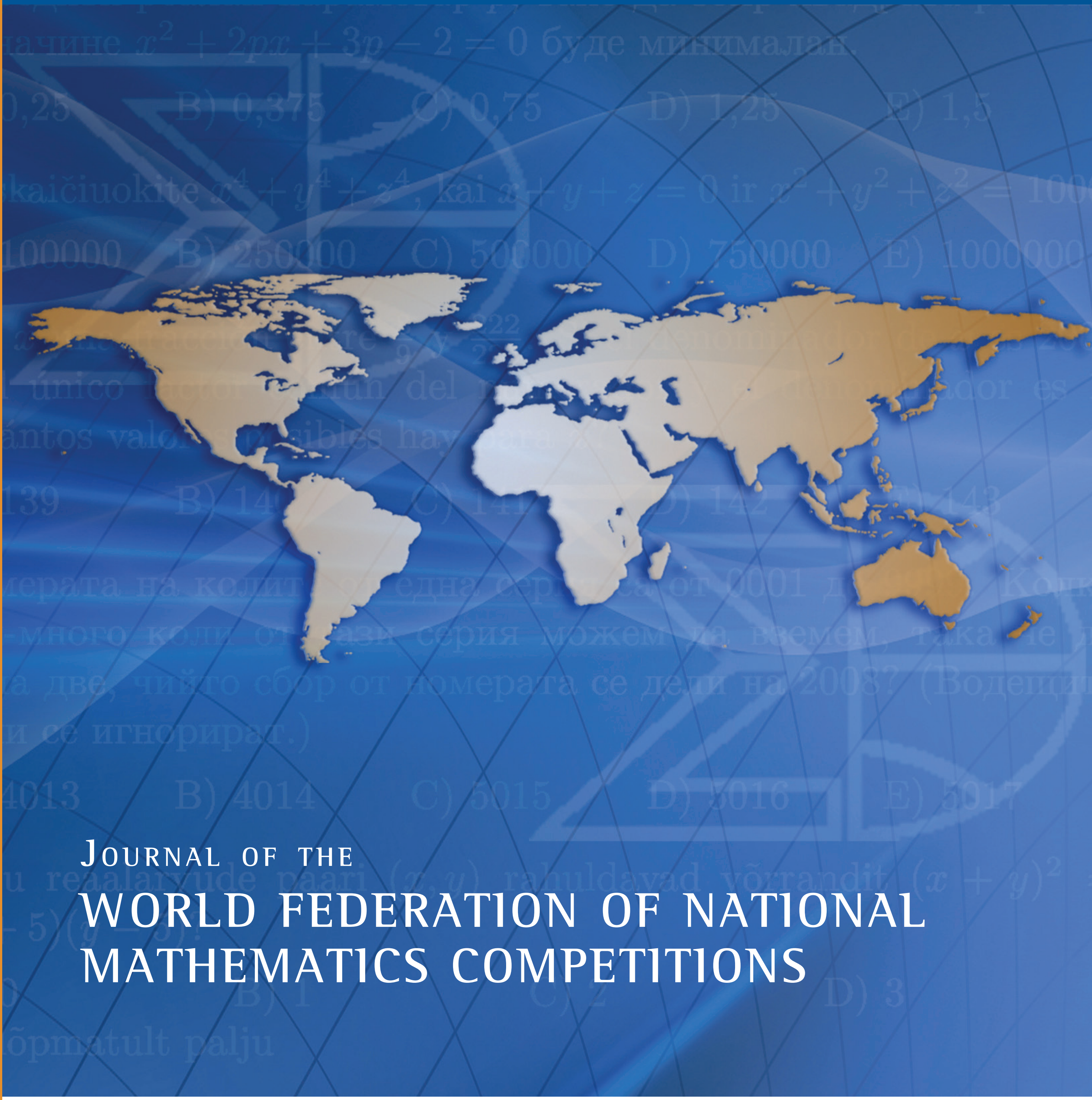
Vol 35 No 1 2022

# MATHEMATICS COMPETITIONS



Journal of the

## WORLD FEDERATION OF NATIONAL MATHEMATICS COMPETITIONS

AUSTRALIAN MATHS TRUST

Vol 35 No 1 2022

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MATHEMATICS COMPETITIONS
JOURNAL OF THE WORLD FEDERATION OF NATIONAL MATHEMATICS COMPETITIONS

(ISSN 1031-7503)
Published biannually by

AUSTRALIAN MATHEMATICS TRUST
170 HAYDON DRIVE
BRUCE ACT 2617
AUSTRALIA

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For WFNMC Standing Committees please refer to ABOUT WFNMC section of the WFNMC website http://www.wfnmc.org/.

# From the President

Dear readers of *Mathematics Competitions* journal!

It is my great pleasure to announce the recipients of the 2022 Paul Erdős Award. The Awards Committee chaired by Alexander Soifer collected and assessed the nominations. The recommended candidates were approved by the Executive Committee of WFNMC. They are (in alphabetic order):

- Géza Kós (Hungary)
- Nairi Sedrakyan (Armenia)
- Sergey Rukshin (Russia)

Congratulations to our distinguished colleagues for their outstanding achievements and meritorious national and international contributions!

We remind all that the Paul Erdős Award has been established to recognize contributions of persons who have played a significant role in the development of mathematical challenges with essential impact on mathematics learning.

The following brief description of the main contributions of our awardees (taken from the report of the Awards Committee) shows that they all completely satisfy the requirements.

Dr. Géza Kós is an Associate Professor of Eötvös University in Budapest. During his early life he participated in the International Mathematical Olympiad (IMO) from 1984–86, winning a silver medal and two gold medals. Dr. Kós has continued to be involved in the IMO. Starting in 2006, with rare exceptions, Dr. Kós has been a coordinator and an important member of the Problem Selection Committee (PSC) of IMO. Starting in 1986, Kós has been involved in the points contest of KöMaL magazine, creating the computer database for the contests and maintaining the website. In 1993, with two colleagues he created a contest in advanced mathematics problems. Kós has created hundreds of contest problems and published 28 articles related to the contest. Starting in 1991, he has also been a member of the contest committee of the József Kürschák Competition. Dr. Kós has been involved in other local competitions as well. In the Miklós Schweitzer competition he was a member of the committee in 1992, 2018 and 2020. In the Romanian Masters in Mathematics he was a coordinator in 2012. Starting in 2016, Kós has been and remains a member of the Jury of the International Olympiad of Metropolises in Moscow (IOM). At the University competitions level, Géza has been helping at the International Mathematics Competitions (IMC) since 1998, and is a key organizer involved in problem selection, preparation of solutions and results, and in some years has been a moderator of the Jury. In 2009, he was the local organizer in Budapest. In 2020 and 2021, when the IMC contest was held online, Kós arranged all the technical details, programmed the website, headed the PSC and led the Jury. He was also invited to be coordinator of the Putnam University examinations in 2020/21 and 2021/2022. During the same years when Olympiads were virtual Géza was a coordinator and a member of the PSC of the European Girls' Mathematical Olympiad (EGMO) for high school students. The same is true for the one-time 2020 Cyberspace Mathematics Competition (CMC), also organized virtually. Dr. Kós was the team leader of his own Eötvös University teams at the Vojtech Jarnik competition for university students in Ostrava most years since 1999. He was elected to the Chair of the Jury 13 times. Starting in 2009, Géza Kós has been a permanent member of the problem committee of the Competencia Iberoamericana Interuniversitaria de Matemáticas (CIIM).

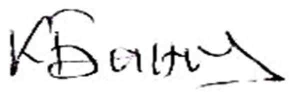
Dr. Nairi Sedrakyan is a Laureate of the highest award of the Ministry of Education and Sciences of the Republic of Armenia: Gold Medal for the Achievement in Teaching. Dr. Sedrakyan has authored 14 books and around 70 articles in different countries (USA, Switzerland, South Korea, Russia) on the topic of problem solving and Olympiad style mathematics, including ”Number Theory through Exercises”, 2019, USA; ”The Stair- Step Approach in Mathematics”, 2018, Springer, USA (550 pages); ”Algebraic Inequalities”, 2018, Springer, USA (256 pages); and ”Geometric Inequalities. Methods of Proving”, 2017, Springer, USA, (464 pages). Starting in 2016, Dr. Sedrakyan has been and remains a member of the Problem Selection Committee (PSC) and a member of the Jury of the International Olympiad of Metropolises, Moscow, Russia. Starting in 2006, Sedrakyan is a member of the problem selection committee and a jury member of International Zhautykov Olympiad, Almaty, Kazakhstan. He was a member of the International Jury and a member of the Problem Selection Committee (PSC) of the 51st International Mathematical Olympiad (IMO), Kazakhstan, 2010. For many years Dr. Sedrakyan was a leader or a deputy leader of the Armenian national team in IMO. He is the author of 11 problems included in the Shortlists of IMO’s. He is a professional coach for IMO (trained 1 Gold Medal winner, 4 Silver Medal winners, and 15 Bronze Medal winners). Dr. Sedrakyan was the President of the Republican Mathematical Olympiads of the Republic of Armenia, 2011-2013, and a Jury member during 1996–2005 and 2009-2013. He was the President and Organizer of International Mathematical Olympiad ”Tournament of Towns” in the Republic of Armenia, 1986–2013. He was the President of the Yerevan’s state Mathematical Olympiad, Republic of Armenia, 1996–2013. Mr. Sedrakyan received Gold Medal for contributions to World’s Mathematical Olympiads and Scientific Activities from the University of Riga and the Latvian Mathematical Committee.

In 2017, Professor Sergey Rukshin became a Laureate of the highest teacher’s award of Russia: People’s Teacher of the Russian Federation. He is a Professor of the Department of Mathematical Analysis of the Russian State Pedagogical University named after A. I. Herzen. Mr. Rukshin is the Scientific Director of the Physics and Mathematics Lyceum No. 239 of Saint Petersburg, one of the only few great mathematics magnet schools of Russia that regularly wins in Russian National Mathematical Olympiad and sends its students to the International Mathematics Olympiad (IMO) on the Team of Russia. At 16, in 1975 he joined the Leningrad Mathematics Center, which he leads still today. In 1981, Mr. Rukshin created the Summer Camp of the Mathematics Center, and in 1992 the Open Olympiad. He is a Member of the Public Council of the Ministry of Education and Science on the reform of the Russian Academy of Sciences. Mr. Rukshin raised generations of first class mathematicians, dozens of IMO medalists, including such world’s top celebrities and Fields Medal Laureates as Grigory Perelman and Stanislav Smirnov. Smirnov was awarded the Order of Honor of Russia. Kazakhstan had not been among the strongest teams in the IMO’s. However, when they hired Rukshin to train and lead their team for the 2001 IMO in Washington, DC, the Kazakhstan Team came in 4th in a field of ca. 90 national teams. It is very important to notice that Mr. Rukshin’s goal has always been to raise students, capable of deep thoughts and solving difficult problems. He considers Olympiad victories as a byproduct of learning mathematics. In all, Sergey Rukshin is a celebrated person in the country of 140 million people, called Russia, with a good number of published papers, book, interviews and appearances on Russian national television.

These short biographies can also be found on the WFNMC web site: http://www.wfnmc.org/awards.html

The awards will be presented at the 9-th Congress of WFNMC that will take place from the 19th to the 25th of July of 2022 in Sofia, Bulgaria.

My best regards,



Kiril Bankov
President of WFNMC
March 2022

# Editor's Page

Dear Competitions enthusiasts, readers of our *Mathematics Competitions* journal!

*Mathematics Competitions* is the right place for you to publish and read the different activities about competitions in Mathematics from around the world. For those of us who have spent a great part of our life encouraging students to enjoy mathematics and the different challenges surrounding its study and development, the journal can offer a platform to exhibit our results as well as a place to find new inspiration in the ways others have motivated young students to explore and learn mathematics through competitions. In a way, this learning from others is one of the better benefits of the competitions environment.

Following the example of previous editors, I invite you to submit to our journal *Mathematics Competitions* your creative essays on a variety of topics related to creating original problems, working with students and teachers, organizing and running mathematics competitions, historical and philosophical views on mathematics and closely related fields, and even your original literary works related to mathematics.

Just be original, creative, and inspirational. Share your ideas, problems, conjectures, and solutions with all your colleagues by publishing them here. We have formalized the submission format to establish uniformity in our journal.

Submission Format
FORMAT: should be LaTeX, TeX, or for only text articles in Microsoft Word, accompanied by another copy in pdf. However, the authors are strongly recommended to send article in TeX or LaTeX format. This is because the whole journal will be compiled in LaTex. Thus your Word document will be typeset again. Texts in Word, if sent, should mainly contain non-mathematical text and any images used should be sent separately.
ILLUSTRATIONS: must be inserted at about the correct place of the text of your submission in one of the following formats: jpeg, pdf, tiff, eps, or mp. Your illustration will not be redrawn. Resolution of your illustrations must be at least 300 dpi, or, preferably, done as vector illustrations. If a text is embedded in illustrations, use a font from the Times New Roman family in 11 pt.
START: with the title centered in Large format (roughly 14 pt), followed on the next line by the author(s)' name(s) in italic 12 pt.
MAIN TEXT: Use a font from the Times New Roman family or 12 pt in LaTex.
END: with your name-address-email and your website (if applicable).
INCLUDE: your high resolution small photo and a concise professional summary of your works and titles.
Please submit your manuscripts to María Elizabeth Losada at
`director.olimpiadas@uan.edu.co`

We are counting on receiving your contributions, informative, inspired and creative. Best wishes,

Maria Elizabeth Losada
EDITOR

# The Art of Proposing Problems in Mathematics Competitions II

*Bin Xiong and Gangsong Leng*

Bin Xiong is a professor of mathematics education in the school of mathematics sciences at East China Normal University. His research interest is in problem solving and gifted education, with an emphasis on methodology of mathematics, theory of mathematics problem solving, mathematics education, and the identification and nurturing of talented students. He has published more than 100 papers and published or edited more than 150 books, both within China and abroad. He served as the leader of the Chinese National Team for the International Mathematical Olympiad for 10 times. He is also involved in the National Junior High School Competition, the National High School Mathematics Competition, the Western China Mathematical Olympiad and the Girls Mathematical Olympiad. In 2018, Prof. Xiong was awarded the Paul Erdős Award by the World Federation of National Mathematics Competitions.

Gangsong Leng is a professor in Department of Mathematics, Founder and Leader of the Convex Geometry group at Shanghai University. His major research interests are convex geometry and integral geometry. In his more than thirty years career, more than 100 academic papers have been published in J. Differential Geom., Adv. Math., Trans. Amer. Math. Soc., Math. Z. and other academic journals. In addition, he has published more than 50 papers on mathematics competition and mathematics education as well. He has been the coach of the National Training Team, and as well a member of the Main Examination Committee of China Mathematics Olympics (CMO). Since 2013, he has been the Chair of Main Examination Committee of the China Western Mathematical Olympiad. He served as the Leader of the China National Team for the International Mathematical Olympiad (IMO) in 2007, and also served as the Deputy Leader in 2006 and 2009. His research won ICCM Best Paper Award (2017), and his achievements on Mathematical Education won him the Paul Erdős Award of the World Federation of National Mathematics Competitions in 2020.

## Introduction

This is the second part of "The Art of Proposing Problems in Mathematics Competitions". In the first part, we give some examples including: a problem originated from Tao's result, the expansion property of pedal triangles, the cardinal number of maximal independent set, finding isosceles trapezoids and Problems on convex sequences.

In this part, we give some more examples.

# The intersections of three nonempty sets

The following problem is from Romania TST in 2004 (see [1]).

**Problem 0.1.1.** *Suppose $n > 1$ is a positive integer, and $X$ is an $n$-term set. $A_1, A_2, \cdots, A_{101}$ are the subsets of $X$ such that the cardinal number of the union of any 50 sets among them is more than $\frac{50}{51}n$. Prove that there exist 3 sets in these 101 subsets such that any two of them have nonempty intersection.*

*Proof.* We consider the graph $G$ with vertices $A_1, A_2, \cdots, A_{101}$. If the intersection of any two sets is nonempty, then we draw an edge between them. This problem requires us to prove the existence of a triangle in this graph $G$.

If there does not exist a triangle in the graph $G$, then there are at least 51 vertices in this graph with degree at most 50. In fact, if the number of vertices with degree at most 50 is at most 50, there exist 51 vertices and the degree for each vertex is at least 51. Therefore, there must be two vertices with edge connecting them, say $A$ and $B$. Note that, there exist edges connecting $A$ and 50 vertices among the remaining 99 vertices, and so is $B$. Therefore, there exists a vertex $C$ with connections to $A$ and $B$. Thus, we get a triangle $ABC$. A contradiction!

Now we assume that $A_1, A_2, \cdots, A_{51}$ are points whose degrees are at most 50. Then each $A_i (i \leq 51)$ has intersections with at most 50 subsets and has no intersections with the remaining 50 subsets. This means that there exist 50 subsets such that $A_i$ is contained in the complement of the union of these 50 sets. Since the cardinal number of the union of any 50 subsets is more than $\frac{50}{51}n$, then the cardinal number of $A_i$ is less than $\frac{1}{51}n$. Hence,

$$|A_1 \cup A_2 \cup \cdots \cup A_{50}| < |A_1| + |A_2| + \cdots + |A_{50}| < \frac{50}{51}n,$$

a contradiction. This means that there must exist 3 sets whose intersection is nonempty in these 101 sets. □

The above approach is very interesting, and it only uses the fact that each $A_i$ is contained in the complement of the union of some 50 subsets. However, if we reconsider this question by counting method, we can find a stronger conclusion.

**Problem 0.1.2.** *Let $n > 1$ be a positive integer and let $X$ be a set with $n$ elements. Suppose $A_1, A_2, \cdots, A_{101}$ are subsets of $X$ and any union of 50 subsets is more than $\frac{50}{51}n$. Prove that there exist 3 subsets whose intersection is nonempty in these 101 subsets.*

*Proof.* We prove it by contradiction. If any 3 subsets of $A_1, A_2, \cdots, A_{101}$ is nonempty, then

$$\sum_{1\le i<j<k\le 101} |A_i \cap A_j \cap A_k| = 0.$$

By the inclusion-exclusion principle,

$$n \ge \left|\bigcup_{i=1}^{101} A_i\right| = \sum_{i=1}^{101} |A_i| - \sum_{1\le i<j\le 101} |A_i \cap A_j|\,. \tag{0.1.1}$$

Without loss of generality, assume that $|A_{101}|$ is the maximal. Observing

$$\begin{aligned} \sum_{i=1}^{50} |A_i| &\ge \left|\bigcup_{i=1}^{50} A_i\right| > \tfrac{50}{51}n, \\ \sum_{i=51}^{100} |A_i| &\ge \left|\bigcup_{i=51}^{100} A_i\right| > \tfrac{50}{51}n, \end{aligned}$$

one can obtain

$$\sum_{i=1}^{101} |A_i| \ge \frac{101}{100}\sum_{i=1}^{100} |A_i| > \frac{101}{100} \times \frac{100}{51} n = \frac{101}{51} n. \tag{0.1.2}$$

It follows from (0.1.1) and (0.1.2) that

$$\sum_{1\le i<j\le 101} |A_i \cap A_j| \ge \sum_{i=1}^{101} |A_i| - n > \left(\frac{101}{51} - 1\right) n = \frac{50}{51} n. \tag{0.1.3}$$

On the other hand, for any $\{k_1, k_2, \cdots, k_{50}\} \subset \{1, 2, \cdots, 101\}$, by inclusion-exclusion principle,

$$\frac{50}{51} n < \left|\bigcup_{i=1}^{50} A_{k_i}\right| = \sum_{i=1}^{50} |A_{k_i}| - \sum_{1\le i<j\le 50} \left|A_{k_i} \cap A_{k_j}\right|,$$

For any subset in $\{1, 2, \cdots, 101\}$ with 50 elements, there exist a similar inequality and the total number of this kind of inequalities is $\binom{101}{50}$. Summing over all these inequalities, we obtain

$$\frac{50}{51} n \cdot \binom{101}{50} < \binom{100}{49} \cdot \sum_{i=1}^{101} |A_i| - \binom{99}{48} \sum_{1\le i<j\le 100} |A_i \cap A_j|\,. \tag{0.1.4}$$

Since any element in $X$ belongs to at most two sets in $A_1, A_2, \cdots, A_{101}$, one has

$$\sum_{i=1}^{101} |A_i| \le 2n. \tag{0.1.5}$$

By (0.1.4) and (0.1.5), one has

$$\begin{aligned} \sum_{1\le i<j\le 100} |A_i \cap A_j| < & \ \tfrac{\binom{100}{49}}{\binom{99}{48}} \cdot 2n - \tfrac{\binom{101}{50}}{\binom{99}{48}} \cdot \tfrac{50}{51} n \\ = & \ \left(\tfrac{200}{49} - \tfrac{100\times 101}{49\times 51}\right) n \\ = & \ \tfrac{100}{49} \cdot \tfrac{1}{51} n. \end{aligned} \tag{0.1.6}$$

It follows from (0.1.3) and (0.1.6) that

$$\frac{100}{49}\cdot\frac{1}{51}n > \frac{50}{51}n,$$

i.e., $100 > 49\times 50$, which is a contradiction. $\square$

The above method is based on the inclusion-exclusion principle. For any subset in $\{1, 2, \cdots, 101\}$, using the inclusion and exclusion principle and then adding the inequalities together is actually taking the "mean value". It is a common method.

The previous problem is a new conclusion based on a new method but still closely related to the original question. In order to design a new problem based on it, it is necessary to make more changes. We first concentrate on the assumption because $\frac{50}{51}n$ may not be optimal. After several attempts, we found that for general $n$ it seems impossible to determine the sharp bound. In retrospect, could we find the maximum for some special smaller $n$? This can assess students' ability on combinational construction. After pondering, we proposed the following problem.

**Problem 0.1.3.** *Let $|X| = 16$. For any 8 subsets of $X$, if the cardinal number of the union of any 4 subsets is not less than $n$, then there must exist 3 subsets among them, whose intersection is nonempty. Find the smallest possible $n$.*

Answer: $n_{\min} = 13$.

*Proof.* Firstly, we prove that the conclusion for $n = 13$ is true. To prove it, we assume the contrary. Suppose that there exist 8 subsets of $X$ satisfying that the number of elements of the union of any 4 subsets of these sets is not less than 13 while any intersection of 3 subsets is empty. Then any 4-subset group of the 8 subsets corresponds least 13 elements in $X$. The number of such elements is at least $13\binom{8}{4}$. On the other hand, each element belongs to at most 2 subsets which means that each element is counted at most $\binom{8}{4} - \binom{6}{4}$ times. Therefore, $13\binom{8}{4} \le 16(\binom{8}{4} - \binom{6}{4})$, i.e., $16\binom{6}{4} \le 3\binom{8}{4}$, which is a contradiction.

Secondly, we will prove $n \ge 13$. If not, assume $n \le 12$ and $X = \{1, 2, \cdots, 16\}$. Let

$$\begin{aligned} A_i = &\quad \{4i-3, 4i-2, 4i-1, 4i\} \quad (i = 1, 2, 3, 4), \\ B_i = &\quad \{j, j+4, j+8, j+12\} \quad (j = 1, 2, 3, 4). \end{aligned}$$

Obviously, the intersection for any 3 subsets is empty. Moreover,

$$\begin{aligned} |A_i \cap A_j| = &\quad 0 \ \ (1 \le i < j \le 4), \\ |B_i \cap B_j| = &\quad 0 \ \ (1 \le i < j \le 4), \\ |A_i \cap B_j| = &\quad 1 \ \ (1 \le i, j \le 4), \end{aligned}$$

Therefore, for any subsets $P, Q, R, S$, if there exist 3 subsets equal to $A_i$(or equal to $B_j$ at the same time), then the number of elements of the union of these subsets is $12 \geq n$. If 2 sets are $A_i$ and the other 2 sets are $B_i$, by inclusion and exclusion principle, we have

$$|P \cup Q \cup R \cup S| = |P| + |Q| + |R| + |S| - 2 \times 2 = 16 - 4 = 12 \geq n,$$

But the intersection of any 3 subsets is empty, which is a contraction.

