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MATHEMATICS COMPETITIONS

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WORLD FEDERATION OF NATIONAL MATHEMATICS COMPETITIONS

AUSTRALIAN MATHS TRUST

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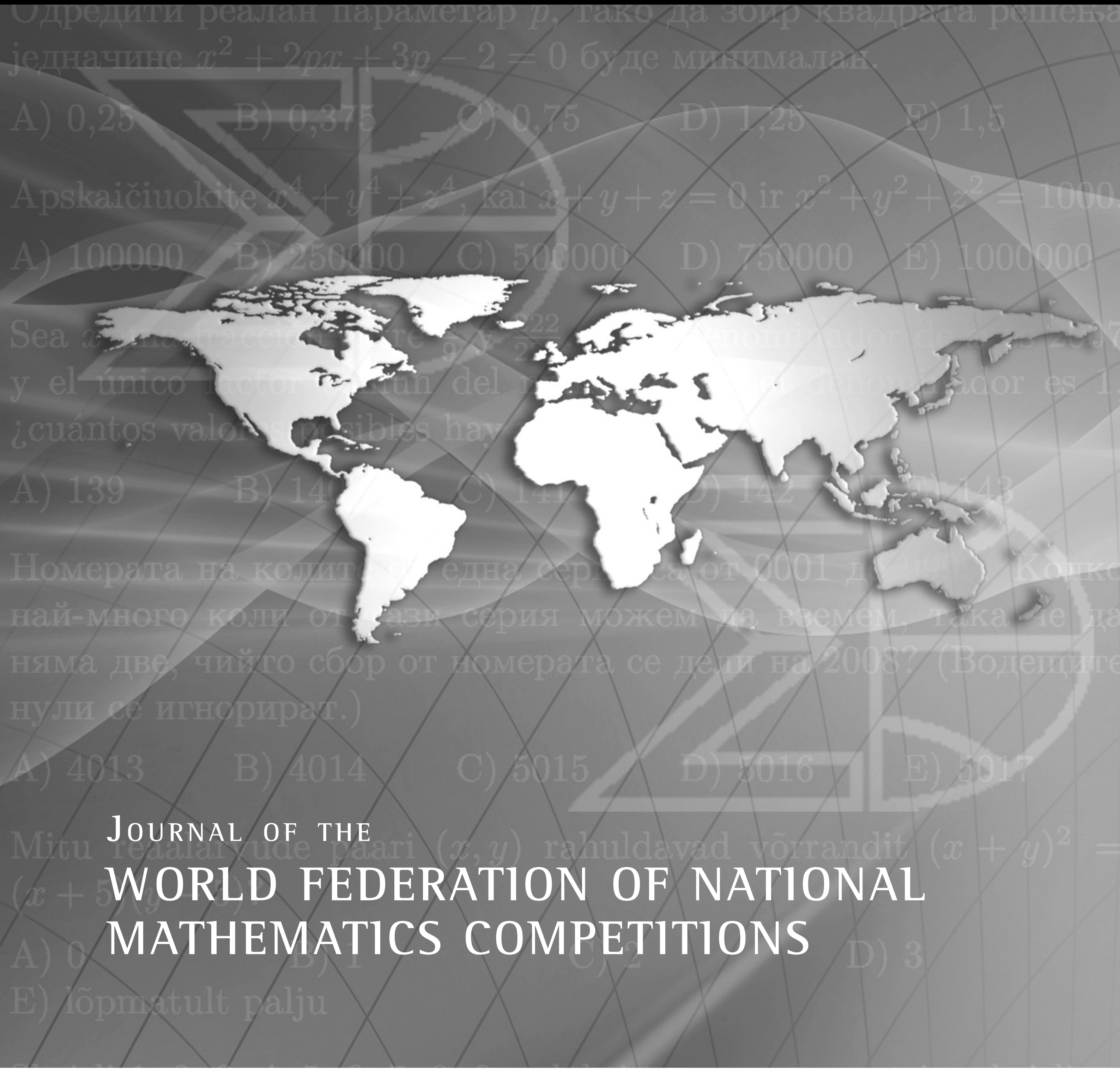

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Contents

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From the President

Dear readers,

As most of you are likely aware, the international mathematics competition scene has been steadily expanding in recent years, and this trend shows no sign of slowing down.

The number of countries participating in the big multi-national competitions continues its upward trend, with the numbers of participating nations at the IMO and the Kangaroo competition now solidly in the three-digit range. Another aspect of this trend is visible in countries just starting to organize competitions, where such activities had not been established previously.

The Covid pandemic led organizers of many regional and even international competitions to experiment with, and sometimes ultmately switch to, online formats. This has had the positive side effect of enabling participation for many students who would not easily be able to travel to central competition sites.

It is very encouraging to see clear signs of growth in Africa, which has long been the area facing the greatest difficulties in establishing a network of national and regional competitions. We are now witnessing strong growth there, with more and more African students gaining the opportunity to take part in all manner of contests. It will be incredibly exciting to see the further developments in this part of the world in the next few years.

Another aspect of the growth we are experiencing right now is the growing interest of the Math Education community in didactical aspects of competitions. The WFNMC is, of course, uniquely positioned to fascilitate communication between the Math Education and Math Competition communities. In this context, I would remind all readers to consider registering both for the International Congress on Mathematical Education (ICME-15; https://icme15.org/), scheduled for Sydney, Australia from July 7th to 14th, 2024 and for the WFNMC mini-conference (http://wfnmc.org/conferences.html) planned for the same venue on the day before ICME begins, namely the 6th of July, 2024.

You are invited to submit a proposal for a talk to be presented at the mini-conference. Proposals should be approximately 50 to 100 words in length and include a title and the name and affiliation of the presenter. The lengths of the talks will depend on the time available, but will likely be 20-30 minutes.

Please send your proposals to the organising committee using the subject line "WFNMC mini conference proposal" to Krzysztof.Ciesielski@im.uj.edu.pl. The deadline for proposals is December 31st, 2023, with titles and abstracts due by March 31st, 2024. (Of course, late proposals will still be considered, depending on time available. Please don't hesitate to submit, even if you happen to miss the deadline by a few days!)

At ICME, you may also consider submitting a talk to Topic Study Group 2.3 with the title **Mathematics and creativity; mathematical competitions; mathematical challenge**. Information on this TSG is available at

https://icme15.org/icme-15-scientific-program/topic-study-groups/\tsg-2-3-mathematics-and-creativity-mathematical-competitions-mathematical-challenge/

Finally, there will be a special session for affiliate organisations on the conference Wednesday, and the WFNMC has prepared a program that we hope will be of interest to anyone working in competitions or curious about learning more about them.

The book **Engaging Young Students in Mathematics through Competitions, World Perspectives and Practices, Volume III: Keeping Competition Mathematics Engaging in Pandemic Times**, based on the presentations at the 2022 WFNMC conference in Sofia, is being readied for print, and should be available for purchase by the time you read this. Information is available at https://worldscientific.com/.

As always, you are invited to submit papers for publication in this journal! Your active participation in all activities of our organisation is greatly appreciated, and I hope to see a great number of you in Sydney!

Robert Geretschläger

Editor's Page

Dear Competitions enthusiasts, readers of our *Mathematics Competitions* journal!

Mathematics Competitions is the right place for you to publish and read the different activities about competitions in Mathematics from around the world. For those of us who have spent a great part of our life encouraging students to enjoy mathematics and the different challenges surrounding its study and development, the journal can offer a platform to exhibit our results as well as a place to find new inspiration in the ways others have motivated young students to explore and learn mathematics through competitions. In a way, this learning from others is one of the better benefits of the competitions environment.

Following the example of previous editors, I invite you to submit to our journal *Mathematics Competitions* your creative essays on a variety of topics related to creating original problems, working with students and teachers, organizing and running mathematics competitions, historical and philosophical views on mathematics and closely related fields, and even your original literary works related to mathematics.

Just be original, creative, and inspirational. Share your ideas, problems, conjectures, and solutions with all your colleagues by publishing them here. We have formalized the submission format to establish uniformity in our journal.

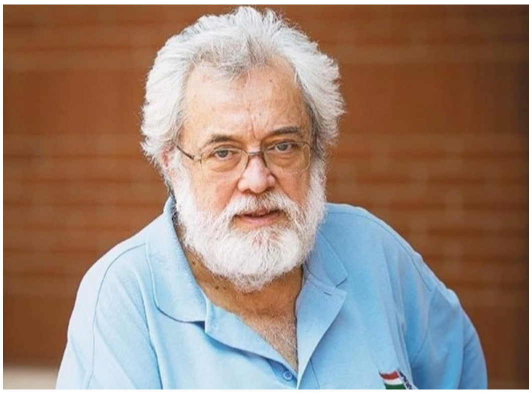
Submission Format
FORMAT: should be LaTeX, TeX, or for only text articles in Microsoft Word, accompanied by another copy in pdf. However, the authors are strongly recommended to send article in TeX or LaTeX format. This is because the whole journal will be compiled in LaTex. Thus your Word document will be typeset again. Texts in Word, if sent, should mainly contain non-mathematical text and any images used should be sent separately.
ILLUSTRATIONS: must be inserted at about the correct place of the text of your submission in one of the following formats: jpeg, pdf, tiff, eps, or mp. Your illustration will not be redrawn. Resolution of your illustrations must be at least 300 dpi, or, preferably, done as vector illustrations. If a text is embedded in illustrations, use a font from the Times New Roman family in 11 pt.
START: with the title centered in Large format (roughly 14 pt), followed on the next line by the author(s)' name(s) in italic 12 pt.
MAIN TEXT: Use a font from the Times New Roman family or 12 pt in LaTex.
END: with your name-address-email and your website (if applicable).
INCLUDE: your high resolution small photo and a concise professional summary of your works and titles.
Please submit your manuscripts to María Elizabeth Losada at
`director.olimpiadas@uan.edu.co`

We are counting on receiving your contributions, informative, inspired and creative. Best wishes,

Maria Elizabeth Losada
EDITOR

József Pelikán 1947–2023

by Géza Kós



In Hungary, his colleagues and students called him "Jocó" (pronounced Yotso, the last o is long), he actually asked his students to call him by this nickname. (People in the International Mathematical Olympiad (IMO) community rather used "Joseph".)

