students, obtaining a total sample of 340 elementary level participants.

Structure and classification of questions by topic

The test set for the second phase of the Puerto Rico Mathematical Olympiad at the elementary level consisted of a total of 15 questions, of which 10 were multiple choice and 5 were open-ended.
The classification of the proposed questions according to topic was based on Alvarado et al (2023) [1]. These authors indicate that for mathematical areas, the most frequent classification topics are the following:

- Arithmetic
- Geometry
- Number theory
- Algebra
- Counting
- Statistics
- Logic

According to Niss (1987) [7], mathematics is a very broad and interdisciplinary field that is applied in different areas of knowledge, and according to Alvarado et al (2023), [1] mathematical questions can be classified into different areas according to their focus and specific application. Therefore, the classification of questions is not unique and could vary.

Classification of the difficulty of the questions

In the difficulty classification, a difficulty index per question is calculated. According to Crocker and Algina (1986) [3], a formula for determining the difficulty index per question i, is given as follows:

$$P_i = \frac{A_i}{N_i}$$

P_i: Difficulty index per question i.
A_i: Number of participants who answer question i correctly.
N_i: Number of people answering correctly plus the number of people answering incorrectly to question i (Total number of participants in question i).
The questions will be classified according to the model proposed by Cárdenas (2013) [2]:

Classification of the question	Difficulty value per question
Very easy	0.81 – 1.00
Relatively easy	0.66 – 0.80
Adequate difficulty	0.51 – 0.65
Relatively difficult	0.31 – 0.50
Difficult	0.11 – 0.30
Very difficult	0.00 – 0.10

Classifying open-ended questions and multiple-choice items by their difficulty index by means of a similar analysis is possible, since Morales (2012) [6] indicates that similar analyses can be performed if all the questions are corrected with the same key or with the same correction system (instead of always having the value of 0 or 1, as in multiple-choice questions, they can score 0 or 1 or also from 0 to 2, from 0 to 5, etc., depending on how the correction key is established). In these cases, the difficulty index is the average of each question regardless of whether it is open-ended or multiple selection because the same key was used for all corrections and scores of 0 if incorrect or 1 if correct are obtained.
To determine in which areas students present greater strengths and weaknesses, an analysis of scores by area in educational assessments was performed, where Sandoval et al (2022) [9] indicate that to take this scale, the average scores in each area should be calculated and then a comparison between the areas should be made, thus determining in which area the students obtained higher scores (Strength in that test) and lower scores (Weakness in this test).

Results

The grading metric applied to elementary school students' tests was based on binary scoring for both the multiple choice and open response parts, with one point for a correct answer and zero for a wrong or blank answer.
As indicated above, a classification of the questions by area was developed, as follows.

Geometry

Questions that have been classified in this area have the following characteristics that have been proposed by [1]:

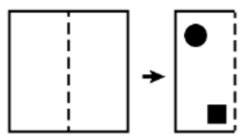
- Area and perimeter of geometric figures
- Geometric transformations
- Shaded area
- Similarity of triangles
- Measurement of internal angles of polygons

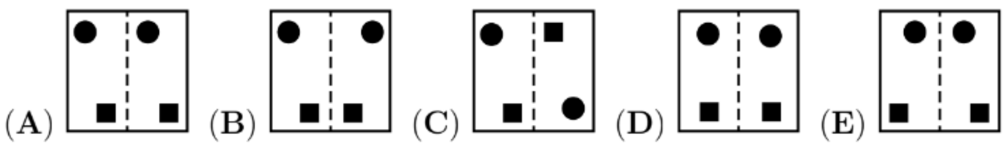
The elementary level exam has a total of 3 questions in the area of geometry corresponding to the multiple-choice section, which are shown below with their respective analysis.

Question 1

Se dobla una hoja de papel por la mitad. Se perfora un agujero cuadrado y uno redondo. ¿Cómo queda la hoja de papel después de ser desdoblada?

A sheet of paper is folded in half. Two holes are punched, one square and one round. How does the sheet look after it is unfolded?





Correct answer: B

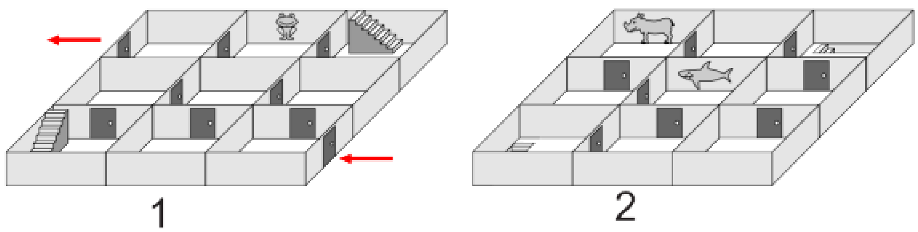
A difficulty analysis was performed according to the scale proposed by Cárdenas (2013) [2] for fourth, fifth and sixth grade levels.
In the case of fourth grade, 55 students answered the question correctly and 64 students answered incorrectly, obtaining a difficulty index $P_1 = 0.46$, classifying it as a relatively difficult question for fourth grade students.
Next, for fifth grade students the study obtained a total of 79 correct answers and 55 incorrect answers, thus obtaining a difficulty index $P_1 = 0.59$, which classifies the question as of adequate difficulty for fifth grade students.
In the case of students in the sixth grade, there are 50 correct answers and 37 incorrect ones, the classification of the question according to its difficulty index $P_1 = 0.57$ corresponds to an item of

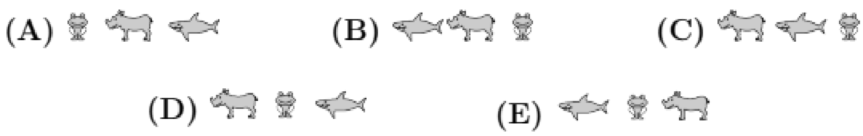
adequate difficulty.
In general, for question 1 the study obtained an index of $P_1 = 0.54$, which places this item as of adequate difficulty.

Question 5

Sofía camina por un laberinto de dos pisos desde la entrada hasta la salida, ambas ubicados en el piso 1. ¿En qué orden encontrará las pegatinas de pared?

Sofía walks through a two-storey maze from the entrance to the exit, both located at floor 1. In what order will she find the wall stickers?

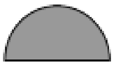
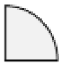


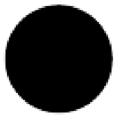


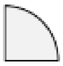
Correct answer: B

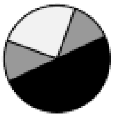
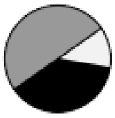
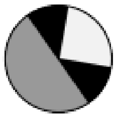
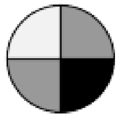
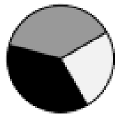
For the difficulty analysis of the question posed above, the study showed that for fourth grade students this item had a difficulty indicator of $P_5 = 0.49$ based off, of 57 students who answered correctly, thus classifying the question as relatively difficult.
There was a total of 84 correct answers from the fifth grade students on this item. The difficulty value for this question is $P_5 = 0.63$, this index places this question as of adequate difficulty for fifth grade students. On the other hand, for 6th grade students, the question is categorized as relatively easy since the associated index is $P_5 = 0.79$, because 69 students responded correctly.
For this question there are 210 correct answers in general, thus obtaining an overall difficulty index $P_5 = 0.62$, which classifies the item as of adequate difficulty.

Question 6

Daniela pegó los 2 trozos de papel   encima del círculo negro de la derecha. ¿Qué no puede obtener ella?



Daniela glued the 2 pieces of paper   *on top of the black circle on the right. What can she not obtain?*

(A)  **(B)**  **(C)**  **(D)**  **(E)** 

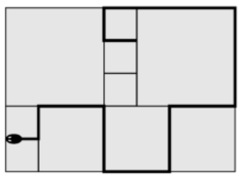
Correct answer: D

In the analysis corresponding to this question for fourth, fifth and sixth grade students, it was determined that the difficulty index for this question corresponds to $P_6 = 0.43$, $P_6 = 0.60$ and $P_6 = 0.70$ respectively, where the question is categorized as relatively difficult, of adequate difficulty and relatively easy respectively.

Question 7

Los Potter tienen un patio con losetas cuadradas de tres tamaños diferentes. Los cuadrados más pequeños tienen un perímetro de 80 cm. Una serpiente descansa en el patio, como se muestra en el diagrama. ¿Cuál es la longitud de la serpiente?

The Potters have a patio with square tiles of three different sizes. The smallest squares have a perimeter of 80 cm. A snake rests on the patio, as shown in the diagram. What is the length of the snake?



(A) 380 cm **(B)** 400 cm **(C)** 420 cm **(D)** 440 cm **(E)** 1680 cm

Correct answer: A

The difficulty indices obtained on this item are $P_7 = 0.1344$, $P_7 = 0.1940$ and $P_7 = 0.2413$ for fourth, fifth and sixth grade students respectively, classifying this question as difficult for the elementary school population.

Globally, a score per area was obtained of 37.61% for fourth grade, 50.37% for fifth grade and 57.76% for sixth grade. The overall score obtained for elementary level students for the area by geometry was 47.79%.

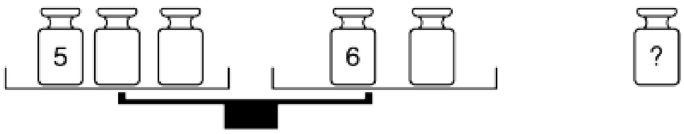
Algebra

In the elementary level test, only one question is classified in this area because it is related to solving an equation. For this question located in the multiple-choice section, the same value will be obtained for the difficulty index and the score per area, because there is only one question related to this area in this test, which is shown below.

Question 8

Hay seis pesas de 1, 2, 3, 4, 5 y 6 kg. Ruth coloca cinco de ellas en la balanza y deja una pesa a un lado. La balanza se equilibra. ¿Qué pesa dejó de lado?

There are six weights of 1, 2, 3, 4, 5 *and* 6 *kg. Ruth puts five of them on the scales and puts one weight aside. The scales balance. Which weight did she put aside?*



(**A**) 1 kg (**B**) 2 kg (**C**) 3 kg (**D**) 4 kg

(**E**) No puede estar segura *(Can't be sure)*

Correct answer: A

For this question a total of 54 correct answers were obtained in the case of fourth grade students, this allows us to determine that the difficulty index for this item corresponds to $P_8 = 0.45$, placing this as a relatively difficult question. For the fifth grade participants there were 75 correct answers, which generates a difficulty index $P_8 = 0.56$, classifying this as a question of adequate difficulty. The difficulty indicator associated with the question in the case of sixth grade students is $P_8 = 0.78$, classifying this question as relatively easy, due to the fact that there were 68 correct answers.
As indicated above, for fourth, fifth and sixth grade students, the scores per area were 45%, 56% and 78% respectively. Overall, there is an area score of 57.94%.

Number theory

Similar to Section 3.2, there is only one question, with the difference that this one corresponds to the open response section.
The following question is classified within the field of number theory because the resolution uses concepts associated with the division algorithm.

Question 15

La línea de Metro tiene seis estaciones: A, B, C, D, E y F. El tren se detiene en cada estación. Cuando llega a una de las dos estaciones finales, cambia de dirección. El conductor del tren empezó a conducir en la estación B y su primera parada fue en la estación C. ¿En cuál estación será su parada número 2023 ?

The Metro line has six stations: A, B, C, D, E, and F. The train stops at every station. When it reaches one of the two end stations, it changes its direction. The train driver started driving at station B and his first stop was station C. Which station will be his 2023th stop?

Correct answer: E

For this question there is a difficulty index $P_{15} = 0.2017$ for fourth grade students, since 24 correct answers were obtained, thus classifying this question as difficult. Fifth grade students archieved a total of 15 correct answers, obtaining a difficulty index $P_{15} = 0.1119$, classifying this question as difficult for these students. In the case of sixth grade students, results showed 25 correct answers and a difficulty index associated with the question of $P_{15} = 0.2874$, placing this question as difficult.

As mentioned above, for this area the difficulty index will be equal to the area score analysis because there is only one question in this section.

Overall, the score for each area is equal to 18.82%.

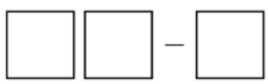
Counting

For this area of mathematics there is only one question, which is located in the open response section. This item was classified in this section because in its resolution concepts such as combinations and permutations are present, making a list of possible cases.

Question 14

Los dígitos 1, 1, 2 y 3 están impresos en cuatro tarjetas diferentes. Se colocan tres tarjetas para hacer una resta, como se muestra en la imagen. ¿Cuántos resultados diferentes se pueden obtener?