In conclusion, the smallest possible $n$ is 13. □

The solution of Problem 0.1.3 needs a construction process, while Problems 0.1.1 and 0.1.2 do not. It is extremely important to assess students' ability of construction.

The following problem is more difficult than Problem 0.1.3.

**Problem 0.1.4.** *Let $|X| = 30$. For any 11 subsets of $X$, if the cardinal number of the union of any 5 subsets is not less than $n$, there exist 3 subsets whose intersection is nonempty. Find the smallest possible $n$.*

The answer is 22. We do not provide a solution here.

Problems 0.1.3 and 0.1.4 are both nice. However, their proofs are still based on taking mean value directly. We hope to transform the proof into two steps: using optimization first and then taking mean value. (Actually, the way that "delete the lowest score and the highest score, then take the mean value" is used a lot, which seems to be a better way of taking the mean value.) As a result, we proposed the sixth problem of the 21th CMO in 2006 (see [4]).

**Problem 0.1.5.** *Let $|X| = 56$. For any 15 subsets of $X$, if the number of elements of the union of any 7 subsets is not less than $n$, there exist 3 subsets among the 15 subsets, whose intersection is nonempty. Find the smallest possible $n$.*

The answer is 41. If we deal with the mean value for 15 subsets, we can only show that the minimum is no more than 42. Therefore, we must use the optimization process: firstly we find the largest subset (the number of elements is at least 8) and delete it; then we take the mean value, which gives the desired result.

*Proof.* The smallest possible $n$ is 41.

We first show that $n$ can be $41$. By contradiction, we assume: There exist 15 subsets of $X$ such that the union of any 7 subsets has no less than 41 elements, but the intersection of any 3 subsets is empty. Since each element belongs to at most 2 subsets, so we can assume without loss of generality that each element belongs exactly to 2 subsets (otherwise we can add some elements in subsets such that the condition above still holds). By the pigeonhole principle, there must be a subset, say $A$, has at least $\lceil \frac{2\times 56}{15} \rceil + 1 = 8$ elements. Denote the other 14 subsets by

$A_1, A_2, \ldots, A_{14}$. Consider any 7 subsets except $A$. They will correspond to $41$ elements in $X$. All of them correspond to at least $41C_{41}^7$ elements. On the other hand, for an element $a$, if $a \notin A$, then there are 2 subsets among $A_1, A_2, \ldots, A_{14}$ containing $a$. So $a$ is counted not more than $\binom{14}{7} - \binom{12}{7}$ times. If $a \in A$, then there is one subset among $A_1, A_2, \ldots, A_{14}$ containing $a$. So $a$ is counted $\binom{14}{7} - \binom{13}{7}$ times. Hence,

$$\begin{aligned} 41C_{41}^7 &\leq (56 - |A|)(C_{14}^7 - C_{12}^7) + |A|(C_{14}^7 - C_{13}^7) \\ &= 56(C_{14}^7 - C_{12}^7) - |A|(C_{13}^7 - C_{12}^7) \\ &\leq 56(C_{14}^7 - C_{12}^7) - 8(C_{13}^7 - C_{12}^7), \end{aligned}$$

which implies $196 \leq 195$, a contradiction.

Next we prove that $n \geq 41$.

We present a counterexample to show that $n$ can not be $\leq 40$. Let $X = \{1, 2, \ldots, 56\}$, and let

$$A_i = \{i, i+7, i+14, i+21, i+28, i+35, i+42, i+49\}, \quad i = 1, 2, \ldots, 7,$$

$$B_j = \{j, j+8, j+16, j+24, j+32, j+40, j+48\}, \quad j = 1, 2, \ldots, 8.$$

Clearly,

$$\begin{aligned} &|A_i| = 8 \ \ (i = 1, 2, \ldots, 7), \\ &|A_i \cap A_j| = 0 \ \ (1 \leq i < j \leq 7), \\ &|B_j| = 7 \ \ (j = 1, 2, \ldots, 8), \\ &|B_i \cap B_j| = 0 \ \ (1 \leq i < j \leq 8), \\ &|A_i \cap B_j| = 1 \ \ (1 \leq i \leq 7, \ 1 \leq j \leq 8). \end{aligned}$$

For any 3 subsets, there must be 2 subsets belonging to $\{A_1, ..., A_7\}$ or belonging to $\{B_1, ..., B_8\}$, and thus their intersection is empty.

For any 7 subsets

$$A_{i_1}, A_{i_2}, \ldots, A_{i_s}, B_{j_1}, B_{j_2}, \ldots, B_{j_t}, \quad (s + t = 7),$$

we have

$$\begin{aligned} &|A_{i_1} \cup A_{i_2} \cup \ldots \cup A_{i_s} \cup B_{j_1} \cup B_{j_2} \cup \ldots \cup B_{j_t}| \\ &= |A_{i_1}| + |A_{i_2}| + \ldots + |A_{i_s}| + |B_{j_1}| + |B_{j_2}| + \ldots + |B_{j_t}| - st \\ &= 8s + 7t - st = 8s + 7(7 - s) - s(7 - s) \end{aligned}$$

$$= (s-3)^2 + 40 \geq 40.$$

while the intersection of any 3 subsets is empty. Thus, the minimum $n$ is not less than $41$.

In conclusion, the smallest possible $n$ is exactly $41$. □

Among the total 150 participants, only 10 of them completely solved this problem. It is a difficult problem, which was voted to be the best problem in the selection activities (voted by participants) sponsored by the Jiuzhang book store in Taiwan in that year.

# A counting problem for sequences

**Problem 0.1.6.** *(1998, Bulgaria; 2010, Hongkong) Let $n$ be a given positive integer. How many sequences of $a_1, a_2, \ldots, a_{2n}$ with $a_i = 1$ or $-1$ such that for any $1 \leq k \leq m \leq n, k, m \in \mathbb{N}^*$,*

$$\left| \sum_{i=2k-1}^{2m} a_i \right| \leq 2 \text{ ?}$$

Question: Could we propose a similar counting problem for sequences of $\pm 1$ with length $2n+1$?

Finally, we modify the assumptions of Problem 0.1.6, and consider a more difficult "dual counting problem":

**Problem 0.1.7.** *Find the number of the different sequences with terms $\pm 1$ and sequence length $2n+1$ such that the absolute value of the sum of any odd successive terms is no greater than 1.*

*Sketch of Proof.* Denote $S_j$ by the sum of preceding $j$ terms. Let $S_0 = 0$. We only consider the case $S_1 = 1$, and the case $S_1 = -1$ is similar. It suffices to find the number of sequences $(S_0, S_1, \ldots, S_{2n+1})$ with $S_0 = 0$, $S_1 = 1$, $S_{j+1} = S_j \pm 1, j \geq 1$ such that, for any $0 \leq k, m \leq n$,

$$|S_{2k} - S_{2m+1}| \leq 1. \tag{0.1.7}$$

Taking $m = 0$ in (0.1.7) yields $S_{2k} = 0$ or 2, and taking $k = 0$ in (0.1.7) yields $S_{2m+1} = -1$ or 1.

There are two cases:

(1) The number of sequences with $S_{2t+1} = 1$ for each $1 \leq t \leq n$ is $2^n$.

(2) The number of sequences with $S_{2t+1} = -1$ for some $t(1 \leq t \leq n)$ is $2^n - 1$.

Then the number of sequences with $S_1 = 1$ satisfying given assumptions is $2^n + 2^n - 1 = 2^{n+1} - 1$.

Thus, the number of sequences satisfying given conditions is $2(2^{n+1} - 1) = 2^{n+2} - 2$. □

# Inequalities for complex numbers from unit modulus

In this section, we will show an example of transforming a math Olympic problem to a research problem. It gives us some enlightenments that how to think, pose and find new problems by mathematical thinking.

**Problem 0.1.8.** *(IMO, 2003) Let $n$ be a positive integer and let $x_1 \leq x_2 \leq \cdots \leq x_n$ be real numbers. Then*

$$\Big(\sum_{i=1}^{n}\sum_{j=1}^{n}|x_i - x_j|\Big)^2 \leq \frac{2(n^2-1)}{3}\sum_{i=1}^{n}\sum_{j=1}^{n}(x_i - x_j)^2.$$

Later, the reverse version of this inequality appeared in NSMath http://www.nsmath.cn/.

**Problem 0.1.9.** *Let $x_1, x_2, \cdots, x_n$ be real numbers. Then*

$$\Big(\sum_{1\leq i<j\leq n}|x_i - x_j|\Big)^2 \geq (n-1)\sum_{1\leq i<j\leq n}(x_i - x_j)^2.$$

Note that the above problem is translation invariant (i.e., invariant under the transformation $x_i \to x_i + t$). Based on translation invariance, it is equivalent to the following:

**Problem 0.1.10.** *Let $x_1 \leq x_2 \leq \cdots \leq x_n$ be real numbers with $\sum_{k=1}^{n} x_k = 0$. Then*

$$\Big(\sum_{k=1}^{n} kx_k\Big)^2 \geq \frac{n(n-1)}{4}\sum_{k=3}^{n} x_k^2.$$

Does it hold for complex numbers? We conjectured that the complex version of Problem 0.1.9 is still true.

**Problem 0.1.11.** *Assume that $z_1, z_2, \cdots, z_n \in \mathbb{C}$. Then*

$$(\sum_{1\leq k<j\leq n}|z_k - z_j|)^2 \geq (n-1)\sum_{1\leq k<j\leq n}(z_k - z_j)^2.$$

Shiquan Li, a student in Yali Middle School, proved our conjecture (see [2]).

If we restrict the complex number in Problem 0.1.11 on the unit circle, then we obtain a weaker inequality. However, we can improve it to the following version.

**Problem 0.1.12.** *Let $z_1, z_2, \cdots, z_n$ be $n$ complex numbers on the unit circle with $\sum_{k=1}^{n} z_k = 0$. Then*

$$\sum_{1\leq k<j\leq n}|z_k - z_j| \geq \frac{n^2}{2}.$$

*Proof.*

$$\sum_{1\le k<j\le n}|z_k - z_j| = \frac{1}{2}\sum_{k=1}^{n}\sum_{j=1}^{n}|z_k - z_j| \ge \frac{1}{2}\sum_{k=1}^{n}|\sum_{j=1}^{n}(z_k - z_j)|$$
$$= \frac{1}{2}\sum_{k=1}^{n} n|z_k| = \frac{n^2}{2}.$$

□

Is the bound in Problem 0.1.12 optimal?

**Problem 0.1.13.** *Let $z_1, z_2, \cdots, z_n$ be $n$ complex numbers on the unit circle with $\sum_{k=1}^{n} z_k = 0$. Denote*

$$S := \sum_{1\le k<j\le n}|z_k - z_j|.$$

*(1) If $n$ is even, the minimum of $S$ is $\frac{n^2}{2}$.*

$(2)^*$ *If $n$ is odd, find the minimum of $S$.*

Note that $(2)^*$ is not completely solved. Yunhao Fu tested the distribution of the minimum point and found a weaker lower bound. Xiaosheng Mu solved the case for $n = 5$ completely.

The following is the reverse version of Problem 0.1.13.

**Problem 0.1.14.** *Let $z_1, z_2, \cdots, z_n$ be $n$ complex numbers on the unit circle. Find the maximum of*

$$\sum_{1\le k<j\le n}|z_k - z_j|.$$

The maximum is $n\cot\frac{\pi}{2n}$. When $n$ points form a regular polygon, the sum attains its maximum.

Problem 0.1.14 is the 2-dimensional version of the following well-known Thompson problem.

**Thompson's problem** *Let $x_1, x_2, \cdots, x_n$ be $n$ points in the unit sphere in $\mathbb{R}^m$. Find the maximum of*

$$\sum_{1\le k<j\le n}|x_k - x_j|.$$

When $m = 3$, the problem is the seventh problem in "Mathematical Problems in 21st Century" proposed by Smale, Fields awards owner, in 1998.

Based on our problems, we proposed the following *reverse Thompson's problem*. Actually Problems 0.1.12 and 0.1.13 are the special cases of this problem in dimension 2.

**Problem 0.1.15.** *Let $x_1, x_2, \cdots, x_n$ be $n$ points on the unit ball in $\mathbb{R}^m$ with $\sum_{k=1}^{n} x_k = 0$ (the centroid is the origin). Find the minimum of*

$$\sum_{1\le k<j\le n}|x_k - x_j|.$$

It seems to be a difficult and new question!

Now it is clear for us that the background of Problem 0.1.8 (problem of IMO) is actually a 1-dimensional (real number) Thompson's problem.

# Problems of short vectors

**Problem 0.1.16.** *(ShahAli, AMM 2010) If a vector $v$ in $\mathbb{R}^n$ satisfies $\|v\| \leq 1$, then we call it a short vector. Let $v_1, v_2, \ldots, v_6$ be 6 short vectors in the plane such that their sum is zero. Prove that there exists three of them satisfying their sum is still a short vector.*

In 2014, Xiaosheng Mu proposed the following generalization of Problem 0.1.16.

**Problem 0.1.17.** *Suppose that the sum of $2n$ vectors in the plane is zero. Prove that there are $n$ vectors among them such that their sum is a short vector.*

It seems that the above two problems are not appropriate for Mathematical contest. Thus, we consider their special cases.

*(1) The case of 1-dimensional real numbers*

**Problem 0.1.18.** *Let $x_1, x_2, \ldots, x_6$ be six real numbers in $[-1, 1]$ such that their sum is zero. How many triples $(i, j, k)$, $1 \leq i < j < k \leq 6$, at least are there, such that $x_i + x_j + x_k \in [-1, 1]$?*

The answer is 12. It is a problem of medium difficulty.

*(2) Problems of short vectors of cubes*

Notice that when $v$ is a short vector, it means that $v$ belongs to an unit disk. If we replace the unit disk by unit cube (the unit ball of the normed space $l_\infty^2$), then we have

**Problem 0.1.19.** *(2017, Summer Olympic Test of NSMath) Let*

$$A = \{z = x + yi \mid |x| \leq 1, |y| \leq 1, x, y \in \mathbb{R}\},$$

*and let $z_1, z_2, \cdots, z_6 \in A$ with $\sum_{i=1}^{6} z_i = 0$. Prove that there exist $1 \leq i < j < k \leq 6$ such that $z_i + z_j + z_k \in A$.*

The common solution is viewing Problem 0.1.18 as a lemma, and then utilizing the counting method. Another elegant proof is to assume the contrary and to analyze the position of the sum of triples.

Next, we introduce these two solutions respectively.

*Proof 1.* The following lemma is needed.

**Lemma** *Let* $x_1, \ldots, x_6 \in [-1,1]$ *be real numbers such that their sum is* $0$*, then there exist* $12$ *triples* $(x_i, x_j, x_k)$ *of* $x_1, \ldots, x_6$ *satisfying that the sum of each triple belongs to* $[-1,1]$.

*Proof of Lemma.* We can assume that the number of nonnegative numbers among $x_1, \ldots, x_6$ is no less than the number of negative ones. Otherwise, we can replace $x_1, \ldots, x_6$ by $-x_1, \ldots, -x_6$. There are following $4$ cases:

*Case 1.* $\{x_1, \ldots, x_6\}$ has $6$ nonnegative numbers. In this case, $x_1 = \cdots = x_6 = 0$. Thus, the sum of any triples of $x_1, \ldots, x_6$ belongs to $[-1,1]$. There are $\binom{6}{3} = 20 > 12$ triples meeting the requirements of the lemma.

*Case 2.* $\{x_1, \ldots, x_6\}$ has exactly $5$ nonnegative numbers. We can assume that the six numbers satisfy $x_1 < 0 \le x_2 \le \cdots \le x_6$. $\forall\, x_i, x_j, x_k \in \{x_1, \cdots, x_6\}$, if $x_1 \notin \{x_i, x_j, x_k\}$, then

$$0 \le x_i + x_j + x_k \le x_2 + \cdots + x_6 = -x_1 \le 1;$$

If $x_1 \in \{x_i, x_j, x_k\}$, then

$$-1 \le x_i + x_j + x_k \le x_1 + \cdots + x_6 = 0.$$

Thus, there are $\binom{6}{3} = 20 > 12$ triples meeting the requirements of the lemma.

*Case 3.* $\{x_1, \ldots, x_6\}$ has exactly $4$ nonnegative numbers. We can assume that the six numbers satisfy $x_1 \le x_2 < 0 \le x_3 \le \cdots \le x_6$. $\forall x_i, x_j \in \{x_3, \ldots, x_6\}$, we have

$$-1 \le x_1 + x_i + x_j \le \frac{x_1 + x_2}{2} + x_i + x_j = -\sum_{t=3}^{6} \frac{x_t}{2} + x_i + x_j \le \frac{x_i + x_j}{2} \le 1.$$

Thus $x_1 + x_i + x_j \in [-1,1]$ and

$$x_2 + x_i + x_j = -(x_1 + \sum_{\substack{3 \le t \le 6,\\ t \ne i,j}} x_t) \in [-1,1].$$

So there are $\binom{4}{2} \times 2 = 12$ triples meeting the requirements of the lemma.

*Case 4.* $\{x_1, \ldots, x_6\}$ has exactly $3$ nonnegative numbers. We can assume that the six numbers satisfy $x_1 \le x_2 \le x_3 < 0 \le x_4 \le x_5 \le x_6$.

Assume that $|x_3| \le |x_4|$. Otherwise, replace $x_1, \cdots, x_6$ by $-x_1, \cdots, -x_6$. Note that

$$-1 \le x_2 + x_5 + x_6 = -x_1 - x_3 - x_4 \le 1 - x_3 + x_3 = 1.$$

i.e. $x_2+x_5+x_6 \in [-1,1]$. Thus, for any $x_i, x_j$ from $x_4, x_5, x_6$ and any $x_k$ from $x_1, x_2$, we have

$$-1 \le x_i + x_j + x_k \le x_5 + x_6 + x_2 \le 1,$$

i.e. $x_i + x_j + x_k \in [-1,1]$. For any $x_i$ in $\{x_4, x_5, x_6\}$ and $x_j$ in $\{x_1, x_2\}$, we have

$$x_i + x_j + x_3 = -((x_4+x_5+x_6-x_i)+(x_1+x_2-x_j)) \in [-1,1].$$

Therefore, there are $\binom{3}{2}\times\binom{2}{1}+\binom{3}{1}\times\binom{2}{1}=12$ triples meeting the requirements of the lemma.

In conclusion, there are always 12 triples meeting the requirements of the lemma. When $x_1 = x_2 = -1, x_3 = x_4 = x_5 = x_6 = \frac{1}{2}$, there are exactly 12 triples meeting the requirements of the lemma. So 12 is optimal.

Back to the original problem. Let $z_k = a_k + ib_k$ $(i = 1, \cdots, 6)$. By the assumptions of the problem, we have $a_k \in [-1,1]$, $\sum\limits_{k=1}^{6} a_k = 0$.

Let $A_1 = \{(j,k,l) \mid 1 \le j < k < l \le 6, a_j + a_k + a_l \in [-1,1]\}$. By the lemma, $|A_1| \ge 12$.

Similarly, letting

$$B_1 = \{(j,k,l) \mid 1 \le j < k \le l \le 6, b_j + b_k + b_l \in [-1,1]\},$$

we also have $|B_1| \ge 12$.

$A_1, B_1$ are the subsets of $S = \{(j,k,l) \mid 1 \le j < k < l \le 6\}$. Since $|S| = C_6^3 = 20 < |A_1| + |B_1|$, $A_1$ and $B_1$ have a nonempty intersection. That is, $\exists\ 1 \le j < k < l \le 6$ such that both $a_j + a_k + a_l$ and $b_j + b_k + b_l$ belong to $[-1,1]$. Therefore $z_j + z_k + z_l \in A$. $\square$

*Proof 2.* Suppose the conclusion is not true. That is, $\forall 1 \le i < j < k \le 6$, the point $z_i + z_j + z_k$ does not belong to the cube $A$.

Let

$$H_1 = \{x + iy \mid y < -1, x, y \in \mathbb{R}\},$$
$$H_2 = \{x + iy \mid y > 1, x, y \in \mathbb{R}\},$$
$$H_3 = \{x + iy \mid x < -1, x, y \in \mathbb{R}\},$$
$$H_4 = \{x + iy \mid x > 1, x, y \in \mathbb{R}\}.$$

We first prove the following lemma.

**Lemma**: *Let* $1 \le i < j \le 6$. *For any* $k_1, k_2$ *satisfying* $1 \le k_1 < k_2 \le 6$, $k_1 \ne i,j$ *and* $k_2 \ne i,j$, *the following cases will not appear.*

*(i)* $z_i + z_j + z_{k_1}$ *belongs to one of* $H_1, H_2$ *and* $z_i + z_j + z_{k_2}$ *belongs to the other;*

*(ii)* $z_i + z_j + z_{k_1}$ *belongs to one of* $H_3, H_4$ *and* $z_i + z_j + z_{k_2}$ *belongs to the other.*

*Proof of Lemma.* If this lemma is not true, we have $|\text{Re}(z_{k_1} - z_{k_2})| > 2$ or $|\text{Im}(z_{k_1} - z_{k_2})| > 2$, which contradicts the definition of $A$.

By the lemma, there are two of $z_1 + z_2 + z_3, z_1 + z_2 + z_4, z_1 + z_2 + z_5$ belonging to the same $H_l$. Without loss of generality, we assume that $z_1 + z_2 + z_3, z_1 + z_2 + z_4 \in H_1$. Combining with $\sum\limits_{i=1}^{6} z_i = 0$, we conclude that $z_4 + z_5 + z_6 \in H_2$.
It follows from the facts $z_1 + z_2 + z_4 \in H_1$, $z_4 + z_5 + z_6 \in H_2$ and the lemma that $z_1 + z_4 + z_5, z_1 + z_4 + z_6, z_2 + z_4 + z_5, z_2 + z_4 + z_6$ belong to neither $H_1$ nor $H_2$. Thus, these four numbers belong to $H_3$ or $H_4$. By the lemma, they must all belong to one of $H_3$ and $H_4$. Without loss of generality, we can assume that they all belong to $H_3$. By the assumption $\sum_{j=1}^{6} z_j = 0$, we conclude that $z_2 + z_3 + z_6 \in H_4$. By the lemma again, $z_2 + z_4 + z_6 \in H_4$. A contradiction. □

A stronger version of Problem 0.1.19 is

**Problem 0.1.20.** *Let*

$$A = \{z = x + yi \mid |x| \le 1, |y| \le 1, x, y \in \mathbb{R}\},$$

*and let* $z_1, z_2, \cdots, z_6 \in A$ *with* $\sum\limits_{i=1}^{6} z_i = 0$*. How many triples* $(i, j, k)$*,* $1 \le i < j < k \le 6$*, at least are there, such that* $x_i + x_j + x_k \in A$*?*

The answer is 6. It seems a difficult question.

# Permutation Problems

First, we look at a simple example.

**Problem 0.1.21.** *Do there exist 4 permutations* $a_1, a_2, \cdots, a_{50}$*;* $b_1, b_2, \cdots, b_{50}$*;* $c_1, c_2, \cdots, c_{50}$*;* $d_1, d_2, \cdots, d_{50}$ *of* $1, 2, \cdots, 50$*, such that*

$$\sum_{i=1}^{50} a_i b_i = 2\sum_{i=1}^{50} c_i d_i \ ?$$

*Sketch of Proof.* The answer is negative. In fact, let $S = \sum\limits_{i=1}^{50} i x_i$. Then

$$S_{\max} = \sum_{i=1}^{50} i^2 = 42925, \quad S_{\min} = \sum_{i=1}^{50} i(51 - i) = 22100.$$

Since $S_{\max} < 2S_{\min}$, it is impossible for equality. □

This problem can be extended trivially to general $n$. However, as a strengthening version of Problem 0.1.21, the following problem is not trivial.