Jocó participated in the KöMaL Magazine’s points contest from his early school years, when he was only 12 years old. Then in 1962 he was selected for the first *special math class* in Fazekas Gimnázium. Beyond KöMaL, I believe, that class was the greatest invention in the last 150 years in Hungarian education of mathematics; the first special math class included Lovász, Pelikán, Laczkovich, Bollobás and Pósa. Excellent teachers and friendly competition between classmates inspired them to learn more and work harder and thus provided the fundamentals for the successes in the next decades of Hungarian mathematics. In particular, these kids were extremely successful in various national and international mathematical competitions; Lovász and Pelikán participated as students of the Hungarian team in the IMO four (4) times.

photo tableau of the first special math class

After finishing university studies, Jocó worked at the Department of Algebra of Loránd Eötvös University in Budapest, Hungary. Beyond his research in algebra (mainly in group theory) and teaching, Jocó was a successful bridge player; in 1986 his team *Taurus*, won a second prize in the European Teams Champions’ Cup.

Although I studied at the same university between 1986 and 1991, I was not a student of his; he taught algebra the previous year and the following year, but not during the year I took Algebra. So I just heard younger and older students expressing their respect for his wide and deep knowledge and the high quality of his lectures, as well as their fear of his extremely difficult tests and exams; he presented a huge amount of mathematics and many students failed.

Later we worked together in the competition committee of the Kürschák contest, in the preparation of the Hungarian IMO team, and from 2006, around the IMO jury; he was the leader of the Hungarian team and usually I was a member of the problem selection committee and a coordinator, so I could follow his activities and help him many times in typing and verifying the Hungarian problem translations. He was proud that he could speak all official languages of IMO, it was normal to find him talking with another team leader in German, French or Russian. He was very popular among the team leaders, and he was elected to be a member, and later the chairman of the Advisory Board (1992–2002 and 2002–2010). In 2014 he was awarded with the Erdős Prize by the World Federation of National Mathematics Competitions.

Jocó among the hungarian participants in IMO 2017

He was single. After losing his mother he lived alone, between piles of books; he had and read many more books than his bookshelves could hold, so whenever I visited him, I had to pass through a labyrinth built from books. In his last years, he kept in touch with colleagues and friends via internet, helped with the two on-line IMOs organized in St. Petersburg and watched many sumo championships, beyond his own sumo game against illness.

Géza Kós
kosgeza@gmail.com

On a St. Petersburg MO[1] problem about distances

Marcin E. Kuczma

Marcin E. Kuczma has been a member of the Regional Committee of the Polish Mathematical Olympiad in Warsaw since 1965. Many of the problems posed at the competition were of his authorship. He has been a lot of times a member of the jury as well as a member of the Coordination Committee at the International Mathematical Olympiad (IMO), the Austrian-Polish Mathematical Competition, and the Baltic Way Mathematical Competition. He was distinguished as a student at IMO and then contributed to IMO in various different roles. He is the author of four excellent books containing olympiad problems. Since 1981 he has been the editor of the mathematical column and competition Klub 44. In 1992 he was awarded the Hilbert Medal by the World Federation of National Mathematics Competitions. In 2023 he was awarded the Dickstein Prize (see page 10 of *Mathematics Competitions* Volume I 2023 http://wfnmc.org/Journal%202023%201.pdf).

Abstract

It is shown that there exist exactly two configurations of five points in the plane whose mutual distances satisfy certain equalities (conditions (1) below). This was inspired by a problem from the 2018 St.Petersburg MO asking whether there exists at least one such configuration.

The result that there are exactly two was anticipated by the problem proposer. A proof is provided in this article.

The St. Petersburg Mathematical Olympiad is without a doubt one of the most interesting math competitions. Browsing e. g. through the report booklet [1] of the 2018 issue one is confronted with no less than 100 attractive problems; some of them hard and demanding, other ones not at all so, yet outstanding in grace and elegance.

The problem referred to in the title of this note bears number 53 (problem statement on p. 20, solution on p. 67). It reads as follows:

Problem. Does there exist a triangle ABC plus two additional points X, Y (in the same plane) such that

$$AX = BY = AB, \quad BX = CY = BC, \quad CX = AY = CA\,? \tag{1}$$

Certainly this is not one of those "hard" problems. The only difficulty lies in deciding if you are looking for an example or for a proof of non-existence. Once you have made up your mind in favour of the first option, it's natural to narrow the search to triangles of some specific shape—say, isosceles ones; the task becomes automatic, and soon you arrive at the following two examples. In each of them you take one of the points A, B, C—say, C—for centre of a circle on which the other four points should be situated.

Example I. Points X, A, B, Y lie on the circle in this order at angles $\angle XCA = \angle ACB = \angle BCY = 30^\circ$.

[1]MO stands for Mathematical Olympiad

Example II. Points X,B,A,Y lie on the circle in this order at angles $\angle XCB = \angle ACY = 60^\circ$, $\angle BCA = 150^\circ$.

The required conditions (1) are satisfied, as is easy to see. These two examples are presented in the official solution in [1] (p. 67), which is concluded with a puzzling remark:

Remark. Probably there are no other examples.

Clearly enough, this is not a statement in probability theory. It looks like a message from the author/proposer that there is reasonable evidence for such conjecture, without however a formal proof. Now, a formal proof is the aim of this note. And thus:

Claim. There exist exactly two configurations of five points A,B,C,X,Y in the plane (up to similitude and possible relabelling within triple A,B,C) satisfying conditions (1).

Proof. Suppose ABC is a triangle as needed, with points X,Y as needed. Let as usual $BC = a$, $CA = b$, $AB = c$. We focus on X (ignoring Y for the moment). According to (1),

$$AX = c, \quad BX = a, \quad CX = b. \tag{2}$$

Toss these points onto complex plane, placing X at the origin. Then A,B,C are represented (respectively) by complex numbers cu, av, bw for some u,v,w with $|u| = |v| = |w| = 1$. Notation $BC = a$, $CA = b$, $AB = c$ gives rise to the equations

$$a = |av - bw|, \quad b = |bw - cu|, \quad c = |cu - av|. \tag{3}$$

The first one, by squaring, yields

$$a^2 = (av - bw)\overline{(av - bw)} = (av - bw)\left(\frac{a}{v} - \frac{b}{w}\right) = a^2 - ab\left(\frac{v}{w} + \frac{w}{v}\right) + b^2,$$

which simplifies to

$$\frac{b}{a} = \frac{v}{w} + \frac{w}{v}; \quad \text{likewise,} \quad \frac{c}{b} = \frac{w}{u} + \frac{u}{w}, \quad \frac{a}{c} = \frac{u}{v} + \frac{v}{u}; \tag{4}$$

the two latter formulas follow from the two last equations in (3). Multiplying out we get (in cyclic sum notation)

$$\begin{aligned} 1 = \left(\frac{v}{w} + \frac{w}{v}\right)\left(\frac{w}{u} + \frac{u}{w}\right)\left(\frac{u}{v} + \frac{v}{u}\right) &= 1 + \sum \frac{v}{w} \cdot \frac{u}{w} \cdot \frac{v}{u} + \sum \frac{w}{v} \cdot \frac{w}{u} \cdot \frac{u}{v} + 1 \\ &= 2 + \sum\left(\left(\frac{v}{w}\right)^2 + \left(\frac{w}{v}\right)^2\right) = 2 + \sum\left(\left(\frac{b}{a}\right)^2 - 2\right); \end{aligned}$$

in the last step we again made use of (4).

Conclusion. The existence of X (with properties (2)) forces the equation

$$\left(\frac{b}{a}\right)^2 + \left(\frac{c}{b}\right)^2 + \left(\frac{a}{c}\right)^2 = 5. \tag{5}$$

Now forget X and let Y enter into play. By (1),

$$BY = c, \quad CY = a, \quad AY = b. \tag{6}$$

If we interchange A with B and a with b, and write X in place of Y, conditions (6) come down exactly to those in (2). In view of the *Conclusion* (above) this shows that the mere existence of Y

(with properties (6)) forces the equations analogous to (5) with a and b interchanged (with the same 5 on the right side). Subtracting the two equations we get

$$\left(\frac{b}{a}\right)^2+\left(\frac{c}{b}\right)^2+\left(\frac{a}{c}\right)^2-\left(\frac{a}{b}\right)^2-\left(\frac{c}{a}\right)^2-\left(\frac{b}{c}\right)^2=0.$$

Multiplication by the common denominator yields

$$b^4c^2+c^4a^2+a^4c^2-a^4c^2-c^4b^2-b^4a^2=0.$$

This factors into

$$(c^2-b^2)(a^2-c^2)(b^2-a^2)=0,$$

showing that among the three positive numbers a,b,c some two are equal. By cyclicity and homogeneity we may assume that $a=b=1$. Plugging into (5) we get an equation (quadratic in c^2) with solutions $c^2=2\pm\sqrt{3}$. The positive values of c are $c=\left(\sqrt{6}\pm\sqrt{2}\right)/2$.

We have shown that—up to similitude and relabelling—the triangle ABC must have sides

$$a=b=1,\quad c=\frac{\sqrt{6}-\sqrt{2}}{2}\qquad\text{or}\qquad a=b=1,\quad c=\frac{\sqrt{6}+\sqrt{2}}{2}.$$

By simple trigonometry, triangle ABC has angles

$$\angle A=\angle B=75^\circ,\quad \angle C=30^\circ\quad\text{or}\quad \angle A=\angle B=15^\circ,\quad \angle C=150^\circ.$$

The equations (2) and (6) determine the position of X and Y. The resulting configurations of the five points are precisely those exhibited in *Example* I and *Example* II.

References

[1] S. L. Berlov, K. P. Kokhas, D. A. Rostovskiĭ, A. I. Khrabrov et al., *Problems posed at the Sankt-Peterburg Mathematical Olympiad for high-school students, year 2018* (in Russian), Moskva 2019.

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Genesis of a Problem: Nesting Parallelepipeds

In memory of Paul Erdős on his 111th Birthday

Alexander Soifer

Alexander Soifer is a Russian-born American mathematician and mathematics author. His works include over 400 articles and 13 books.

Every spring since 1983, Soifer, along with other mathematician colleagues, sponsor the Colorado Mathematical Olympiad (CMO) at the University of Colorado in Colorado Springs. In May 2018, in recognition of 35 years of leadership, it was renamed the Soifer Mathematical Olympiad.

In 1991 Soifer founded the research quarterly Geombinatorics, and publishes it with the Geombinatorics editorial board.

In July 2006 at the University of Cambridge, Soifer was presented with the Paul Erdős Award by the World Federation of National Mathematics Competitions. His Erdős number is 1.