Digits 1, 1, 2 and 3 are printed on four different cards. Three cards are laid out to make a subtraction, as shown in the picture. How many different results can be obtained?



correct answer: 15

Results for this question yielded a difficulty index $P_{14} = 0.0168$ for fourth grade students, $P_{14} = 0.0448$ for fifth grade and $P_{14} = 0.1339$ for sixth grade students. Classifying this question as very

difficult for the fourth and fifth grade population while it is considered difficult for sixth grade students.
Since this area of mathematics has only one question, the difficulty index is considered as the score per area. Therefore, the score per area for fourth grade is 1.68%, for fifth grade it is 4.48% and for sixth grade it corresponds to 13.37%, thus obtaining an overall area score of 5.88%.

Arithmetic

In this section, 4 questions will be analyzed, 2 of which correspond to the multiple-choice part, while the remaining ones are open-ended.
These questions are classified in this area of mathematics because their solution uses concepts related to:

- Basic operations
- Combined operations (mathematical expressions involving two or more operations such as addition, subtraction, multiplication, and/or division)
- Problems with percentages

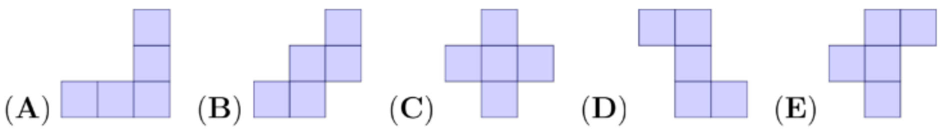
1. Multiple choice questions

Question 4

Mauricio coloca una de las cinco piezas en la cuadrícula. ¿Qué pieza de las siguientes debería usar para cubrir los números con la suma mayor?

Mauricio places one of the five pieces on the grid. Which piece of the following should he use to cover the numbers with the largest sum?

1	6	7
9	5	4
2	8	3



Correct answer: E

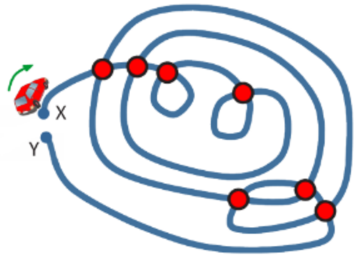
The results obtained by elementary school students are as follows:
The fourth grade students achieved a total of 90 correct answers, which categorizes this question as relatively easy because the associated difficulty index is $P_4 = 0.7563$. Fifth grade students have a total of 107 correct answers, this population obtained a difficulty index $P_4 = 0.7985$, placing this question as relatively easy. In the case of sixth grade students, a difficulty scale $P_4 = 0.8620$, was determined, classifying this question as very easy for these participants.

2. Questions corresponding to open answer

Question 11

Esteban conduce desde X hasta Y. En cada intersección se detiene antes de seguir recto. En total, ¿cuántas veces se detiene?

Esteban drives from X to Y. At each intersection, he stops before going straight ahead. In total, how many times does he stop?



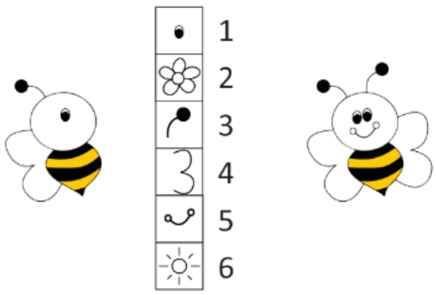
Correct answer: 11

For this question it was found that fourth grade students obtained 17 correct answers generating a difficulty index $P_{11} = 0.1429$ giving it a ranking of difficult; for fifth grade participants after having 48 correct answers the value associated with the difficulty of the question is $P_{11} = 0.3582$, placing this as a relatively difficult question, similarly the sixth grade students obtained 32 correct answers resulting in a difficulty index $P_{11} = 0.3678$ which as relatively difficult.

Question 12

Rosa quiere terminar la abeja de la izquierda según el modelo de la derecha. Rosa necesita ganar puntos para desbloquear partes de la abeja. ¿Cuántos puntos necesita ganar para completar la abeja?

Rosa wants to finish the bee on the left according to the model on the right. Rosa needs to win points to unlock parts of the bee. How many points does she need to win to complete the bee?



Correct answer: 11

For this question the fourth grade students achieved a total of 65 correct answers, while the fifth grade students achieved 99 correct answers. In the sixth grade population, 70 students responded correctly, thus obtaining difficulty indexes $P_{12} = 0.5462$, $P_{12} = 0.7388$ y $P_{12} = 0.8046$ respectively, classifying this question as of adequate difficulty for the fourth grade and relatively easy for the fifth and sixth grade groups.

In general, it was found that the scores per area for the participating students in fourth, fifth and sixth grades are 48.18%, 63.18% y 67.82% respectively. Finally, the subject of arithmetic was given an area score of 59.12%.

Logic

The following is an analysis of a total of 5 items, 4 of which are located in the multiple-choice section and 1 in open response, the questions classified in this area have in their resolution concepts such as:

- Solving problems that are solved by analyzing a limited number of cases.
- Classification of objects
- Completion of puzzles.

1. Multiple choice questions

Question 2

Cinco niños comparten un cumpleaños y cada niño tiene su propio bizcocho. Lila es dos años mayor que José, pero un año menor que Alan. Víctor es el más joven. ¿Cuál es el bizcocho de Sara?

Five children share a birthday and each child has their own cake. Lila is two years older than José, but one year younger than Alan. Víctor is the youngest. Which is Sara's cake?

(A) 

(B) 

(C) 

(D) 

(E) 

Correct answer: C

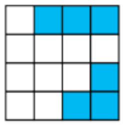
The following was found based on elementary school student results:

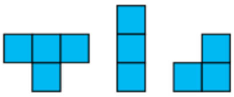
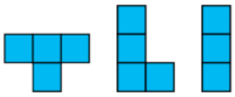
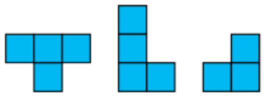
- The difficulty index associated with fourth grade students is $P_2 = 0.4202$, placing this question as relatively difficult.
- Fifth grade students managed to obtain 82 good answers, this indicates they have a difficulty score $P_2 = 0.6119$, where this score classifies the question as of adequate difficulty.
- With a total of 61 correct answers generating a difficulty value $P_2 = 0.7011$ which indicates that the question is relatively easy.

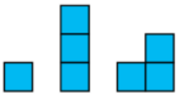
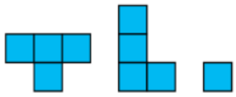
Question 3

Max quiere completar el rompecabezas que se muestra. ¿Qué piezas de las siguientes tiene que usar para completar el rompecabezas?

Max wants to complete the puzzle shown. Which of the following pieces does he have to use to complete the puzzle?



(A)  (B)  (C) 

(D)  (E) 

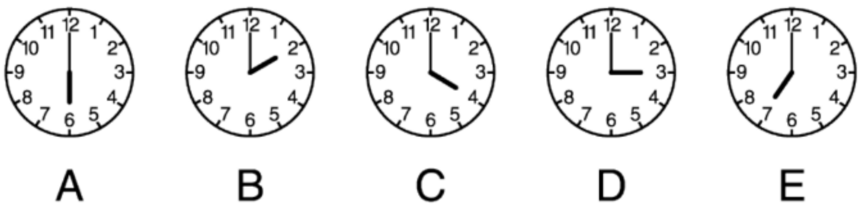
Correct answer: A

It was found that sixth grade students obtained the highest difficulty index $P_3 = 0.7931$ indicating that this question is placed as relatively easy, while for fifth grade a value $P_3 = 0.7388$was found placing this as relatively easy. The lowest index obtained in this question corresponds to fourth grade students, which is $P_3 = 0.6387$, classifying it as of adequate difficulty.

Question 9

Hay cinco relojes en la pared. Se sabe que un reloj va adelantado una hora, un reloj va atrasado una hora, un reloj muestra la hora correcta y dos relojes se han detenido. ¿Qué reloj muestra la hora correcta?

There are five clocks on the wall. It is known that one clock is an hour fast, one clock is an hour slow, one clock shows the correct time and two clocks have stopped. Which clock shows the correct time?



(A) A (B) B (C) C (D) D (E) E

Correct answer: D

For the analysis of this question, a total of 169 correct answers were collected, of which 45 corresponded to fourth grade students, 72 to fifth grade and finally 52 correct answers for sixth grade.

The associated difficulty index for fourth grade students is $P_9 = 0.3782$ which indicates that the question is relatively difficult, while for fifth and sixth grade the indices obtained are $P_9 = 0.5373$ and $P_9 = 0.5977$, respectively, placing this question as of adequate difficulty for both groups.

Question 10

María ha sombreado exactamente 5 cuadraditos en una cuadrícula de 4×4. Ella desafía a 5 de sus amigos a adivinar qué cuadraditos ha sombreado. Las cuadrículas que dibujaron sus amigos se muestran en las opciones. María los mira y dice:

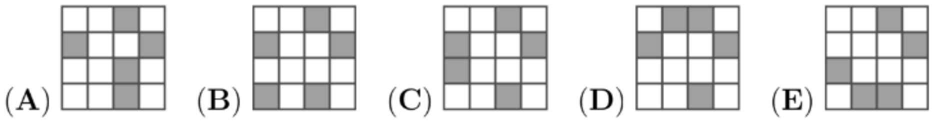
> “Uno de ustedes tiene razón y cada uno de los demás tiene cuatro cuadraditos correctos.”

¿Cuál es la respuesta correcta?

Maria has shaded exactly 5 small squares in a 4×4 grid. She challenges 5 of her friends to guess which squares she has shaded. The grids drawn by the friends are shown in the options. Maria looks at them and says:

> *“One of you is right and each of the rest of you has four squares correct.”*

Which is the correct answer?



Correct answer: D

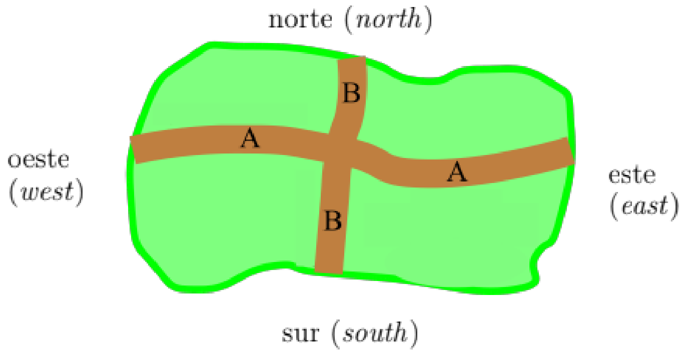
The difficulty indices found in this question are $P_{10} = 0.2605$, $P_{10} = 0.2612$ and $P_{10} = 0.4023$, for fourth, fifth and sixth grade students respectively. With the results found above, it can be determined that for fourth and fifth grade students the question is difficult, while for sixth grade students the item is considered relatively difficult.

2. Questions corresponding to open response.

Question 13

Hay 7 casas al norte del camino A, 8 casas al este del camino B y 5 casas al sur del camino A. ¿Cuántas casas hay al oeste del camino B?

There are 7 *houses north of road* A, 8 *houses east of road* B *and* 5 *houses south of road* A. *How many houses are west of road* B*?*



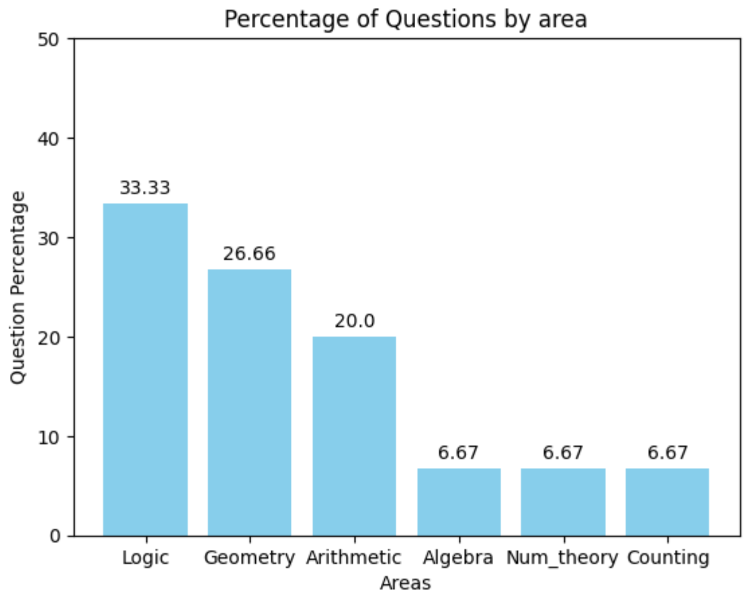
Correct answer: 4

This question obtained a total of 51 correct answers of which 14 were from fourth graders, 25 from fifth graders and 12 from sixth graders, placing this item for the three levels as difficult, but with difficulty indexes $P_{13} = 0.1176$, $P_{13} = 0.1866$ and $P_{13} = 0.1379$ respectively.
In general terms, there is a score per area for the fourth grade of 36.30%, The fifth and sixth grade students have a score of 46.72% and 54.02%, respectively. Globally, for the elementary education population, a score of 44.45%.