**Problem 0.1.22.** *Let $x = \{x_1, x_2, \cdots, x_n\}$ be a permutation of $1, 2, \cdots, n$. Denote $f(x) = x_1 + 2x_2 + \cdots + nx_n$. Can the value $f(x)$ be any integer in the interval between $\sum_{i=1}^{n} i(n+1-i)$ and $\sum_{i=1}^{n} i^2$?*

In 2009, we studied this question. We first considered the case $n = 3$: Let $\{x_1, x_2, x_3\}$ be a permutation of $1, 2, 3$. Then the range of $f(x) = x_1 + 2x_2 + 3x_3$ is $\{10, 11, 13, 14\}$, without 12. The number 12 is a "discontinuous point"!

It inspired us to think: For what kind of $n$, the range of $f(x)$ has discontinuous points? For what kind of $n$, the range of $f(x)$ is a set of successive positive integers?

By case studyp, we found that when $n \geq 4$ the range of $f(x)$ does not have discontinuous points.

Therefore, we proposed the following problem, which appeared in the Chinese Southeast Mathematical Olympiad in 2009 (see [5]).

**Problem 0.1.23.** *Let $n \geq 4$ be a positive integer and denote by $A$ all the permutations of $1, 2, \cdots, n$. For any $x_n = (a_1, a_2, \cdots, a_n) \in A$, let $f(x_n) = a_1 + 2a_2 + \cdots + na_n$. Determine the cardinal number of the set $\{f(x_n) \mid x_n \in A\}$.*

*Proof.* The solution is $|M_n| = \frac{n^3-n+6}{6}$.

We will prove

$$M_n = \left\{ \frac{n(n+1)(n+2)}{6}, \frac{n(n+1)(n+2)}{6} + 1, \cdots, \frac{n(n+1)(2n+1)}{6} \right\},$$

by induction.

When $n = 4$, by the rearrangement inequality, the minimal element and maximal element of the set $M$ is

$$f(\{4,3,2,1\}) = 20 \text{ and } f(\{1,2,3,4\}) = 30$$

respectively.

Together with

$$f(\{3,4,2,1\}) = 21, \ f(\{3,4,1,2\}) = 22,$$

$$f(\{4,2,1,3\}) = 23, \ f(\{3,2,4,1\}) = 24,$$

$$f(\{2,4,1,3\}) = 25, \ f(\{1,4,3,2\}) = 26,$$

$$f(\{1,4,2,3\}) = 27,\ f(\{2,1,4,3\}) = 28,$$
$$f(\{1,2,4,3\}) = 29,$$

we have that $M_4 = \{20, 21, \cdots, 30\}$ has $\frac{4^3-4+6}{6} = 11$ elements. Thus, the conclusion is true when $n = 4$.

Assume that the conclusion holds for $n-1 (n \geq 5)$. Next we prove it for $n$. Consider a permutation $X_{n-1} = \{x_1, x_2, \cdots, x_{n-1}\}$ of $1, 2, \cdots, n-1$.

First, let $x_n = n$. We get a permutation $x_1, x_2, \cdots, x_{n-1}, n$ of $1, 2, \cdots, n$. In this case,

$$\sum_{k=1}^{n} kx_k = n^2 + \sum_{k=1}^{n-1} kx_k.$$

By induction hypothesis, $\sum_{k=1}^{n} kx_k$ can attain any positive integer in the interval of

$$\left[\frac{n(n^2+5)}{6}, \frac{n(n+1)(2n+1)}{6}\right].$$

Anther way is to take $x_n = 1$. We have

$$\begin{aligned}\sum_{k=1}^{n} kx_k &= n + \sum_{k=1}^{n-1} kx_k \\ &= \frac{n(n+1)}{2} + \sum_{k=1}^{n-1} k(x_k - 1).\end{aligned}$$

By induction hypothesis, $\sum_{k=1}^{n} kx_k$ can attain any positive integer in the interval of

$$\left[\frac{n(n+1)(n+2)}{6}, \frac{2n(n^2+2)}{6}\right].$$

Noticing that

$$\frac{2n(n^2+2)}{6} \geq \frac{n(n^2+5)}{6},$$

we have that $\sum_{k=1}^{n} kx_k$ can attain any positive integer in the interval of

$$\left[\frac{n(n+1)(n+2)}{6}, \frac{n(n+1)(2n+1)}{6}\right].$$

Therefore, the conclusion holds for $n$, and our proof is completed by induction.

Hence, the cardinal number of $M_n$ is

$$\frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)(n+2)}{6} + 1 = \frac{n^3 - n + 6}{6}.$$

□

It is a question of medium difficulty.

# A discrete Wirtinger type inequality

The celebrated Wirtinger inequality states that:

*If $f, f' \in L^2[0,\pi]$, and $f$ is a function with a period of $2\pi$ satisfying*

$$\int_0^{2\pi} f(x)dx = 0,$$

*then*

$$\int_0^{2\pi} f'(x)^2 dx \geq \int_0^{2\pi} f^2(x)dx.$$

In 1950, Schoenberg [3] established the following discrete version of Wirtinger inequality.

*If $z_1, z_2, \ldots, z_n$ $n \geq 2$ are complex numbers such that $\sum\limits_{k=1}^{n} z_k = 0$, then*

$$\sum_{k=1}^{n} |z_{k+1} - z_k|^2 \geq 4\sin^2 \frac{\pi}{n} \sum_{k=1}^{n} |z_k|^2,$$

*where $z_{n+1} = z_1$.*

In 1992, Alzer [7] obtained a variant version of the discrete Wirtinger inequality.

*If $z_1, z_2, \ldots, z_n$ $n \geq 2$ are complex numbers such that $\sum\limits_{k=1}^{n} z_k = 0$, then*

$$\sum_{k=1}^{n} |z_{k+1} - z_k|^2 \geq \frac{12n}{n^2-1} \max_{1\leq k\leq n} |z_k|^2,$$

*where $z_{n+1} = z_1$. The constant $\frac{12n}{n^2-1}$ is best possible.*

Notice that the condition of $z_{n+1} = z_1$ is called the *period condition*. By this condition, the sum of $n$ differences $z_2 - z_1, z_3 - z_2, \ldots, z_n - z_{n-1}, z_1 - z_n$ is zero. Without the period condition, is there an analogous Alzer type inequality? After a deal of contemplation, we proposed the following problem.

**Problem 0.1.24.** *Let $n \geq 2$ be a given positive integer. Find the maximum of $\lambda(n)$ such that, for any complex numbers $z_1, z_2, \ldots, z_n$ satisfying $\sum\limits_{k=1}^{n} z_k = 0$, one has*

$$\sum_{k=1}^{n-1} |z_{k+1} - z_k|^2 \geq \lambda(n) \max_{1\leq k\leq n} |z_k|^2.$$

This question for real numbers is actually the first problem of CMO in 2006 (see [4]), while the solution is completely different from the Alzer inequality.

Another natural question is whether there exists a reserve Alzer inequality. So we have the following two problems.

**Problem 0.1.25.** *(The reverse Alzer inequality under the non-period condition) If $z_1, z_2, \ldots, z_n$, $n \geq 2$, are complex numbers such that $\sum\limits_{k=1}^{n} z_k = 0$, then*

$$\sum_{k=1}^{n} |z_k|^2 \geq \frac{1}{n}\left[\frac{n^2}{4}\right] \min_{1\leq k\leq n-1}\{|z_{k+1}-z_k|^2\},$$

*where the coefficient $\frac{1}{n}\left[\frac{n^2}{4}\right]$ is best possible.*

**Problem 0.1.26.** *(The reverse Alzer inequality under the period condition) Let $n$ be a given positive integer. Find the maximum of $\lambda(n)$ such that, for any $n$ complex numbers $z_1, z_2, \ldots, z_n$, one has*

$$\sum_{k=1}^{n} |z_k|^2 \geq \lambda(n) \min_{1\leq k\leq n}\{|z_{k+1}-z_k|^2\},$$

*where $z_{n+1} = z_1$.*

*Solution.* Let

$$\lambda_0(n) = \begin{cases} \dfrac{n}{4} & \text{when } n \text{ is even,} \\ \dfrac{n}{4\cos^2\frac{\pi}{2n}} & \text{when } n \text{ is odd,} \end{cases}$$

we will prove that $\lambda_0(n)$ is the maximum of $\lambda(n)$.

If there exists a positive number $k$ $(1 \leq k \leq n)$ such that $|z_{k+1} - z_k| = 0$, then the conclusion holds. Thus, we can assume that

$$\min_{1\leq k\leq n}\{|z_{k+1}-z_k|^2\} = 1. \tag{0.1.8}$$

Under this condition, it suffices to prove that the minimal of $\sum_{k=1}^{n} |z_k|^2$ is $\lambda_0(n)$.

when $n$ is even, since

$$\begin{aligned}\sum_{k=1}^{n} |z_k|^2 &= \frac{1}{2}\sum_{k=1}^{n}(|z_k|^2 + |z_{k+1}|^2) \\ &\geq \frac{1}{4}\sum_{k=1}^{n}|z_{k+1}-z_k|^2 \\ &\geq \frac{n}{4}\min_{1\leq k\leq n}\{|z_{k+1}-z_k|^2\} = \frac{n}{4},\end{aligned}$$

the equality holds if $(z_1, z_2, \cdots, z_n) = \left(\frac{1}{2}, -\frac{1}{2}, \cdots, \frac{1}{2}, -\frac{1}{2}\right)$. Thus, the minimal of $\sum_{k=1}^{n} |z_k|^2$ is $\frac{n}{4} = \lambda_0(n)$.

We now consider the case that $n$ is odd. Let

$$\theta_k = \arg\frac{z_{k+1}}{z_k} \in [0, 2\pi),\ k = 1, 2, \cdots, n.$$

For each $k$ $(k = 1, 2, \cdots, n)$, if $\theta_k \le \frac{\pi}{2}$ or $\theta_k \ge \frac{3\pi}{2}$, by (0.1.8), we have

$$\begin{aligned} |z_k|^2 + |z_{k+1}|^2 &= |z_k - z_{k+1}|^2 + 2|z_k||z_{k+1}|\cos\theta_k \\ &\ge |z_k - z_{k+1}|^2 \ge 1. \end{aligned} \tag{0.1.9}$$

If $\theta_k \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$, it follows from $\cos\theta_k < 0$ and (0.1.8) that

$$\begin{aligned} 1 &\le |z_k - z_{k+1}|^2 \\ &= |z_k|^2 + |z_{k+1}|^2 - 2|z_k||z_{k+1}|\cos\theta_k \\ &\le (|z_k|^2 + |z_{k+1}|^2)(1 - 2\cos\theta_k) \\ &= (|z_k|^2 + |z_{k+1}|^2) \cdot 2\sin^2\frac{\theta_k}{2}. \end{aligned}$$

Therefore,

$$|z_k|^2 + |z_{k+1}|^2 \ge \frac{1}{2\sin^2\frac{\theta_k}{2}}. \tag{0.1.10}$$

Now, we consider the following two cases.

(i) If $\theta_k \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$, $\forall 1 \le k \le n$, by (0.1.10),

$$\sum_{k=1}^{n} |z_k|^2 = \frac{1}{2}\sum_{k=1}^{n}\left(|z_k|^2 + |z_{k+1}|^2\right) \ge \frac{1}{4}\sum_{k=1}^{n}\frac{1}{\sin^2\frac{\theta_k}{2}}. \tag{0.1.11}$$

Since $\prod\limits_{k=1}^{n}\frac{z_{k+1}}{z_k} = \frac{z_{n+1}}{z_1} = 1$,

$$\sum_{k=1}^{n}\theta_k = \arg\left(\prod_{k=1}^{n}\frac{z_{k+1}}{z_k}\right) + 2m\pi = 2m\pi, \tag{0.1.12}$$

where $m \in \mathbb{N}^*$ and $m < n$. Since $n$ is odd, it follows that

$$0 < \sin\frac{m\pi}{n} \le \sin\frac{(n-1)\pi}{2n} = \cos\frac{\pi}{2n}. \tag{0.1.13}$$

Let $f(x) = \frac{1}{\sin^2 x}$, $x \in \left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$, then $f(x)$ is a convex function. By (0.1.11), the Jensen inequality, (0.1.12) and (0.1.13),

$$\begin{aligned} \sum_{k=1}^{n}|z_k|^2 &\ge \frac{1}{4}\sum_{k=1}^{n}\frac{1}{\sin^2\frac{\theta_k}{2}} \ge \frac{n}{4}\cdot\frac{1}{\sin^2\left(\frac{1}{n}\sum_{k=1}^{n}\frac{\theta_k}{2}\right)} \\ &= \frac{n}{4}\cdot\frac{1}{\sin^2\frac{m\pi}{n}} \ge \frac{n}{4}\cdot\frac{1}{\cos^2\frac{\pi}{2n}} = \lambda_0(n). \end{aligned}$$

(ii) If there exists a $j$ $(1 \le j \le n)$ such that $\theta_j \notin \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$, denote by

$$I = \left\{ j \,\middle|\, \theta_j \notin \left(\frac{\pi}{2}, \frac{3\pi}{2}\right),\ j = 1, 2, \cdots, n \right\}.$$

By (0.1.9), we have that $|z_j|^2 + |z_{j+1}|^2 \geq 1,\ \forall j \in I$; By (0.1.10), for any $j \notin I$,

$$|z_j|^2 + |z_{j+1}|^2 \geq \frac{1}{2\sin^2\frac{\theta_j}{2}} \geq \frac{1}{2}.$$

Therefore,

$$\begin{aligned}\sum_{k=1}^{n} |z_k|^2 &= \frac{1}{2}\left(\sum_{j\in I}(|z_j|^2 + |z_{j+1}|^2) + \sum_{j\notin I}(|z_j|^2 + |z_{j+1}|^2)\right)\\ &\geq \frac{1}{2}|I| + \frac{1}{4}(n - |I|)\\ &= \frac{1}{4}(n + |I|) \geq \frac{n+1}{4}. \end{aligned} \tag{0.1.14}$$

Note that

$$\begin{aligned}&\frac{n+1}{4} \geq \frac{n}{4}\frac{1}{\cos^2\frac{\pi}{2n}}\\ \Leftrightarrow &\cos^2\frac{\pi}{2n} \geq \frac{n}{n+1}\\ \Leftrightarrow &\sin^2\frac{\pi}{2n} = 1 - \cos^2\frac{\pi}{2n} \leq 1 - \frac{n}{n+1} = \frac{1}{n+1}. \end{aligned} \tag{0.1.15}$$

when $n = 3$, (0.1.15) holds; When $n \geq 5$,

$$\sin^2\frac{\pi}{2n} < \left(\frac{\pi}{2n}\right)^2 < \frac{\pi^2}{2n}\cdot\frac{1}{n+1} < \frac{1}{n+1}.$$

(0.1.15) also holds. Thus, for any odd number $n \geq 3$,

$$\frac{n+1}{4} \geq \frac{n}{4}\cdot\frac{1}{\cos^2\frac{\pi}{2n}}.$$

Together with (0.1.14), we have

$$\sum_{k=1}^{n} |z_k|^2 \geq \frac{n}{4}\cdot\frac{1}{\cos^2\frac{\pi}{2n}} = \lambda_0(n).$$

Note that if we assume

$$z_k = \frac{1}{2\cos\frac{\pi}{2n}}\cdot e^{\frac{i(n-1)k\pi}{n}},\ k = 1, 2, \cdots, n,$$

we have

$$|z_k - z_{k+1}| = 1,\ k = 1, 2, \cdots, n.$$

Thus, $\sum_{k=1}^{n} |z_k|^2 = \lambda_0(n)$.

In conclusion, the maximum of $\lambda(n)$ is $\lambda_0(n)$. □

The Problem 0.1.26 is actually the fifth problem of China TST in 2014 (see [6]). Among 60 students, only 6 students got it right.

*Acknowledgements*

We would like to thank Ruixiang Zhang for their careful reading and suggestions to improve the original draft. Research of the second author is supported by a research grant from Shanghai Key Laboratory of PMMP 18dz2271000.

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# The Road from Submission to Perfection

## The problem-selection process for a Popular High School Mathematics Competition

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**Abstract**
In this article, we address the problem-selection process of the Mathematical Kangaroo, which is an international, popular multiple-choice mathematics competition. We describe the necessary steps starting with problem suggestions and ultimately reaching a finalized national version of the competition. The intention here is to illustrate the dynamics typical to such a selection as well as pointing out the multivariate possibilities of modification of submitted problems. We discuss and reflect these modifications by analyzing various examples of competition problems of recent years.
**Keywords: Problem selection, Mathematics competitions, Mathematical Kangaroo**

# Introduction

Mathematics competitions help identify students with higher abilities in mathematics, motivate these students and therefore "have positive impact on education" (Kenderov, 2006, p.1583). Both by solving problems during the competition and via discussions after the competitions, participants (not just winners) increase their knowledge significantly (ibid., p.1589). Competitions differ from other learning opportunities by an important property: "Problems for competitions are usually composed externally to any particular school, and can test a student's ability to use the mathematics they have learned in the classroom in new contexts" (Taylor, 2017, p. 303). Therefore, it is worth taking a closer look at the process of composition of competitions.

It is typically a multilevel process to finally present students a mathematics competition they work on. It usually starts with suggestions of problems, which are then selected and modified by a small group of experts. We show this process of problem selection vicariously using the Mathematical Kangaroo, an international, popular mathematics competition. By popular or inclusive mathematics competitions the "integral impact on the learning of mathematics becomes significant for the overall development of the contemporary society" (Kenderov, 2009, p. 1589) and from this point of view the contribution of the Mathematical Kangaroo "is difficult to overestimate" (ibid.). Therefore, it is worth taking a closer look at this particular competition. Although the problem-selection process certainly differs from contest to contest, there are some aspects of this specific instance worth underlining, that might well be considered typical. We will do our best to point out these aspects, as they manifest themselves in the annual work of this particular competition. Both authors are part of the international selection process and furthermore responsible for one particular country of this international competition as being members of the Austrian organization.

Furthermore, in this paper we would like to address a rarely discussed issue of mathematical competitions. The selection process allows modifications on the problems at several points. Sometimes this is necessary (e.g. for linguistic reasons), but sometimes this is also a matter of taste of the people involved or due to the appropriateness of the problems for the students. We will deal with each of these aspects of editing problem suggestions in the body of the paper.

Before we get to the description of the process, we give some general information about the Mathematical Kangaroo and the annual AKSF (Association Kangourou sans Frontières) meetings. The Mathematical Kangaroo is an international competition that has been organised annually since 1991. In recent years, more than 70 countries have been organising the competition and over six million students from grade 1-12 (or 13) participated in 2019 (before the Covid-19 pandemic).

The competition problems are developed for six levels, according to the ages of the participants. The category for grades 1 and 2 is called Pre-Écolier, for grades 3 and 4 Écolier, for grades 5 and 6 Benjamin, for grades 7 and 8 Cadet, for grades 9 and 10 Junior and for grade 11 and up Student. The number of problems in each group varies from 15 to 30 (becoming greater for the older groups), as does the time allowed to solve them (from 60 to 75 minutes). In each level, there are equally many problems with a value of 3, 4 or 5 points for a correct answer, with one quarter of the available points deducted for incorrect answers, and no deductions for problems not attempted.

The AKSF organization, although originally conceived as a European group, is currently composed of 76 members from all over the globe, with about a dozen applications for membership status currently pending. The members of AKSF are national (or, under certain special circumstances, regional) institutions, with the right and responsibility to organize the competition in their geographical (or sometimes linguistic) area as part of a general program designed to popularize mathematics at school level. More information about this is readily available at the public AKSF website (see AKSF-website, 2021).

The article is organized as follows. Section 2 deals with the annual international problem selection of the Mathematical Kangaroo and particular issues of cooperation. In particular, in sections 2.1 and 2.2 we describe the main selection of the problems, which is undertaken by representatives of the national Kangaroo organizations at an annual meeting in the late fall from a general perspective. Afterwards, we focus on the problem selection of one particular group, the Student working group, and point out the dynamics of the working group in section 2.3 and sources of conflicts during the problem selection in section 2.4. This section ends with some thoughts on modifications of problems in section 2.5. Section 3 first describes the preparation of the national versions of the Mathematical Kangaroo competition, by taking the Austrian working group as an example in sections 3.1-3.3. After giving some insight on the modification of a particular task by the national group in section 3.4 we focus on differences between comparable national versions of the Mathematical Kangaroo in section 3.5 by exemplarily analysing a particular task

# 2. International Problem Selection – Round 1

## 2.1 From suggestions to international competition papers: A short overview of the problem selection of the Mathematical Kangaroo

In October or November of each year, there is a meeting in one of the member states, where various organizational aspects of the group's work are dealt with. Most important of all, this is where the problems for the annual competition are selected.

In the months leading up to the annual meeting, members are asked to submit problem suggestions to the internal website of the organization. Each year, hundreds of problems are submitted here for consideration, and the problems that appear on the next year's papers are chosen from this pool. At the time of submission, problems are given labels of several types. Each problem is suggested for a specific level and a specific points-category. These may well change during the course of the problem-selection process. It is not unusual for a problem suggested as a 5-point problem for Benjamin to wind up being selected as a 4-point problem for Cadet, for instance. Also, problems are categorized by topic - these are Algebra, Logic, Geometry and Number - and for the more difficult or witty of the problems, there is an option to add a solution available to the posers, that is sometimes made use of. (This option has, in fact, become more of a suggestion than an option recently.)

The members then have the opportunity to rate (and solve) the suggested problems in the database of the website. There are several levels to this rating, according to how appropriate the problem seems, how nice or interesting it is felt to be, and also how difficult. These ratings are designed to be used as a form of pre-selection, to reduce the number of problems to be discussed in detail at the actual meeting.

At the annual meeting itself, six groups are convened, one for each of the six levels, with each participant at the annual meeting taking part actively in one of these groups. Therefore, the problem selection takes place for every level of the competition independently with some universal rules and selective exchange between designated coordinators of the different levels. During several hours of intense discussion, the merits of the individual problems are discussed, the wording of the problems is refined somewhat, and 15 to 30 problems, depending on the requirements of the particular level, are eventually selected for the paper, along with a handful of "reserve problems" for every level. The motivation behind the reserve problems lies in the fact that some countries may choose to eliminate some problems due to national curriculum considerations or for some other local reasons, and it is therefore considered to be advantageous to offer some alternatives that could easily have been chosen for the main paper, but just failed to make the cut at the last moment. According to the by-laws of the Mathematical Kangaroo, each member has the right to exchange up to five problems in each level if they are deemed to be inappropriate for their local purposes for whatever reason. Also, problems are often modified subtly at the national level during the translation process. More about this aspect will be explained in Section 3.

It should be mentioned that every effort is undertaken to make the selected problems wide-ranging, thought-provoking and intellectually entertaining for participants from all backgrounds and all skill levels. In Geretschläger & Donner (2021), the authors describe the requirements on problems posed for the Mathematical Kangaroo from several perspectives in detail.

The final round in the problem-selection process of the several groups at the meeting consists of polishing the selected problems. While there is no attempt at perfection made at this stage, since the so-called "post-processing" is yet to follow after the meeting, this is still done as well as possible. The phrasing of the problems is improved in the English-language originals that will be given to the countries for translation into their own languages, for instance. This is obviously a step that the group members not so comfortable with English as a working language do not participate in actively. Also, some thought is given to lesser issues, such as the distribution of correct answers, the quality of the graphics, and so on. Issues of this type will all be dealt with after the meeting, as will be described in the following section.

### 2.2 Post-processing: Finalizing the international version of the Mathematical Kangaroo

After the annual meeting, the problem sets that have been selected by the working groups are fine-tuned somewhat before they are handed over to the various national groups for translation into the national versions. There are several reasons for the implementation of this process.