Soifer's book of his life is coming out on January 29, 2024: https://www.amazon.com/New-Mathematical-Coloring-Book-Mathematics/dp/1071635964

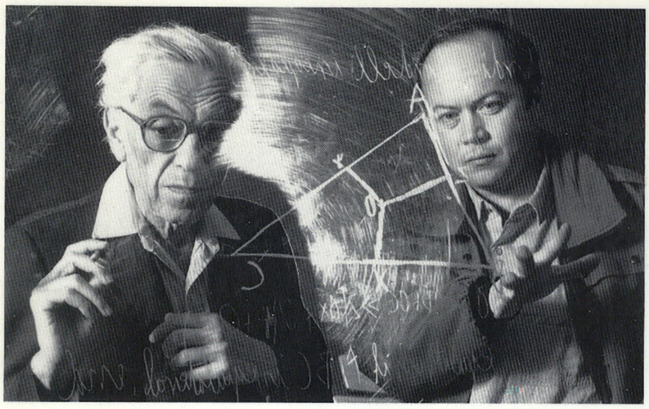
Abstract
New Olympiad problems occur to us in mysterious ways. This paper describes how a 1989 problem that appeared in "Kvant" in 1990 as problem 1188 was transformed for a current olympiad.

Preliminaries

I started thinking about "The Famous Five," as the journalists named the sets of problems for the Soifer (formerly Colorado) Mathematical Olympiad, as usual, on board a plane taking me to the 9th Congress of the World Federation of National Mathematics Competitions in Sofia, Bulgaria.

In an old obscure Russian booklet, I noticed a 1989 problem that appeared in "Kvant" in 1990 as problem 1188, and liked it very much and decided to use it as problem 5A below; I simply increased the sizes. I then substantially improved the result of problem 5A while creating and solving problems 5B and the ultimate result, problem 5C.

During the Congress, I shared an outdoor table with the Armenian mathematician Nairi Sedrakyan (to whom I was to present the Paul Erdős Award a few days later) and Deputy Editor of "Kvant" Sergei Dorichenko of Moscow. I asked Sergei whether he can get me the name of the author of problem 1188. He searched a file in his laptop and to everyone's disbelief disclosed the author: sitting across the table from me Nairi Sedrakyan!



Paul Erdős (left) and Alexander Soifer. *Photograph by Tom Kimmel*

The problems

5. Nesting Rectangles
A. (N. Sedrakyan). Given 2022 distinct rectangles whose lengths and widths are positive integers not exceeding 100. Are there necessarily 41 rectangles that could be nested in each other: the first rectangle in the second one, the second in the third, $\cdots$, the fortieth in the forty-first?
In order to nest a rectangle $a \times b$ in a rectangle $a_1 \times b_1$, we require that $a \leq a_1$; $b \leq b_1$ and at least one of these inequalities is strict.

B. (A. Soifer). Given 2022 distinct rectangles whose lengths and widths are positive integers not exceeding 100. Are there necessarily 46 rectangles that could be nested in each other?

C. (A. Soifer) Find the maximum number of nestable rectangles that are certain to appear in any collection of 2022 distinct rectangles with integral dimensions not exceeding 100.

Solution of 5A. A rectangle is defined by its dimensions, which form an ordered pair of integers, and an ordered pair (x, y) can be displayed by a unit square, whose upper-right coordinates are (x, y) in the Cartesian Plane. We now split all such unit squares into 50 disjoint sets, each in the form of "Γ", as shown in Figure 5.1.

We get 2022 pigeons (given rectangles) sitting on 50 Γ-shaped pigeonholes. Since $2022 > 50 \times 40$, by the Pigeonhole Principle, there is a pigeonhole with at least 41 pigeons on it. The pigeons-rectangles of the same pigeonhole can easily be nested one in the other. ■

Solution of 5B. You may have noticed that some pigeonholes are smaller than others. Let us eliminate the 10 smallest pigeonholes. Given 2022 rectangles, the smallest 10 pigeonholes may contain at most $3 + 7 + \cdots + 39 = 210$ pigeons (rectangles). This leaves $2022\text{˘}210 = 1812$ pigeons sitting on 40 pigeonholes. Since $1812 > 45 \times 40$, by the Pigeonhole Principle, there is a pigeonhole with at least 46 pigeons on it. These pigeons-rectangles can be nested one in the other. ■

Nairi Sedrakyan & Alexander Soifer, Sofia, Bulgaria, July 22, 2022

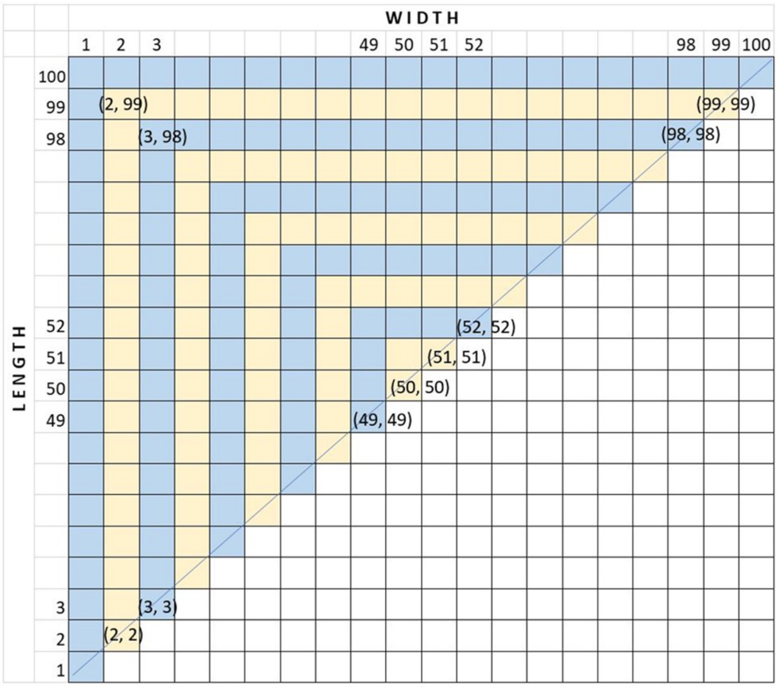


Figure 5.1

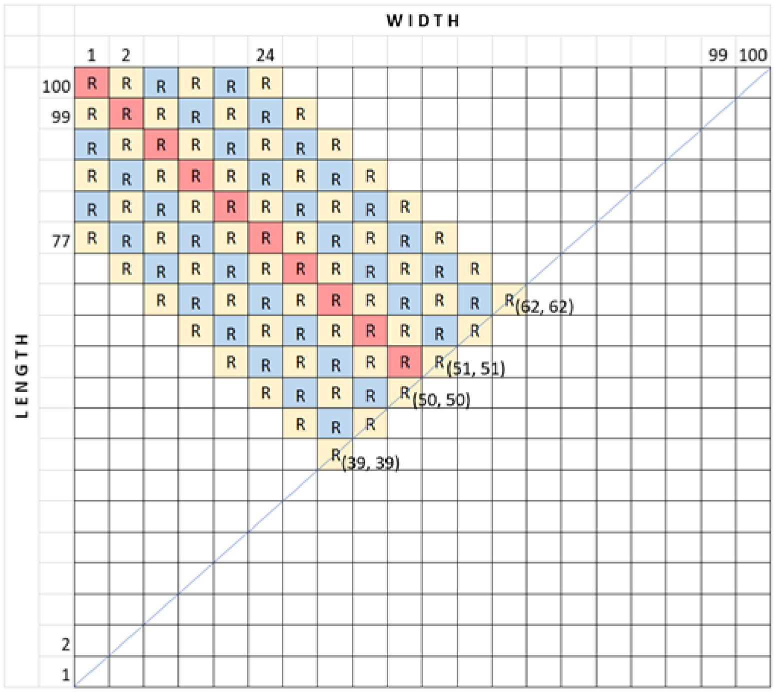


Figure 5.2

Solution of 5C. As in the solutions of problems 5A and 5B, we represent a rectangle of dimensions (x,y) by a unit square of coordinates (x,y) in the Cartesian Plane (Figure 5.2). Without loss of generality, we assume that $x \leq y$, so we need only the half of the first quadrant, above or on the main diagonal. ■

Let us place 2086 rectangles on 47 diagonals sloping down from the upper left to lower right. A rectangle can only nest in another rectangle which is above it and/or to its right in the grid; therefore, a chain of nested rectangles can contain at most one rectangle from each of these diagonals. The main diagonal D going from $(1,100)$ to $(50,51)$, depicted in red, and 23 diagonals on each side of D, carrying the colors of the Ukrainian flag, can accommodate the following number of unit squares (i.e., our distinct rectangles):
$50+2(50+249+248+\ldots+239) = 50+2(50+8811) = 2086$.

Remove one of the 'last' diagonals of length 39, for example the diagonal going from $(1,77)$ to $(39,39)$, and we will get 46 diagonals with 2086˘39 $= 2047$ rectangles on them. This set of 2047 rectangles contains at most 46 nestable rectangles. By the way, $2047 > 2022$. Thus, the required maximum is 46. ■

The number 2047 of 'accommodated' rectangles shows that we could have saved this problem for the 63rd Soifer Mathematical Olympiad in the year 2047. Bob Ewell will be 101, and I will be 99.

Paul Erdős often preferred to look at a particular case, to see what works and why. Then he posed general problems and general conjectures. We can generalize this 'mini-max' problem in the plane, and further in the Euclidean 3-space, and the Euclidean n-space.

Definition. Given positive integers n, k; for each set S of k parallelepipeds with linear dimensions not exceeding n, we calculate the maximum number of nesting parallelepipeds, and then take the minimum over all S. We define the function $R(n,k)$ as follows:

$$R(n,k) = \min \max S.$$

Problem in E^2. Find $R(n,k)$ for the Euclidean plane E^2.

Problem in E^3. Find $R(n,k)$ for the Euclidean 3-space E^3.

Problem in E^n. Find $R(n,k)$ for the Euclidean n-space E^n.

Kudos. I thank Bob Ewell for translating my doodles into nice computer-aided illustrations.

Dr. Alexander Soifer
asoifer@uccs.edu

Examples of some simple-difficult problems in the history of the Baltic Way competition

Dominik Bysiewicz

Dominik is a current student of Pure Mathematics at the Faculty of Mathematics and Computer Science of the Jagiellonian University in Kraków, Poland. He obtained his bachelor's degree in Pure Mathematics there in 2023. Since 2022, he has been the leader of the Polish Team in the Baltic Way mathematical competition. He is also actively involved in the organization of the Mathematical Olympiad and several olympiad training camps each year. Dominik enjoys teaching olympic math to younger students, especially geometry problems.