Conclusions

A total of 15 questions from the second phase test of the 2023-2024 Puerto Rico Mathematics Olympiad were classified in different areas of mathematics, the percentage of questions per area are shown below:

- The logic area had a total of 5 questions which is equivalent to 33.33% of the test questions
- For the geometry topic, there were a total of 4 questions, equalling 26.66% of the test.
- The area of arithmetic was present in the test with 3 questions which correspond to a 20% of the test.
- With only one question on the test, algebra, number theory and counting each account for 6.67% of the test.



The percentages by number of questions per area in this exam are approximately aligned with the percentages of exercises by Alvarado et al. (2023) [1], which are based on elementary school exams conducted in Latin America. Now, the level of demand presented by the test will be determined individually for the three grades analyzed and globally.

Fourth grade

Percentage of items	Classification of the question	Difficulty value per question
0%	Very easy	0.81 – 1.00
7%	Relatively easy	0.66 – 0.80
13%	Adequate difficulty	0.51 – 0.65
40%	Relatively difficult	0.31 – 0.50
33%	Difficult	0.11 – 0.30
7%	Very difficult	0.00 – 0.10

It can be observed that for fourth grade students the test was very demanding because 80% of the items are relatively difficult or of a larger scale.

Fifth grade

Percentage of items	Classification of the question	Difficulty value per question
0%	Very easy	0.81 – 1.00
20%	Relatively easy	0.66 – 0.80
40%	Adequate difficulty	0.51 – 0.65
7%	Relatively difficult	0.31 – 0.50
26%	Difficult	0.11 – 0.30
7%	Very difficult	0.00 – 0.10

It can be observed that for fifth grade students the demand of the exam was not as intense as for fourth grade students, 60% was in the adequate difficulty or relatively easy.

Sixth grade

Percentage of items	Classification of the question	Difficulty value per question
7%	Very easy	0.81 – 1.00
40%	Relatively easy	0.66 – 0.80
13%	Adequate difficulty	0.51 – 0.65
13%	Relatively difficult	0.31 – 0.50
27%	Difficult	0.11 – 0.30
0%	Very difficult	0.00 – 0.10

The percentage of relatively difficult or more difficult questions is less than 50%, therefore, we can determine that the test was not very demanding for this population.
According to Escudero et al (2000)[4], the average level of difficulty of an exam should be distributed as follows: 5% easy questions, 20% relatively easy, 50% with adequate difficulty, 20% relatively difficult and 5% difficult or very difficult.
Due to the above, it can be concluded that the group that comes closest to having a medium difficulty in the test was the fifth grade students, since the test presents a strong demand for the fourth grade and a constantly lower demand for sixth grade. This is an expected result because Piaget et al (1986) [8], indicates that the evolutionary development of students in mathematics is characterized by an orderly sequence of stages, each stage includes the previous ones and is reached around certain ages:

- Period of Concrete Operations (7-11 years):
  During this stage, children acquire the ability to perform concentrated mental operations, they can solve mathematical problems involving conservation, classification and seriation, their thinking is more logical.
  At this stage students are generally in the fourth and fifth grades.
- Formal Operations Period (11-15 years):
  At this stage, adolescents develop abstract thinking and the ability to reason hypothetically, can solve more complex mathematical problems and understand abstract concepts. Their thinking becomes more systematic and reflective.
  The fifth and sixth grade students are at this stage, so a clearer difference can be seen between fourth and sixth grade students.

In general terms, it can be observed that:

Elementary school students

Percentage of items	Classification of the question	Difficulty value per question
2%	Very easy	0.81 – 1.00
22%	Relatively easy	0.66 – 0.80
22%	Adequate difficulty	0.51 – 0.65
20%	Relatively difficult	0.31 – 0.50
29%	Difficult	0.11 – 0.30
5%	Very difficult	0.00 – 0.10

In general, it is determined that the level of demand of the test is medium to high, although it would be expected since it is an eliminatory competition where only a percentage of this population passes to the third round. It is important to mention that the competition among these students is

by grade and not in a general way since they all take the same test.
It is known beforehand that performance is not necessarily the same and although the objective is not to compare performance between grades, it is important to measure these difficulty indices in order to determine the level of difficulty presented by the test. Finally, overall scores were established for the six areas of mathematics to determine strengths and weaknesses in the test. The following will present, from highest to lowest percentage difficulty, the areas present in this test by grade:

Fourth and fifth grade

1. Counting
2. Number theory
3. Logic
4. Geometry
5. Algebra
6. Arithmetic

As can be seen, the strengths for these students are arithmetic, geometry and algebra, while the weaknesses are counting and number theory. These results are similar to those found for sixth grade, except that they differ in the order of the strengths, since these students have a higher performance in algebra, then in arithmetic and geometry.
In general, the following results are obtained for elementary school students.

Elementary school students

1. Counting (5.88%)
2. Number theory (18.82%)
3. Logic (44.45%)
4. Geometry (47.79%)
5. Algebra (57.94%)
6. Arithmetic (59.12%)

In general, it is determined that the placement of strengths per area arithmetic in first place and algebra in second place, although the difference between them is not very significant. Counting and number theory hold the first and second positions, respectively, in terms of test difficulties. Therefore, the second phase of the Puerto Rico Mathematics Olympiads helps to determine the topics or areas of mathematics that should developed so that students can achieve greater knowledge and experience in solving these problems, therefore, it is important that the organizing committee and student coaches take into account these parameters to emphasize training within these areas of mathematics.

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Luis Cáceres
University of Puerto Rico, Mayaguez Campus
luis.caceres1@upr.edu

Junior Arauz-Aguilar
University of Puerto Rico, Mayaguez Campus
junior.arauz@upr.edu

Characterizing particularly frequently selected distractors at the Mathematical Kangaroo

Lukas Donner and Evita Lerchenberger

Lukas Donner (né Andritsch) is a former participant of national and international mathematical competitions and has now been working in the field of Mathematical Olympiads for several years. He finished his PhD in pure maths in February 2019 in Graz (Austria) and is now a postdoctoral researcher at the University of Duisburg-Essen (Germany). His research interests lie in various fields of mathematical education, especially in cognitive-psychological aspects of mathematical problem solving, mathematical competitions and subject matter didactics.

Evita Lerchenberger (née Hauke) studied music and mathematics and taught these two subjects at a secondary school in Graz, Austria. She is currently working at the University of Graz, where she completed her PhD studies in the didactics of mathematics. She is interested in the matter of the motivation and interest of students in mathematics as a subject as well as in competitions, especially the Kangaroo competition, in which she herself enthusiastically participated some years ago.

Abstract
In terms of the number of participants, the Mathematical Kangaroo is the world's largest annual mathematical competition. It is held in multiple-choice format (1 out of 5), which gives the distractors (i.e. wrong answer-options) a special role. The aim of this article is to analyse those tasks of the Kangaroo competition in which a particularly large proportion of students have chosen exactly one of the distractors of the task. We attempt to characterise these distractors by taking into account the research on mathematical intuition and the dual-process theory. To this end, we introduce the concept of the empirical trap. Based on the response frequencies of participants in the Mathematical Kangaroo in Austria between 2015 and 2019, five different types of such empirical traps can be identified, namely (1) external task characteristics, (2) misleading mathematical intuitions, (3) special distractors, (4) inadequate calculations and (5) partially correct solutions. Based on the results, some thoughts of the authors and suggestions for task creators are formulated as basis for discussion.
Keywords: Problem selection, Mathematics competitions, Mathematical Kangaroo

Introduction

Every year, around 6 million participants from all school levels take part in the Mathematical Kangaroo, making it the largest international math competition for pupils and students. As with any multiple-choice format, the answer-options are a central component of the tasks of the Mathematical Kangaroo in addition to the item stem (consisting of text and often also of graphics) itself. The wrong answer-options (distractors) must be designed in such a way that they are wrong, but they should still appear as plausible as possible (Michelsen, 2015). Thus, in the process of creating tasks and compose them to whole competitions, thought is always given to how the distractors should be formulated (Donner and Geretschläger, 2023). As the competition consists of mathematical tasks, the distractors can depict plausible but incorrect trains of thought or address a misconception of a mathematical object. They can also hide the correct solution, for example by simply being an ascending series of numbers without further intention.

We start our discussion of potentially particularly frequently chosen distractors by presenting two example tasks from past competitions (see website of Austrian Mathematical Kangaroo, n.d.), which illustrate the wide range of ideas that can lie behind the choice of certain distractors.

One method used by task creators is to adopt numbers from the item stem as answer-options. This is also the case in the following task:

Problem "blackboard" (grade 9-10, high level of difficulty, 2018)
Some integers are written on a board, amongst them the number 2018. The sum of all these integers is 2018. The product of these integers is also 2018. Which of the following could be the number of integers written on the board?
(A) 2016 (B) 2017 (C) 2018 (D) 2019 2020

The number 2018 appears several times in the item stem, while the correct answer is 2017. At first glance, 2018 appears to be an attractive number for those working on the task, as it appears several times within the item stem. However, the correct answer is 2017.

The following example illustrates an example of a distractor for the above-mentioned taking up of typical trains of thoughts that can occur in working on a task, but which do not correspond to a complete solution:

Problem "sunshine" (grade 7-8, medium level of difficulty, 2018)
A hotel in the Caribbean correctly advertises using the slogan: "350 days of sun in the year!" How many days does Mr. Happy have to spend in the hotel in a year with 365 days to be guaranteed to have two consecutive days of sunshine to enjoy?
(A) 17 (B) 21 (C) 31 (D) 32 35

An apparent (optimal) number of days is given by distractor (A), but since the question asks how many days must be *certainly* spent in the hotel in order to fulfil the given condition, answer-option (D) is correct (for a detailed analysis of the distractors, see Andritsch et al., 2020).

It is known that the considerations just described can play a role in the creation of distractors (Donner and Geretschläger, 2022, Geretschläger and Donner, 2022). The spectrum of such potentially particularly frequently selected distractors can range - as the two example tasks just shown make clear - from the direct adoption of numbers from the item stem to typical errors that are only expected to occur in certain parts of a solution approach.

However, it is not clear whether and to what extent such or other temptations in distractors are actually chosen by an extraordinarily large proportion of participants and whether these distractors or the possible thought processes that lead to their choice can be further characterized.

The aim of this article is to use a suitable conceptualization of the term *empirical trap* and to include the response frequencies of Austrian students in tasks of the Mathematical Kangaroo to examine whether the assumptions about the occurrence of traps (cf. Andritsch et al., 2020; Geretschläger and Donner, 2022), which have been described solely theoretically so far, can be empirically proven. Those tasks that are obtained in this way are to be examined and analyzed according to certain criteria. Conclusions for competition participants and in particular for task creators will be formulated, and the potential of the findings and ideas for further research directions will be demonstrated. To this end, some information concerning the tasks of the Mathematical Kangaroo, and research findings on (misleading) mathematical intuition and dual-process theory, which form the frame of reference for the categories developed for these particular tasks, will first be described.

Theoretical Background

Tasks of the Mathematical Kangaroo

The Mathematical Kangaroo is held in six age levels, with usually two grades being combined into one age level. Every task consists of an item stem (text and often graphics) and five possible answer-options, of which exactly one is correct. Depending on the age level, the competition consists of a different number of tasks that have to be completed in a relatively short time: For instance, all participants from the 7th grade and above have 75 minutes to complete 30 tasks. The tasks are rated with 3, 4 or 5 points according to the difficulty assessed by the task creators, whereby there are the same number of tasks of each value in each age level. If the correct answer-option is chosen, a student receives the points specified for the particular task (between 3 and 5); for choosing a distractor, a quarter of the points to be achieved for the task are deducted from the current score. It is also possible to give no answer, which is awarded 0 points. The 3-point tasks are designed in such a way that they should, in principle, be solvable even for the weakest students in the total time allowed, the 4-point tasks are of medium difficulty and the 5-point tasks are intended in particular as appealing tasks for specialists (Akveld et al. 2020). The analysis of several years of the Mathematical Kangaroo in Austria conducted by the authors of this article in Lerchenberger and Donner (2024) clearly shows that the task creators' assessment of how many points a task should be worth corresponds very well with the average points achieved per task, and hence the "empirical difficulty" of the tasks.

Geretschläger and Donner (2022) describe the typical characteristics of the tasks in the Mathematical Kangaroo: In addition to taking up exciting phenomena from the curricular content fields (which are algebra, logic, geometry and numbers), text comprehensibility and the use of pictures to support task comprehensibility, the distractors in particular occupy a central role due to the fact that a large number of tasks have to be solved - but not exactly justified - in a very short time. The basic concept of the competition, such as the possibility of guessing or the limited time resource, favors the fact that, in addition to mathematical knowledge, other factors can also influence participants' actions as well as their strategic approach in the competition situation (cf. Donner et al.