First of all, there are a few countries that use the official (English-language) version for their competition as is, without creating a separate national version. For these countries, a version must be created that is as free of errors as is humanly possible, with appropriately good versions of graphics and optimized formulations of the problems and distractors. Also, the graphics are created in a uniform version that is of a high quality, so that each country can use the official graphics in their own print versions, without having to redundantly create their own. Finally, some problems may have errors in their preliminary versions that may have been missed during group work, and such errors can be corrected before the translation process carries such errors over into multiple versions internationally. A small group of experts in such matters create the official wording of each problem as well as the official graphics in high-quality formats, and solutions are written for some of the more difficult problems to make the national groups aware of particularly tricky and interesting aspects of some of the more difficult questions.

In the last few years, this part of the process has become more important. Previously, it was assumed that the tweaking of the problems would be done at a national level anyway, as the various language versions are created. With so many member countries taking part in the Mathematical Kangaroo now, it has become increasingly clear that the officially sanctioned versions of the papers need to be as close as possible to a usable print version. Nevertheless, the completion of the process is still done mostly at the national levels. Before we describe the national processes in detail however, we turn our attention to the dynamics of the selection process, the sources of conflicts during problem selection and to what extent suggested problems and selected problems sometimes differ.

## 2.3 The Dynamics of Group Problem Selection, illustrated by the Student working group

In this section we take a closer look at the process of selection using the example of the Student working group. The details of the problem-selection process in the Student group have evolved over time, and each year brings some new wrinkles. Still, the basic method of discussion and selection has remained much the same over the course of the last two decades.

One important factor that may make the process in this particular case different from the work done in other similar competitions is the sheer size of the group. With so many countries participating actively in the problem-selection process, and many countries sending multiple delegates to the annual meetings, all six working groups at the AKSF meetings tend to be quite large. In 2019, for instance[1], the Student working group was made up of 30 active members. It is not hard to imagine that a detailed discussion of each individual problem is not easy in a group of this size. Once you also take the language difficulties in such a group into account, it becomes clear that some forms of systematic voting are required to organize the workflow. This is due to the fact that the work in the group is done in English, and it is assumed that all participants have adequate working knowledge of English for this purpose. Unfortunately, this is far from being the case, and some aspects of the process must be organized accordingly.

In order to deal with this, much work is done in written form. Active participation in the discussions of the group is generally restricted to the group members who feel comfortable speaking in English in such a venue, and the others take part mostly through their votes and with occasional mathematical contributions presented at a blackboard or flipchart, such things being somewhat less language dependant. When the group first convenes, the group chair has already prepared a short list of

[1]The 2020 meeting was held on-line due to the effects of the Covid-19 pandemic, and the dynamics of the meeting were therefore somewhat atypical.

about 50 to 60 problems (recall that 30 problems are required for the paper) according to the online voting that was done in advance. Problems that were generally agreed to be unpopular, too hard or ill-suited for the competition are therefore no longer under discussion at this point. An exception occurs when some specific problem finds a champion who will argue its unique merits. In this case, it may be decided after some discussion that such a problem may be added to the short list, but a large number of proposed problems are then no longer in contention at all.

At this point, the short list has already been distributed to the group members, and they have had a few days to take a (second) close look at the short-listed problems. Since all problems that were considered appropriate by a majority of the people taking part in the advance rating are generally included here, a small number of problems at most will be added to this version of the list. One reason why the short list must often be augmented nevertheless, lies in the fact that it typically does not include enough really easy problems. Interesting easy problems are few and far between, and there are never enough of these proposed by the countries, despite the fact that about 200 problems are proposed each year in the Student group. What easy problems there are, are then often marked as uninteresting in the first round of rating. Of course, many of them are indeed quite uninteresting from a strictly mathematical point of view. The members of the group are all seasoned mathematicians and educators, and they have typically encountered variants of some of these problems many times in various circumstances. It is no wonder that they find them dull. Nevertheless, some "uninteresting" problems (in this sense) must be included in the paper in order to offer some initial motivation for the slightly less interested participant to engage with it.

The discrepancy between "easy" and "interesting" is well known to the group, of course. There are generally not enough appropriate easy problems submitted to the Student group by the problem posers in the first place. Most problem posers that develop easier problems for the Mathematical Kangaroo tend to suggest them for the lower age levels, as easier problems will usually also be suitable for these. This means that the Student group will often have to resort to poaching easy problems from the younger levels. This is not at all easy, as the more interesting of these problems will typically be selected by the groups they were submitted to (as they are also dealing with a dearth of good problems of this type), and so a typical tactic is to resort to creating variants. An idea for a nice easy problem the group finds among the suggestions for one of the younger levels is changed, with different numbers, different structures, or whatever is possible, as long as the result is still easy and interesting, but sufficiently different that the two versions are not immediately recognisable as clones.

This is a constant topic of discussion in the working group developing the problem sets for the Student group. Some find it inappropriate to ask questions that could also, in principle, be solved by much younger students, while others would argue that this is a good thing.

One good reason to limit the content of the problems to topics close to the school curriculum for the age is motivational. It can be argued that students will engage more deeply with the material in their regular classes if they have previously encountered closely related problems they considered to be entertaining at a competition. For this reason, some would argue that the topics chosen for the competition should all reflect some aspect of the students' classroom work, albeit in a somewhat entertaining setting. Unfortunately, strict adherence to this concept severely limits the possibilities at our disposal. It therefore follows that there will always be intense discussions concerning the inclusion of problems that could just as easily be posed at a younger age level.

Once the structure of the short list has been agreed on, concrete problem selection begins. In a first step, the most popular problems from the rating are pre-selected. These are generally problems that no one rated as being unsuitable, and that received more ratings of "very nice" than just "acceptable". There are surprisingly few such problems in a typical list. If there are five such problems among the 200 proposed problems, it is a good year; in some years there are just one or two. These problems are considered individually by the group, and if there are no objections, they are included in the paper. The next step is to choose the problems that will have the numbers 1 to 5 on the paper. In many ways, this is the most difficult part of the process. Sometimes there are not even five candidates for this part of the paper available yet, and the group must go on a search for candidates to be poached, but soon it is possible to start the voting process.

The working group votes on the problems by groups according to perceived difficulty. Once a list of candidate problems has been made for a certain part of the paper, each member of the group has as many votes as there are problems to be chosen. If there are nine candidates for the five problems from 1 to 5, each member of the group gives their votes to their five preferred problems among the nine, and the number of votes for each problem is then tallied. The problems obtaining the most votes are then briefly discussed individually, and the problems 1 to 5 of the competition are then chosen. At this point, the discussion already allows for the consideration of other aspects. Some may be mathematical. For instance, the group will try to avoid choosing two problems involving pure calculation among these five, or two similar problems involving triangles. Other considerations may be practical. For instance, type-setting is always at the back of everyone's heads, although this is a minor consideration, of course. Still, two problems requiring very large graphics will seldom be chosen to follow one another, for instance.

This general method is then followed for other groupings of problems in the paper. Next, an analogous method is generally applied to select problems 6 to 10, then the problems from 11 to 20, and finally the 5-point problems 21 to 30. If some problems have already been pre-selected at the beginning, some specific category may have less slots available. For instance, if there were already two obvious 4-point problems selected at the outset, there may be only eight slots left for the segment of 4-point items from 11 to 20.

Once the problems have been selected for each of the groupings, the problems are put into an order that seems to be reasonable for the competition. There are several factors that can enter into the discussion at this point. Primarily, the problems are sorted by rising levels of perceived difficulty. An attempt is made to mix up the subject matter in as colorful and entertaining a way as possible. The first few problems should be really, really easy and the last few (generally the last three) can be somewhat difficult, giving the students with some background in higher level mathematics competitions a chance to shine. These problems should still be solvable in a short period of time, but some knowledge of Olympiad-style thinking might be rewarded here. There is an active attempt made to avoid any such requirement in the rest of the paper, but it is assumed that only the really knowledgeable students will get this far, because of the limited time available to solve all the problems. Also, this is meant to limit the number of perfect scores. In actual fact, it is not that unusual for there not to be any perfect scores at all in some countries in some levels, especially the more advanced ones. For this reason, problem selection is often criticized as tending toward a too difficult paper overall.

### 2.4 Sources of Conflict

Now that we have finished our description of the process of problem selection in the Student group, which is quite specific to this particular competition, we can turn our attention again to a more general topic.

The Mathematical Kangaroo is meant to transcend countries, languages and cultures. There is an ideological principle at the very foundation of the undertaking, in assuming the fact that the fascination emanating from abstract mathematical thought is a universally applicable constant independent of any cultural barriers. A popular trope often encountered in this context is that of the "universal language of mathematics". This is also an epithet commonly applied to the arts or to sports. In the case of mathematics however, there is the implied added benefit of the utility of the undertaking in the mix, since mathematics has proven itself so useful, particularly in the modern technological world. For this reason, much is made of the fact that students all over the world can be given the opportunity to work on the same problems, and generate the same positive feelings when they make the same intellectual discoveries along the way, giving them a common goal in the pleasant pursuit of a highly worthwhile pastime.

This is a wonderful unifying idea, and years of experience with this in practice do seem to confirm its validity to a large extent. Nevertheless, finding a common base for the competition problems is not as easy as one might think. In fact, there are several important differences to be dealt with in reaching agreement on the problems of the common paper.

On closer inspection, one type of difference will usually turn out to be quite superficial. These are the divergences resulting from the curricula specific to the various countries. Of course, such matters need to be dealt with, but many of the difficulties resulting from this can be dealt with at a national level. Still, there are some topics that need to be discussed again and again.

One such matter is the role of calculators in the classroom. While it is certainly the case that some kinds of calculators are now in common use more or less everywhere, the way they are used is not at all uniform. In some countries, they are simply tools used whenever very large or unwieldy numbers arise. In many countries, however, the widespread use of calculators has led to a situation where students are used to relying on their ubiquitous machines to such an extent that they no longer develop any intuition for number properties of the type that would have been the norm before calculators became common, and are still so elsewhere. An argument (that the authors of this paper are highly skeptical of) can be made, of course, that such a type of number intuition is not worth the time spent in developing and cultivating it. In some countries, this argument has been widely (and often just implicitly) accepted in the schools, and because of that, students from such countries have a completely different view of problems concerning number properties than would students habitually used to doing some mental calculation.

An example of such a problem is the following 4-point item, taken from the Student paper 2019:
A) What is the integer part of

$$\sqrt{20+\sqrt{20+\sqrt{20+\sqrt{20+\sqrt{20}}}}}?$$

**(A) 4** (B) 5 (C) 6 (D) 20 (E) 25

At this point, we should recall that the use of calculators during the competition is not allowed in the Mathematical Kangaroo at all. The very existence of this rule is a bit contentious, although it is accepted as given in the AKSF organization. There is great resistance to the idea that a large number of traditional types of problems like this one could no longer be reasonably posed if calculators were available. If students were allowed to use a calculator to answer this question, there would be nothing of interest to it at all, as the expression could simply be typed in and the result read out in the calculator's display. Students used to calculators being available for any and all number-related activities will therefore need to give this problem more thought than students more schooled in the recognition of number structure. Such students may more readily have the essential idea of working from the inside-out and realizing that all the numbers we are taking roots of are all slightly less than 25, and their roots therefore always slightly less than 5, making the answer obvious. Students whose only idea is to calculate the result with a machine will find it much harder to overcome the urge to calculate from the outside inward, in the way one would input such an expression with the aid of a keyboard.

This example should give an idea of the difficulties the international group must deal with in determining the relative difficulty of a selected problem. Numbers are only one example of the discrepancy resulting from differing use of technology. Another such example is that of functions. If students are very used to (and possibly dependant on?) graphing calculators to visualize functions, they will have a completely different attitude to problems concerning functions than will students not used to relying on such technology. This will also play a role in the rating process, of course. Another area of concern is geometry. In some countries, elementary Euclidean geometry has been all but completely phased out of the schools, while it is still an important part of school mathematics elsewhere. It is quite clear that this will make it difficult to agree of the levels of difficulty of problems in this topic area in the Mathematical Kangaroo. This is actually an omnipresent point of contention, as problems in Euclidean geometry are quite popular among the problem posers.

This last point brings us to the matter of differing traditions in the teaching of mathematics. This is closely related to the question of curricular differences, but not exactly the same. An example of this would be the matter of solid geometry. In some curricula, solid geometry is a topic dealt with in an exclusively analytic fashion, while others place more value in visualization of three-dimensional objects and positions. Because of this, any question set in three-dimensional space is considered quite difficult by most problem posers, even though some of these problems are of a very elementary type for students used to this type of thinking.

Finally, there are also differences in personality and experience among the members of the problem-selection group to be considered in this context. The different members of the working groups come from quite different backgrounds. Some are research mathematicians, some are math education researchers, and some are active teachers. Many at this level have a background in working with gifted students, and specifically they often have backgrounds in working on Mathematical Olympiads. It is easy to imagine the disparate standpoints that result from such diversity. This is, of course, one of the strengths of the Mathematical Kangaroo. Having all of these different points of view actively involved in the problem-selection process is a great asset in creating a high-quality paper. The discussions leading to this result are, however, not always easy.

## 2.5 Why suggested problems are often not equivalent to selected problems

All suggested problems which are selected for the competitions are automatically edited with respect to linguistic aspects for the official English-language version.

Sometimes the experts decide to additionally modify some content related aspects of certain tasks, as the following example of the Junior paper 2021 shows. The problem was originally suggested as a 3-point problem in the following formulation:

B) Ahmad walks up 8 steps going up either 1 or 2 steps at a time. There is a lion on the $6^{th}$ step, so he cannot stop on this step. In how many different ways can Ahmad reach the top step?
(A) 6 (B) 7 **(C) 8** (D) 9 (E) 10

The problem was selected by the international Junior group, and the formulation was slightly modified (modifications are highlighted):

B1) Ahmad walks up 8 steps going up either 1 or 2 steps at a time. There is a hole on the 6th step, so he cannot use this step. In how many different ways can Ahmad reach the top step?

Nobody would argue that replacing the lion by a hole really changes the intention of the author of this problem. But at the same time, the expert group increased the plausibility of the task (why would somebody climb over a lion?). Furthermore, it was selected as a 4-point problem as students need a multi-step argument by first noting the necessity of splitting the path into two parts and then counting the number of possibilities for each of the subpaths, consisting of steps 1-4 and 6-8 respectively, and finally multiplying these numbers.

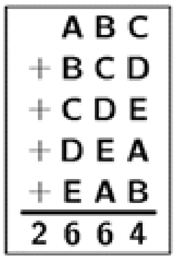
This example shows that, beside linguistic smoothing, the contents of problems are occasionally modified as well (in this particular example in a minimal way).

In addition, a crucial fact in any popular competition, and this is especially true for multiple choice competitions, is that the structure of the problems is just as important as their intrinsic mathematics and its formulation. The selection of options given for the solutions (in other words, the "multiple choices") play a central role. Some such problems are structured in such a way that the specific distractors offer aid in solving the problem. On the other hand, certain specific incorrect answers are often offered specifically with the intent to entice contest participants making predictable types of logical errors (so called traps).

Therefore, it is comprehensible that major modifications are being made by the changes of distractors implemented in some problems by the working groups, either to fulfill general agreements on distractors (controlling the difficulty of the task) or in order to enrich the problem.

We will now show examples for each of these types of modifications of distractors:

First, let's have a look at the following 3-point problem which was chosen for the Student paper of 2020 as an example:
C) The sum of five three-digit numbers is 2664, as shown on the board. What is the value of A+B+C+D+E?
(A) 4 (B) 14 **(C) 24** (D) 34 (E) 44




These distractors are different to those in the problem as it was originally submitted:
(A) 14 (B) 24 (C) 104 (D) 124 (E) none of the previous

There is a broad consensus within the international expert's group that 3-point problems (of every level) should not contain distractors which open the problem. Hence the distractor (E) of the submission had to be modified in order to choose the problem as a 3-point problem for the competition. Furthermore, the international group was of the opinion that the distractors (C) and (D) were not plausible as the sum of five digits can obviously not be greater than $9 \cdot 5 = 45$. But the problem stated as in C), is easier than with the originally suggested distractors, because all of the incorrect answer options are part of the distractor "none of the previous". The fact that only 24 is in the right range of approximately achieving 2664 as the sum should be clear at first glace without any calculation. This seems to be a deficiency of the modified version (and hence of the chosen problem). The reason why - for instance – 26 or 22 was not included in the list of the answer options may be explained by the fact that the task is just worth 3 points – hence it should not contain any traps (e.g. answer options representing certain attractive but incorrect trains of thought) and give all participants a realistic chance of solving the problem, either by calculation or by estimation.

Beside the special requirements on distractors of 3-point problems, the international experts sometimes just think that the suggested distractors do not fulfill the requirement of being "distracting". As an illustration of this, we consider the following 5-point problem of the Junior paper of 2021:

D) Let $N$ be the smallest positive integer whose sum of its digits is 2021. What is the sum of the digits of $N + 2021$?
**(A) 10** (B) 12 (C) 19 (D) 28 (E) 2021

As $N$ is to be as small as possible, it must contain as few digits as possible. This means it contains as many nines as possible, and the number in question is therefore $N = 59 \ldots 9$ (a 5 followed by 224 copies of the digit 9).
The simple calculation $N + 2021 = (N + 1) + 2020$ gives us $60\ldots02020$, and therefore the sum of the digits of the number in question is 6+2+2=10.
The other answer options are the consequence of the following train of thought: It is easy to conclude very quickly that the number we are looking for must be very small. If the solver takes into account that the number has the remainder 1 when dividing by 9 (as $N$ and 2021 both have the remainder 5), it then remains to carefully check the answer options (A), (C) and (D).

When the task was originally proposed by one of the two authors of this article, the distractors were quite different, namely
(A) 10 (B) 12 (C) 2021 (D) 2023 (E) 4042

The number 2023 was given as a distractor to make the option 2021 seem reasonable, as this distractor would otherwise seem to be implausible. 4042 was meant to attract inaccurate readers of the task, who might think that adding 2021 and 2021 would solve the problem. The idea of considering residue classes modulo 9 - which excludes all of these original incorrect answers apart from 4042 - was not considered as the key strategy for attacking the task by the poser. As always, the way of attacking and solving such a problem is based on the knowledge of the students, which varies in the participating countries. In Austria, only very few students are aware of residual classes. Without any doubt, the task was made more difficult by changing the distractors, by simultaneously letting 2021 be an outstanding and therefore unattractive option.

This example shows that much more consequential changes can take place during the problem-selection process than simple issues of formulation.

# 3. National problem selection – Round 2

### 3.1 Introductory remarks

So far, we have described the process of choosing a problem set in an international context, using the example of the Student working group. Of course, this process is quite similar in every other subgroup dealing with the other levels of the Mathematical Kangaroo. Now we will take the next step and consider the work that follows at the national level in order to finalize a particular national version of the competition – versions that students are eventually working on. We will discuss this using the example of the national working group in Austria, as both authors have taken part actively in this process for many years. It should be mentioned that due to organizational, curricular and minor linguistic differences this process takes place independently from the analogous process in Germany and Switzerland, despite the common language. A detailed description of the German (and Swiss) process can be found in Noack & Unger (2020).

### 3.2 Translation and the first impression

The multi-step process starts immediately after the final international version is made available. The first step is the initial translation of the problems. One person is responsible for providing a preliminary translation, including the answers and a rudimentary version of the figures, for each level. From this point on, national experts are consulted and these accompany all remaining steps. This group consists of active and retired teachers, researchers, university students and former participants of Mathematical Olympiads. The high quality of the competition can only be ensured through these different experiences and the multi-layered fields of expertise of the members of the group. Additionally, various detailed discussions of individual problems and multi-step engagement with the tasks greatly reduces the likelihood of any serious errors.

First of all each member of the group is asked to solve the problems of levels they feel the greatest affinity for (perhaps because of the grades they teach or because of their Mathematical Olympiad background) and to make any suggestions for improving the (German) formulations. Greatest attention must be paid to the individual language during translation of the items (to actively avoid creating ambiguities), due to the huge amount of text involved. Simultaneously, they are asked to assess the level of difficulty of every individual problem and of the problem sets in total. Their proposals are then incorporated into a second version, with conflicting opinions marked as such.

### 3.3 Preparing a preliminary national version

Next, the national group meets in person to discuss certain critical formulations and to finalize the problems from a linguistic point of view. At this juncture, items considered inappropriate for whatever reason are replaced by items from the reserved list. This is actually quite a common occurrence, as not all problems fit in the particular curriculum of the intended grades. Sometimes there is also broad agreement that two specific items should be switched within the problem set because the assessed relative difficulties differ from the initially intended order. At the end of the meeting, the order of the items within the problem sets is considered final.

For several years, this step was carried out at one meeting. Since there are now six levels, the workload is too large and therefore it was split into two meetings (at different locations) and each of these meetings deals with three levels, with each meeting typically lasting half a day. (It should be noted that this will be different in the year this is being written, unfortunately. Due to the constrictions of the Covid-19 pandemic, the meeting had to be replaced by online communication, as is the case with so many activities in 2020 and 2021.)

The revised version is then once again made available to the entire group. New figures are created at this juncture, if necessary. Despite the excellent quality of the figures now made available internationally during the finalization process described in section 1.2., this is often still required to some extent. It is quite common for pictograms and figures to include labels or even words that are in common use in English-speaking countries, but would be considered unusual for Austrian students. For instance, the label “O” for the circumcircle of a triangle is always replaced by “U” (short for “Umkreismittelpunkt”) in the Austrian version. Sometimes, such minor changes can even make it necessary to redraw the entire figure.

In addition to all the editorial changes and new figures being incorporated into the files as required, there are also some other details to deal with. Cover sheets are added, containing fields for students to enter their solutions as well as some information about the competition. Furthermore, at this time, a solution grid containing the letters for the correct solutions to all problems in all the levels is written.

### 3.4 Finalizing the national competition: Preparing the competition for Austrian students

At a third - and crucial – step, the entire group meets in one place for two days. At the beginning of this meeting, everyone solves problems in levels they are not yet familiar with. This ensures that more people (and, critically, not always the same people) read and solve the problems. Some of the items may still contain typos or some minor residual issues of formulation, and these can be fixed quickly. At this stage, we would hope that there are no more issues concerning ambiguity due to the wording, but this can occasionally still be the case, and it is important that they be discovered. Linguistic ambiguities must absolutely be avoided, but at the same time care must be taken that no new ones arise at this point because of hasty changes made before fully reflecting on their consequences. Therefore, despite the rather large workload, this process is carried out very carefully.

There follows a further round of discussion and final versions of the wording are fixed for each item as the consensus of a large part of the group. Rare incidents of incorrect interpretation of problems during this penultimate phase show how important it is for as many different people as possible to read and check the formulations of each item at different stages of development.

Finally, the problem sets are formatted. The main problem at this point is inevitably: How can we possibly place the huge number of figures on the limited number of pages determined by the print format? At least three people then independently confirm the solution grid for each level, because these will ultimately serve as the sole basis for grading the students' submissions and therefore may not contain any errors. On the second day, in addition to completing these tasks, all the problem sets, solution grids and accompanying peripheral documents are subjected to a last round of proofreading and solutions for as many problems as possible are written. The latter are made available on the public website after the competition has been closed (see Kangaroo Austria, 2021).

This multi-step process shows that the national group is very much aware of both the linguistic and mathematical challenges of the competition. Beside linguistic issues, the levels of difficulty must be wisely adjusted to the specific level appropriate for Austrian students by interchanging and rearranging some of the problems. Sometimes problems are even modified in a major way, as we point out in the following section.