Abstract
The Baltic Way mathematical competition has been organised since 1990. Since 1992 at least 8 countries of the Baltic region (and Iceland) have taken part each year. There were also some guest appearances, like Ireland, Israel, or even South Africa. The competition usually takes place in November in tribute to the Baltic Chain demonstration in 1989, a peaceful political manifestation against Soviet Union occupation of Estonia, Latvia and Lithuania.

Introduction

The Baltic Way mathematical competition is a team competition. Each team of 5 students has 4.5 hours to solve 20 problems arranged in 4 main groups of olympic maths: Algebra, Combinatorics, Geometry and Number Theory. Because of the style of the competition, there are some problems expected to be less difficult than others. However, as all mathematicians know, sometimes simpler solutions are not necessarily the easier ones. In the following article I would like to show some simple-difficult problems from the past editions of the Baltic Way.

. . . .

Problem 1

Let's begin with the problem that was my inspiration to write this article. In the current edition one of the combinatorics problems was about cutting a triangle into smaller ones. Only one out of 10 teams solved it properly.

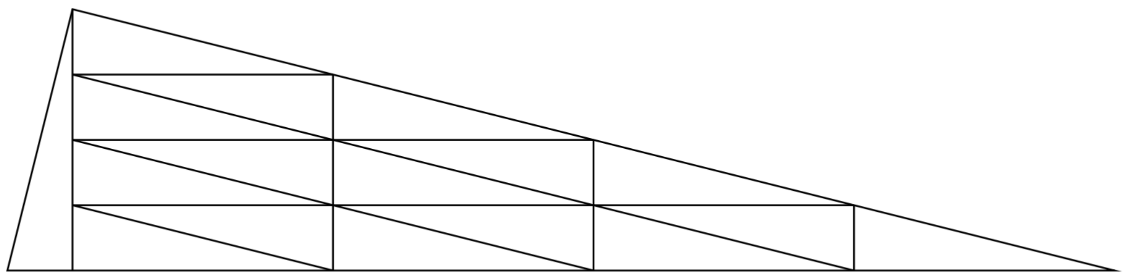
Baltic Way 2023 - problem 9 **(points: 5 out of 50)**

Determine if there exists a triangle that can be cut into 101 congruent triangles.

Solution: We will show that such a cut is indeed possible. Moreover, we will show this is possible for any $n = m^2 + 1$, where $m \in \mathbb{N}$.

Choose an arbitrary positive integer m and draw an altitude in a right-angled triangle with a ratio between sides of $1 : m$. This altitude cuts the triangle into two similar triangles with a scale factor of m. The largest of them can further be cut into m^2 smaller equal triangles by splitting all sides into m equal parts and connecting corresponding points with parallel lines. Thus a triangle can be split into $m^2 + 1$ equal triangles.

The figure shows this for $m = 4$, but in our problem we must take $m = 10$. ■



As we can see, the problem was all about determining whether the statement is true or not. The most difficult part was that the answer is "yes" and the solution is about finding such a cut. I believe that most of the students thought this problem was about negation, one of the misleading properties is that 101 is a prime number.

. . . .

Problem 2

A similar situation, but with a more difficult problem happened the year before. The following Algebra problem also begins with a decision on whether to prove or disprove the statement. Again, only one team solved it.

Baltic Way 2022 - problem 3 **(points: 5 out of 50)**

We call a two-variable polynomial $P(x,y)$ secretly one-variable, if there exist polynomials $Q(x)$ and $R(x,y)$ such that $deg(Q) \geqslant 2$ and $P(x,y) = Q(R(x,y))$ (e.g. x^2+1 and x^2y^2+1 are secretly one-variable, but $xy+1$ is not).

Prove or disprove the following statement: If $P(x,y)$ is a polynomial such that both $P(x,y)$ and $P(x,y)+1$ can be written as the product of two non-constant polynomials, then P is secretly one-variable.
Note: All polynomials are assumed to have real coefficients.

Solution: We will prove that $P(x,y) = 4x^2y^2 - (x+y)^2$ is a counterexample. First, we show that $P(x,y)$ and $P(x,y)+1$ can be written as products:

$$P(x,y) = (2xy)^2 - (x+y)^2 = (2xy - x - y)(2xy + x + y),$$

$$P(x,y)+1=(2xy-1)^2-(x-y)^2=(2xy-x+y-1)(2xy+x-y-1).$$

Now we need to prove it is not secretly one-variable. Assume that it is.

As $deg(P)=4$ and $deg(Q)\geqslant 2$, then $deg(Q)=2=deg(R)$ or $deg(Q)=4$, $deg(R)=1$.

Case 1: $deg(Q)=deg(R)=2$

If $R(x,y)$ contains x^2 or y^2, then $Q(R(x,y)$ contains x^4 or y^4, contradiction, so $R(x,y)$ has to have xy. But P has x^2 and y^2, so R has to have also x and y. Unfortunately if it has xy, x and y, then P has x^2y and xy^2, contradiction.

Case 2: $deg(Q)=4$, $deg(R)=1$

Naturally R has to be of two variables, so it contains both x and y, so $R(x,y)=ax+by+c$. Then $Q(R(x,y)$ contains x^4 and y^4, contradiction. ■

This is not a simple problem, but it is like playing darts, we hope we can find an example that ruins the statement. The one-variable condition looks horrible at first sight, but to be honest it is pretty intuitive. The whole point is about finding such a polynomial P which is a product of polynomials, $P+1$ is a product of polynomials as well and P contains clearly different components of x and y.

. . . .

Problem 3

Now we change the climate a bit. Again, an Algebra problem, however this time it's just to solve a set of equations. At first glance, it looks really difficult and discouraging. This time the problem defeated all the teams, the highest score was just 1 point.

Baltic Way 2020 - problem 5 **(points: 2 out of 50)**

Find all real numbers x,y,z such that

$$x^2y+y^2z+z^2=0,$$
$$z^3+z^2y+zy^3+x^2y=\frac{1}{4}(x^4+y^4).$$

Solution: The answer $x=y=z=0$ is obvious. We will show this is the only one possible. If $y=0$, then $z=0$ and $x=0$. If $y\neq 0$, but $z=0$, then $x=0$ and $y=0$, contradiction.

Now, we solve the first equation as a quadratic one for variables x,y,z, respectively:

$$x=\pm\frac{\sqrt{-4y^3z-4yz^2}}{2y},$$
$$y=\frac{-x^2\pm\sqrt{x^4-4z^3}}{2z},$$
$$z=\frac{-y^2\pm\sqrt{y^4-4x^2y}}{2}.$$

Thus to have any solution, the three square roots have to be real, so:

$$\begin{aligned} -4y^3z - 4yz^2 &\geqslant 0, \\ x^4 - 4z^3 &\geqslant 0, \\ y^4 - 4x^2y &\geqslant 0. \end{aligned}$$

But now, when we sum up those inequalities and order it up:

$$\frac{1}{4}(x^4 + y^4) \geqslant z^3 + z^2y + zy^3 + x^2y.$$

Thus by the second equality of the problem, all inequalities have to be equalities, so the only possibility is that all square roots are equal to 0, which gives us $x = 0$, then $y = 0$, then $z = 0$. ■

As one can notice the problem is solved in such a way that every high school student, not even interested in mathematical contests, can understand and provide a solution. The method used above is a kind of punishment for overcomplicated mathematical thinking. However it also contains a really interesting idea.

. . . .

Let us go back to problems in Combinatorics, the following one contains a construction that cuts hexagons into congruent figures. It's very similar to the first problem of this article. This time the problem was solved properly by two teams, three more gained some points.

Problem 4

Baltic Way 2017 - problem 9 **(points: 15 out of 55)**

A positive integer n is Danish if a regular hexagon can be partitioned into n congruent polygons. Prove that there are infinitely many positive integers n such that both n and $2^n + n$ are Danish.

Solution: At first we note that $n = 3k$ is Danish for any positive integer k, because a hexagon can be cut in 3 equal parallelograms as shown by Figure 1. Furthermore a hexagon can be cut into two equal trapezoids (Fig. 2) each of which can afterward be cut into 4 equal trapezoids of the same shape (Fig. 3) and so on. Therefore any number of the form $n = 2 \cdot 4^k$ is also Danish.

If we take any Danish number $n = 2 \cdot 4^k$ of the second type, then

$$2^n + n = 2^{2\cdot 4^k} + 2 \cdot 4^k \equiv 1 + 2 \equiv 0 \pmod 3$$

showing that $2^n + n$ is also a Danish number. ■

The idea here was to find two different formulas for a Danish n. One of them is simple, just divide into trios, the second one is more complicated, but allows us to use it for simple numbers n and then use an easier one for a lot more complicated numbers $2^n + n$. Thanks to that we were able to finish the solution. The difficulty here was probably the fact that in the text there was no word about what the polygons look like. Most probably, students found Fig. 1, but couldn't find Fig. 3 (or similar), which was essential to finish.

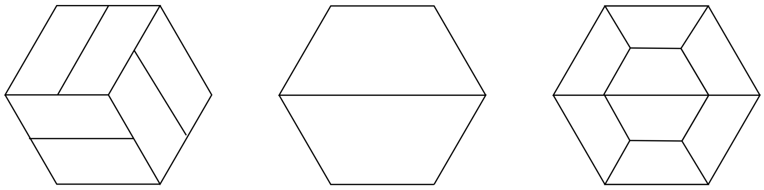


Figure 1 Figure 2 Figure 3

. . . .

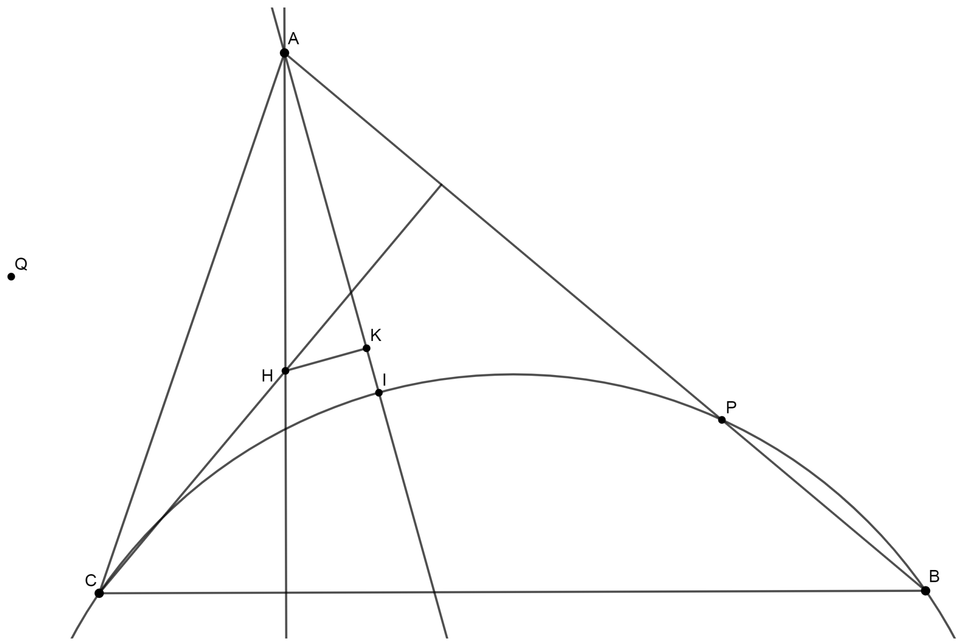
The last, but not the least, problem that we discuss here concerns geometry. It was solved by only two teams, while four other obtained nonzero points.