2021).

In keeping with the spirit of the competition, tasks are deliberately set that favor original, strategic approaches. Sometimes even involving the answer-options as the starting point of an argument or as parts of an argument can result in an efficient solution approach (cf. Andritsch et al. 2020, Donner et al., 2021). At the same time the primary focus on answer-options can lead to incorrect and hasty conclusions, and therefore to the selection of a seemingly favorable distractor (Donner and Geretschläger 2022, Geretschläger and Donner, 2022), like the distractors of the examples discussed in the introduction. Successful participants are aware of the phenomenon of misleading distractors and tend to check tasks that are worth at least 4 points due to this phenomenon (Donner et al. 2021).

The two tasks “blackboard” and “sunshine” of the Mathematical Kangaroo discussed in the introduction indicate that there are very different ways in which particularly frequently selected distractors can potentially appear: Do they correspond to a first (visual, superficial) impression when reading the task, or do they represent viable but incomplete approaches. In order to obtain a better, more differentiated description of the possible characteristics of such potentially particularly frequently selected distractors, it is helpful to take a look at mathematics didactics and cognitive psychology research on the topic of (misleading) mathematical intuition.

Misleading intuition: Insights of research in mathematics education and in cognitive psychology

Since the 1980s, mathematics didactics research has increasingly and systematically focused on the phenomenon of intuition in mathematical tasks. The work of Fischbein has been influential in this regard (Tirosh, D. and Tsamir 2020). According to Fischbein (1987), intuitive knowledge means immediate, self-evident, uncritically accepted knowledge with a feeling of imminent certitude. Intuitions can arise from everyday experiences (i.e. primary intuitions), but also from educational experiences (i.e. secondary intuition), through repeated exposure to a specific topic (Fischbein, 1999). Fischbein (1999) explicitly emphasizes that some errors when working on mathematical tasks are not due to logical inadequacies or a lack of domain-specific knowledge, but can primarily be the result of misleading intuitions that even hinder correct reasoning due to the feeling of overconfidence.

Vinner (1997) contributed to the explanation of misleading problem-solving processes by characterizing *pseudo-analytical thought processes*: Pupils identify superficially seemingly relevant aspects of a task and then spontaneously carry out typical solution approaches for these features. Vinner sees the reasons why students use these strategies in the fact that they are not always motivated to solve problems and that these strategies sometimes lead to correct results. One example is the incorrect application of the so-called ‘keyword strategy’: ‘It seems that children are strongly inclined to start conducting a calculation that is explicitly shown in the problem (i.e. the subtraction) even when inhibiting that tendency and first considering which strategy to apply would be desirable’ (Torbeyns et al. 2009, p.714). The strategy ‘add if more, subtract if less’ for the signal words ‘more’ or ‘less’ in text tasks is considered a special form of pseudo-analytical thought processes (cf. Lewis and Mayer 1987). The intuitive rule theory (Stavy and Tirosh, 2000) also confirms that intuitive answers to tasks can be determined to a large extent by external characteristics of the task, in particular by salient features of objects that are not decisive for the reasoning behind the solution.

As an alternative approach to explaining the phenomenon of (misleading) intuition, the following basic distinction between two types of cognitive processes has been described in *cognitive psycho-*

logy for decades, parallel to the mathematical characterization of intuition, within the framework of *dual-process theory* (cf. Kahnemann 2003, Evans and Over 1996, Stanovich 1999): A process is classified as Type 1 (intuitive) if it does not require any working memory resources, otherwise it is classified as a Type 2 process (reflective) (Evans and Stanovich 2013). In short: Type 2 processes require a certain amount of effort. Typically, an intuitive solution comes to mind before a task has been fully analyzed, with great conviction of its correctness and with the feeling that it does not need to be checked further (Van Dooren and Inglis 2015). Decisions are therefore primarily made on the basis of Type 1 processes (Kahnemann 2012). A widespread phenomenon for content-related error-prone Type 1 processes is thinking in proportional contexts and the associated tendency to overgeneralize these (cf. Gillard et al. 2009). However, Evans and Stanovich (2013) explicitly emphasize that not every Type 2 process necessarily leads to success and that the view of a fundamental superiority of Type 2 processes over error-prone Type 1 processes therefore is wrong.

For many tasks, it would therefore be necessary to pause the solution approach and to recognize that an initially intuitive answer based on a Type 1 process is incorrect and that a task should be analyzed in more detail. This can be illustrated with the help of a task from the so-called Cognitive Reflection Test (Frederick 2005): '*A bat and a ball cost $1.10. The bat costs $1.00 more than the ball. How much does the ball cost?*'. The intuitive answer, which according to Kahnemann (2012) is also primarily given, is that the ball would cost 10 cents. However, this intuition is wrong and it requires a check in the sense of a (repeated) analysis of the second condition, namely the difference in costs for the two objects, in order to expose the misleading intuition and obtain the correct result. The salient – in this case syntactic – feature of the task ('one dollar more') leads to a certain reaction ('the bat costs 1 dollar'), but it is precisely this reaction that needs to be inhibited (in the sense of controlled) (De Neys et al. 2013). The above explanations can also be understood in terms of pseudo-analytic thought processes as an example of the incorrect application of the keyword strategy.

Merging theory and competition practice: Expected categories of potentially frequently selected distractors

Based on the conceptual design of the Mathematical Kangaroo described above, it should be assumed that the limited working time, the multiple-choice format and the problem-solving nature of the tasks are three main reasons why intuition must be given an important role in the Mathematical Kangaroo when working on tasks: for example, by answering quickly, the important, scarce resource "time" can be conserved so that it is available for challenging tasks with difficult content and, conversely, checking the own solution is essential if you want to avoid selecting a distractor that seems correct (at a first glance).

The tasks "blackboard" and "sunshine" presented in the introduction reflect two completely different types of challenges for the competition participants: In the first problem (blackboard), the number 2018 appears both in the text and as a distractor; careless solvers could select this number directly after a superficial examination without further consideration of the task. In the second problem (sunshine), a possible, constructive first approach, in which parts of the mathematical structure are recognized, but where not all conditions are taken into account, leads to the selection of a distractor. In conjunction with the compilation of existing theoretical findings from both scientific disciplines that describe the phenomenon of misleading intuition, it can be surmised that two fundamentally distinguishable forms of misleading approaches can occur in tasks of the Mathematical Kangaroo:

I. *Is the answer found intuitively or does it require a willingness to make an effort* (Type 1 process vs. Type 2 process)?

II. *Is an (apparent) mathematical structure recognized or is the task handled on the basis of purely superficial characteristics?*

These characteristics could also be reinforced and further refined due to distractors: The following three distinct characteristics for potentially particularly frequently selected distractors in the Mathematical Kangaroo can be deductively described on the basis of these theoretical findings.

The intuitive rules theory as well as Vinner's description of pseudo-analytical thought processes suggest that some distractors are frequently chosen that are not (predominantly) dependent on the mathematical content, but are due to external characteristics of a task. In particular, the so-called keyword strategy is taken into account: a direct hint from the task is chosen as a distractor. We refer to this category as *answer direct (AD)*. Distractor 2018 of the problem "blackboard" is an instance for this category AD.

In addition, it is to be expected that distractors are chosen intuitively from the incorrect application of mathematical properties (symmetry, proportionality, patterns) etc., i.e., solutions are based in particular on Fischbein's secondary intuitions, but which are misleading at the particular task. This category is referred to as the *wish for pattern recognition (WP).*

A third, excepted type of potentially particularly frequently selected distractors, is based on the intention of the task creators of the Mathematical Kangaroo. In some tasks, distractors are deliberately set that appear plausible due to careless behavior approaching the task (cf. Donner and Geretschläger, 2022). For example, participants could be tempted to consider only one special case of the task instead of solving it in general, such as distractor 17 in the problem "sunshine". Since this type of discussion is already part towards a solution, but still a distractor is chosen, we refer to this category as *sustainable approach towards a solution (SA).*

Research questions

The considerations outlined above suggest that certain distractors are chosen particularly frequently in the tasks of the Mathematical Kangaroo. However, to the best of our knowledge, this commonly known concept of a "*trap*", especially in multiple-choice formats, has not yet been conceptualized and therefore cannot yet be empirically recorded. The question therefore arises as to how this concept can be empirically recorded on the basis of actual response frequencies in tasks and thus how those tasks with *empirical traps* can be determined.

It is also known that both external task characteristics and domain-specific reasons (like misconceptions) can provoke (intuitively) incorrect answers and that misleading Type 2 processes can also lead to common errors. It is necessary to investigate which of these types of theoretical, meaningfully describable "traps" actually occur in Mathematical Kangaroo and whether other - or further - characteristics of empirical traps can be identified.

Summing up, two research questions can be formulated for this article:

RQ 1: For which tasks do a large proportion of participants in the competition choose a certain distractor, which therefore represents an *empirical trap*?

RQ 2: To what extent can possible reasons for the choice of these distractors be characterized and categorized?

Method

Data

The data basis was the response frequencies of all Austrian participants from the third grade onwards for all tasks of the Mathematical Kangaroo in the years 2015 to 2019. Depending on the age level and competition year, this is between 5000 and over 40000 participants. The category for first and second grade was excluded from the analyses as teachers are allowed to read out the tasks for first grade pupils, which may not ensure the comparability of answer frequencies. For each of the 690 tasks in this period, it is known how many participants ticked an answer from (A) to (E) and how many left the task unanswered.

Analysis of data

The distribution of responses to answers (A) to (E) was analyzed for each of the 690 tasks from the period 2015 to 2019. The proportion of those students who did not give an answer was not considered for this article, so that the relative proportion of answers to (A) to (E) can always be considered.

Our concept of "empirical traps"

A particularly attractive distractor of a task in the Mathematical Kangaroo is referred to as an empirical trap. Particularly attractive means that it was chosen much more frequently than the correct answer-option and that this frequency is not (superficially) due to guessing (i.e. even distribution of 20% across all answer-options). In order to exclude extreme cases (e.g. only one answer given) and a probable random occurrence (e.g. guessing of all participants) as far as possible, a sufficient number of answers must be available.

In order to analyze the tasks with particularly attractive distractors to which the required conditions apply, the following terminology was used:

The concept of an empirical trap. Assume that at least 100 answers are available for a task of the Mathematical Kangaroo (multiple-choice format 1 out of 5). A distractor of the task is called an *empirical trap* if it accounts for at least 40 % of the answers given and is chosen at least twice as often as the correct answer-option.

All 690 investigated tasks between 2015 and 2019 were answered by more than 1000 participants, which ensures that the first condition (at least 100 answers are available) is met.

A task in the Mathematical Kangaroo is called a *trap task* if it contains at least one empirical trap as defined above.

Analysis of the tasks and suitable categories

We extensively analyzed the subset of tasks that contain an empirical trap.

To address research question 2, three categories for empirical traps were derived deductively (see section for details) and further categories were developed inductively. The methodological approach was based on the method of content-structuring qualitative content analysis according to Kuckartz (2018).

As a coding unit, a complete task containing an empirical trap (i.e. item stem, including any illustrations, as well as answer-options) was used, since in the vast majority of cases a distractor only appears particularly attractive in comparison to the other answer-options or the item stem (cf. Andritsch et al. 2020).

Since the actual students' thoughts that lead to the error in the competition situation are not known, the categorization was based on the following consideration '*Is there a plausible explanation as to why this distractor was chosen?*'. Based on this question, the two authors initially viewed 50% of the material. In doing so, they attempted to assign the tasks to the deductively determined categories, whereby the need for further categories became apparent. Hence, by mutual agreement, two further categories were developed inductively. In the course of coding the entire material, the definitions of the categories were sharpened to such an extent that each trap task could be assigned to exactly one category and there was consensus between the authors on this assignment.

A coding manual was developed for the five deductively-inductively developed categories, which contains the descriptions of the respective category, anchor examples and distinctions from other categories.

In order not to limit the view of the tasks and their assignment to the categories to the two researchers alone, three experts were asked for their assessment of the most plausible path to the particularly attractive distractor. Their assessments were compared with the authors' categories (section provides more details). The expert group covers key perspectives on the Mathematical Kangaroo, as it comprises a former participant, a task creator, and a teacher who is very experienced in preparing students for Mathematical competitions in general and the Mathematical Kangaroo in particular.

Results

Empirical traps in the Mathematical Kangaroo

In total, there are 64 problems from 2015 to 2019 in which one of the distractors can be described as an empirical trap according to our specification of the term (see section 4.3). This number corresponds to around 9.3% of all tasks for the analyzed age levels in the Mathematical Kangaroo during the period under investigation. The analysis shows that the theoretical possibility of obtaining two empirical traps in one task did not occur.