## 3.5 Major modification of individual problems – analysis of a particular problem

The following 5-point problem was chosen for the Junior level 2021 at the international meeting:

E) How many five-digit positive numbers have the product of their digits equal to 1000?
(A) 10 (B) 20 (C) 30 **(D) 40** (E) 60

We start our analysis with a possible solution of the problem: On the one hand, such a number must contain the digit $5$ three times. On the other hand, the product of the remaining two digits must be eight in order for the product to equal $1000 = 5$ş $\cdot\, 8$. There are two options that yield this, as we have $8 = 1 \cdot 8 = 2 \cdot 4$. It follows that there are two variants of how a number with the required property can be assembled. Either it consists of the digits $1, 5, 5, 5, 8$ or of the digits $2, 4, 5, 5, 5$. Both cases are of the same combinatorial type, and there are $\binom{5}{3,1,1} = \frac{5!}{(3!1!1!)} = 20$ different numbers of each. A total of 40 numbers have the desired property, and so (D) is the correct answer.

Analysing a problem of the Mathematical Kangaroo, it is always necessary to look at the answer options. In this case, only distractor (B) seems to be an attractive alternative, as two possible errors lead to this answer: Either a solver forgets one of the two cases or he or she overlooks the fact that a number is not already determined by the positioning of the three fives among the digits - there are two possibilities for the remaining two digits - and considers both cases. The further options can only be chosen by pure guessing, by unsystematic approaches or by multiple errors in an unclear way.

At the Austrian version of the Kangaroo, the following was one of the 5-point problems at the Junior level in 2021[2]:

E1) How many four-digit positive numbers have the product of their digits equal to 100?
(A) 6 (B) 12 (C) 16 **(D) 18** (E) 24

Although the problem seems to be very similar to the version above, it differs in a fundamental way: Starting with the same initial ideas, the student deduces that the number must contain the digit 5 twice and the product of the remaining digits must be $4 = 1 \cdot 4 = 2 \cdot 2$. Hence, there are two different cases, as one of them requires the multiple occurrence of a second digit. This reduces the number of solutions by the factor 2 in this particular case (6 solutions with digits $2, 2, 5, 5$ and 12 solutions with digits $1, 4, 5, 5$). As the number of solutions is smaller altogether, a systematic way of counting (without using combinatorial formulas) is feasible in a reasonable amount of time. At the same time, the combinatorial arguments are more complex, as both cases are of different types. The problem is therefore enriched through the adaptation to the new version with two cases which can not be dealt with analogously, while simultaneously reducing the complexity, because the respective number of solutions is reduced. Looking at the distractors of E1), we see that each of the incorrect answer options is initially plausible: (A) and (B) are achieved as solutions when the student forgets to consider one of the two cases (or miscalculating the case $1, 4, 5, 5$), (E) matches with the error of considering both cases like $1, 4, 5, 5$. Finally, distractor (C) is meant to attract students who create a list of solutions in an unstructured way.

Summing up, the Austrian experts took the problem as a basis for developing an even more attractive problem,

- by widening the possible methods of finding a solution (by reducing the number of solutions),
- by mathematically enriching the task, as both cases must be dealt with in different ways and
- by creating more plausible distractors.

Very few problems are modified in that way, but it should be noted that this type of change is not a specifically Austrian phenomenon, and is actually done in many of the countries participating in the Mathematical Kangaroo, as we discuss in the following section.

### 3.6 National competitions: is there an unambiguous Kangaroo competition?

Whereas the vast majority of the problems are just translated within the national rounds of fine-tuning of the competition, as pointed out in the previous section, major modifications of the

[2]The German translation of the problem can be found in Mathematical Kangaroo Austria – Junior (2021), p.4.

internationally chosen problems can sometimes take place during this process.

Along with these obvious changes, however, we also want to demonstrate that the usual way of editing the internationally chosen problems can sometimes lead to problems which seem somehow different at first glance.

As an example of this, we will now discuss problem B) of section 2.5. again and compare the German and the Austrian versions of the problem with the problem as it was selected by the international group.

First of all, here is (again) the problem as it was chosen by the international Junior problem group:

B1) Ahmad walks up 8 steps going up either 1 or 2 steps at a time. There is a hole on the 6th step, so he cannot use this step. In how many different ways can Ahmad reach the top step?

(A) 6 (B) 7 **(C) 8** (D) 9 (E) 10

The Austrian version of the competition reads as follows (Mathematical Kangaroo Austria - Junior, 2021, p.3.):

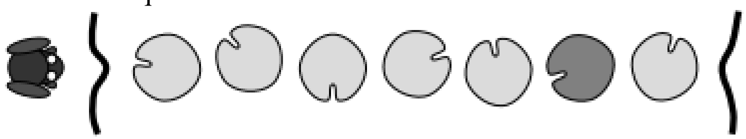
B2) Andrea steigt acht Stufen hinauf. In jedem Schritt nimmt sie entweder eine Stufe oder zwei Stufen auf einmal. Die sechste Stufe kann sie nicht benutzen, weil sie kaputt ist. Auf wie viele verschiedene Arten kann Andrea die achte Stufe erreichen?

(A) 6 (B) 7 **(C) 8** (D) 9 (E) 10

Translation of the task:
Andrea walks up eight steps. At each move, she takes either one step or two steps at once. She can't use the sixth step because it's broken. In how many different ways can Andrea reach the eighth step?

The German version of the problem[3] is stated in this way (Mathematical Kangaroo Germany, 2021, p.2.):

B3) Ein Frosch möchte einen Teich überqueren. Er nutzt 7 Seerosenblätter in einer Reihe. Er springt immer nur 1 oder 2 Seerosenblätter vorwärts. Das 6. Seerosenblatt muss er überspringen, weil es welk ist. Wie viele verschiedene Varianten gibt es für den Frosch, den Teich auf diese Weise zu überqueren?



[3]In Germany, the name for this level is "Klassenstufen 9 und 10" (class 9 and 10) instead of "Junior".

(A) 6 (B) 7 (C) 8 (D) 9 (E) 10

Translation of the task:
A frog wants to cross a pond. It uses 7 water lily leaves in a row. He only jumps 1 or 2 water lily leaves forward at a time. He has to leave out the 6th water lily leaf because it is wilted. How many different ways are there for the frog to cross the pond in this way?

The reason why the Austrian formulation of the problem includes the “eighth step” is that the term “top step” may be misleading. The German version offers an elegant way to get rid of the this potential inaccuracy by assessing the other end of the pond as the “top step”.

The example points out that the representation of the internationally selected problems at the national competitions differ – ranging from minor differences in notation (or terms) to major differences concerning the content of the task or even additional figures. These modifications definitely even result in different options toward understanding, attacking and solving the problem.

Revisions of internationally selected problems are intentionally carried out by the German group. The goal is to preserve the mathematical content of the tasks, and change the ”storyline” of often inner-mathematical or implausible contents in order to motivate the students working on the problems (see Noack & Unger, 2020, section 4.5).

# 4. Conclusion

It is a long and multifaceted path from the submission of a problem to its final appearance in each of the national versions of the Mathematical Kangaroo. In this article we have depicted this path chronologically, starting from submitted problem suggestions and ending with the versions as they eventually appeared in the Austrian competition papers. There are multiple options for the modification of chosen problems that can be done without actually replacing it altogether. This ranges from linguistic revision to translation and from changing particular distractors to modifying the complexity of the given numbers within the tasks. These modifications are generally undertaken for good reasons. Perhaps the goal may be improvement of the cognitive demand of the task (by changing distractors, for instance), or changing the complexity of calculations required to solve the task. As described in section 2.4., in some cases it is possible to simplify calculations and simultaneously increase the attraction of the incorrect answer options, which definitely enriches the problem. There are two main criticisms that apply to major modifications of this kind.

First, modifications that go beyond linguistic aspects change the suggested problem in such a way that it may not really represent the problem poser’s initial concept and idea anymore. Second, if the distractors are modified by one group of international experts (and not accepted as they are by the bigger group of international experts suggesting the problems), this unification of preferences may increase the chance for students to solve problems via test-wiseness (see Donner et al. 2021) or convergence strategy (see Andritsch [Donner] et. al. 2020). Sometimes, the idea of the suggested problem is fabulous, but modifications must be done due to linguistic issues or because of typos within the distractors. Perhaps deeper discussion is warranted with respect to the extent to which the group should be allowed to redesign problems. There is, of course, always the option to choose a different problem out of the extensive amount of problem suggestions, which may not have to be modified at all. With regard to this aspect, the board of the AKSF organization is constantly trying to increase the quality of the problem suggestions of their members by providing templates and pointing out the need of high-quality problem suggestions whenever the opportunity to do so arises. Additionally, in recent years, more and more effort has been put into a finalization process meant to result in an optimal final version of all problems that could, in principle, be used as is, if an English-language version is acceptable in a participating country.

It should be pointed out that most of the suggested problems are not modified in such a large scale as the examples cited in this article. As major modifications are only performed on a very small number of problems and as at most 5 problems can be replaced in the national paper altogether, (usually from the "reserved" list of pre-approved alternates) the core and heart of the competition, consisting of the problems which had been suggested in advance and then chosen within the international meeting, remain – even in the national versions of the competition. Both the experienced international group and the national groups of experts try to create a competition which is as attractive as possible for the participating students in each country. All changes along the path from suggestions to the final competition are made for this reason and can therefore also be justified in a comprehensible manner. Effects of modifications of problems on the motivation of the students on the one hand and the approaches of the students when facing different versions of the "same" task on the other hand are part of current investigations of the authors. Preliminary results have already been obtained. It should be emphasized that the wealth of experience regarding the Mathematical Kangaroo and the varied backgrounds of the members of the international and national groups ensure the attainment of the ultimate goal of offering a competition that is both exciting and challenging on multiple levels. After all, the ultimate purpose of the competition is to enable as many young people as possible to enjoy mathematics for the duration of the competition - and beyond.

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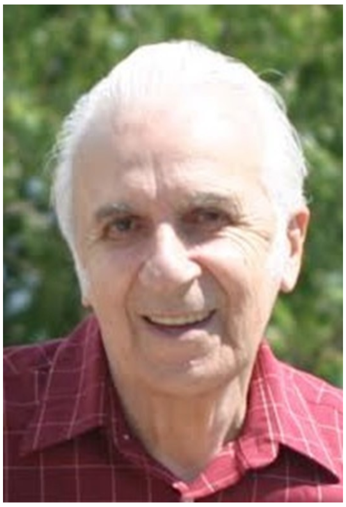
# In praise and support of
# Year-Round Mathematical Competitions

*George Berzsenyi*



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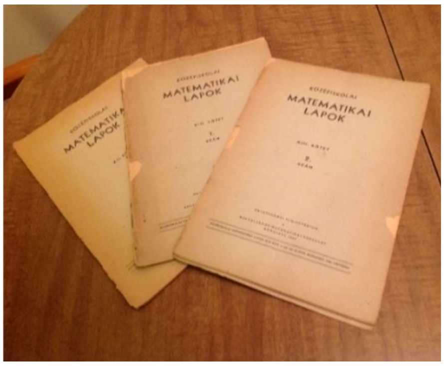
**Abstract**
The purpose of this article is to describe Hungary's high school mathematics journal, ***KöMaL***, its young sister-journal, ***Abacus*** for students in Grades 3-8, and my own efforts in the United States to emulate these publications – thereby providing an opportunity for year-round creative and competitive problem solving for students year after year. I strongly recommend the creation of similar journals in other countries as well, especially in smaller countries, where the graders might not be overwhelmed by the huge number of responses. Conducted properly with challenging and interesting problems, one can be ensured that in 2-3 generations problem solving will become an intellectual habit among the students, strongly supported by their parents and even their grandparents. That's what happened in Hungary and that led to Hungary's excellence in mathematics and the sciences.

## Preliminaries

As an immigrant to the United States, I always considered my foremost duty to introduce to my adoptive land the treasures that I brought with me from "the old country". Therefore, after I settled into the teaching profession and became familiar with the American competition scene in the area of mathematics, it was natural for me to reflect upon its weaknesses and the strengths of the Hungarian system I left behind. I was missing ***KöMaL***, Hungary's high school mathematics journal and its challenging problems month after month.




***KöMaL*** stands for ***Középiskolai Matematikai Lapok*** (in English, High School Mathematics Journal), to which I subscribed as a high school student. It was so dear to me that even when I left Hungary, crossing the border to Austria on foot with a small bundle on the 29th of November 1956, I brought three of its issues with me since I did not yet manage to read them thoroughly. They served as my talisman during my many years of struggle to create something similar in the United States and beyond. I show them here as a reminder of those intentions.

In the present note, I will reflect on two of my attempts of 30-40 years ago to emulate ***KöMaL***. I will also introduce my readers to ***Abacus***, the sister-journal of ***KöMaL***, wishing her belatedly a happy 30th birthday. By doing so, my purpose is to encourage small countries with demographics similar to that of Hungary, to create their own **Competition Corner / USAMTS** programs. If they do so and if they keep their programs alive and well, then I can promise that in 2 or 3 generations they will rival the rest of the world in mathematical power. That's what Hungary has done ever since the birth of ***KöMaL*** more than 125 years ago.

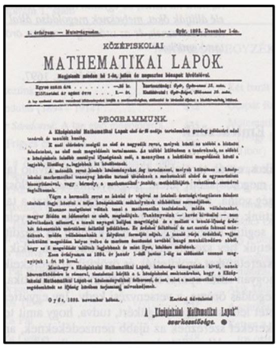
## Historical sojourn

Hungary was established with the Conquest of the Carpathian Basin by the Magyars in 896. Thereafter Hungary (in Hungarian, Magyarország, i.e., the country of the Hungarians) was recognized as a sovereign country until 1526, when in the Battle of Mohács, the Ottoman Empire subdued the Kingdom of Hungary. The occupation of most of Hungary by the Turks ended in 1686, but rather than regaining its independence, Hungary became a vassal state of the Austrian Habsburg Empire. Following several unsuccessful wars for independence, it was not until the **1867 Compromise** that Hungary regained much of its independence.

At that time, huge efforts were made by Hungary to catch up with the rest of Europe in nearly every walk of life. Roads and railroads were built, bridges were erected across the Danube, and the navigation of her rivers and Lake Balaton were made possible. Hospitals and other public buildings were built, including 400 schools across the country. Nevertheless, centuries of foreign exploitation and the lack of even rudimentary industry and commerce, made Hungary a '3rd – world' agricultural country with an outdated feudal system and mentality, not much different than many countries of Africa and the Middle East, artificially created by the colonizing powers after they extracted most of the valuables of the land and disrupted the governance of the people.

Left to her own devices, Hungary had to lift herself by her own bootstraps in order to catch up with the rest of Europe in nearly every walk of life. That's when the famous Kürschák Competition was initiated, the Eötvös College for properly trained teachers of mathematics and the sciences was established, as well as ***KöMaL*** was launched. Along with the formation of scientific organizations, these were the instruments that made Hungary into a mathematical super-power in the 20th Century. Of these, I will write about the humble beginnings of ***KöMaL***, believing that many other countries have dedicated teachers like the late Dániel Arany was, and hence other developing countries can come up with their own versions of ***KöMaL***.

## The birth of *KöMaL*



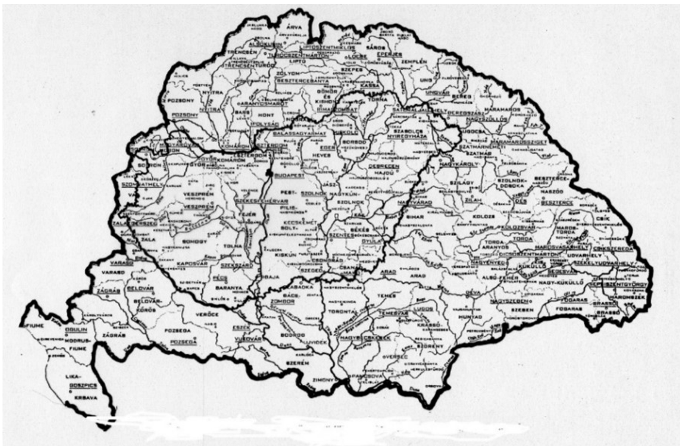
KÖZÉPISKOLAI
MATHEMATIKAI LAPOK.

PROGRAMMUNK.

***KöMaL*** was launched in 1893 in the small, but progressive town of Győr by Dániel Arany, a teacher at the local science high school. Arany sent a copy of it to every high school in Hungary. Nevertheless, relative to the many other happenings, it was a small event that probably went unnoticed by everyone except for the students who responded to its problems and the teachers who subscribed to the publication.

From the outset, ***KöMaL*** appeared 10 times a year, with each issue consisting of 16 or more pages, and in its first 3 years, the publication featured 239 problems. To 208 of those problems a total of 1055 solutions were submitted by 151 students. And even in its first year, there were 132 subscribers to it due to the tireless correspondence of Dániel Arany with many teachers throughout the country. He was eminently cultured, with fluency in German, French and English and knowledge of Latin and Greek too. His knowledge of French made it possible for him to become familiar with the Journal de Mathematiques Elementaires for talented high school students. While his ***KöMaL*** emulated that publication, he went much further. For example, by featuring the matriculation examinations of the high schools across the land, he helped in the uniformization thereof.

Admittedly, Hungary was a larger country then, as indicated by the map on the right, and Budapest was not the only city of importance in the country. Pozsony (now Bratislava in Slovakia), Kolozsvár (now Cluj-Napoca in Romania), Kassa (now Kosice in Slovakia) and Nagyvárad (now Oradea in Romania), to mention just a few, were rival cultural centers. With their loss, the importance of Budapest increased, and it was fast emerging as an equal of most European capitals.



Thus, when Dániel Arany turned over the editorship to László Rátz, a highly regarded mathematics teacher in Budapest, it became possible (and necessary too) to centralize the entire operation. But it was only after its second rebirth following World War II that ***KöMaL***'s problems became the ingredients of year-round nationwide competitions. Nowadays and 4 generations later, thousands of students submit solutions to the problems of ***KöMaL***, to be graded by former successful contestants studying at the universities of Budapest. Everyone, who might start such a program must aim for such a support system in the long run.

## The ‘Competition Corner’ in the *Mathematics Student* journal

In the United States it was not until 1952 that the National Council of Teachers of Mathematics (NCTM) published the first issue of the ***Mathematics Student Journal***. It was just a pamphlet of 4 pages, which, unfortunately didn’t grow into more than 6 pages over the years. Its Problems Section was edited by Mannis Charosh, who stayed with it until 1964, at which time he published a selection of its problems under the title ***Mathematical Challenges***. Later the editor of the journal became Thomas Hill, who published ***Mathematical Challenges Plus 6***, covering the problems proposed between 1965 and 1973 (the ‘plus 6’ in the title referred to 6 articles from the journal, 3 of which were authored by students). By the time my friend, Dr. David (Dave) Logothetti took over the editorship in 1978, the name of the journal had been shortened to ***Mathematics Student (MS)***; that’s how I will refer to it in the future. By naming me editor of the Problem Section and giving me a free hand with it, Dave gave me my first opportunity to emulate ***KöMaL*** in America. I renamed the Problem Section ‘Competition Corner’; that’s how I will refer to it in the sequel.

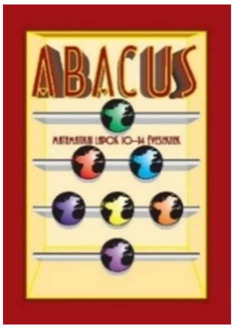
While the circulation of the ***MS*** was at 30,000, it was not available to individual student subscribers, but was sent mostly to teachers in bundles of 5 or 30. Seeing the relatively small number of submissions to my predecessor, Steve Conrad, as the editor of the Problem Section, and wanting to make sure that the best students in the country were also aware of the Competition Corner, I made arrangements to obtain the home addresses of the students who made the national Honor Roll on the American High School Mathematical Examination (AHSME), i.e., scored 100 or more out of 150 points and did not yet graduate from high school. I wrote a personal invitation to all of them and was pleased that many of them accepted the invitation. I did the same in the next two years and hence, we ended up with well over 500 participants during the 3 years of my involvement in the program

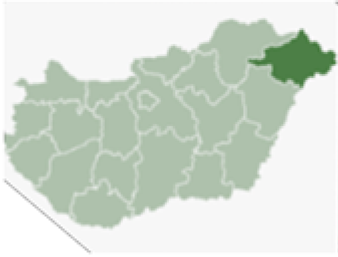
I used the space allotted to me by Dave (exactly half of the 4 or 6 pages) to conduct year-round competitions with 5 problems per round, i.e., per issue of the year, which was 8 in the first year (8 issues of 4 pages) and 6 in the next two years (6 issues of 6 pages). The students were always given a month to solve the problems and I tried to report back to them with an evaluation of their submissions in a month also, though the solutions to the problems did not appear in the***MS*** until later.

Being a one-man operation (with some essential help by my wife, Kay) in addition to a heavy teaching load at a state university, not only did I have to select the problems to be posed (with the help of a ‘Call for Problems’ to mathematical friends), but I did all the grading and correspondence, and kept the scores in a ‘hand-operated database’. After preparing the solutions, Kay typed them, and along with the new set of problems and some additional materials, I submitted them to Dave.

## The arrival of *Abacus*

A decade later in Hungary, working behind the scenes in a remote region of Hungary, a young mathematician by the name of Sándor Róka decided that the students in middle schools should also have a journal. Hence, he and his wife, Bea, an award-winning teacher of physics, created ***Abacus*** – shown below, along with a picture of Sándor, a map of present-day Hungary and of the region, where they tested the prototype of ***Abacus*** on the students.




By ‘prototype’ above, I mean 4 years of sending out a set of 8 problems, different sets for students in Grade 4, in Grades 5 & 6 and in Grades 7 & 8 for students in the region, grading the submissions by the students, and sending them an evaluation of their work along with the official solutions to the problems and the new set of problems – 8 times a year. They handled it as a small business, and hence there was additional accounting, reporting and lots of correspondence as well – handled most efficiently by Sándor and Bea Róka.

Their correspondence course in problem solving became a bonified journal in 1994, just in time for the 100th anniversary of the birth of ***KöMaL***. They named it ***Abacus*** and added a number of new columns. I will comment on those new columns later; presently, I want to focus only on the columns of mathematical problems, emphasizing that the scores for the solutions of the problems in ***Abacus*** are also accumulated over the year, a report on them is published in ***Abacus***, and the pictures of the most outstanding problem solvers, separated by their grade levels are featured, just like in ***KöMaL***. At the end of the year 20 students per grade are honored in such manner.

Needless to say, the publication of ***Abacus*** was well received not only by the students and their parents, but by the educational community as well. As a consequence, the János Bolyai Mathematical Society accepted the challenge of taking over the editorial duties, as well as the numerous other responsibilities of publication in 1998, and ***Abacus*** soon grew up to be a proper sibling to ***KöMaL***. Presently, ***Abacus*** appears 9 time a year on 48 pages in each issue.

## The USA Mathematical Talent Search (USAMTS)

My next opportunity to create something ***KöMaL***-like came after moving to Terre Haute, IN, in 1988 to chair the Department of Mathematics at Rose-Hulman Institute of Technology (RHIT). As I was proofreading a congratulatory piece that I wrote on the 25th anniversary of the Wisconsin Talent Search in my regular ‘Problems, Puzzles and Paradoxes’ column in ***Consortium***[4], it seemed reasonable to ponder on the possibility of launching a similar program nationwide. After discussing the matter with members of my faculty and key personnel of the school’s administration, as well as with my friends Sol Garfunkel[5] and Walter Mientka[6] and after making the appropriate mathematical preparations, we were ready to launch the USAMTS under the auspices of COMAP via a separate column in ***Consortium***.

[4]A quarterly publication by the Consortium for Mathematics and Its Applications (COMAP)

[5]Executive Director of COMAP

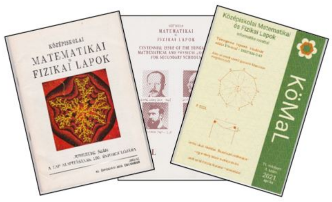
[6]Executive Director of the CAMC (Committee on the American Mathematics Competitions) in the MAA (Mathematical Association of America)

We decided on 4 rounds of 5 problems per year. I asked Walter Mientka for the home addresses of the participants of AIME, wrote my letters of invitation to the students, and we were ‘ready to roll’.