Problem 5

Baltic Way 2017 - problem 11 **(points: 16 out of 55)**

Let H and I be the orthocentre and incentre, respectively, of an acute angled triangle ABC. The circumcircle of the triangle BCI intersects the segment AB at the point P different from B. Let K be the projection of H onto AI and Q the reflection of P in K. Show that B, H and Q are collinear.

Solution: Let H' be the reflection of H in K. The reflection about the point K sends Q to P, and the line BH to the line through H' and orthogonal to AC. The reflection about the line AI sends P to C, and the line through H' orthogonal to AC to the line through H orthogonal to AB, but this is just BH. Since composition of the two reflections sends B, H and Q to the same line, it follows that B, H and Q are collinear. ■



As one can see, the problem above uses only basic geometric transformations which are symmetries.

This is a really nice example of the fact that one needs to understand geometry, not only use complex theorems, but also know how to improvise with the most basic tools. There are many more examples of such problems, which are difficult because of the basic nature of the tools needed to complete the proof.

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Wielkopolska Liga Matematyczna

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Abstract
The article concerns *Wielkopolska Liga Matematyczna* – a mathematics competition for secondary school students and older grades of primary school.

Introduction

Wielkopolska Liga Matematyczna (WLM) popularizes mathematics by conducting each year a correspondence mathematics competition for students from schools in Wielkopolska, as well as by organizing mathematics workshops and popular science lectures discussing issues of higher mathematics in an illustrative manner. The Faculty of Mathematics and Computer Science of Adam Mickiewicz University and the Poznań Division of the Polish Mathematical Society are involved in organizing the league. The competition involves solving problems independently at home, divided into three sets. In each set, participants have four problems to solve.

The creation of WLM took place at the end of 2009 and was a joint idea of prof. Krzysztof Pawałowski and Bartłomiej Bzdęga, then a student, now a lecturer and the chairman of WLM. Initially, it was a competition only for high school students. In 2015, on the initiative of members of the StuDMat Scientific Club, work began on launching WLM for Junior High School Students (WLMJ). The competition at this level was held for the first time in 2016. In 2020, WLM and WLMJ were merged. From now on, WLM has three age categories: junior, senior and veteran.

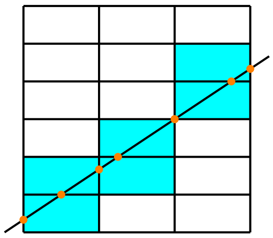
In the following sections we show selected problems from WLM. Some are just the result of trying to answer the question “what if...?”. Some have a bit of higher mathematics behind them. Others are interesting because of the numerous, significantly different ways of solving them.

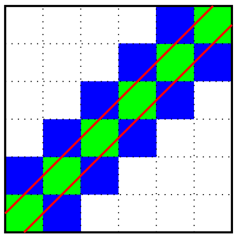
What if...?

Mathematicians ask this question frequently – what can be proven with the given assumptions? Most often this doesn't lead to anything interesting, but there are exceptions. They may become problems that are difficult for modern science to solve, or slightly easier ones that can be given to students to solve. The problems presented in this section were created this way.

Problem. Depending on the positive integer n, find the largest number c with the following property:
For each tessellation of the $2n \times 2n$ square into $2n^2$ domino pieces there is a line that intersects at least c of them.

Solution. Consider the arrangement of dominoes shown in the illustration on the left. There are $2n-1$ vertical and $n-1$ horizontal dividing lines. Each of them can be cut at most once, so in this arrangement it is impossible to intersect more than $3n-1$ dominoes.





Now we will prove that it is always possible to cut $3n-1$ dominoes, regardless of the tiling. In the illustration on the right, we see the $2n$ green squares of the diagonal and the $4n-2$ blue squares adjacent to them. To cover all the colored squares, we need $2n$ dominoes (each covering one green square – and one blue square) and $2n-2$ dominoes to cover the remaining blue squares. This means that on some side of the diagonal there will be at least $n-1$ dominoes covering only the blue squares. So one of the red lines will cross at least $3n-1$ dominoes. $\square$

Problem. Does there exist a positive integer N and a sequence $\{a_n\}$ of positive integers such that

$$a_n < a_{n+1} < a_1 + a_2 + \ldots + a_n, \qquad a_{n+1} \mid a_1 + a_2 + \ldots + a_n$$

for all $n \geq N$?

Solution. The answer is no. Suppose such a number and sequence exist. Then for each $n \geq N$ we have

$$a_1 + a_2 + \ldots + a_n = d_n a_{n+1},$$

where $d_n > 1$ is a positive integer. It follows that $d_{n+1}a_{n+2} - d_n a_{n+1} = a_{n+1}$, equivalently

$$d_{n+1}a_{n+2} = (d_n + 1)a_{n+1}.$$

Since $a_{n+2} > a_{n+1}$, it must be $d_{n+1} < d_n + 1$, i.e. $d_{n+1} \leq d_n$. It follows that for all sufficiently large n – let's say $n \geq M$ – the equality $d_n = d$ holds, where $d > 1$ is some integer constant.

We conclude that for $n > \max\{N, M\}$ the equality $a_{n+1} = \frac{d+1}{d} \cdot a_n$ holds, but a sequence of positive integers cannot have this property. $\square$

Problem. There is an infinite chessboard with n rooks on it. Let's call a rook calm if it attacks at most two other rooks. Depending on n, determine the smallest possible number of calm rooks.

Solution. Let $s(n)$ be the smallest possible number of calm rooks for n rooks placed on the board. We will show that

$$s(n) = \begin{cases} 1 & \text{for } n = 1, \\ 2 & \text{for } n = 2, \\ 3 & \text{for } n \in \{3,4\}, \\ 4 & \text{for } n \geq 5. \end{cases}$$

The cases $n = 1$, $n = 2$ and $n = 3$ are immediate because in them each rook is calm.

For $n = 4$ and $n = 5$ we will show that the number of non-calm rooks is at most 1. Assume, for the sake of contradiction, that R_1 and R_2 are two different non-calm rooks. Let us denote by $\mathscr{X}$ the set of rooks attacked by R_1, and $\mathscr{Y}$ those by R_2. If rooks R_1 and R_2 attack each other, then $\mathscr{X} \cap \mathscr{Y} = \emptyset$, so the number of rooks is at least $|\mathscr{X}| + |\mathscr{Y}| \geq 6$ – a contradiction. Otherwise $R_1, R_2 \notin \mathscr{X} \cup \mathscr{Y}$. Note that $|\mathscr{X} \cap \mathscr{Y}| \leq 2$. The number of rooks is therefore at least

$$2 + |\mathscr{X} \cup \mathscr{Y}| = 2 + |\mathscr{X}| + |\mathscr{Y}| - |\mathscr{X} \cap \mathscr{Y}| \geq 2 + 3 + 3 - 2 = 6$$

and again we have a contradiction. It follows that for $n = 4$ we have at least 3 calm rooks, and for $n = 5$ – at least 4.

For $n = 4$ we will get three calm rooks by placing the rooks on squares of coordinates $(1,1)$, $(2,1)$, $(3,1)$, $(2,2)$. For even $n = 2k > 5$ we get four calm rooks by selecting squares

$$(1,1),\ (2,1),\ (1,2),\ (2,2),\ (1,3),\ (2,3),\ \ldots,\ (1,k),\ (2,k).$$

In the case of odd $n = 2k + 1 \geq 5$ we add one rook on the square $(1, k+1)$.

It remains to show that for $n \geq 6$ there will always be at least 4 calm rooks. We showed earlier that this is true for $n = 5$ and let this be the initial condition of inductive reasoning. Consider an arbitrary arrangement of $n \geq 6$ rooks.

`Case 1.` Some calm rook W_1 attacks at most one other rook – let's call it W_2, if there is one. Let's take away rook W_1 – we then have $n - 1$ rooks, among which there are at least four calm rooks by the inductive assumption. When restoring the W_1 rook back, we add one calm rook and replace at most one calm rook with a non-calm one (this can only be the W_2 rook). It follows that in this case for n rooks we also have at least 4 calm rooks.

`Case 2.` Each calm rook attacks exactly two other rooks. Then it is impossible for all rook to be in one row. Consider the "topmost" non-empty row of the chessboard – there are at least two rooks in it, because if there was only one there, it would attack at most one other rook. The two outermost rooks in this row are calm, as one of them can only attack down and right, and the other can only attack down and left. Similarly, in the "lowest" row also the two outermost rooks are calm, which in total gives us at least 4 different calm rooks. $\square$

Higher mathematics behind

Problem. Each of the subsets of the set $\{1, 2, \ldots, n\}$ are written on one of n cards numbered from 1 to n. Prove that for some m, on the mth card there is a set containing m and a set that does not contain m.

Solution I. Let A denote the set of all those $k \in \{1,2,\ldots,n\}$ for which there is at least one set on the kth card that does not contain k. The set A is located on one of the cards, let it be the card with the number a. If $a \notin A$, then there are no sets on the a-th card that do not contain a, so there is no set A there – a contradiction. Then $a \in A$. The definition of the set A implies that there is at least one set on the a-th card that does not contain a. On the same card there is the set A, which contains a, so the proof is complete. $\square$

This proof can also be carried out for any, not necessarily finite, set X. We assign to each subset of the set X a certain element of the set X (in the problem it was the number of the card on which this subset was written). From the assertion of the problem we conclude that two subsets are assigned the same element, so there is no bijection between the set X and the set of its subsets. This is Cantor's famous theorem.