Empirical traps were found in all age categories, with grades 3-4 and 5-6 containing the fewest empirical traps (see Table 1). However, there are also fewer tasks in these categories overall because there are only 24 tasks per competition (and not 30 as in the other age categories).

While only 7 of the 3-point tasks analyzed contain an empirical trap, 23 of the 4-point tasks and 34 of the 5-point tasks do, which means that half of the empirical traps are worth five points.

Categorization of empirical traps (and trap tasks)

In addition to the three categories derived from theory (see section 2.3), two further categories were formed inductively, i.e. on the basis of the data:

In some tasks, the application of an obviously unsuitable arithmetic operation that is not directly specified by the task leads to an empirical trap, which is why the need for such a category which we call *pointless calculations (PC)* became clear when analyzing the data. Due to the execution of an operation not directly specified by the task, these are also Type 2 processes.

It has also been shown that *special distractors (SD)* often represent empirical traps. These distractors can represent an outlier within the answer-options (e.g. in relation to the order of magnitude of the numerical values), which appears particularly attractive due to this unusualness. Furthermore, answer-options such as ‘This cannot be determined’ or ‘Another number’, which regularly occur in the Mathematical Kangaroo, also frequently appeared as empirical traps. As they are of a different nature to the other four answer-options, they are also assigned to the category SD.

Based on the deductive-inductive content analysis, the trap tasks can therefore be divided into the following five categories.

There are tasks in which the distractor corresponding to the empirical trap:

1. can be read directly from the item stem (answer direct “AD“).
2. originates from a wish to recognize a pattern, a symmetry or continuing a sequence (wish for pattern recognition “WP“).
3. is apparent of a completely different nature than the other answer-options (special distractor “SD“).
4. results from a (more or less) pointless/inadequate operation performed without considering the concrete situation; but which cannot be explained by “answer direct” (pointless calculation “PC“).
5. can be explained by a viable solution approach (e.g. sustainable ideas leading to a solution) but incorrectly executed calculation or overlooking/adding some conditions (sustainable approach “SA“).

The distribution of the 64 tasks to the five categories was strongly confirmed by the three experts: A Fleiss-κ of 0.84 was calculated using the statistical software datatab. This means that there is an almost perfect match (according to Landis and Koch, 1977) between the coding of the authors and that of the group of experts.

Table 1 shows how the trap tasks are distributed across the different age levels and the five categories.

		Grade					
		3-4	5-6	7-8	9-10	11-13	Total
Cat. of empirical traps	AD	4	2	0	4	3	13
	WP	0	1	3	3	3	10
	SD	0	1	2	4	3	10
	PC	1	1	6	7	0	15
	SA	2	5	3	2	4	16
	Total	7	10	14	20	13	64

Table 1: Distribution of the trap tasks across the age levels and the trap category

In order to show the characteristics of the five categories and to make the distinctions between them clear, these are explained using the following tasks (see Figure 1). In the tasks given, the correct answer is framed and the empirical trap is printed in bold. Any notes for the readers of this article are highlighted in grey in the task template. The grade at which the task was set is noted above each task, as well as the points to be achieved and the competition year. All tasks originate from the Austrian Mathematical Kangaroo. They can be retrieved in German as well as in English on the website of Austrian Mathematical Kangaroo (n.d.).

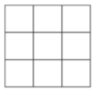
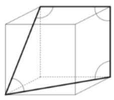
category	**task**	**remark (plausible justification)**
AD (answer direct)	(9-10, 5 points, 2018) Some integers are written on a board, amongst them the number 2018. The sum of all these integers is 2018. The product of these integers is also 2018. Which of the following could be the number of integers written on the board? (A) 2016 (B) 2017 **(C) 2018** (D) 2019 (E) 2020	The number that appears several times in the item stem can be selected as an answer-option.
WP (wish for pattern recogn.)	(7-8, 4 points, 2018)  Jakob writes one of the natural numbers from 1 to 9 into each cell of the 3x3-table. Then he works out the sum of the numbers in each row and in each column. Five of his results are 12, 13, 15, 16 and 17. What is his sixth sum? (A) 17 (B) 16 (C) 15 **(D) 14** (E) 13	It is assumed that the missing sum must be the missing number 14 in the sequence 12, 13, 15, 16, 17. In contrast to category AD, this 'wish for pattern recognition' cannot be taken directly from the task, but requires a conclusion based on this assumption. However, the answer must not require an additional calculation/operation.
WP (wish for pattern recogn.)	(9-10, 4 points, 2016)  In the diagram we see a cube and four marked angles. How big is the sum of those angles? (A) 315° (B) 330° (C) 345 ° **(D) 360°** (E) 375°	The acquired knowledge that the sum of angles of a quadrilateral is 360° is intuitively applied here without reflecting on the fact that the figure drawn is not a deltoid in the plane.
SD (special distractor)	(7-8, 4 points. 2015) During a thunder storm it rained 15 litres per square meter. By how much did the water level of an outdoor swimming pool increase? (A) 150 cm (B) 0,15 cm (C) 15 cm (D) 1,5 cm **(E) it depends on the size of the swimming pool.**	The special distractor (E) is apparently different from all further answer-options. It is also possible that the distractor is chosen because the student cannot solve the problem despite trying.
SD (special distractor)	(9-10, 5 points, 2016) Two heights of a triangle have lengths 10 cm and 11 cm. Which of the following lengths cannot be the length of the third height? (A) 5 cm (B) 6 cm (C) 7 cm (D) 10 cm **(E) 100 cm**	Answer-option (E) is the only number that is way larger than the other numbers in the item stem and in further answer-options.
PC (pointless calc.)	(5-6,5 points, 2017). In a bag there are only red and green marbles. If one randomly takes out five marbles, there is at least one red one. If one randomly takes out six marbles, there is at least one green one. What is the maximum number of marbles in the bag? **(A) 11** (B) 10 (C) 9 (D) 8 (E) 7	The numbers 5 and 6 are added. However, the addition is not a direct 'instruction' from the item stem, but an incorrect assumption by the person working on the task.
SA (sustain. approach)	(7-8, 4 points, 2018) A hotel in the Caribbean correctly advertises using the slogan: **"**350 days of sun in the year!" How many days does Mr. Happy have to spend in the hotel in a year with 365 days to be guaranteed to have two consecutive days of sunshine to enjoy? **(A) 17** (B) 21 (C) 31 (D) 32 (E) 35	The assumption that the 15 "rainy days" (days without sunshine) occur consecutively leads to the special case 17, but if the rainy days are interrupted by days of sun, this leads in the extreme case to the weather changing daily for 30 days - this would be the last necessary step to the solution in order to fulfil the required condition with certainty.

Figure 1: Exemplary tasks for each category.

Discussion

In this article, an empirically tangible framing of the well-known phenomenon of traps in multiple-choice competitions is first developed. Based on this definition of an empirical trap, the tasks of the Austrian version of the Mathematical Kangaroo in the years 2015-2019 were analyzed. There are 64 empirical traps among the 690 tasks in this period. The empirical traps are mainly distributed among the medium and hard tasks of the competition, only 7 of the trap tasks are easy competition tasks (e.g. worth 3 points). This largely, but not completely, corresponds to the competition organizers' specification that traps should only occur in medium-difficulty tasks as well as hard tasks respectively (Donner and Geretschläger 2022). Empirical traps were found in all age levels, with grades 3-4 and 5-6 containing the fewest (see Table 1). This can at least be partly explained by the fact that there are only 24 tasks per competition (and not 30) as in the other age levels. The theoretical possibility of one task containing two traps did not materialize in practice.

In the Mathematical Kangaroo, it is always possible to tick none of the answer-options, which correspond to receiving zero points for this task, while incorrect answers result in a point deduction. This indicates that many students should be at least relatively sure of their answer and accept the risk of a point deduction. In combination with the fact that the criteria for being denoted as an empirical trap are quite strict, we were very surprised that nearly 10% of all competition tasks are trap tasks. This suggests that deeper knowledge about the phenomenon of empirically determined, particularly frequently selected distractors might be an important element for the development of the competition.

Therefore, the second research objective was the deductive-inductive identification of various characteristics of the trap tasks. In particular, the associated frequently chosen distractors were characterized with regard to the intended solution approaches, which could likely be the cause of the occurrence of the respective empirical trap. The reasons assumed from the literature for the occurrence of potentially particularly frequently selected distractors are *external task characteristics*, *misleading secondary intuitions* and only *partially correct solution approaches*. All three causes were also identified on the basis of the tasks analyzed. In addition, the analysis revealed two further reasons for empirical traps, namely *special distractors* and *pointless calculations* (not intended by the item stem).

The concept of empirical trap and the possible intention behind potentially frequently selected distractors in the Mathematical Kangaroo

Semantically, the term 'trap' suggestively implies a certain intention on the part of the task creator, but the chosen research method only allows a very limited statement to be made as to the extent to which there was an *intention* in the trap tasks identified. In the case of traps in the three deductively obtained categories, for example, it was already assumed a priori that distractors that can be taken directly from the item stem, that follow a given pattern, or that only correspond to partially solved tasks, were deliberately built in as such. Conversely, however, it is equally possible that in many cases the occurrence of the empirical trap is unintentional or, in contrast, that such deliberately built in temptations do not appear as such in the answer frequencies determined empirically, as students are not misled (and therefore the requirements for a trap task are not met). In order to investigate the connection between intention and the occurrence of an empirical trap, the task creators have to be interviewed in advance and asked for their assessment of the intended temptations for the tasks set.

For multiple choice tasks, it is essential that distractors appear plausible, and at least in part particularly tempting: 'Identifying distractors is an essential step in MCQ [multiple choice question, remark by the authors] construction because distractors need to be misleading and plausibly incorrect' (Kumar 2023). Some distractors depict possible mathematical misconceptions or other obvious sources of error, not only in mathematical performance tests, but also in the Mathematical Kangaroo. However, not every distractor that appears plausible turns out to be an empirical trap. If a distractor was actually chosen exceptionally often, we speak of an empirical trap in this article, but we do not understand this as the intention of the task creators or as a certain mathematical misconception which may be known to be very common among pupils.

Possible answer-options can also be those that do not correspond to intuitive but incomplete solution approaches or typical, expectable errors. The task creators may deliberately offer distractors to indicate which (incorrect) ways of thinking the students might correspond to in the task. In these cases, the presence of their own solution among the answer-options may appear to the students as confirmation. This is also favored by the time pressure in the competition, as there might be no need to check the result. So there exist several reasons as to why a particular distractor might manifest as an empirical trap.

How do our findings fit with the existing literature?

Since the Mathematical Kangaroo is a time-limited multiple-choice competition, it is obvious that superficial task features can have a major influence on the choice of an answer-option. The search for topic-independent categories for temptations follows the tradition of Stay and Tirosh (2000) that salient but irrelevant task features are central to misleading, intuitive responses. At the Mathematical Kangaroo, two of the categories we determined are based on superficial task features: the direct adoption of parts of the item stem (AD) and the category PC, e.g. performing meaningless calculations (in relation to the task). However, in contrast to AD category, the latter is not characterized by misleading intuitions.

As either certain features of the task are adopted directly (AD) or supposed patterns are recognized directly that are not subject to further reflection (WP), from a cognitive psychology perspective, the empirical traps in the AD and WP categories can be attributed to Type 1 processes. On the other hand, anticipated approaches of empirical traps in the PC and SA categories (pointless calculation and sustainable approach, resp.), which lead to the particularly frequently chosen distractor, make use of working memory - albeit only rudimentarily in some cases - which makes them Type 2 processes by definition. In the first case, due to the surface structure of the task (given numbers, addressed content), calculations are carried out that are not directly intended by the item stem and lead to a distractor, and which do not initiate any links to the necessary thoughts towards a solution. In the case of SA, however, a (large) part of the solution path is followed, but a wrong turn is taken at a 'critical' point, as explained in in section with reference to the task "sunshine" discussed in the introduction. It may be that only a special case of the problem is considered or that one of the conditions of the problem is overlooked. In any case, the approach is viable and could possibly be corrected in a further review or reflection on the task.

Only in the SD category (special distractor) is the situation less clear with regard to agreement with the theory: Two plausible reasons can lead to there being outliers among the distractors or, for example, distractors of the type '(E) this is not solvable' or '(E) it depends' being chosen by the participants. For some participants, the distractor might be attractive because it differs from the other answer-options that it may lead to a direct, superficial choice without a holistic analysis of the entire task, including its content (Type 1 process). At the same time, however, a Type 2 process (in which mathematical relationships in the task are recognized) might also lead to the selection of the distractor if the task does not appear solvable from the participant's point of view despite analytical consideration, or if a different result is assumed than with the other answer-options and therefore the selection of the distractor is justified. An assignment to the existing findings of the theory of mathematical intuition also does not appear possible because the majority of tasks in the SD category are hidden-proof tasks in a multiple-choice format, because options such as 'another number' or 'this is not solvable' require further justification than the direct choice of an answer-option. These are therefore not typical multiple-choice tasks. By providing such an "open distractor", the participants are also deprived of many possible tactical approaches with the help of the distractors (Geretschläger and Donner, 2022). For instance, answer-option-based working backwards (see Andritsch et al., 2020 for details) is no longer possible in such tasks.