My main regret is that I did not change the term ‘Search’ to ‘Development’ in the name of the program in order to reflect its true nature. Moreover, I also regret that in the second year, when we started to send the students a ***USAMTS Newsletter***, I didn’t coin a fancy name like The ***Mathematics Competitor*** for it. Then it might have had a chance to grow into a proper journal.

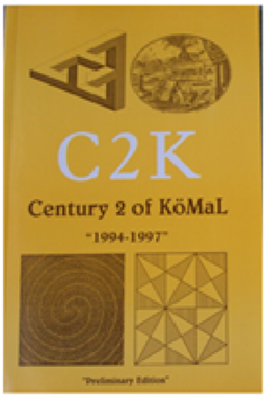
## More about *KöMaL*

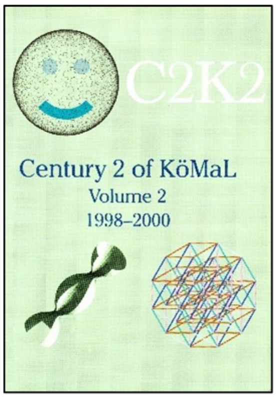
Since ***KöMaL*** predated the other three start-ups by 85 or more years, clearly, I must explain the situation in Hungary at the end of the 19th Century more comprehensively. While it is true that the country was jubilant after the Compromise of 1867, it is also true that there were many different nationalities living within the borders of Hungary, and not all were elated. In particular, they did not welcome the ‘Magyarization’ efforts of the Hungarians towards the end of the 19th Century, which can be explained only as an unfortunate imitation of centuries of unsuccessful ‘Germanization’ efforts in Hungary by the Austrians.
Even at the time of The Conquest of 896, the Hungarians were accompanied by tribes of other nationalities who settled in various parts of the country including the bordering areas, which they agreed to defend in case of war. Later, after the Mongolian invasion during the first decade of the 13th Century, other nationalities were invited to settle parts of the country, where the most people were lost. Hungary was also home to many Gypsies and Jews, as well as Serbs and other people of the Balkans fleeing the Turks. Many Austrians, Saxons, Swabians and other Germanic people also found a home in Hungary after the Ottoman Empire was forced to give up Hungary’s occupation after 150 years but left much of the countryside torched and with hardly any population.
Unfortunately, the otherwise peaceful coexistence with the minorities within Hungary’s borders was drastically changed by the Dictate of Trianon at the end of World War I, which robbed Hungary of more than two-thirds of its land, two-thirds of its population, much of its natural resources (forests, mines, etc.) and many of her major cities / cultural centers. Thus, the present-day Central-European countries are not very different from many of the countries of Africa and Asia whose borders were drawn arbitrarily by the former colonizing powers, and fail to reflect the national, tribal, religious, ethnic and linguistic divisions of the land.
Consequently, it was near-miraculous that ***KöMaL*** could be revived and that by the 1930s it was once again the main mathematical talent-development program in Hungary.




## Yet more about *KöMaL*

Jumping ahead by several years, I show on the right the Centennial Issue of ***KöMaL***, published in December 1993, along with its English equivalent (in the center) and a second English issue, published in August 1994. In view of the fact that I needed some prizes for the winners of the USAMTS, and that they were affordable, I ordered over a hundred of the English language issues, thereby subsidizing their publication. Using them as prizes, I spread the information about ***KöMaL*** among the USAMTS participants.




Later I came up with the idea of collecting into a special volume the problems and solutions that appeared in ***KöMaL*** during the first few years of its second century. Hence, ***C2K*** was born, covering the years 1994 to 1997, and once again, I ordered many copies thereof. Next, a follow-up to that was also published and called ***C2K2***, and again, I managed to make use of it as a prize for the winners of the USAMTS. In the Introduction to ***C2K***, that was addressed to the winners of the USAMTS, I gave a detailed description of ***KöMaL*** and its history in the hope that someday someone would succeed better than I did in emulating ***KöMaL***. Both of the above volumes were edited by Vera Oláh, who was the editor of ***KöMaL*** at that time. I served as one of the assistant editors to ***C2K***. Some years later, under the editorship of Gyula Nagy, several more issues of ***KöMaL*** appeared in English. Along with an article that Gyula wrote in the present publication[7], they also provide a glimpse into this wonderful publication.

## The year-round problem-solving competitions in *KöMaL* and *Abacus* today

In addition to articles of interest to high school students and reports on various competitions, ***KöMaL*** presently conducts ten different year-round problem-solving competitions at different grade- and difficulty-levels. They are as follows:

- At the lowest level there are 35 problems in the year marked by K; only students in Grades less than 10 are allowed to submit solutions to them. The first 3 problems are the same as the last 3 in ***Abacus***.
- At the next level, there are 7 problems marked by C in each issue; of them the first 5 are for students in Grades 10 and less, while the last 5 are for students in Grades 11 and 12. Of the first 5, two are the same as the last two K-problems Here the students are separated by grade level into the following three groups: students in Grades less than 9, students in Grades 9 and 10 and students in Grades 11 and 12. This level has a total of 45 problems during the year for each of the three grade levels.
- At the next level there are 8 problems marked by B in each issue, but the students are restricted to submitting solutions to only 6 of them. This time the choice is theirs. They are separated by grade level into the following five groups: students in Grades less than 9, students in Grade 9, students in Grade 10, students in Grade 11 and students in Grade 12. They all have 54 problems for the year for each of the five grade levels.
- And finally, there are 2 or 3 problems marked by A in each issue of ***KöMaL***; they are for students preparing for international competitions and/or careers in mathematical research, regardless of their grades.

[7]Mathematics Competitions, Vol. 29, No. 2 (1996), pp.26-41

In its turn, ***Abacus*** features 45 problems per year for the students in Grades 3 to 6 and 54 for those in Grades 7 and 8. Since the point gathering competition is broken down by grades, there are separate competitions for Grades 3 or less, Grades 4, 5, 6, 7 and 8 – six competitions per year. Accordingly, in 7 of its 9 issues

- For students in Grades 3 & 4, there are 7 problems; of them the first 5 are for students in Grades 3 and less, while the last 5 are for students in Grade 4.
- For students in Grades 5 & 6, there are 7 problems; of them the first 5 are for students in Grade 5, while the last 5 are for students in Grade 6.
- For students in Grades 7 & 8, there are 8 problems; of them the first 6 are for students in Grade 7, while the last 6 are for students in Grade 8.

In addition to these, following the example of ***KöMaL***, there is a physics section in ***Abacus*** too, with 5 problems per issue. There are also 2 harder mathematics problems marked MP, 3 logic problems, 4 problems in chess, and 1 each in sudokus and nonograms. Furthermore, there are 3 problems in German and 2 in English in order for the students to hone their language skills by submitting solutions in the appropriate languages. In all of these, the students compete year-round for points. And whenever there is a gap, the journal is filled with a mathematical curiosity, a puzzle, a joke or a historical fact.

Naturally, ***KöMaL***'s physics problems are much more advanced and include experimental ones (marked by M), easier ones for the lower grades (G) and harder ones (P) for the upper grades, and year-round competitions in their solutions. In informatics, similarly, there are programing problem that are easier (I), harder (I/S) and hard (S) and competitions in solving them. Moreover, ***KöMaL*** has mathematical articles on topics of interest to high school students, reports on competitions as well as on other events, like the recognitions given to outstanding teachers of mathematics and physics. More specifically, ***Abacus*** reports on the results of the nationwide problem-solving competition of mathematics teachers at the elementary and middle school level, while ***KöMaL*** reports on the Distinguished Teaching Awards named after László Rátz (Rátz Tanár Úr Életműdíj) each year.

## Synopsis and Recommendations

While I hugely admired the incredible riches of ***KöMaL***, I had to settle for just one set of 5 problems per round in both of my attempts to emulate their year-round mathematical competitions. Neither did I have the physical manpower to do more, nor did I have a reservoir of ingenious problems created by a standing committee of 15 superb 'problemists' utilized by ***KöMaL***. Occasional 'Calls for Problems' were a poor substitute for that.

While the problems featured in my programs were much easier than those in a national Olympiad, I followed the style of the Olympiads and that of the Wisconsin Talent Search, where similarly, all of the contestants are offered the same problems regardless of grade and maturity levels. With 5 problems per round, one can afford to have one or even two relatively easy ones, suitable for beginners and younger participants. One of the problems can also be harder to make sure that even the best and most advanced participants are challenged. It is also possible to make sure that at least one of the problems is geometrical, and that at least in every other round, the geometric problem is

3-dimensional. Five problems also allow for a healthy mix of algebraic, number theoretic, logical, trigonometric and combinatorial problems – all at the pre-calculus level.

As to the number of rounds per year, I recommend 4 at the outset, to be extended to 5 or 6, but not to be reduced to 3 or less. In Year 1 (1978-1979) of the Competition Corner, the ***MS*** appeared 8 times, and hence, I had to cope with 8 rounds. It was a lot, but doable. In Years 2 & 3 (1979-1980 & 1980-1981), it appeared only 6 times, and hence, the pace was more modest. Nevertheless, not wanting to exhaust my colleagues and since ***Consortium*** appeared 4 times, I reduced the USAMTS to 4 rounds, which we managed with relative ease.

Ideally, the grading should be done by university students, who took part and did well in the program while in high school. That's how it is done in Hungary when it comes to the ***KöMaL*** submissions, but at the outset, there are no 'graduates' of the program. Hence, at the outset one must rely on faculty help, as I did. Fortunately, thanks to Dr. Gene Berg and his successors, I could also rely on the mathematicians at the National Security Agency (NSA), when the number of submissions got huge.

Back in 1978-1981, e-mailing was not yet available, and even in 1989-1998 only a few high school students were accessible via e-mail. Nowadays, however, it should be possible to keep up with the former participants of programs via e-mail, and in more developed countries one should be able to organize grading via electronic communication with carefully selected former participants. Thereby, after the initial 4-5 years of working with the program, it should be possible to turn over – with proper supervision – the grading of the submissions to the earlier winners of the program.

## The advantages of year-round competing

Based on the experiences and reflections of the former participants and on my personal views developed during my days as a student in Hungary, I summarize below some of the benefits of year-round competing

- Mathematically bright students need appropriate challenges regularly, rather than occasionally.
- Problem-solving should be made into an intellectual habit, rather than a once-a-year experience.
- Clever students are often bored in the classroom; challenging them with interesting mathematical problems gives an outlet for their creativity.
- Such programs can help the students improve their writing skills and presentation styles via developing complete, well-written solutions.
- Having a month to develop one's solutions is much more realistic then doing so in a timed situation.
- Meeting stringent deadlines is best-learned and mastered at an early age.
- The solution of an interesting problem is a wonderful event, a most satisfactory accomplishment in itself. Possibly seeing one's work in print, makes it even more special.

- Voluntarily observing an honor system in developing their own solutions will serve them well.
- Seeing the often more clever or elegant solutions of others develops an appreciation of each other, a better sense of one's own capabilities, and at times even some much-needed modesty.
- The anticipation of seeing the evaluation of one's submissions and the joy of receiving the next set of problems is also a worthwhile experience. Many parents recalled the positive reactions of their kids upon receiving all of my mailings to them. Seemingly, they disappeared with my letters and were not seen for the rest of the day and beyond.
- The students developed a sense of ownership in the program by seeing their names among the 'Commended Solvers' and/or having a solution attributed to them. They were rightly proud of their accomplishments and wanting more of the same, they worked harder.
- Creative mathematical problem solving is an excellent preparation for nearly every profession, including law, medicine, business, engineering, economics, and the other sciences.
- Mathematics is unique among the sciences in that one can state difficult problems simply and solve them without extensive technical background by applying clever ideas time and again in an unexpected manner.
- Most of the Hungarian mathematicians, physicists, engineers and scientists, in general, credit ***KöMaL*** for the basic training they received while in high school.

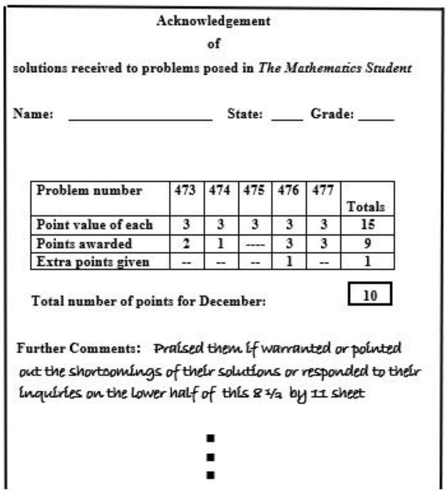
In fact, many of the 43 of us featured in a neat little book edited by Sándor Róka credited ***KöMaL*** in answer to the question:
**Why did I become a mathematician?**
(That is the English translation of the title of the book)


## Concluding Remarks

Clearly, with the 16 different year-round competitions conducted by ***KöMaL*** and ***Abacus***, Hungary has reached the Nirvana of mathematical rivalry of all students, regardless of grade-level and prior experiences. But one must remember that it took them more than a century to reach their present status, and that more than 4 generations have grown up on ***KöMaL***. Thus, it is likely that the father or mother, or one of the grandfathers, grandmothers, aunts, or uncles of the student working on a ***KöMaL*** or ***Abacus*** problem was also a ***KöMaL*** - enthusiast a generation or two earlier. Thereby in Hungary such an involvement often is already a family-trait.

Toward that lofty goal, I recommend humble beginning with four rounds of 5 carefully chosen problems. Then, if and when a stronger support system is established, one might expand the offering to 5 or 6 rounds of 7 problems, with the first 5 for younger participants and the last 5 for students in the upper grades. As a next step, one might designate one of the problems appropriate for extensions and/or generalizations and give an extra point for such accomplishments. I found that many students love such opportunities and for most of them it is a novel experience.

I also recommend a ‘personal touch’ via the evaluation of the students’ submissions, making them recognize that the grader is a fellow enthusiast in problem-solving. A few words of encouragement can go a long way!
A typical Evaluation Form is shown on the right, where Problem 476 allowed for an extra point for meaningful generalizations. Time and again I used the space to answer their questions, and at times I even entered into regular correspondence with some of them. They told me about their readings, about their accomplishments on other competitions, and they shared with me even some confidences.
I also recommend a pictorial tribute to the best of them at the end of the school year, as well as Certificates of Participation to the regular participants even if their results are modest.
Finally, I strongly believe in appropriate prizes to the winners, preferably, books of appropriate content.



Acknowledgement
of
solutions received to problems posed in *The Mathematics Student*

Name: ________________ State: ____ Grade: _____

| Problem number | 473 | 474 | 475 | 476 | 477 | Totals |
|---|---|---|---|---|---|---|
| Point value of each | 3 | 3 | 3 | 3 | 3 | 15 |
| Points awarded | 2 | 1 | ---- | 3 | 3 | 9 |
| Extra points given | -- | -- | -- | 1 | -- | 1 |

Total number of points for December: 10

Further Comments: Praised them if warranted or pointed out the shortcomings of their solutions or responded to their inquiries on the lower half of this 8 1/2 by 11 sheet

I strongly believe that following Hungary’s example in year-round competing in creative mathematical prooblem solving is the best approach toward developing a culture of mathematical excellence. The capital of most countries can serve as a mathemacical center, and at the outset it is perfecly suitable to use snail-mail for conducting such a program. A website should also be utilized and electronic submissions and the use of LaTeX should be encouraged from the outset.

## Add-On

I was ready to submit this article when my friend, Tünde Kántor neè Varga shared with me Vojtech Bálint’s excellent article on “Hungarian mathematics development stimuli”[8], which gives a much-deeper historical background to the birth of ***KöMaL*** than the one given in the present article. Professor Bálint also gives lists of the most successful solvers of the ***KöMaL*** problem. He was the leader of the Slovakian team to the IMOs 11 times between 1996 and 2013.

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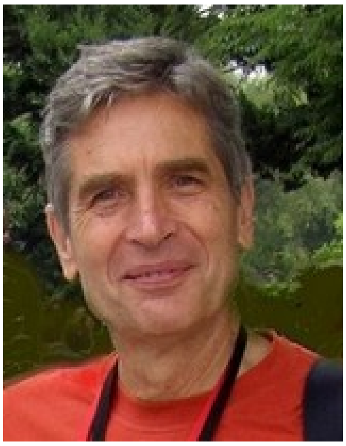
gberzsenyi@gmail.com

[8] Antiquitates Mathematicae,Vol. 14(1) 2020, p. 1-15

# Cutting a Polygon: from Mathematics Competition Problems to Mathematical Discovery[9]

*Kiril Bankov*



Kiril Bankov prepares future mathematics teachers as a professor of mathematics education at the University of Sofia. He graduated and received his PhD in mathematics at the same university. Prof. Bankov is also a member of the Bulgarian Academy of Sciences. He has been working for several international projects in mathematics education. Kiril Bankov has been involved in mathematics competitions in Bulgaria for more than 20 years as an author of contest problems and as a member of juries. He was the Secretary of World Federation of National Mathematics Competitions (WFNMC), then was elected as the Senior Vice President and in July 2018 he became the President of WFNMC.

**Abstract**
Problems are the intellectual product of mathematics competitions. Many of these problems lead to interesting generalizations. Sometimes the generalizations are beyond elementary mathematics and they are real challenges for professional mathematicians. Such generalizations are a basis of some of the discoveries in mathematics. This paper presents the development of ideas, inspired by problems from mathematics competitions, as beautiful mathematical discoveries. The problems are about the partitions of a polygon (mainly a rectangle) into different shapes – a topic that is fruitful for both creation of problems for mathematics competitions and for mathematical discoveries.

## Introduction

Cutting a given polygon into a finite number of certain shapes is a topic that offers a variety of tasks for mathematics competitions. For example, here is a beautiful problem from the Saint Petersburg Mathematical Olympiad, 1968 (Fomin, 1994).

**Problem** 1. Prove that an equilateral triangle cannot be cut into a finite number of equilateral triangles in such a way that any two triangles are not congruent.

Cutting a polygon into triangles is studied in a variety of papers, for example in Bankov (2011), where a solution to the above problem is given. There are contest problems for cutting a polygon into shapes that are not triangles. For example, the following problem from the 16th Mathematical Olympiad in Poland, 1964-65 (Straszewicz, S. 1972, Problem 79; see also Straszewicz, S. &

[9]This paper is based on the author's keynote talk at TSG46 of ICME-14.

Browkin, J., 1978, Problem 95).

**Problem** 2. Prove that each square can be cut into squares whose number is an arbitrarily natural number greater than 5 but cannot be cut into 5 squares.

**Solution**. Notice that if a square is cut into $k$ squares, it can also be cut into $(k+3)$ squares by connecting the midpoints of the opposite sides of one square. Also, for every natural number $n, n > 1$, a square can be cut into $2n$ squares as shown in Figure 1.

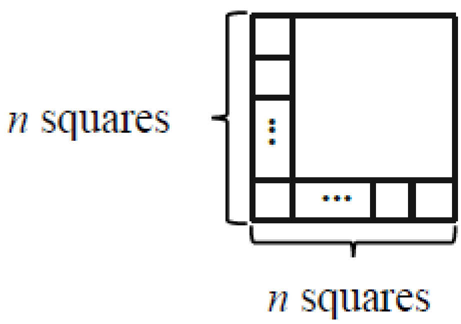



Figure 1: Partition of a square into $2n$ squares

With combinations of these partitions a square can be cut into squares whose number is an arbitrarily natural number greater than 5. For the second part of the problem, assume that a square with side $a$ can be cut into 5 squares. Obviously, four of these squares should have a common vertex with one of the vertices of the given square. Let the sides of these four squares be $a_1, a_2, a_3, a_4$. There are two possible locations of the fifth square (denote its side by $a_5$):

1. It is inside the given square. Then, assuming an appropriate numbering, from $a = a_1+a_2 = a_2+a_3 = a_3+a_4 = a_4+a_1$ it follows that $a_1 = a_3$ and $a_2 = a_4$. The area of the given square can be expressed in two different ways: $(a_1+a_2)^2 = 2a_1^2 + 2a_2^2 + a_5^2$. The last equation is not possible.
2. It has a side lying on a side of the given square. Then, assuming an appropriate numbering, $a = a_1 + a_2 = a_2 + a_3 = a_3 + a_4 = a_4 + a_5 + a_1$. This is also not possible.

Problems 1 and 2 give rise to further discovery. For example:

**Question** 1. Is it true that if a square is cut into a finite number of squares, there are at least two congruent squares?

The answer to Question 1 is negative. The earliest historical traces of this task go back to the years 1923-1924 (Mauldin, 2015). At about this time, Stanisław Ruziewicz, professor of mathematics at the university of Lwów, proposed to his students to find out if a rectangle could be made up of squares of different sizes. In 1925 Zbigniew Moroń, a junior assistant of Ruziewicz presented in a publication (Moroń, 1925) two such partitions of a rectangle.

The first partition of a square into squares of different sizes was described by Sprague (1939). After then, some mathematicians worked on the problem to find the smallest number of squares of different sizes in such a partition. Duijvestijn, A. (1962) proved that it is not possible to cut a square into less than 21 squares of different sizes. Later, Duijvestijn, A. (1978) found such a partition with 21 squares with the aid of a computer. This rose interest in the development of computer algorithms for the partitioning of a square into squares of different sizes with additional properties about the partition (Gambini, 1999).

A similar question arises from the obvious fact that a square can be cut into a finite number of isosceles right angled triangles.

**Question** 2. Is it true that if a square is cut into a finite number of isosceles right angled triangles, there are at least two congruent triangles?

The answer to Question 2 is negative. Sergey Dorichenko sent me this drawing (Figure 2) showing a partition of a (7 by 7) square into 7 isosceles right angled triangles without a congruent pair of triangles.

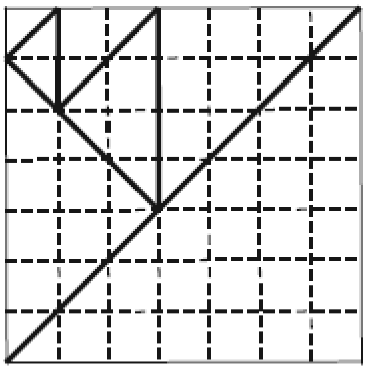


Figure 2: Partition of a square into isosceles right angled triangles without a congruent pair of triangles

## Possible and Impossible Partitions

Triangulation is a special case of cutting a polygon. It is a partition of a polygon into a finite number of triangles such that: (1) none of the vertices of the triangles lies on a side of the polygon (except the vertexes of the polygon); (2) any two triangles either do not have a common point, or they have a common vertex, or they have a common side (Figure 3).

Main properties of the triangulations are discussed in different articles, for example in Bankov (1991). The number $t$ of the triangles in any triangulation is $t = 2n + k - 2$, where $k$ is the number of the vertices (the sides) of the polygon, and $n$ is the number of all vertices of the triangles in the partition that are in the interior of the polygon. This follows from the equation

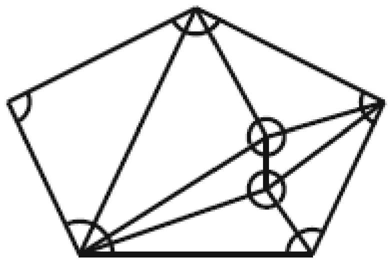


Figure 3: Triangulation of a polygon

$180° = 360° + 180°(k - 2)$.

The equation $t = 2n + k - 2$ shows that the parity of the number $t$ of the triangles in any triangulation equals to the parity of the number $k$ of the sides of the polygon. Therefore, for the special case of a square, we obtain the following result, a variation of which was given in the 32th Bulgarian Mathematical Olympiad, 1983 (Kenderov, Tabov, 1990).