Solution II (by Kosma Kasprzak). We will prove the result by induction. Let $T(n)$ denote the hypothesis for given n. $T(1)$ is clearly true. For fixed $n > 1$ we assume that $T(n-1)$ is true. In the second inductive step we have two cases to deal with:

(1) all the subsets of $\{1,2,\ldots,n\}$ not containing n are written on the cards from 1 to $n-1$,

(2) all the subsets of $\{1,2,\ldots,n\}$ containing n are written on the cards from 1 to $n-1$,

otherwise $T(n)$ holds.

If (1) is true, then the conclusion follows by the inductive assumption. If (2) is true, then we can remove n from all sets and do the same. $\square$

Problem. Let p and q be different primes and $a > 1$ be an integer. Let

$$M = 1 + a^p + a^{2p} + \ldots + a^{(q-1)p}, \qquad N = 1 + a^q + a^{2q} + \ldots + a^{(p-1)q}.$$

Prove that $\gcd(M,N) > 1$.

Solution I. We have $M,N \mid 1 + a^{pq}$, so

$$\gcd(M,N) = \frac{MN}{\operatorname{lcm}(M,N)} > \frac{a^{(q-1)p+(p-1)q}}{1+a^{pq}} > 1,$$

completing the proof. $\square$

The second solution uses the properties of cyclotomic polynomials. By the definition, $\prod_{d|n} \Phi_d(x) = x^n - 1$. Moreover $\Phi_n(x) = \prod_\zeta (x - \zeta)$, where the product runs over all the primitive nth complex roots of unity. In particular, it follows easily from the last equality that $\Phi_n(x) > 1$ for $n \geq 2$ and $x \geq 2$.

Solution II. We have

$$M = \frac{a^{pq}-1}{a^p-1} = \frac{\Phi_{pq}(a)\Phi_p(a)\Phi_q(a)\Phi_1(a)}{\Phi_p(a)\Phi_1(a)} = \Phi_{pq}(a)\Phi_q(a)$$

and similarly $N = \Phi_{pq}(a)\Phi_p(a)$. Therefore $\gcd(M,N) \geq \Phi_{pq}(a) > 1$. $\square$

Problem. Let A be a finite set of integers. For each positive integer n, let us denote by A_n the set of numbers that can be written as the sum $a_1 + a_2 + \ldots + a_n$, in which $a_1, \ldots, a_n \in A$ are not necessarily different. Let $r_n = |A_{n+1}| - |A_n|$. Prove that the sequence $\{r_n\}$ stabilizes.

This problem is a particular case of the following theorem by Askold Khovanskii. For any Abelian group G consider $A \subseteq G$ and create the sequence of sets $\{A_n\}$, as in the problem. Then there exists a polynomial $P(n)$ such that $|A_n| = P(n)$ for all sufficiently large n.

Solution. For $|A| = 1$ the conclusion is obvious, so let $|A| > 1$. If all elements of the set A are increased by a certain constant or multiplied by a certain non-zero constant, the number of elements of the sets A_n will not change. We can therefore assume that $\min A = 0$ and $\gcd(A) = 1$.

The set of positive integers that cannot be written with the use of the sum of a number of necessarily different elements of the set A is then finite. Let us denote this set by X. The elements of the set A_{n+1} arise from pairwise addition of the elements of the sets A_n and A, in particular $A_n \subseteq A_{n+1}$, since $0 \in A$. Let $m = \max A$ and $x = \max X$. For a sufficiently large n_0 we have:

$$\{0,1,2,\ldots,x\} \setminus X \subset A_{n_0} \quad \text{and} \quad x+1, x+2, \ldots, x+m \in A_{n_0}.$$

Inductively, for natural t we have

$$\{0,1,2,\ldots,x\} \setminus X \subset A_{n_0+t} \quad \text{and} \quad x+1, x+2, \ldots, x+(t+1)m \in A_{n_0+t}.$$

For large $n = n_0 + t$ – let's say for $n > N$ – the following inequality holds:

$$x + (t+1)m > \tfrac{1}{2}(n_0 + t)m = \tfrac{1}{2}\max A_n.$$

This means that for $n > N$ the only natural numbers smaller than $\frac{1}{2}\max A_n$ and not belonging to A_n are the elements of the set X.

Now consider the set $B = \{m - a : a \in A\}$ and create a sequence of sets $\{B_n\}$ analogous to the sequence $\{A_n\}$, but for the set B. It is clear that $0 \in B$ and $\gcd(B) = 1$. Applying the same argument, we come to the conclusion that for sufficiently large n – let's say $n > N'$ – the only natural numbers smaller than $\frac{1}{2}\max B_n$ and not belonging to B_n are elements of the set Y, which we define similarly to the set X.

Since $B_n = \{mn - \alpha : \alpha \in A_n\}$, we conclude that for $n > \max\{N, N'\}$ the set A_n contains the numbers $0, 1, 2, \ldots, mn$ with the following exceptions: elements of the set X and numbers of the form $mn - \gamma$, where $\gamma \in Y$. In that case

$$|A_n| = mn - |X| - |Y| \text{ for } n > \max\{N, N'\}.$$

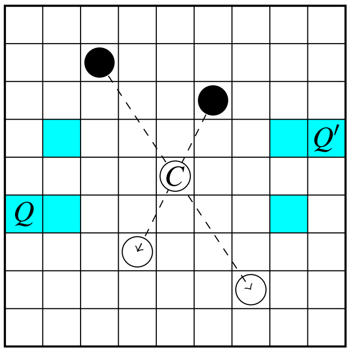
This means that for such n the equality $r_n = m$ holds, i.e. the sequence $\{r_n\}$ is constant from a certain point. $\square$

Open problems

From time to time, it happens that when composing problems, questions arise that the author of the problem (nor his colleagues) cannot answer.

Problem (proposed by Kacper Bem)**.** Two players alternately place queens on an $n \times n$ chessboard, where n is an odd integer. The queen can only be placed on a free square that is not attacked by any previously placed queen. The first player unable to make a move loses. Which player has a winning strategy?

Solution. The first player places the first queen in the middle of the board. Then, each subsequent move of the first player is symmetrical to the last move of the second player with respect to the center of the board.



After each of the first player's moves, the set of available fields is centrally symmetrical. If the squares Q and Q' are available squares symmetrical with respect to the center, then the queen placed on the square Q does not attack the square Q' – otherwise it would have to attack the queen placed in the middle of the board. Therefore, when the second player places a queen on the Q square, then the first player will be able to respond by placing another queen on the Q' square. The game has to end sometime, so the first player wins. $\square$

Open problem: For $n = 2$ and $n = 4$ the first player also wins. What is the result for even $n \geq 6$?

Problem. Find positive integer solutions of the equation $m^{m^2} = (2n^2)^{n^2}$.

Solution. The function $f(x) = x^{x^2}$ is increasing on the interval $[1, +\infty)$. The given equation is equivalent to $f(m) = f(n\sqrt{2})$, so $\frac{m}{n} = \sqrt{2}$. The equation has no positive integer solutions. $\square$

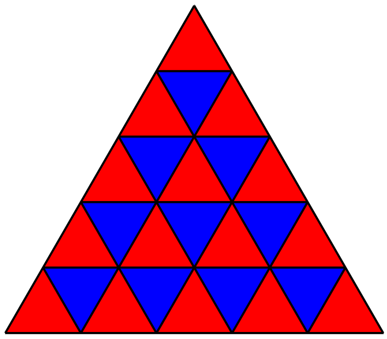
Open problem: Find a pure number-theoretical solution.

Multiple solutions

Here we present one problem that has at least three surprisingly different solutions.

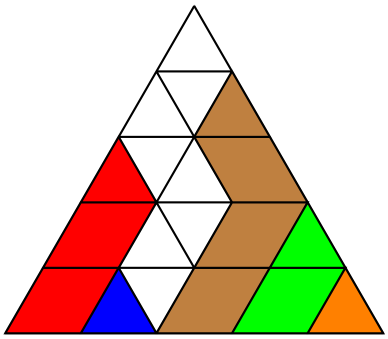
Problem. Let n be a positive integer. An equilateral triangle with side n is cut into t equilateral triangles with side 1 and a number of diamonds (rhombi with angle 60°) with side 1. Prove that $t \geq n$.

Solution I. Let's color the triangle as in the picture below.



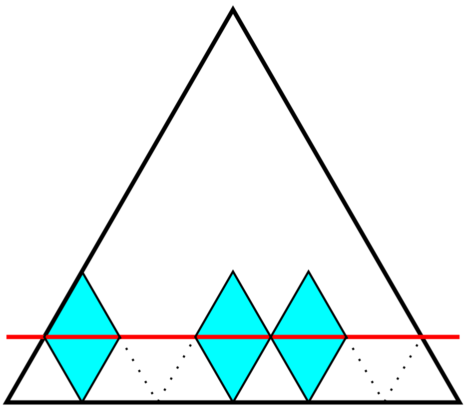
Each diamond covers two triangles of different colors. There are n more red triangles than blue ones, so $t \geq n$. $\square$

Solution II (by Klaudia Tarabasz). Let's call one of the sides of the triangle the base. Serpents (see the picture below) are figures made of diamonds arranged one above the other, starting from the base.



On top of each diamond there must be either another diamond or an equilateral triangle. Since the number of diamonds in a serpent must be finite (perhaps equal to 0), each serpent has a "head" in the shape of an equilateral triangle. The number of equilateral triangles is not less than the number of serpents, i.e. n. $\square$

Solution III (by participants). We prove the result by induction. For $n = 1$ the statement is true. Now assume that it holds for some $n \geq 1$. Let's consider an equilateral triangle with side $n+1$ and draw a red line as in the figure below.



Let's say the red line passes through exactly k diamonds. By the inductive assumption, we have at least $n-k$ equilateral triangles above the red line. There must be at least $k+1$ of them below the red line, which gives at least $n+1$ triangles altogether. $\square$

Acknowledgment

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A Problem in Area

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Trained as a physicist with a BS in 1962 at Caltech and a PhD in 1966 at UC Berkeley. After a career at Logicon in the Aerospace industry Dick has authored four books on recreational mathematics and enjoys traveling to tennis tournaments as an avid fan. His wife of 61 years and he enjoy their three children and four grandchildren.

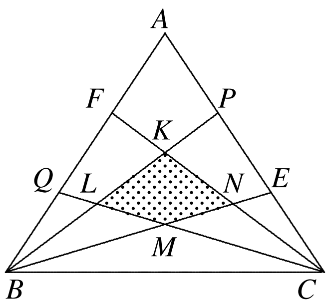
Dr.Andy Liu is a regular contributor to *Mathematics Competitions* producing the bi-annual International Mathematics Tournament of Towns article and co-authoring a wealth of other articles.