A summary of the connections between these two strands of theory on intuition and empirical traps is shown in Table 2.

Math. educ. / cogn. psych.	Type 1 process	Typ 2 process
Superficial treatment	AD	PC
Math. contents are recognised	WP	SA

Table 2: Categories of empirical traps and research on intuition

Limitations of the study and open questions

There are a number of reasons for the limitations of the study.

The requirements for a distractor to be defined as an empirical trap are very high. If, for example, two distractors appear much more attractive than the correct solution, it is possible that both represent a temptation, but the task is not declared a trap task because the effects influence or overlap each other. The reason for the narrow definition of the term empirical trap is that it allows a more precise characterization to be made if there is an outstanding distractor that can be investigated in each case. With the characterization used, which only allows special 'extreme cases' of tasks to be filtered out, over 9% of tasks were identified as trap tasks in the period from 2015 to 2019. This means that on average two to three items per competition contained such an empirical trap. Tasks in which, for example, the correct answer was ticked more often than required in our characterization, or tasks with more than one tempting distractors, are not included in the analysis in this article. In Lerchenberger and Donner (2024), some further empirically proven phenomena of frequently chosen distractors were described and discussed with regard to particularly high and low, conspicuous solution frequencies, such as double traps, twin traps or non-sellers. All these terms help to describe tasks in which certain distractors appear particularly attractive and the correct answer-option, on the other hand, appears particularly unattractive to the participants. These discussions on tasks can contribute to reflection when creating the Mathematical Kangaroo.

Based on theoretical, preliminary work from different scientific traditions and the principles of the specific competition design of the Mathematical Kangaroo, it is possible to identify genuinely different and characteristic possible causes for empirical traps by comparing them with the theoretical preliminary work and to search a posteriori for possible reasons that led the vast majority of participants to choose a certain distractor. In addition, according to the authors' assessment, both the inductively and deductively derived categories of empirical traps are confirmed by a group of experts with excellent interrater reliability when they were asked for the reason that seems most plausible to them, which most likely led the students to choose the particular distractor.

It cannot be said with certainty a priori whether the unattractiveness of all further answer-options (including the correct one) could lead to a distractor proving to be an empirical trap. This cannot be ruled out, but it can be stated that for all the trap tasks found within this study, the experts were largely able to find plausible explanations in the form of potential and obvious solution approaches for ticking the distractor in question.

A counterpart to trap tasks in particular are tasks in which superficial processing (possibly unintentionally by the task creator) leads to the correct answer. Such tasks in the Mathematical kangaroo have so far only been described exemplarily (see discussion of 'bestsellers' in Lerchenberger and Donner 2024). A more detailed task analysis in this regard, including the solution frequencies, could provide a worthwhile empirical contribution to the conscious, reflective design of competition tasks.

In addition, three further impulses can be provided.

The open question of whether selecting a distractor in the category special distractor (SD) is a Type 1 or 2 process, or whether and in what proportion both possibilities can actually be identified, can only be answered by examining specific processes of solution approaches. This is an obvious target for ongoing research.

The question of whether omitting or replacing a potentially particularly frequently selected distractor would lead students to revisit the task and thus better achieve the competition's goal of encouraging students to puzzle requires empirical studies with concrete comparisons between tasks with and without the specific distractor.

By expanding the current sample size, it would also be possible to quantitatively interpret age-dependent trends in the frequency of each category of empirical traps. In addition, by identifying and analyzing trap tasks in different countries, transregional statements can be made for each individual trap category. For instance, Stavy et al. (2006) were able to demonstrate the occurrence of substantial parts of the intuitive rule theory in different cultures. This aspect appears to be of particular interest with regard to the global character of the competition and its further development.

All the open questions mentioned show the need for further research in this area, which can promote discussion in the creation of competition tasks.

Implications for participants of the competition and task creators

The categories found offer pupils and students a decisive added value when working on the tasks in the Mathematical Kangaroo, for example when preparing for the competition. Until now, participants could only be generally warned of the phenomenon of tempting distractors and thus encouraged to exercise caution when working on the usually tricky tasks during the competition. However, Dewolf et al. (2014) impressively show that general warnings ('Be careful, the tasks may be more difficult than expected') have no positive effect on the frequency of solutions. It can therefore be assumed that a general warning about possible occurrence of traps in the Mathematical Kangaroo does not protect pupils from falling into them when working on the task itself, as they do not know whether the specific task they are working on contains a tempting distractor. In contrast to general warnings, knowledge of the specific types of traps and explicitly working on ideal-typical tasks in the preparation to the competition can possibly protect against choosing traps of categories AD or WM if, for example, this special feature of the 'direct' answer or direct pattern recognition in a task is noticed.

It should be discussed to what extent and what types of trap tasks should occur in a competition at all.

The aim of the Mathematical Kangaroo is to popularize mathematics (Akveld et al., 2020). With this in mind, a recommendation is made for the flavor of the tasks: 'The flavour of the tasks should be as varied as possible, but also accessible to as many participants as possible' (Akveld et al., 2020). Therefore, a major interest of this competition is to support serious problem-solving approaches of students by means of the mainly easily accessible tasks. From our point of view, based on the results presented here, the question arises whether there should be traps based on *misleading (secondary) intuitions (category WP)* at all. In the tradition of Polyá (1969), inductive reasoning in mathematics education, especially the discovery of connections, plausible argumentation and even the 'guessing' of proofs, is given more space today in curriculum than deductive reasoning. By providing a potential stimulus based on a seemingly recognizable pattern, many participants are led to believe that they have made a kind of intuitive 'discovery'. According to the findings of dual-process theory, a deeper examination of the task in question - and the analytical drawing of any deductive conclusions - is made more difficult. In contrast, if the suspected answer is not among the possible answers, a student is practically forced to reconsider his or her solution and forced to start a new problem-solving approach.

From the authors' point of view, the situation is different with empirical traps from the category SA (sustainable approach). These tasks enable a meaningful mathematical discussion, as the temptation is only such when parts of the solution have already been found. The multiple-choice format strongly supports incomplete approaches, as certain distractors (only supposedly) suggest an additional confirmation of the correctness and completeness of the approach. In competition, this type of task can ensure that strategies such as (error) control and reflection - used in the right place - can help to avoid choosing such a distractor.

In principle, the authors are of the opinion that suitable traps used in the right place represent a particular challenge for the participants of the Mathematical Kangaroo. As this type of multiple-choice competition is also intended to encourage participants to engage with successful strategies of solving a problem, traps can in some cases lead to participants becoming more aware of the possible answers and recognizing that some answer-options could be misleading. This in turn can stimulate the thought process and, particularly in the sense of a competition, also represents a certain attraction for participants.

In conclusion, the authors would like to mention three points which, based on the findings of this research project, they believe should at least be discussed within the community interested in the Mathematical Kangaroo (and similar competitions as well):

- *3-point tasks should not contain empirical traps*: This is attempted anyway (cf. Donner and Geretschläger 2022), but care should be taken to ensure that distractors could also not unintentionally become traps. Knowledge about the different empirically found categories can contribute to this.
- *Some types of empirical traps seem to prevent engagement with mathematical content*: Without any doubt, mathematical misconceptions and typical errors are a main source for plausible distractors (Geretschläger and Donner, 2022). However, especially the empirical traps in the categories AD (Answer direct) and at least to some extent, empirical traps of the category WP (wish for recognizing a pattern), may prevent the problem solvers even from engaging with the mathematical content of the problem. This might not be perfectly in line with the aim of the Mathematical Kangaroo to promote the popularization of mathematics and encourage pupils to solve problems.
- Open answer-options change the multiple-choice competition's character: Distractors like "another number" or "it cannot be determined" are hidden proof tasks. They are therefore

not typical multiple-choice tasks. Apart from that, such distractors can enrich a task and make it more interesting, as it reduces the tactical options a solver has at their disposal in dealing with a task (Geretschläger and Donner, 2022). The fact that such distractors also appear as empirical traps in the category SD (special distractor) stresses that they represent a certain attraction for a large proportion of participants in the Mathematical Kangaroo. If their appeal were superficial, we believe this would be a reason to question their use: If they distract pupils from thinking seriously about the content of the tasks, it would at least be questionable whether the goal of promoting the popularization of mathematics can be achieved in this way.

Lukas Donner
University of Duisburg-Essen, Faculty of Mathematics, Essen, Germany
lukas.donner@uni-due.de. ORCID: 0000-0003-1263-1478

Evita Lerchenberger
University of Graz. Institute for Mathematics and Scientific Computing, Graz, Austria
evita.lerchenberger@uni-graz.at. ORCID: 0000-0003-0102-7160

An application of vector projections to a concurrency theorem in a triangle

Quang Hung Tran

Quang Hung Tran graduated with a Bachelor's and Master's degree in Pure Mathematics from Vietnam National University, Hanoi (VNU). He is currently a teacher at the High School for Gifted Students, VNU University of Science (HSGS), one of the leading institutions for training gifted Mathematics students in Vietnam. With over 12 years of experience teaching Olympic Mathematics to Vietnam and international teams, he is a coach (specializing in geometry) for the Vietnamese team participating in the International Mathematical Olympiad (IMO) (from 2012 to present) and for the Saudi Arabian team participating in the IMO (in the years 2015, 2016, and 2017). He is particularly interested in various branches of geometry such as Olympiad geometry (problem-solving), Euclidean geometry, non-Euclidean geometry, geometry in higher dimensions, geometric inequalities, and complex analysis. His hobbies include reading books, listening to classical music, and always making time for his family whenever possible.

Abstract
In this paper, we will demonstrate an intriguing application of parallel vector projection within a plane, coupled with metric relationships within a triangle, to establish a captivating theorem regarding concurrency within a triangle.

Introduction

In the realm of plane geometry, the concept of parallel projection, involving the transformation of vectors onto a line in a linear fashion, holds significant theoretical importance. Euclidean vectors, denoted by $\overrightarrow{PQ}$ for the vector connecting an initial point P to a terminal point Q, and represented as $\vec{a}$, $\vec{b}$, and so forth, play a fundamental role in this context. In this section, we revisit the definition of parallel projection and lay the groundwork for its application in our subsequent theorem.

Definition (See [1]). Let d be a line in a plane, and consider an arbitrary point P. Now, let ℓ represent a line that is not parallel to d. The mapping $\mathbf{pj}_d^\ell$ transforms point P into another point P^* on line d in such a way that the lines PP^* and ℓ become parallel. This mapping, characterized by the direction line ℓ onto line d, is known as parallel projection (refer to Figure 2). We denote the result of this operation as:

$$P^* = \mathbf{pj}_d^\ell(P).$$

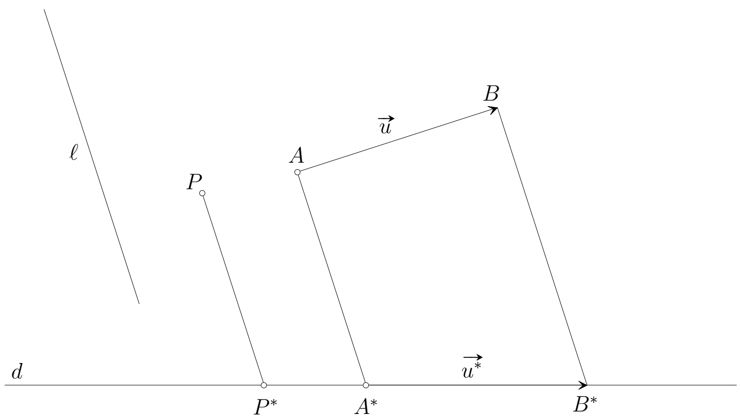


Figure 2: Illustration of parallel projection.

Definition. Let d be a line in a plane, and consider an arbitrary point P. Again, let ℓ be a line that is not parallel to d. Now, suppose we have a parallel projection $\mathbf{pj}_d^\ell$ in place. Let $\vec{u} = \overrightarrow{AB}$, and let $\overrightarrow{u^*} = \overrightarrow{\mathbf{pj}_d^\ell(A)\,\mathbf{pj}_d^\ell(B)}$. We introduce a mapping $\overrightarrow{\mathbf{pj}}_d^\ell$, which transforms vector $\vec{u}$ into vector $\overrightarrow{u^*}$, denoted as $\overrightarrow{u^*} = \overrightarrow{\mathbf{pj}}_d^\ell(\vec{u})$. This mapping is known as the parallel projection of vectors, with the direction line ℓ onto line d. The association mapping of $\overrightarrow{\mathbf{pj}}d^\ell$ is represented as $\mathbf{pj}_d^\ell$ (refer to Figure 2).