**Statement** 1. A square can be triangulated into any even number of triangles but cannot be triangulated into an odd number of triangles.

There are situations dealing with the so-called semi-triangulation of a polygon. This is a partition of a polygon into finite number of triangles that satisfies only property (2) of the triangulation, i.e. it is allowed that some of the vertices of the triangles lie on the sides of the polygon. The number $t$ of the triangles in any semi-triangulation is $t = 2n + m + k - 2$, where $k$ and $n$ are as above, and $m$ is the number of all vertices of the triangles on the sides of the polygon excluding its vertices.

In some cases the question is about cutting a polygon into special type of triangles.

**Problem** 3. Prove that there is a semi-triangulation of a square into 8 acute-angled triangles but not into less than 8 acute-angled triangles.

**Solution**. Figure 4 presents a semi-triangulation of a square $ABCD$ into 8 acute-angled triangles. To do this, let $E$ and $F$ be the midpoints of $AB$ and $CD$ respectively. Draw five semi-circles with diameters $AB, BC, AD, CF$, and $DF$ inside the square. Choose points $G$ and $H$ in the region of the square that is outside these semi-circles such that $G$ and $H$ be symmetrical about $EF$. Draw the solid lines. Prove that the obtained triangles are acute-angled.
For the second part of the problem, assume that there is a semi-triangulation of a square into less than 8 acute-angled triangles. Each interior vertex of the semi-triangulation must be a common point for at least 5 segments. Each vertex that lies on a side of a square must be a common point for at least 2 segments that are inside the square. From $t = 2n + m + 2$ and $t \leq 7$ it follows that $2n + m \leq 5$. The case $n = 1$ is not possible. The other cases: $\{n = 2.m = 0\}$ and $\{n = 2, m = 1\}$ are also impossible.
Here is another example. It is obvious that a square can be cut into a finite number of right angled triangles. However, if we make some restrictions on the shape of the right angled triangles it may happen that such a partition cannot be done.

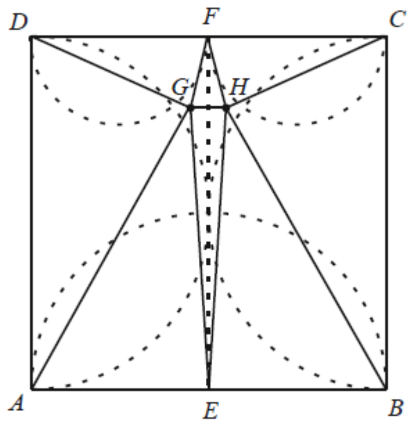



Figure 4: Semi-triangulation of a square into 8 acute-angled triangles

**Problem** 4. Prove that there is no semi-triangulation of a square into a finite number of right-angled triangles each having an angle of 30°.

**Solution**. Assume that there is such a semi-triangulation of a square. Let the length of the side of the square be 1. Denote by $a$ the shortest side of all triangles in the semi-triangulation. Then the length of the shortest side of any of the triangles can be expressed as $2^m\sqrt{3^n}a$ where $m$ and $n$ are whole numbers. The area of such a triangle is $2^{m-1}3^n\sqrt{3}a^2$. The sum of the area of all triangles of the semi-triangulation equals the area of the square. This equation can be written as $M\sqrt{3}a^2 = 1$, where $M$ is a natural number. On the other hand, one of the sides of the square equals the sum of the lengths of the sides of the triangles of the semi-triangulation. This equation is of the form $(N + P\sqrt{3})a = 1$ where $N$ and $P$ are natural numbers. Then, the area of the square is $(N^2 + 3P^2 + 2NP\sqrt{3})a^2 = 1$. Therefore, we obtain $M\sqrt{3} = N^2 + 3P^2 + 2NP\sqrt{3}$ which is impossible.

The condition that the partition in Problem 4 is a semi-triangulation is not important. The Statement in Problem 4 can be generalized in the following way.

**Statement** 2. A square cannot be cut into a finite number of right-angled triangles each having an angle of 30°.

There is not an elementary proof of Statement 2. It is proven by Laczkovich (1990) using such tools as fields, vector spaces, isomorphism between fields, and the complex roots of the unity. Actually, Laczkovich proved a more general statement.

**Statement** 3. A square cannot be cut into a finite number of right-angled triangles all of whose angles, when measured in degrees, are even integers.

Figure 5 shows that a $1 \times 4$ rectangle can be cut into three right-angled triangles each having an angle of $15°$.

It follows that a square can be cut into 12 right-angled triangles each having an angle of $15°$.

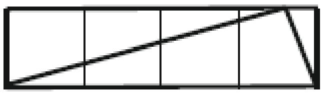


Figure 5: Partition of a $1 \times 4$ rectangle into three right-angled triangles with an angle of $15^\circ$ each

## Partition into Triangles of Equal Area

The source of this section is not a problem from competitions but from an examination. However, the story is so interesting and instructive that it deserves to be told.
It is easy to see that a square can be cut into any even number of triangles of equal area. Intuitively, one may expect that this is also true for any odd number of triangles. Perhaps, this is the reason that in 1965 Fred Richman from the University of New Mexico intended to pose the problem for cutting a square into any odd number of triangles of equal area on an examination in a master's program. He tried to solve it prior the exam, but he did not succeed and the problem was not posed on the exam. The continuation of the story shows that this is a good example when intuition is lying. Cutting a square into an odd number of triangles of equal area was posed as a problem in the American Mathematical Monthly (Richman, F. & Thomas. J., 1967) and nobody solved it. In 1968 John Thomas published a paper (Thomas, J., 1968) where he proved that it is not possible to cut a square into an odd number of triangles of equal area for a special case of the position of the vertices of the triangles. This wonderful proof combines combinatorial and analytical methods. It is understandable for more able students in the last grades of school or in the beginning courses of the university. Here is a brief overview of Thomas' evidence. He is looking for an answer to the following

**Question** 3. Can a square be cut into an odd number of triangles of equal area?

Denote by $R(a, b)$ the rectangle with vertices $(0; 0)$, $(a; 0)$, $(0; b)$, and $(a; b)$ in a Cartesian coordinate system $O_{xy}$. The transformation $T : \{x' = \lambda x, y' = \mu y\}$ maps $R(a, b)$ onto $R(\lambda a, \mu b)$ and multiplies the area by the constant factor $\lambda\mu$. Therefore, Question 3 is equivalent to

**Question** 4. Can a rectangle be cut into an odd number of triangles of equal area?

Note that a triangle of integer area $s$ can be cut into $s$ triangles of unit area. Let $N = mn$ where $m$ and $n$ are odd integers. If Question 4 has a positive answer for $N$ triangles, then $R(m, n)$ can be cut into triangles of unit area. Therefore, Question 4 reduces to

**Question** 5. If $m$ and $n$ are odd integers, can $R(m, n)$ be cut into finite number of triangles of integer area?

Thomas made two restrictions: (1) he considers semi-triangulations of the rectangle; (2) the vertices of the semi-triangulations can only be the so-called lattice-points, i.e. points in the plane whose coordinates are both integers.

There are four types of lattice-points depending on the parity of their coordinates, namely: $A$ = (even, even), $B$ = (even, odd), $C$ = (odd, even), and $D$ = (odd, odd). For each segment having vertices at lattice points $v_1 = (x_1, y_1)$ and $v_2 = (x_2, y_2)$ consider the determinant $\Delta(v_1, v_2) = \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}$. A segment is called even or odd depending on whether $\Delta(v_1, v_2)$ is even or odd, respectively. The area of a triangle whose vertices are lattice-points $v_1$, $v_2$ and $v_3$ is equal to $\frac{1}{2}(\Delta(v_1, v_2) + \Delta(v_2, v_3) + \Delta(v_3, v_1))$. Such a triangle has an integer area if and only if it has exactly null or two odd sides. It is easy to check that out of ten types of segments only three are odd and they are $\{B, C\}$, $\{C, D\}$ and $\{B, D\}$.

It follows that a lattice triangle has an integer area if and only if its vertices are not of three different types. Let $m$ and $n$ be odd integers. Assume that there is a semi-triangulation of $R(m, n)$ into triangles of integer area. Since the vertices of $R(m, n)$ are all of different types, its sides are of types $\{A, B\}$, $\{C, D\}$ $\{A, C\}$ and $\{B, D\}$. The sides of type $\{A, B\}$ of the triangles in the semi-triangulation may appear either inside $R(m, n)$ or on its $\{A, B\}$ side. The number of these that are on the boundary of $R(m, n)$ is odd. These that are inside $R(m, n)$ are common for two triangles of the semi-triangulation. Therefore, the total number of $\{A, B\}$ sides in all triangles is odd. The triangles whose vertices are not of three different types contribute an even number of sides $\{A, B\}$. Hence, there must be a triangle whose vertices are of three different types. This triangle does not have an integer area. Therefore there is no lattice-point semi-triangulation of $R(m, n)$ into triangles of integer area for odd integers $m$ and $n$.

Thomas also considered partitions that are not semi-triangulations. His final result is the following

**Statement** 4. It is not possible to cut the unit square into an odd number of triangles having the same area, for which all vertices have rational numbers with odd denominators as coordinates.

The story ended when finally, Paul Monsky (1970) proved the following statement without any restrictions of the vertices of the triangles.

**Statement** 5. A square cannot be cut into an odd number of triangles having the same area.

An elementary proof of Statement 5 is not known. Monsky's proof is beautiful but it uses topological tools (Sperner's lemma) and 2-adic valuations. In some sense, Monsky extended Thomas' idea presented above. He begins with the so-called "2-adic valuation function" $\phi : \mathbb{Q} \to \mathbb{Q}$ defined for any rational number $q \neq 0$ , $q = \frac{2^a b}{c}$ where $b$ and $c$ are odd integers, as $\phi(q) = a$. Then he uses the non-trivial assertion that this function can be extended to a function $\Phi : \mathbb{R} \to \mathbb{R}$ that satisfies some useful properties. Using this last function it is possible to color the points of the square in three colors. Combinatorial arguments show that there is a triangle in the partition whose vertices are in three different colors. This leads to the conclusion that the number of the triangles is even.

## Tiling

In some situations it is more convenient to express the tasks as tiling rather than cutting.
Mathematically, this is all the same. Many contest problems use the idea of tiling. One of the most popular problem is to show that an chessboard with two diagonally opposite corners removed

cannot be tiled with dominos. In this and many other cases coloring is a useful method.

Let $m$ and $n$ be natural numbers. Coloring is not needed to see that an $m \times n$ rectangle can be tiled with dominos if and only if either $m$ or $n$ is even. The $m \times n$ rectangle can by tiled with $3 \times 1$ tiles if and only if either $m$ or $n$ is divisible by 3, because the area $mn$ of the rectangle must be divisible by 3. The same argument can be used for tiling an $m \times n$ rectangle with $k \times 1$ tiles for any prime number $k$. To investigate which $m \times n$ rectangle can be tiled with $k \times 1$ tiles for any natural number $k$, coloring in $k$ colors may be used. A more powerful method, however, is to use the complex roots of unity. Assume that a rectangle $m \times n$ can be tiled with $k \times 1$ tiles. "Color" the cell in row $i$ and column $j$ in the rectangle with "color" $z^{i+j-2}$, where $1$, $z$, $z^2, \dots, z^{k-1}$ are the $k$th complex roots of the unity (Figure 6).

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1 | $z$ | $z^2$ | … | … | $z^{k-1}$ | 1 | $z$ | $z^2$ | … |
| $z$ | $z^2$ | $z^3$ | … | $z^{k-1}$ | 1 | … | | | |
| $z^2$ | $z^3$ | … | | | | | | | |
| ⋮ | ⋮ | | | | | | | | |
| ⋮ | ⋮ | | | | | | | | |
| $z^{k-1}$ | 1 | | | | | | | | |
| 1 | $z$ | | | | | | | | |
| ⋮ | ⋮ | | | | | | | | |

Figure 6: Using $k$th complex roots of the unity as $k$ colors

Each $k \times 1$ tile covers all "colors" exactly once. The sum of the covered "colors" by each $k \times 1$ tile is $1 + z + z^2 + \cdots + z^{k-1} = 0$. Therefore, the sum of all entries in the cells of the tiled rectangle must also be 0, that is $0 = \sum_i^m \sum_j^n z^{i+j-2} = (\sum_i^m z^{i-1})(\sum_j^n z^{j-1})$. It follows that one of the sums in the brackets must be 0. But $\sum_i^m z^{i-1} = 0$ if and only if $z^m = 1$, which means that $m$ is divisible by $k$. If the other sum is 0, it means that $n$ is divisible by $k$. We have proved:

**Statement** 6. An $m \times n$ rectangle can be tiled with $k \times 1$ tiles if and only if $k$ divides either $m$ or $n$.

Using similar arguments one may prove

**Statement** 7. An $m \times n$ rectangle can be tiled with $k \times r$ tiles if and only if $k$ divides either $m$ or $n$ and $r$ divides either $m$ or $n$.

We will consider now a generalization of the last statement. In some sense, the generalization is a transition from a discrete to a continuous case. Let $a$ and $b$ be positive real numbers.

**Statement** 8. If an $a \times b$ rectangle can be tiled by rectangles each of which has at least one side of integer length, then either $a$ or $b$ is an integer.

It seems that it was popular at the second half of the last century to find different proofs of Statement 8. For example, Wagon (1987) describes 14 proofs. Here is one of them that shows

the analogy between the discrete and continuous case.
Assume that the coordinates of the vertices of the tiled rectangle are $(0; 0)$, $(a; 0)$, $(0; b)$, and $(a; b)$ in a Cartesian coordinate system $O_{xy}$. In the reasoning before Statement 6 (discrete case) we assigned each square $(i; j)$ the value $z^{i+j-2}$. We will now assign each point $(x; y)$ of the square (continuous case) the value $f(x, y) = \sin 2\pi x \sin 2\pi y$. The "sum" of the values of all points that each tile $T$ covers is $\iint_T f(x, y) = 0$ because each tile has a side of integer length and the integral of $\sin 2\pi x$ over any interval with integer length is zero. Therefore, the integral of $f(x, y)$ over the whole tiled rectangle must also be zero, that is $0 = \int_0^a \int_0^b \sin 2\pi x \sin 2\pi y dx dy = \frac{1}{(2\pi)^2}(1 - cos 2\pi a)(1 - cos 2\pi a)$. Hence, either $cos 2\pi a = 1$ or $cos 2\pi b = 1$, which means that either $a$ or $b$ is an integer. A beautiful generalization, isn't it?

In line with what has been said here, it is worth noting a result of David Hilbert's student Max Dehn. In the beginning of the 20th century Dehn contributed to the solution to the third Hilbert's problem. Soon after this he published a paper (Dehn, 1903) where he proved the following

**Statement** 9. A rectangle can be tiled with finitely many squares if and only if the ratio of its sides is a rational number.

The statement is not surprising itself but the proof is quite difficult. Dehn's original proof is very complicated. Improvements of the proof have been made over the years. However, until now no relatively easy to understand proof has been found.

The proofs of many of the generalizations described in this paper are not on elementary mathematics level. Even if the statements sound geometric, most of the proofs use advanced algebra tools.

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# A Trigonometric Inequality

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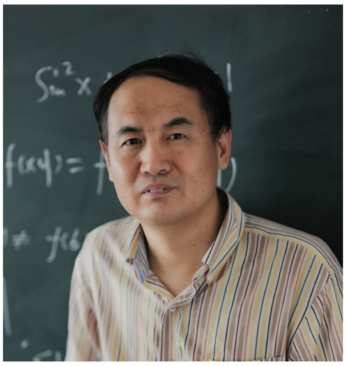


Zhu Huawei has, since 2016, been the Principal of Shenzhen Middle School in Shenzhen, one of the top-ranked middle schools in China. His highest academic degree is a Doctor of Philosophy in Science Education, obtained from the Central China University of Science and Technology in 2005. His most recent award was as the Best Educator of the Year 2019 in China. He had served as the Leader of the Chinese national team in the International Mathematical Olympiad in 2009. He was on the Problem Committee of the Chinese Mathematical Olympiad from 2005 to 2016, and since 2011, he has been the Director of the International Mathematics Assessment Services. He has authored over one hundred books and over one hundred articles on mathematics competitions in Chinese. He also has publications in English outside China.

## Introduction

The following problem appears in 1993 in the Sixth Irish Mathematical Olympiad.

**Problem 1.**

Prove that for all integers $n \geq 1$ and all real numbers $x$ such that $0 < x < \pi$,

$$\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots + \frac{\sin(2n-1)x}{2n-1} > 0.$$

This trigonometric inequality has a simple uniform structure. It has a lot of depth, and its proof is anything but simple. In this paper we present our proofs, consider some variations, explore what lie behind it and give some generalizations. There are also some exercises with solutions at the end.

# Proofs

We denote the left side of the above inequality by $f(x)$. We present three proofs.

**First Proof:** We use product-to-sum trigonometric formula,

$$2\sin x\sin\left(2k-1\right)x=\cos\left(2k-2\right)x-\cos 2kx$$

Then

$$\begin{aligned}
&2f\left(x\right)\sin x\\
&=1-\cos 2x+\frac{\cos 2x-\cos 4x}{3}+\frac{\cos 4x-\cos 6x}{5}+\cdots+\frac{\cos\left(2n-2\right)x-\cos 2nx}{2n-1}\\
&=1-\left(1-\frac{1}{3}\right)\cos 2x-\left(\frac{1}{3}-\frac{1}{5}\right)\cos 4x-\cdots-\left(\frac{1}{2n-3}-\frac{1}{2n-1}\right)\cos\left(2n-2\right)x-\frac{\cos 2nx}{2n-1}\\
&\geqslant 1-\left(1-\frac{1}{3}\right)-\left(\frac{1}{3}-\frac{1}{5}\right)-\left(\frac{1}{5}-\frac{1}{7}\right)-\cdots-\left(\frac{1}{2n-3}-\frac{1}{2n-1}\right)-\frac{1}{2n-1}
\end{aligned}$$

The above equality holds if and only if $\cos 2kx=1\left(k=1,2,\cdots,n\right)$. Since $0<x<\pi$, we have $\cos 2x\neq 1$. So $f\left(x\right)\sin x>0$, and $f\left(x\right)>0$. Hence we have proved the original inequality.
**Remark:** The key step is to use trigonometric identities to change the product $2\sin x\sin(2k-1)x$ into the difference $\cos(2k-2)x-\cos 2kx$. Although we do not achieve cancellations directly, this is possible after recombining the resulting terms, yielding the result $2f(x)\sin x>0$.

**Second Proof:** Let $a_k=\frac{1}{2k-1}$, $b_k=\sin\left(2k-1\right)x$, then $f\left(x\right)=\sum\limits_{k=1}^{n}a_kb_k$. Note that by product-to-sum trigonometric formula we have

$$\begin{aligned}
\sum_{k=1}^{n}b_k&=\sum_{k=1}^{n}\sin\left(2k-1\right)x\\
&=\frac{\sum\limits_{k=1}^{n}2\sin x\sin\left(2k-1\right)x}{2\sin x}\\
&=\frac{\sum\limits_{k=1}^{n}\left[\cos\left(2k-2\right)x-\cos 2kx\right]}{2\sin x}\\
&=\frac{1-\cos 2nx}{2\sin x}\\
&=\frac{2\sin^2 nx}{2\sin x}
\end{aligned}$$

Hence by Abel Transform

$$f\left(x\right)=\sum_{k=1}^{n}a_kb_k=a_n\sum_{k=1}^{n}b_k+\left(a_{n-1}-a_n\right)\sum_{k=1}^{n-1}b_k+\cdots+\left(a_1-a_2\right)b_1$$

Since $0<x<\pi$, and $\{a_k\}$ is strictly decreasing, we have $a_{t-1}-a_t>0$, and $\sum\limits_{k=1}^{t}b_t=\frac{2\sin^2 tx}{2\sin x}>0$, where $t=2,3,\cdots,n$, and $a_n,b_1>0$. Thus $f\left(x\right)>0$. We have proved the original inequality.

**Remark:** We use the Abel Transforms to separate, in $f(x) = \sum_{k=1}^{n} a_k b_k$, the summable trigonometric series $b_k = \sin(2k-1)$ from the strictly decreasing sequence $a_k = \frac{1}{2k-1}$. This allows us to deal with them separately. Particularly useful is the fact that $\sum_{k=1}^{t} b_k = \dfrac{2\sin^2 tx}{2\sin x} > 0$ for $\leq t \leq n$. Hence after the Abel Transform, every term in the sum of $f(x)$ is positive, leading to $f(x) > 0$.

**Third Proof:**

$$\begin{aligned} f'(x) &= \cos x + \cos 3x + \cos 5x + \cdots + \cos(2n-1)x \\ &= \frac{1}{2\sin x}\left[2\sin x\cos x + 2\sin x\cos 3x + 2\sin x\cos 5x + \cdots + 2\sin x\cos(2n-1)x\right] \\ &= \frac{1}{2\sin x}\left[\sin 2x + \sin 4x - \sin 2x + \sin 6x - \sin 8x + \cdots + \sin 2nx - \sin(2n-2)\right] \\ &= \frac{\sin 2nx}{2\sin x}, \end{aligned}$$

Let $f'(x) = 0$. Since $0 < x < \pi$, we have $x_k = \frac{k\pi}{2n}$, where $k = 1, 2, \cdots, 2n-1$. In each interval $(0, x_1), (x_2, x_3), \cdots, (x_{2n-2}, x_{2n-1})$ the function $f$ is strictly increasing, as $f'(x) > 0$ there; in each interval $x \in (x_1, x_2), (x_3, x_4), \cdots, (x_{2n-1}, \pi)$ the function $f$ is strictly decreasing, as $f'(x) < 0$ there.

Note that $f(0) = f(\pi) = 0$. If we want to prove that $f > 0$, we only need to prove that all minima of $f$ are positive, i.e. $f(x_{2k}) = f\left(\frac{k\pi}{n}\right) > 0\,(k = 1, 2, \cdots, n-1)$. It is easily to prove that $f(x) = f(\pi - x)$, i.e. the graph of $f$ is symmetry with respect to the line $x = \frac{\pi}{2}$. So we only need to prove that $f\left(\frac{k\pi}{n}\right) > 0\,\left(k = 1, 2, \cdots, \left[\frac{n}{2}\right]\right)$.

Now we prove a stronger statement: the sequence $f\left(\frac{k\pi}{n}\right)\,\left(k = 0, 1, 2, \cdots, \left[\frac{n}{2}\right]\right)$ is strictly increasing.