Problem
The following was proposed by Dario Uri. Two sides of a triangle are trisected. The points of trisection are joined to the opposite vertex. These four lines define a convex quadrilateral. What is the area of this quadrilateral in terms of the area of the triangle?

Since area relations are preserved by affine transformation, a possible approach is to take the vertices of the triangle to be (0,0), (1,0) and (0,1). The coordinates of the four vertices of the quadrilateral can be obtained via analytic geometry. There are standard formulas which yield the area of the quadrilateral.

The following is an alternative approah.

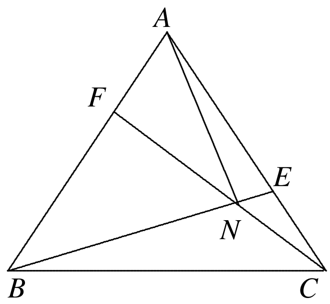
Solution:
Let the triangle be ABC. Using affine transformation if necessary, we take $AB = AC$. Let E and P be the points of trisection of AC, with E closer to C. Let F and Q be the points of trisection of AB, with F closer to A. Let K, L, M and N be the respective points of intersection of BP with CF, BP with CQ, BE with CQ and BE with CF.



We use $[T]$ to denote the area of the polygon T. Then

$$[KJMN] = [KBC] - [LBC] - [NBC] + [MBC].$$

We first compute $[NBC] = [LBC]$.



Let $[ABC] = 1$, $[FAN] = x$ and $[EAN] = y$. Since $BF = AF$, $[FBN] = 2x$ so that $[NBA] = 3x$. Since $AE = 2CE$, $[NBC] = \frac{3x}{2}$. In an analogous manner, we can prove that $[NBC] = 3y$. It follows that $x = 2y$, so that $[NBC] = \frac{2}{7}$.

By analogous arguments with simpler details, we can prove that $[KBC] = \frac{1}{2}$ and $[MBC] = \frac{1}{5}$. Hence

$$[KLMN] = \frac{1}{2} - 2 \times \frac{2}{7} + \frac{1}{5} = \frac{9}{70}.$$

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International Mathematics Tournament of the Towns

Andy Liu

In 1976, Andy Liu received a Doctor of Philosophy in mathematics and a Professional Diploma in elementary education, making him one of very few people officially qualified to teach from kindergarten to graduate school. He was heavily involved in the International Mathematical Olympiad. He served as the deputy leader of the USA team from 1981 to 1984, and as the leader of the Canadian team in 2000 and 2003. He chaired the Problem Committee in 1995, and was on the Problem Committee in 1994, 1998 and 2016. He had given lectures to school children in Canada, the United States, Colombia, Hungary, Latvia, Sweden, Tunisia, South Africa, Sri Lanka, Nepal, Thailand, Laos, Malaysia, Indonesia, the Philippines, Hong Kong, Macau, Taiwan, China and Australia. He ran a mathematics circle in Edmonton for thirty-two years, and continued his book-publishing after his retirement from the University of Alberta in 2013. He is currently writing his twentieth mathematics book, which is based on Greek Mythology.

Selected Problems from the Fall 2022 Papers

1. Is it possible to paint each integer greater than 1 in one of three colors so that two of them have different colors, and that if a and b have different colors, then the product ab is of the third color?

2. Alice knows that 7 of her 8 coins are genuine and have the same weight. The eighth coin is counterfeit, and is either heavier or lighter than a genuine coin. Bob has a balance. For a fee of one coin, he will perform a weighing for Alice behind her back. He will tell her the correct result if the coin paid is genuine, but tell her anything if the coin paid is counterfeit. How can Alice identify and keep 5 genuine coins?

3. The incircle ω of a rhombus $ABCD$ is tangent to AB at P, BC at Q and CD at R. A small circle tangent to ω is tangent to DA at E and AB at F. Another small circle tangent to ω is tangent to CD at G and DA at H. Prove that the lines PQ, QR, EF and GH determine a square.

4. Half of the integers from 1 to 2022 inclusive on the number line are painted red, while the others are painted blue. A segment of any length with a red left endpoint and a blue right endpoint is said to be of the first type. A segment of any length with a blue left endpoint and a red right endpoint is said to be of the second type. Is it possible that the total length of the segments of the first type is equal to the total length of the segments of the second type?

5. Consider the infinite sequence $\{1,19,199,1999,\dots\}$.
   (a) Is it possible to pick at least three terms such that all digits in their sum are 2s except for a single digit?
   (b) All digits of the sum of several terms are 2s except for a single digit. What can this digit be?

6. A $2n \times 2n$ chessboard is covered by $2n^2$ dominoes. A rookie is a chess piece which can only move to an adjacent square in the same row or column. A rookie visits every square exactly once on a continuous path. What is the

   (a) maximum;

   (b) minimum

   number of moves from one square of a domino to the other square of the same domino?

7. Let $n > 1$ be an integer. On the infinite plane, we paint black all lattice points both coordinates of which are multiples of n. We start from the origin and move to the first black point along a row or a column in either direction. We make the same choice at each black point, never visiting the same one more than once. After several moves, we return to the origin. Prove that the number of lattice points inside the closed path is 1 more than a multiple of n.

8. Baron Munchausen claims that he has drawn a polygon and a point inside it in such a way that any line passing through this point divides the polygon into three polygons. Could the baron be right?

9. X, Y, Z and T are points outside a regular nonagon $ABCDEFGHI$ such that

$$\angle AXB = \angle BYC = \angle CZD = \angle DTE = 20^\circ.$$

   Prove that the points X, Y, Z and T lie on the same circle if we also have

$$\angle TDE = \angle ZCD + 20^\circ = \angle YBC + 40^\circ = \angle XAB + 60^\circ.$$

10. There are several banknotes whose face values are distinct positive integers. It is known that exactly n of them are counterfeit. Any set of banknotes may be submitted to a detector, which will determine the sum of the face values of all the counterfeit banknotes submitted. Prove that all the counterfeit banknotes can be identified in n checks, if

    (a) $n = 2$;

    (b) $n = 3$.

11. The chord AD divides a circle ω into two parts, and P is a point inside the smaller part. The circle λ is concentric with ω and tangent to AD. The tangents from P to λ intersect the major arc AD of ω at B and C. and the segments BD and AC intersect at Q. Prove that PQ bisects the segment AD.

12. At most 100 straight lines are drawn, cutting the interior of a circle into pieces. Is it possible to divide the pieces into n groups with the same total area, where

    (a) $n = 201$;

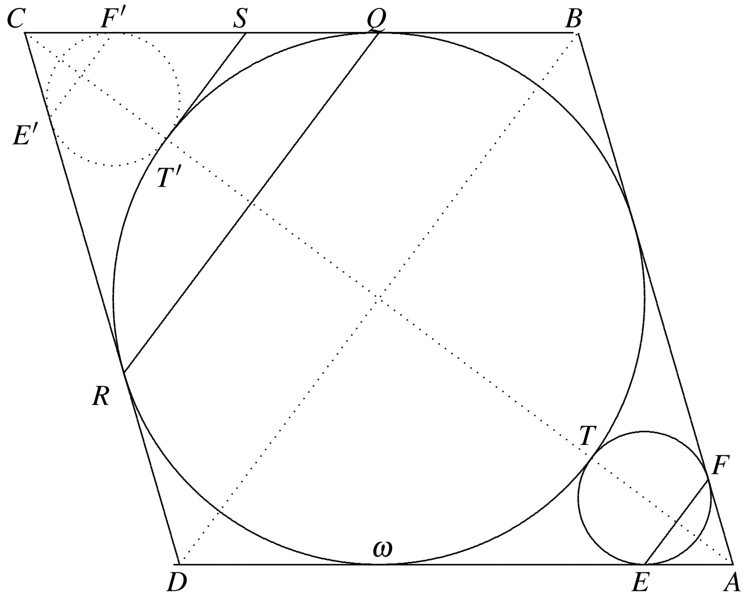
    (b) $n = 400$?

Solutions

1. We claim that the task is not possible. Assume to the contrary that it is. Let a be azure, b be black and c be crimson. Then ab is crimson and a^2b is black. Similarly, ac is black and a^2c is crimson. If a^2 is azure, then a^2b is crimson and a^2c is black. If a^2 is black, then a^2c is azure. If a^2 is crimson, then a^2b is azure. These are all contradictions.

2. Let the coins be A, B, C, D, E, F, G and H. Alice first pays H to Bob and asks him to weigh D and E against F and G. Suppose Bob tells her that they bakance. Then Alice knows that D, E, F and G are all genuine. Next she pays A to Bob and asks him to weigh C against D. If they also balance, then C is her fifth genuine coin. If they do not, then B is her fifth genuine coin. Suppose Bob tells Alice that D and E do not have the same total weight as F and G. By symmetry, we may assume that Bob says D and E are the heavier pair. Then Alice knows that A, B and C are genuine. Next she pays G to Bob and asks him to weigh B and C against D and F. If they balance, then D and F are the fourth and the fifth genuine coins. If B and C are the heavier pair, then D and E are the fourth and the fifth genuine coins. If D and F are the heavier pair, then E and F are the fourth and the fifth genuine coin.

3. Note that both QR and EF are perpendicular to AC so that they are parallel to each other. Similarly, PQ is parallel to GH so that the four lines determine a rectangle. We claim that this is actually a square. Let the small circle at A be tangent to ω at T and reflect it across BD. The points E, F and T are carried to E', F' and T' respectively, as shown in the diagram below. Let the common tangent of the reflected circle with ω at T' intersect BC at S. Then ST' is perpendicular to AC and $SQ = ST' = SF'$. Hence PQ and $E'F'$ are equidistant from T', the same as the distance from EF to T. It follows that the distance between PQ and EF is equal to the diameter of ω. Similarly, so is the distance between PQ and GH, and our claim is justified.