We recall a theorem that is known as Thales' theorem in [1, pp.23].

Theorem 1 (Thales' theorem)**.** The parallel projections are affine mappings.

Theorem 2. Furthermore, parallel projections of vectors are linear mappings.

With these foundational concepts in place, we are prepared to apply parallel projections to establish a new theorem in plane geometry:

Theorem 3. Consider an acute-angled triangle ABC with orthocenter H and incenter I. Let HD, HE, and HF represent the bisectors of triangles HBC, HCA, and HAB, respectively. Additionally, let X, Y, and Z denote the midpoints of segments EF, FD, and DE, respectively. Our theorem states that the four lines AX, BY, CZ, and IH are concurrent (see Figure 3).

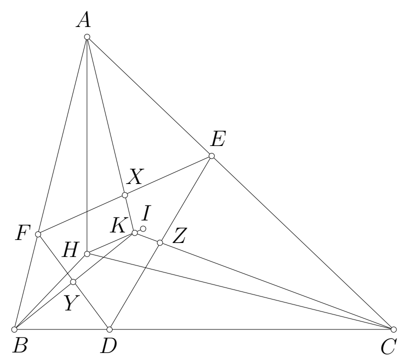


Figure 3: Illustration of Theorem 3.

Through the utilization of parallel projections, we will provide a proof for this theorem in the subsequent sections of this paper.

Proof of Theorem 3

In this solution, we will adopt the following notations and definitions:

1. When referring to the length segment $\overline{XY}$, we will follow Newton's concept of directed line segments (as described in [3, p. 30]), which implies that $\overline{XY} = -\overline{YX}$.
2. The notation XY or YX will represent the Euclidean distance between two points X and Y.
3. We will use the following notations for the sides and circumradius of triangle ABC:
   - Let $BC = a$, $CA = b$, and $AB = c$.
   - The circumradius of triangle ABC will be denoted as R, and it is also the circumradius of triangles HBC, HCA, and HAB.
4. We will denote the areas of triangles HBC, HCA, HAB, and ABC as S_a, S_b, S_c, and S, respectively. Additionally, we have the following area formulas:
   i) $S_a = \frac{a \cdot HB \cdot HC}{4R}$,
   ii) $S_b = \frac{b \cdot HC \cdot HA}{4R}$,
   iii) $S_c = \frac{c \cdot HA \cdot HB}{4R}$,
   iv) $S = \frac{abc}{4R}$.
5. The inradius of triangle ABC will be denoted as r.
6. We will use r_a, r_b, and r_c to represent the exradii at the vertices A, B, and C of triangle ABC.

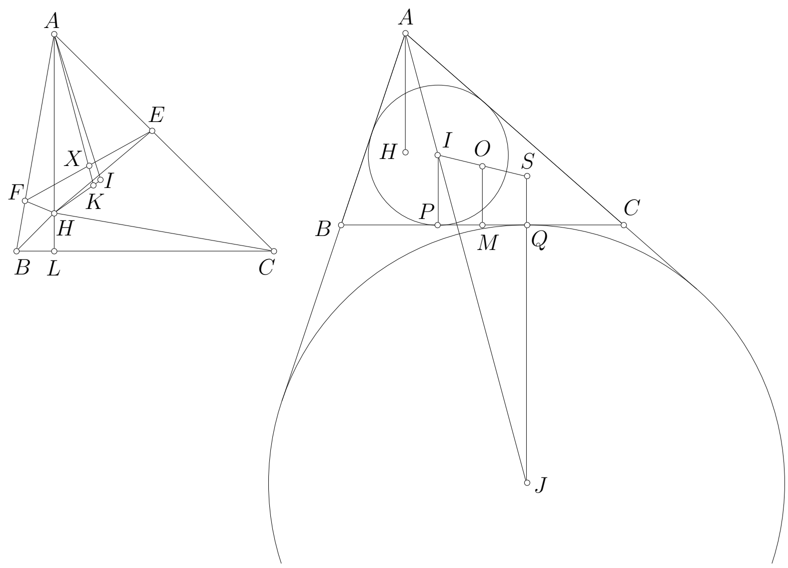


Figure 4: Illustration of the proof for Theorem 3.

Proof. (See Figure 4):

Let AL be the altitude of triangle ABC (with L lying on side BC). Utilizing the Pythagorean theorem and various metric relations, we can derive the following equation:

$$\begin{aligned}\frac{b}{HB}-\frac{c}{HC}&=\frac{AL}{LB}-\frac{AL}{LC}\\&=\frac{AL}{LB\cdot LC}(LC-LB)\\&=\frac{1}{LH}\cdot\frac{(LB+DC)(LC-LB)}{a}\\&=\frac{LC^2-LB^2}{aLH}\\&=\frac{b^2-c^2}{aLH}.\end{aligned}\tag{1}$$

Considering that the barycentric coordinates of the incenter I are (a,b,c), we have:

$$a\overrightarrow{IA}+b\overrightarrow{IB}+c\overrightarrow{IC}=\vec{0}.\tag{2}$$

From this equation, we obtain:

$$b\overrightarrow{AB}+c\overrightarrow{AC}=(a+b+c)\overrightarrow{AI}.\tag{3}$$

Now, as the areas of triangles HBC, HCA, HAB, and ABC are denoted as S_a, S_b, S_c, and S (with H lying inside triangle ABC), and the barycentric coordinates of H are (S_a,S_b,S_c), we have:

$$S_a\overrightarrow{HA}+S_b\overrightarrow{HB}+S_c\overrightarrow{HC}=\vec{0}.\tag{4}$$

Hence, we can write:

$$S_b\overrightarrow{AB}+S_c\overrightarrow{AC}=S\overrightarrow{AH}.\tag{5}$$

Combining equations (3) and (5), we derive:

$$(cS_b-bS_c)\overrightarrow{AC}=S_b(a+b+c)\overrightarrow{AI}-bS\cdot\overrightarrow{AH}\tag{6}$$

and

$$(bS_c-cS_b)\overrightarrow{AB}=S_c(a+b+c)\overrightarrow{AI}-cS\cdot\overrightarrow{AH}.\tag{7}$$

Given that X is the midpoint of EF, we can express:

$$2\overrightarrow{AX}=\overrightarrow{AE}+\overrightarrow{AF}=\frac{HA}{HA+HC}\overrightarrow{AC}+\frac{HA}{HA+HB}\overrightarrow{AB}.\tag{8}$$

Combining equation (8) with equations (6) and (7), we deduce:

$$\begin{aligned}&\frac{2(HA+HB)(HA+HC)}{HA}(cS_b-bS_c)\overrightarrow{AX}\\&=(HA+HB)(cS_b-bS_c)\overrightarrow{AC}-(HA+HC)(cS_b-bS_c)\overrightarrow{AB}\\&=[(HA+HB)S_b(a+b+c)-(HA+HC)S_c(a+b+c)]\overrightarrow{AI}+\\&\quad+[b(HA+HB)S-c(HA+HC)S]\overrightarrow{AH}.\end{aligned}\tag{9}$$

Let K be the intersection of AX and IH. Utilizing the parallel projection of vectors with the direction line AX onto line IH, denoted as $\overrightarrow{\mathbf{pj}}_{IH}^{AX}$, we can establish:

$$\overrightarrow{\mathbf{pj}}_{IH}^{AX}\left(\overrightarrow{AX}\right)=\vec{0},\overrightarrow{\mathbf{pj}}_{IH}^{AX}\left(\overrightarrow{AI}\right)=\overrightarrow{KI},\overrightarrow{\mathbf{pj}}_{IH}^{AX}\left(\overrightarrow{AH}\right)=\overrightarrow{KH}.\tag{10}$$

From equations (9) and (10), considering that $\overrightarrow{\mathbf{pj}}_{IH}^{AX}$ is a linear mapping, we obtain:

$$\begin{aligned}\vec{0} = [(HA+HB)S_b(a+b+c)-(HA+HC)S_c(a+b+c)]\,\overrightarrow{KI}\\ +[b(HA+HB)S-c(HA+HC)S]\,\overrightarrow{KH}\end{aligned} \tag{11}$$

or

$$\frac{\overline{KH}}{\overline{KI}} = -\frac{(HA+HB)S_b(a+b+c)-(HA+HC)S_c(a+b+c)}{b(HA+HB)S-c(HA+HC)S}. \tag{12}$$

Additionally, note that:

$$2HA\cdot HL = R^2-OH^2 \quad \text{and} \quad \frac{a\cdot AD}{(b+c-a)} = r_a, \tag{13}$$

where r_a represents the A-exradius of triangle ABC, and O is the circumcenter of ABC. We can easily observe:

$$HA+r_a = 2OM+JS-QS = 2OM+2R-(2OM-r) = 2R+r, \tag{14}$$

where M is the midpoint of BC, P and Q are the points of tangency of BC with the incircle and A-excircle, respectively, and S is the reflection of I in O.

Combining equations (12), (1), (13), and (14), we arrive at the following expression:

$$\begin{aligned}\frac{\overline{KH}}{\overline{KI}} &= -\frac{(HA+HB)S_b(a+b+c)-(HA+HC)S_c(a+b+c)}{b(HA+HB)S-c(HA+HC)S}\\
&= \frac{(a+b+c)}{S}\cdot\frac{\frac{b\cdot HA\cdot HC}{4R}(HA+HC)-\frac{c\cdot HA\cdot HB}{4R}(HA+HB)}{b(HA+HB)-c(HA+HC)}\\
&= \frac{HA\cdot HB\cdot HC}{2Rr}\cdot\frac{(\frac{b}{HB}-\frac{c}{HC})HA+(b-c)}{HA(b-c)+bHB-cHC}\\
&= \frac{HA\cdot HB\cdot HC}{2Rr}\cdot\frac{\frac{b^2-c^2}{a\cdot LH}HA+(b-c)}{HA(b-c)+S_c-S_b}\\
&= \frac{HA\cdot HB\cdot HC}{2Rr}\cdot\frac{\frac{b^2-c^2}{a\cdot LH}HA+(b-c)}{HA(b-c)+\frac{c\cdot HA\cdot HB}{4R}-\frac{bHA\cdot HC}{4R}}\\
&= \frac{HA\cdot HB\cdot HC}{2Rr}\cdot\frac{\frac{b^2-c^2}{aLH}HA+(b-c)}{HA(b-c)-\frac{HA\cdot HB\cdot HC}{4R}(\frac{b}{HB}-\frac{c}{HC})}\\
&= \frac{HA\cdot HB\cdot HC}{2Rr}\cdot\frac{\frac{b+c}{a\cdot LH}HA+1}{HA-\frac{HA\cdot 2S_a}{a}\frac{b+c}{a\cdot LH}}\\
&= \frac{HA\cdot HB\cdot HC}{2Rr}\cdot\frac{\frac{b+c}{a\cdot LH}HA+1}{HA-\frac{HA(b+c)}{a}}\\
&= -\frac{HA\cdot HB\cdot HC}{2Rr}\cdot\frac{(b+c)HA+a\cdot LH}{LH\cdot HA(b+c-a)}\\
&= -\frac{HA\cdot HB\cdot HC}{(R^2-OH^2)Rr}\cdot\frac{(b+c-a)HA+a\cdot AL}{(b+c-a)}\\
&= -\frac{HA\cdot HB\cdot HC}{(R^2-OH^2)Rr}(HA+r_a)\\
&= -\frac{HA\cdot HB\cdot HC}{(R^2-OH^2)Rr}(2R+r).\end{aligned} \tag{15}$$

Similarly, from equation (15), we can conclude that lines BY and CZ must pass through point K. Therefore, the proof of Theorem 3 is complete. □

Conclusion

In this conclusion, we underscore the significance of vector projections and metric relations as essential elements of our solution. Notably, our approach aimed to steer clear of trigonometric methods in pursuit of an elegant solution.

It is worth noting that proving the concurrency of the three lines, namely AX, BY, and CZ, is a relatively straightforward task, as it can be accomplished using Ceva's Theorem. However, the real challenge lies in demonstrating that the point of intersection lies precisely on the line IH.

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Quang Hung Tran
tranquanghung@hus.edu.vn

The 64th International Mathematical Olympiad

Angelo Di Pasquale
IMO Team Leader, Australia

Angelo was twice a contestant at the International Mathematical Olympiad. He completed a PhD in mathematics at the University of Melbourne studying algebraic curves. He is currently Director of Training for the Australian Mathematical Olympiad Committee (AMOC), and Australian Team Leader at the International Mathematical Olympiad.
He enjoys composing Olympiad problems for mathematics contests.

The 64th International Mathematical Olympiad (IMO) was held 2–13 July 2023 in the city of Chiba, Japan. This was the second time that Japan has hosted the IMO.

A total of 618 high school students from 112 countries participated. Of these, 67 were female.

As per normal IMO rules, each participating country may enter a team of up to six students, a Team Leader and a Deputy Team Leader.[1]

Participating countries also submit problem proposals for the IMO. This year there were 167 problem proposals from 52 countries. The IMO Problem Selection Committee shortlisted 30 of these for potential use on the IMO exams.

At the IMO the Team Leaders, as an international collective, form what is called the *Jury*. The Jury makes the important decisions that shape each year's IMO. Their first task is to set the two IMO competition papers from the aforementioned shortlist and approve marking schemes. During this period the Leaders and their observers are trusted to keep all information about the contest problems completely confidential.

The six problems that ultimately appeared on the IMO exam papers may be described as follows.

1. A very easy number theory problem proposed by Colombia.
2. A medium classical geometry problem proposed by Portugal.
3. A difficult algebra problem about sequences satisfying a certain polynomial property proposed by Malaysia.
4. An easy algebraic inequality proposed by the Netherlands.
5. A medium combinatorics problem proposed by the Netherlands.
6. A very difficult beautiful classical geometry problem proposed by the United States of America.

[1] The IMO regulations also permit countries to enter a small number of additional staff as Observers. These may fulfil various roles such as meeting child safety obligations, assisting with marking and coordination, or learning about how to host an IMO.

These six problems were posed in two exams held on Saturday 8 July and Sunday 9 July. Each exam paper had three problems. The contestants worked individually. They were allowed four and a half hours per paper to attempt the problems. Each problem was scored out of a maximum of seven points.

After the exams, the Leaders and their Deputies spent about two days assessing the work of the students from their own countries, guided by the marking schemes which had been agreed to earlier. A local team of markers called Coordinators also assessed the papers. They too were guided by the marking schemes but are allowed some flexibility if, for example, a Leader brought something to their attention in a contestant's exam script that is not covered by the marking scheme. The Team Leader and Coordinators must agree on scores for each student of the Leader's country in order to finalise scores. Any disagreements that cannot be resolved in this way are ultimately referred to the Jury. No such referrals occurred this year.

The contestants found Problems 1 and 4 to be the easiest with average scores of 5.85 and 4.72, respectively. Problem 6 was the hardest, with only 6 contestants receiving full marks on it. It averaged just 0.28 overall. The score distributions by problem number were as follows.

Mark	**P1**	**P2**	**P3**	**P4**	**P5**	**P6**
0	26	202	396	86	219	555
1	19	100	102	100	29	11
2	67	6	7	32	174	36
3	9	62	23	8	52	4
4	9	20	8	4	4	1
5	6	7	6	1	13	1
6	8	6	3	3	9	4
7	474	215	73	384	118	6
Mean	5.85	3.16	1.26	4.72	2.42	0.28

The medal cuts were set at 32 points for Gold, 25 for Silver and 18 for Bronze. The medal distributions[2] were as follows.

	Gold	**Silver**	**Bronze**	**Total**
Number	54	90	170	314
Proportion	8.7%	14.6%	27.5%	50.8%

These awards were presented at the closing ceremony. Of those who did not get a medal, 192 contestants received an Honourable Mention for scoring full marks on at least one problem.

Five contestants achieved the most excellent feat of a perfect score of 42.

The 2023 IMO was organised by the Mathematical Olympiad Foundation of Japan.

Hosts for future IMOs have been secured as follows.

11-22 July, 2024	Bath, United Kingdom
2025	Sunshine Coast, Australia
2026	People's Republic of China

Much of the statistical information found in this report can also be found on the official website of the IMO.

www.imo-official.org

[2]The total number of medals is approved by the Jury and should not normally exceed half the total number of contestants. The numbers of Gold, Silver and Bronze medals should be approximately in the ratio 1:2:3.

IMO 2023
Chiba, JAPAN 64th

English (eng), day 1

Saturday, 8. July 2023

Problem 1. Determine all composite integers $n > 1$ that satisfy the following property: if $d_1, d_2, \ldots, d_k$ are all the positive divisors of n with $1 = d_1 < d_2 < \cdots < d_k = n$, then d_i divides $d_{i+1} + d_{i+2}$ for every $1 \leqslant i \leqslant k-2$.

Problem 2. Let ABC be an acute-angled triangle with $AB < AC$. Let Ω be the circumcircle of ABC. Let S be the midpoint of the arc CB of Ω containing A. The perpendicular from A to BC meets BS at D and meets Ω again at $E \neq A$. The line through D parallel to BC meets line BE at L. Denote the circumcircle of triangle BDL by ω. Let ω meet Ω again at $P \neq B$.
Prove that the line tangent to ω at P meets line BS on the internal angle bisector of $\angle BAC$.

Problem 3. For each integer $k \geqslant 2$, determine all infinite sequences of positive integers $a_1, a_2, \ldots$ for which there exists a polynomial P of the form $P(x) = x^k + c_{k-1}x^{k-1} + \cdots + c_1 x + c_0$, where $c_0, c_1, \ldots, c_{k-1}$ are non-negative integers, such that

$$P(a_n) = a_{n+1}a_{n+2}\cdots a_{n+k}$$

for every integer $n \geqslant 1$.

Language: English

Time: 4 hours and 30 minutes.
Each problem is worth 7 points.

IMO 2023
Chiba, JAPAN 64th

English (eng), day 2

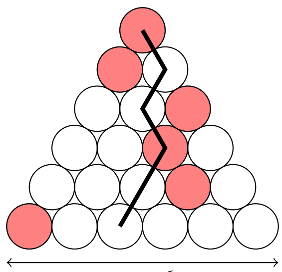
Sunday, 9. July 2023

Problem 4. Let $x_1, x_2, \dots, x_{2023}$ be pairwise different positive real numbers such that

$$a_n = \sqrt{(x_1 + x_2 + \cdots + x_n)\left(\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}\right)}$$

is an integer for every $n = 1, 2, \dots, 2023$. Prove that $a_{2023} \geqslant 3034$.

Problem 5. Let n be a positive integer. A *Japanese triangle* consists of $1 + 2 + \cdots + n$ circles arranged in an equilateral triangular shape such that for each $i = 1, 2, \dots, n$, the i^{th} row contains exactly i circles, exactly one of which is coloured red. A *ninja path* in a Japanese triangle is a sequence of n circles obtained by starting in the top row, then repeatedly going from a circle to one of the two circles immediately below it and finishing in the bottom row. Here is an example of a Japanese triangle with $n = 6$, along with a ninja path in that triangle containing two red circles.



$n = 6$

In terms of n, find the greatest k such that in each Japanese triangle there is a ninja path containing at least k red circles.

Problem 6. Let ABC be an equilateral triangle. Let A_1, B_1, C_1 be interior points of ABC such that $BA_1 = A_1C$, $CB_1 = B_1A$, $AC_1 = C_1B$, and

$$\angle BA_1C + \angle CB_1A + \angle AC_1B = 480^\circ.$$

Let BC_1 and CB_1 meet at A_2, let CA_1 and AC_1 meet at B_2, and let AB_1 and BA_1 meet at C_2. Prove that if triangle $A_1B_1C_1$ is scalene, then the three circumcircles of triangles AA_1A_2, BB_1B_2 and CC_1C_2 all pass through two common points.

(Note: a scalene triangle is one where no two sides have equal length.)

Language: English

Time: 4 hours and 30 minutes.
Each problem is worth 7 points.

Some IMO Country Totals

Rank	Country	Total
1	People's Republic of China	240
2	United States of America	222
3	Republic of Korea	215
4	Romania	208
5	Canada	183
6	Japan	181
7	Vietnam	180
8	Turkey	176
9	India	174
10	Taiwan	173
11	Islamic Republic of Iran	172
12	Singapore	171
13	United Kingdom	167
14	Israel	163
14	Mexico	163
16	Brazil	161
17	Belarus	159
17	Italy	159
19	Thailand	158
20	Germany	156
21	Kazakhstan	154
22	Hungary	153
23	Australia	52
24	Hong Kong	151
25	Bulgaria	149
26	Greece	145
26	Philippines	145
28	France	142
29	Netherlands	139
30	Mongolia	138

Distribution of Awards at the 2023 IMO

Country	Total	Gold	Silver	Bronze	HM
Albania	48	0	0	0	4
Algeria	100	0	0	2	4
Argentina	96	0	1	1	3
Armenia	133	0	2	3	1
Australia	152	1	2	2	1
Austria	106	0	1	1	4
Azerbaijan	102	0	1	1	4
Bangladesh	110	0	0	3	2
Belarus	159	0	4	2	0
Belgium	92	0	0	2	4
Bolivia	53	0	0	0	4
Bosnia and Herzegovina	130	0	1	4	1
Botswana	15	0	0	0	1
Brazil	161	1	2	3	0
Bulgaria	149	1	1	4	0
Burkina Faso	8	0	0	0	0
Cameroon	6	0	0	0	0
Canada	183	1	4	1	0
Chile	20	0	0	0	2
Colombia	78	0	0	2	3
Costa Rica	69	0	0	1	4
Croatia	103	0	0	4	2
Cuba	11	0	0	0	1
Cyprus	103	0	1	1	4
Czech Republic	112	0	0	4	1
Denmark	97	0	0	2	3
Dominican Republic	13	0	0	0	0
Ecuador	27	0	0	0	1
El Salvador	35	0	0	0	3
Estonia	108	0	0	3	3
Finland	91	0	1	1	4
France	142	0	1	5	0
Georgia	129	1	0	4	1
Germany	156	0	3	3	0
Ghana	12	0	0	0	0
Greece	145	1	1	3	1
Guatemala	5	0	0	0	0

Country	Total	Gold	Silver	Bronze	HM
Honduras	17	0	0	0	1
Hong Kong	151	1	1	4	0
Hungary	153	1	2	3	0
Iceland	58	0	0	1	4
India	174	2	2	2	0
Indonesia	128	0	1	3	2
Iraq	11	0	0	0	0
Ireland	63	0	0	1	2
Islamic Republic of Iran	172	1	4	1	0
Israel	163	1	3	2	0
Italy	159	1	2	3	0
Japan	181	2	3	1	0
Kazakhstan	154	0	2	4	0
Kenya	4	0	0	0	0
Kosovo	34	0	0	0	2
Kyrgyzstan	79	0	0	0	6
Latvia	101	0	0	3	3
Liechtenstein	17	0	0	0	2
Lithuania	91	0	1	1	4
Luxembourg	14	0	0	0	1
Macau	107	0	1	2	3
Malaysia	114	1	0	3	2
Mauritania	24	0	0	0	1
Mexico	163	1	3	2	0
Mongolia	138	1	0	4	1
Montenegro	20	0	0	0	1
Morocco	83	0	0	1	5
Myanmar	32	0	0	0	2
Nepal	26	0	0	0	1
Netherlands	139	0	1	5	0
New Zealand	91	0	0	1	5
Nicaragua	22	0	0	0	2
North Macedonia	127	1	1	2	2
Norway	94	0	0	2	4
Oman	27	0	0	1	0
Pakistan	42	0	0	1	1
Panama	31	0	0	0	3
Paraguay	41	0	0	0	4
People’s Republic of China	240	6	0	0	0

Country	Total	Gold	Silver	Bronze	HM
Peru	133	0	2	3	0
Philippines	145	0	3	3	0
Poland	133	0	1	4	1
Portugal	73	0	0	1	4
Puerto Rico	24	0	0	0	1
Republic of Korea	215	4	2	0	0
Republic of Moldova	101	0	0	3	3
Romania	208	5	1	0	0
Rwanda	35	0	0	0	3
Saudi Arabia	130	0	1	3	2
Serbia	120	0	1	3	2
Singapore	171	2	3	0	1
Slovakia	128	0	1	4	1
Slovenia	91	0	0	1	5
South Africa	105	0	1	1	4
Spain	131	0	1	4	1
Sri Lanka	80	0	0	1	5
Sweden	96	0	1	1	3
Switzerland	117	1	0	3	1
Syria	95	0	0	2	4
Taiwan	173	1	4	1	0
Tajikistan	75	0	0	1	4
Tanzania	0	0	0	0	0
Thailand	158	1	3	1	1
Tunisia	72	0	0	1	4
Turkey	176	1	5	0	0
Turkmenistan	83	0	0	1	4
Uganda	7	0	0	0	0
Ukraine	134	0	1	5	0
United Arab Emirates	36	0	0	1	1
United Kingdom	167	2	2	2	0
United States of America	222	5	1	0	0
Uruguay	37	0	0	0	3
Uzbekistan	106	0	0	3	3
Venezuela	11	0	0	0	1
Vietnam	180	2	2	2	0
Total (112 teams, 618 contestants)		**54**	**90**	**170**	**192**

N.B. Not all countries entered a full team of six students.

Angelo Di Pasquale
Angelo.DiPasquale@amt.edu.au

Australian Maths Trust
170 Haydon Drive
Bruce ACT 2617 AUSTRALIA
Tel: +61 2 6201 5136