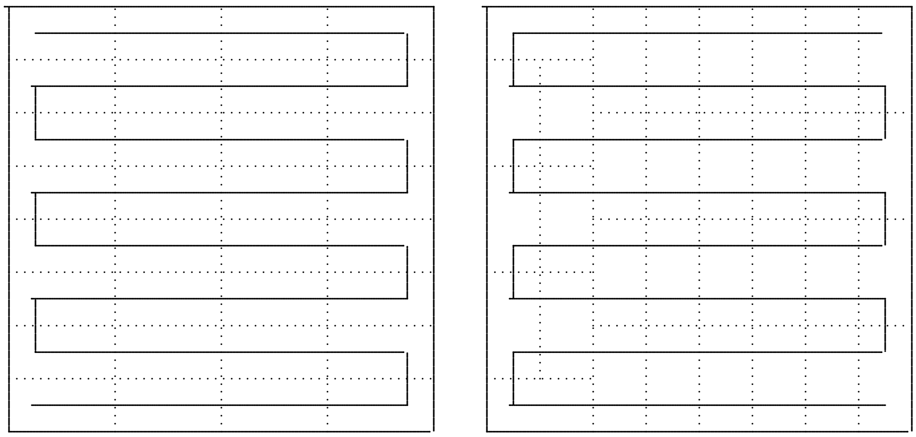


4. **Solution by Sasha Shapovalov:**
A segment of the first or the second type is called a special segment. It is any segment with endpoints of different colors. The measure of a segment $[a,b]$ is defined to be $a+b$. Since each point is an endpoint of exactly 1011 special segments, the total measure of all special segments is $1011(1+2+\cdots+2022)$, which is an odd number. The length of $[a,b]$ is $b-a$, which has the same parity as $a+b$. Hence the total length of all special segments is also odd. Since each segment has integral length, the special segments cannot be divided in any way into two types such that the total length of those of each type is the same.

5. **Solution by Sasha Shapovalov:**
   (a) We have 19+199+1999+19999+199999+1999999+19999999+199999999=222222212.
   (b) We see from (a) that the extra digit can be 1. The example 1+19=20 shows that it can also be 0. We claim that it cannot be any other digit. Since $10^n \leq 199\cdots99 < 2\times10^n$,

the sum s of k distinct terms satisfies $t \leq s < 2t$, where the digits of t consist of k copies of 1 and some copies of 0. Then s and $2t$ have the same number of digits, and at least one digit of s is smaller than the corresponding digit of $2t$. Hence this digit is less than 2.

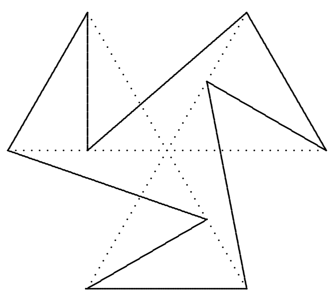
6. **Solution by Alexandr Semenov:**
   (a) The rookie can make as many as $2n^2$ same-domino moves, as illustrated in the diagram below on the left for the case $n = 4$. Clearly, $2n^2$ is maximum since this is the number of dominoes.



   (b) For $n = 1$, obviously the minimum number of same-domino moves is 1. For $n \geq 2$, it is 2, as illustrated in the diagram above on the right for the case $n = 4$. Indeed 2 is minimum. The chessboard has 4 corner squares. At most 1 can be the starting square and at most 1 can be the finishing square. In passing through either of the other 2 corner squares, the rookie must make a same-domino move.

7. We use mathematical induction on the length of the closed path. The minimum length is $4n$, and the number of lattice points enclosed is $(n-1)^2 \equiv 1$ modulo n. Consider a closed path of greater length. Then there exists a row or column containing black lattice points, and with black lattice points on both sides. This row or column intersects the closed path in at least one segment, which must have black endpoints. This segment separates the closed path into two shorter closed paths. By the induction hypothesis, the number of lattice points inside each closed path is congruent to 1 modulo n. The number of lattice points in the separating segment, excluding the endpoints, is congruent to -1 modulo n. Hence the number of lattice points inside the original closed path is congruent to $1 + 1 + (-1) = 1$ modulo n.

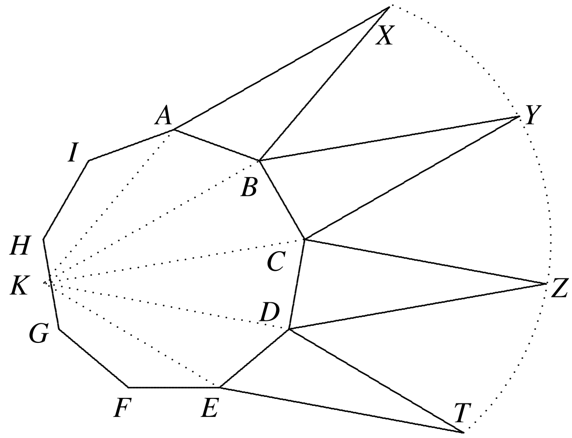
8. **Solution by Sasha Shapovalov:**
   As usual, the Baron is right. The diagram below shows a possible nonagon with three-fold rotational symmetry. The centre of symmetry is collinear with three sets of three vertices.



9. **Solution by Alexandr Semenov:**
Let K be such that $KAXB$ is a parallelogram. Then $\angle AKB = 20^\circ = \angle ACB$, so that K lies on the circumcircle of the nonagon. We have $BC = CB$, $\angle CKB = 20^\circ = \angle BYC$ and

$$\angle KCB = \angle KCA + \angle ACB = \angle KBA + 20^\circ = \angle XAB + 20^\circ = \angle YBC.$$

Hence triangles KCB and YBC are congruent, so that $KBYC$ is also a parallelogram. Similarly, so are $KCZD$ and $KDTE$. Then X, Y, Z and T all lie on the circle obtained from the incircle of the nonagon by a homothety with centre K and coefficient 2.

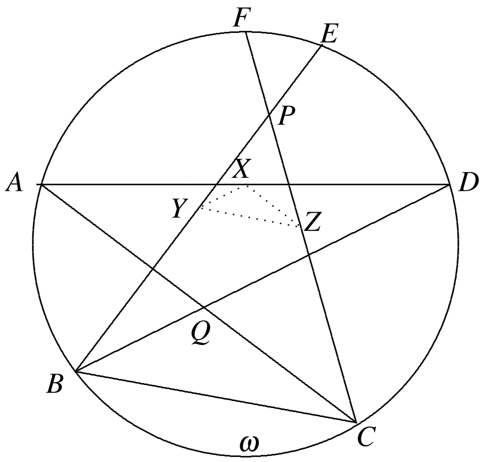


10. **Solution by Alexandr Semenov:**
   (a) In the first detection, test all the banknotes. We will know the sum of the face values of the two fakes. Distinct pairs of banknotes with this total face value must be disjoint. In the second detection, test one banknote from each pair.

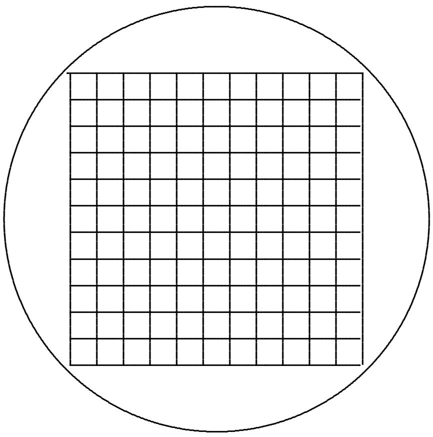
   (b) In the first detection, test all the banknotes. We will know the total t of the face values of the three fakes. A banknote is classify as high or low according to whether its face value exceeds $\frac{t}{3}$ or is at most that. If all three fakes are high, then their total face value will exceed t. If all three are low, then their total face value will at less than t. Hence the number of fakes among the high banknotes is one or two. In the second detection, test all the high banknotes. We will know the sum s of the face values of the fakes among them. If $s \leq \frac{2t}{3}$, then there is only one fake among the high banknotes and its face value is s. The two fakes among the low banknotes can be identified in the third detection as in (a). Suppose $s \geq \frac{2t}{3}$. There are two possibilities.
   First Scenario. Only one high banknote is fake.
   Its face value is s, and the sum of the face values of the two fake low banknotes is $t-s$. Hence exactly one of them has face value between 1 and $\frac{t-s}{2}$.
   Second Scenario. Only one low banknote is fake.
   Its face value is $t-s$, and the sum of the face values of the two fake high banknotes is s. Hence exactly one of them has face value between $\frac{t}{3}$ and $\frac{s}{2}$.
   At this point, we do not know in which scenario we are. In the third detection, test the low banknotes with face values between 1 and $\frac{t-s}{2}$ along with the high banknotes with face values between $\frac{t}{3}$ and $\frac{s}{2}$. Since $s \geq \frac{2t}{3}$, $\frac{t}{3} > \frac{t-s}{2}$, so that there is only one fake banknote among those tested. We can deduce the face values of all three fake banknotes.

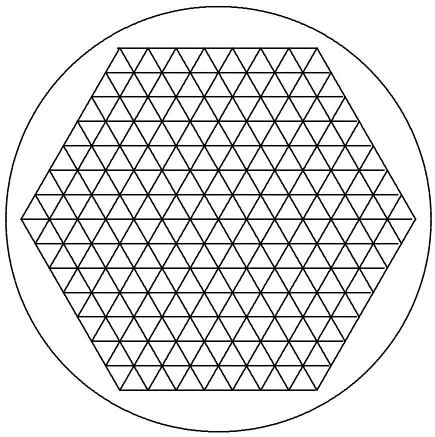
11. **Solution by Alexandr Semenov:**
Extend BP and CP to cut ω at E and F respectively. Then AD, BE and CF are chords of ω with equal lengths since they are all tangent to λ which is concentric with ω. Let their

respective midpoints be X, Y and Z. By symmetry, $PY = PZ$ and $PB = PC$. Hence YZ is parallel to BC. Similarly, ZX is parallel to CA and XY is parallel to DB. Hence triangles XYZ and QBC are similar. In fact, they are homothetic with centre P, so that P, X and Q are collinear.



12. **Solution by Sasha Shapovalov:**
    (a) The task is possible using only 24+40=64 straight lines. Take the area of the circle to be 201. Then the area of the square inscribed in it is $\frac{402}{\pi} > 121$. First draw 24 straight lines forming a centrally placed square of side length 11, divided into 121 unit squares, as shown in the diagram below on the left. Then draw 40 diameters so that adjacent diameters enclose a region of the outer ring with area 1 at each end. We will then have 121+80=201 groups of pieces with the same total area in each group.





    (b) The task is possible using 45+53=98 straight lines. Take the area of the circle to be 400. Then the area of the regular hexagon inscribed in it is $\frac{600\sqrt{3}}{\pi} > 294$. First draw 45 straight lines forming a centrally placed regular hexagon of side length 7, divided into 294 equilateral triangles each with area 1, as shown in the diagram above on the right. Then draw 53 diameters so that adjacent diameters enclose a region of the outer ring with area 1 at each end. We will then have 294+106=400 groups of pieces with the same total area in each group.

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