Since $f'(x) = \frac{\sin 2nx}{2\sin x}$, we have $f(x) = \int_0^x \frac{\sin 2nt}{2\sin t}\mathrm{d}t$. Then

$$f\left(\frac{k\pi}{n}\right) - f\left(\frac{(k-1)\pi}{n}\right) = \int_{\frac{(k-1)\pi}{n}}^{\frac{k\pi}{n}} \frac{\sin 2nt}{2\sin t}\mathrm{d}t$$

Using integration by substitution we have

$$\int_{\frac{(k-1)\pi}{n}}^{\frac{k\pi}{n}} \frac{\sin 2nt}{2\sin t}\mathrm{d}t = \int_{2(k-1)\pi}^{2k\pi} \frac{\sin t}{2\sin\frac{t}{2n}}\mathrm{d}\frac{t}{2n} = \frac{1}{4n}\int_{2(k-1)\pi}^{2k\pi} \frac{\sin t}{\sin\frac{t}{2n}}\mathrm{d}t$$

Since

$$\begin{aligned} \int_{2(k-1)\pi}^{2k\pi} \frac{\sin t}{\sin\frac{t}{2n}}\mathrm{d}t &= \int_{2(k-1)\pi}^{(2k-1)\pi} \frac{\sin t}{\sin\frac{t}{2n}}\mathrm{d}t + \int_{(2k-1)\pi}^{2k\pi} \frac{\sin t}{\sin\frac{t}{2n}}\mathrm{d}t \\ &= \int_{2(k-1)\pi}^{(2k-1)\pi} \frac{\sin t}{\sin\frac{t}{2n}}\mathrm{d}t + \int_{2(k-1)\pi}^{(2k-1)\pi} \frac{\sin(t+\pi)}{\sin\frac{t+\pi}{2n}}\mathrm{d}(t+\pi) \\ &= \int_{2(k-1)\pi}^{(2k-1)\pi} \frac{\sin t}{\sin\frac{t}{2n}}\mathrm{d}t - \int_{2(k-1)\pi}^{(2k-1)\pi} \frac{\sin t}{\sin\frac{t+\pi}{2n}}\mathrm{d}t \\ &= \int_{2(k-1)\pi}^{(2k-1)\pi} \sin t\left(\frac{1}{\sin\frac{t}{2n}} - \frac{1}{\sin\frac{t+\pi}{2n}}\right)\mathrm{d}t \end{aligned}$$

When $1 \leqslant k \leqslant \left[\frac{n}{2}\right]$, $t \in [2(k-1)\pi, (2k-1)\pi]$, $\frac{t}{2n}, \frac{t+\pi}{2n} \in \left[\frac{(k-1)\pi}{n}, \frac{k\pi}{n}\right] \subseteq \left[0, \frac{\pi}{2}\right]$, we have

$$\sin t\left(\frac{1}{\sin\frac{t}{2n}} - \frac{1}{\sin\frac{t+\pi}{2n}}\right) \geqslant 0,$$

Thus

$$\int_{2(k-1)\pi}^{(2k-1)\pi} \sin t \left( \frac{1}{\sin \frac{t}{2n}} - \frac{1}{\sin \frac{t+\pi}{2n}} \right) \mathrm{d}t > 0,$$

so

$$f\left(\frac{k\pi}{n}\right) - f\left(\frac{(k-1)\pi}{n}\right) > 0 \left(k = 1, 2, \cdots, \left[\frac{n}{2}\right]\right),$$

Therefore $f\left(\frac{k\pi}{n}\right) > f\left(\frac{(k-1)\pi}{n}\right) > \cdots > f(0) = 0$. We have proved the original inequality.

**Remark:** From a functional point of view, in order to show that we always have $f(x) > 0$, it is only necessary to show that the minimum value of $f$ is positive. It is natural to investigate the monotonicity of the function and then determine its extremal values. However, $f$ has a very large number of critical points in the interval $(0, \pi)$, and the comparison of their values is difficult. Using integration to determine the monotonicity of the function seems an overkill. Perhaps the reader may have noticed that the condition $f > 0$ depends also on $n$, and perhaps induction may have a role to play. However, our effort in this direction is unsuccessful.

# Variations

The great American-Hungarian mathematician and mathematics educator György Pólya once said, "When you find a mushroom, keep your eyes open because they tend to grow in bunches." The seeking of related problems is an interesting and fruitful exercise in mathematics.

We notice that in Problem 1 we have omitted all the fractions with even denominators. What would happen if we bring them into the picture?

**Problem 2.**
Prove that for all integers $n \geq 1$ and all real numbers $x$ such that $0 < x < \pi$,

$$\sin x + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \cdots + \frac{\sin nx}{n} > 0.$$

**Proof:**

Let $f_n(x) = \sin x + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \cdots + \frac{\sin nx}{n}$, $0 \leqslant x \leqslant \pi$.

(1) When $n = 1$, $f_1(x) = \sin x > 0$ always holds for $0 < x < \pi$;

(2)Assume it holds when $n = k$, that is, assume that $f_k(x) = \sin x + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \cdots + \frac{\sin kx}{k} > 0$ always holds for $0 < x < \pi$. We want to show that it still holds when $n = k + 1$, Now

$$\begin{aligned} {f_{k+1}}'(x) = \sum_{t=1}^{k+1} \cos tx &= \frac{2 \sin \frac{x}{2} \sum\limits_{t=1}^{k+1} \cos tx}{2 \sin \frac{x}{2}} \\ &= \frac{\sum\limits_{t=1}^{k+1} \left[\sin\left(t + \frac{1}{2}\right)x - \sin\left(t - \frac{1}{2}\right)x\right]}{2 \sin \frac{x}{2}} \\ &= \frac{\sin\left(k + \frac{3}{2}\right)x - \sin \frac{x}{2}}{2 \sin \frac{x}{2}} \\ &= \frac{\sin \frac{(k+1)x}{2} \cos \frac{(k+2)x}{2}}{\sin \frac{x}{2}} \end{aligned}$$

Let ${f_{k+1}}'(x) = 0$. Since $0 < x < \pi$, we have ${x_m}' = \frac{2m\pi}{k+1}$, ${x_m}'' = \frac{(2m+1)\pi}{k+2}$, where $2m = 1, 2, \cdots, k$.

Note that $f_n(0) = f_n(\pi) = 0$, and $f_{k+1}(x)$ is continuous on $(0, \pi)$, so we only need to prove that $f_{k+1}(x)$ is positive at critical points ${x_m}', {x_m}''$. By inductive hypothesis

$$
\begin{aligned}
f_{k+1}({x_m}') &= f_k({x_m}') + \frac{\sin(k+1){x_m}'}{k+1} = f_k({x_m}') > 0 \\
f_{k+1}\left({x_m}''\right) &= f_k\left({x_m}''\right) + \frac{\sin(k+1){x_m}''}{k+1} = f_k\left({x_m}''\right) + \frac{\sin\frac{(k+1)(2m+1)\pi}{k+2}}{k+1} \\
&= f_k\left({x_m}''\right) + \frac{\sin\frac{(2m+1)\pi}{k+2}}{k+1} \\
&= f_k\left({x_m}''\right) + \frac{\sin {x_m}''}{k+1} > 0
\end{aligned}
$$

Thus $f_{k+1}(x) > 0$ always holds for $0 < x < \pi$.

Combining (1) and (2) and using the principle of mathematical induction, we have solved problem 2.

**Remark:** From the literature, we discover that this is the well-known Fejer-Jackson Inequality, first proposed by Fejer in 1910 and proved by Jackson in 1911. Since then, many proofs have been presented, but most of them require advanced knowledge and techniques. Our inductive argument makes use of the derivative to determine the critical points of $f_n(x)$, avoiding the concepts of monotonicity and extremal value determination. The trick is going from the case $n = k$ to the case $n = k + 1$ is to use the fact that the added term, $\frac{\sin(k+1)x'_m}{k+1}$ or $\frac{\sin)k+1)x''_m}{k+1}$, is positive. It should be mentioned that similar approaches are used in the three proofs of Problem 1 have not led to success. Interested readers may wish to pursue this.

What would happen if we replace $x$ by $\pi - x$ in Problem 2?

**Problem 3.**

Prove that for all integers $n \geq 1$ and all real numbers $x$ such that $0 < x < \pi$,

$$\sin x - \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots + (-1)^{n-1}\frac{\sin(2n-1)x}{2n-1} > 0.$$

Note that we can obtain a solution to Problem 3 from the solution to Problem 2 via a simple transformation, without having to go through an inductive argument again. This idea has many applications.

# Explorations

Pólya had said that after working on a mathematical problem, we should try to think of related problems. It is often fruitful to check on the history of the problem, to learn from the experience of people who had worked on it before.

We know from the work of Fourier that almost all periodic functions, including some very complicated and esoteric ones, can be expressed as limiting cases of summations of the sine and cosine function, or Fourier series. Problems 1, 2 and 3 all involve the sine function with period $2\pi$. When we let $n$

approach infinity, the limiting cases should be the Fourier series of certain functions with period $2\pi$.

To be specific, according to Fourier expansion, the function $g(x) = \begin{cases} -\frac{\pi}{4}, -\pi < x < 0, \\ \frac{\pi}{4}, 0 < x < \pi. \end{cases}$ with period $T = 2\pi$ can be expanded to

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

where

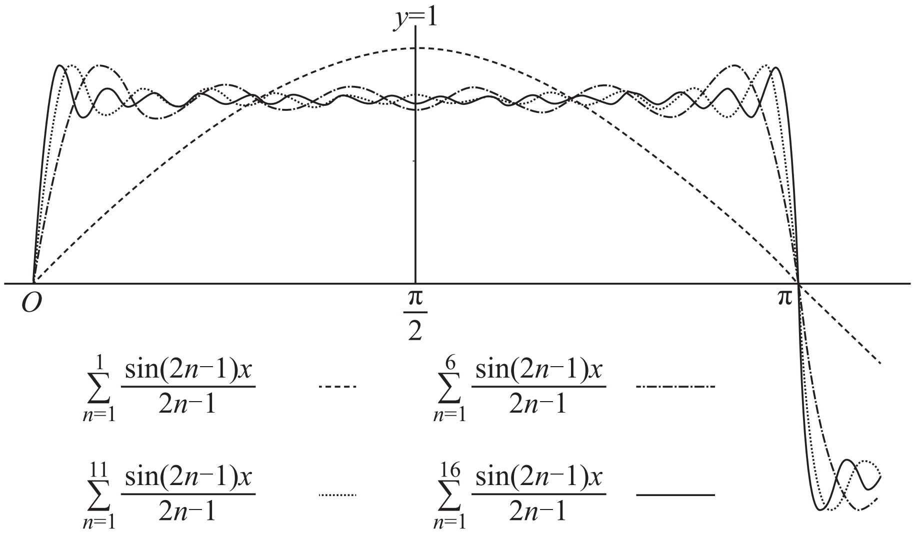
$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} g(x)\, \mathrm{d}x = 0, a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} g(x) \cos nx \mathrm{d}x = 0$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} g(x) \sin nx \mathrm{d}x = \frac{2}{\pi} \int_{0}^{\pi} \frac{\pi}{4} \sin nx \mathrm{d}x = \frac{1}{2n} [1 - (-1)^n] = \begin{cases} 0, n \text{is even,} \\ \frac{1}{n}, n \text{is odd.} \end{cases}$$

So

$$g(x) = \sin x + \frac{\sin 3x}{3} + \cdots + \frac{\sin(2n-1)x}{2n-1} + \cdots = \sum_{k=1}^{\infty} \frac{\sin(2k-1)x}{2k-1}$$

Hence, Problem 1 is to prove that $f(x) = \sum_{k=1}^{n} \frac{\sin(2k-1)x}{2k-1}$, which is the partial sum of $g(x)$ , is always positive on $(0, \pi)$. As shown in next picture, we draw the graph of the function $f(x)$ on $(0, \pi)$ in the same rectangular coordinate system when $n = 1, 6, 11, 16$. Obviously, when $n$ is increasing, the graph of $f$ is getting closer to the graph of $g$ on $(0, \pi)$. Thus by Fourier's theorem we have $g(x) = \sum_{k=1}^{\infty} \frac{\sin(2k-1)x}{2k-1} > 0$ on the interval $(0, \pi)$.




Similarly, the function $g(x) = \begin{cases} \frac{-\pi-x}{2}, -\pi < x < 0, \\ \frac{\pi-x}{2}, 0 < x < \pi. \end{cases}$ with period $T = 2\pi$ can be expanded to

$$g(x) = \sin x + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \cdots + \frac{\sin nx}{n} + \cdots = \sum_{k=1}^{\infty} \frac{\sin kx}{k}$$

Problem 2 is to prove that $f(x) = \sum_{k=1}^{n} \frac{\sin kx}{k}$, which is the partial sum of $g(x)$, is always positive on $(0, \pi)$.

The Fourier expansion of the function $g(x) = \frac{x}{2}$, for $-\pi < x < \pi$ is

$$g(x) = \sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \cdots + (-1)^{n+1}\frac{\sin nx}{n} + \cdots = \sum_{k=1}^{\infty}(-1)^{k+1}\frac{\sin kx}{k}.$$

Problem 3 is to prove that $f(x) = \sum_{k=1}^{n}(-1)^{k+1}\frac{\sin kx}{k}$, which is the partial sum of $g(x)$, is always positive on $(0, \pi)$

# Generalizations

Pólya had also said that no problem can ever be said to be completely solved; there are always issues worthy of more research. He regarded the considerations of general cases and special cases as a great source which leads to further discovery.

Following his advice, we come up with the following.

**Problem 4.**

Prove that for all positive integers $n$, all real numbers $x$ such that $0 < x < \pi$, and real numbers $a_1 \geq a_2 \geq a_3 \geq \cdots \geq a_n > 0$ we have

$$a_1 \sin x + a_2 \sin 3x + a_3 \sin 5x + \cdots + a_n \sin(2n-1)x > 0.$$

**Proof:**

Let $b_k = \sin(2k-1)x$, then $f(x) = \sum_{k=1}^{n} a_k b_k$. Similarly to the second proof of Problem 1, we have,

$$\sum_{k=1}^{n} b_k = \sum_{k=1}^{n} \sin(2k-1)x = \frac{2\sin^2 nx}{2\sin x}$$

By Abel Transform we have

$$\sum_{k=1}^{n} a_k b_k = a_n \sum_{k=1}^{n} b_k + (a_{n-1} - a_n)\sum_{k=1}^{n-1} b_k + \cdots + (a_1 - a_2)\, b_1$$

Since $0 < x < \pi$, and $\{a_k\}$ is weakly decreasing, we know that $a_{t-1} - a_t \geqslant 0$, and $\sum_{k=1}^{t} b_t = \frac{2\sin^2 tx}{2\sin x} > 0$, where $t = 2, 3, \cdots, n$, and $a_n, b_1 > 0$. Thus $f(x) > 0$ always holds. We have proven the original inequality.

Note that Problem 1 is the special case of Problem 4 where $a_k = \frac{1}{2k-1}, k = 1, 2, \cdots, n$.

**Problem 5.**

Prove that for all positive integers $n$, all real numbers $x$ such that $0 < x < \pi$, and real numbers $a_1 \geq a_2 \geq a_3 \geq \cdots \geq a_n > 0$,

$$a_1 \sin x + a_2 \frac{\sin 2x}{2} + a_3 \frac{\sin 3x}{3} + \cdots + a_n \frac{\sin nx}{n} > 0.$$

**Proof:**

Note that

$$\frac{d}{dt}\frac{\sin at}{a(\sin t)^a} = -\frac{\sin(a-1)t}{(\sin t)^{a+1}}$$

Let $a = 2k, t = \frac{x}{2}$, then

$$\frac{d}{dx}\frac{\sin kx}{k(\sin\frac{x}{2})^{2k}} = -\frac{\sin\frac{(2k-1)x}{2}}{(\sin\frac{x}{2})^{2k-1}}$$

By integrating both sides of the equation on $(x, \pi)$ we can get

$$\frac{\sin kx}{k(\sin\frac{x}{2})^{2k}} = \int_x^\pi \frac{\sin\frac{(2k-1)x}{2}}{(\sin\frac{x}{2})^{2k-1}}dx = \int_{\frac{x}{2}}^{\frac{\pi}{2}} \frac{2\sin(2k-1)\theta}{(\sin\theta)^{2k-1}}d\theta$$

So

$$\frac{\sin kx}{k} = 2\int_{\frac{x}{2}}^{\frac{\pi}{2}} \left(\sin\frac{x}{2}\right)^{2k}\frac{\sin(2k-1)\theta}{(\sin\theta)^{2k-1}}d\theta = 2\int_{\frac{x}{2}}^{\frac{\pi}{2}} \left(\frac{\sin\frac{x}{2}}{\sin\theta}\right)^{2k}\frac{\sin(2k-1)\theta}{\sin\theta}d\theta$$

Then

$$\sum_{k=1}^{n}\frac{a_k\sin kx}{k} = 2\int_{\frac{x}{2}}^{\frac{\pi}{2}}\sum_{k=1}^{n} a_k\left(\frac{\sin\frac{x}{2}}{\sin\theta}\right)^{2k}\frac{\sin(2k-1)\theta}{\sin\theta}d\theta$$

Let $b_k = a_k\left(\frac{\sin\frac{x}{2}}{\sin\theta}\right)^{2k}\frac{1}{\sin\theta}$. Since $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_n > 0$ we have $b_1 \geqslant b_2 \geqslant \cdots \geqslant b_n > 0$. Now by the result from Problem 4 we know that

$$\sum_{k=1}^{n} a_k\left(\frac{\sin\frac{x}{2}}{\sin\theta}\right)^{2k}\frac{\sin(2k-1)\theta}{\sin\theta} = \sum_{k=1}^{n} b_k\sin(2k-1)\theta > 0$$

Thus

$$\sum_{k=1}^{n}\frac{a_k\sin kx}{k} > 0$$

We have proven the original inequality.

Note that Problem 2 is the special case of Problem 5, where $a_k = 1, k = 1, 2, \cdots, n$.

**Remark:** In the past decades there were presented many elegant, rigorous, innovative, deep and widely-spread problems springing up among the native and international competitions. These problems are brilliant resources to improve people's mathematics capability. During the solving, practicing and researching progress of those problems, we are getting deeper into the insight of the world. Therefore, we not only gain methods of studying knowledge but also happiness by practicing mathematics, which is beneficial to our professional growth.

# Exercises

**Exercise 1.** (1949 Kürschák Competition, Hungary)

Prove that for each real number $x$ such that $0 < x < \pi$, we have $\sin x + \dfrac{\sin 2x}{2} + \dfrac{\sin 3x}{3} > 0$.

**Exercise 2.**

Prove that all any real number $x$, we have$\sin x + \sin 2x + \sin 3x < \frac{5}{2}$.

**Exercise 3.**

Prove that for all integers $n$ and all real numbers $x$ such that $0 < x < \frac{\pi}{2}$

$$\cos x - \frac{\cos 3x}{3} + \frac{\cos 5x}{5} + \cdots + (-1)^{n-1}\frac{\cos(2n-1)x}{2n-1} > 0.$$

**Exercise 4.** (1967 Putnam Competition, USA)

Let $f(x) = a_1 \sin x + a_2 \sin 2x + \cdots + a_n \sin nx$, where $n$ is a positive integer and $a_1, a_2, \ldots, a_n$ are real numbers. If $|f(x)| < |\sin x|$ for any real number $x$, prove that $|a_1 + 2a_2 + \cdots + na_n| \leq 1$.

**Exercise 5.**

Prove that for all positive integers $n$ and all real numbers $x$

$$|\sin x| + \frac{|\sin 2x|}{2} + \frac{|\sin 3x|}{3} + \cdots + \frac{|\sin nx|}{n} \geqslant |\sin nx|.$$

**Exercise 6.**

Prove that for all integers $n \geq 1$ and all real numbers $x$ such that $0 < x < \pi$,

$$\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots + \frac{\sin(2n-1)x}{2n-1} \leq 2\sqrt{\pi}.$$

**Solution to Exercise 1.**

This is of course just a special case of Problem 2. We present a solution without reference to Problem 2.

**Proof:** Let $x \in (0, \pi)$. Then

$$\begin{aligned}
&\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x \\
&= \sin x + \sin x \cos x + \frac{1}{3}\left(3\sin x - 4\sin^3 x\right) \\
&= \sin x \left(2 + \cos x - \frac{4}{3}\sin^2 x\right) \\
&= \frac{\sin x}{3}\left(4\cos^2 x + 3\cos x + 2\right) \\
&= \frac{\sin x}{3}\left[(1+\cos x)^2 + (1+\cos x) + 3\cos^2 x\right] \\
&> 0
\end{aligned}$$

**Solution to Exercise 2.**

**Proof:** We have

$$\begin{aligned}
&\sin x+\sin 2x+\sin 3x\\
&\leqslant \left|\sin 2x+(\sin x+\sin 3x)\right|\\
&=2\left|\sin x\cos x+\sin 2x\cos x\right|\\
&\leqslant 2\left(\left|\sin x\cos x\right|+\left|\sin 2x\cos x\right|\right)\\
&\leqslant 2\sqrt{\left(\sin^2 x+\cos^2 x\right)\left(\cos^2 x+\sin^2 2x\right)}\\
&=2\sqrt{\cos^2 x+4\left(1-\cos^2 x\right)\cos^2 x}\\
&=2\sqrt{-4\left(\cos^2 x-\frac{5}{8}\right)^2+\frac{25}{16}}\\
&\leqslant 2\sqrt{\frac{25}{16}}\\
&=\frac{5}{2}.
\end{aligned}$$

**Solution to Exercise 3.**

**Proof:** In Problem 1, let $t=\frac{\pi}{2}-x\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$, then

$$\cos t-\frac{\sin 3t}{3}+\frac{\sin 5t}{5}+\cdots+(-1)^{n-1}\frac{\sin\left(2n-1\right)t}{2n-1}>0$$

Hence the inequality holds.

**Solution to Exercise 4.**

**Proof:** Let $M=|a_1|+|a_2|+\cdots+|a_n|$. For a positive integer $k$ $(1\leqslant k\leqslant n)$, since $\lim\limits_{x\to 0}\frac{\sin kx}{\sin x}=k$, by definition of the limit of a function, for any $\varepsilon>0$ there exists $x$ such that $\sin x\neq 0$, and

$$\left|\frac{\sin kx}{\sin x}-k\right|<\frac{\varepsilon}{M}, k=1,2,\cdots,n.$$

Now we have

$$\begin{aligned}
1\geqslant\left|\frac{f\left(x\right)}{\sin x}\right|&=\left|\sum_{k=1}^{n}\frac{a_k\sin kx}{\sin x}\right|=\left|\sum_{k=1}^{n}ka_k-\sum_{k=1}^{n}\left(\frac{\sin kx}{\sin x}-k\right)a_k\right|\\
&\geqslant\left|\sum_{k=1}^{n}ka_k\right|-\sum_{k=1}^{n}\left|\left(\frac{\sin kx}{\sin x}-k\right)\right|\left|a_k\right|\geqslant\left|\sum_{k=1}^{n}ka_k\right|-\varepsilon
\end{aligned}$$

Since $\varepsilon>0$ is arbitrary, $|a_1+2a_2+\cdots+na_n|=\left|\sum\limits_{k=1}^{n}ka_k\right|\leqslant 1$.

**Solution to Exercise 5.**

**Proof:** First, note that

$$\sin|x+y|=|\sin x\cos y+\cos x\sin y|\leqslant|\sin x\cos y|+|\cos x\sin y|\leqslant|\sin x|+|\sin y|\,.$$

We proceed by induction on $n$.

(1) When $n=1$, it is trivial;

(2) Assume it holds when $n\leqslant k$, i.e.

$$|\sin x|\geqslant|\sin x|\,,$$

$$|\sin x| + \frac{|\sin 2x|}{2} \geqslant |\sin 2x|,$$
$$\cdots\cdots,$$
$$|\sin x| + \frac{|\sin 2x|}{2} + \frac{|\sin 3x|}{3} + \cdots + \frac{|\sin kx|}{k} \geqslant |\sin kx|.$$

Adding these inequalities above we have

$$k|\sin x| + (k-1)\frac{|\sin 2x|}{2} + \cdots + 1 \cdot \frac{|\sin kx|}{k} \geqslant |\sin x| + |\sin 2x| + \cdots + |\sin kx|.$$

If we add $|\sin x| + |\sin 2x| + |\sin 3x| + \cdots + |\sin kx|$ to both side of the inequality, then

$$\begin{aligned}&(k+1)\left(|\sin x| + \frac{|\sin 2x|}{2} + \frac{|\sin 3x|}{3} + \cdots + \frac{|\sin kx|}{k}\right)\\ &\geqslant (|\sin x| + |\sin kx|) + (|\sin 2x| + |\sin(k-1)x|) + \cdots + (|\sin kx| + |\sin x|)\\ &\geqslant k|\sin(k+1)x|.\end{aligned}$$

Thus

$$|\sin x| + \frac{|\sin 2x|}{2} + \frac{|\sin 3x|}{3} + \cdots + \frac{|\sin kx|}{k} + \frac{|\sin(k+1)x|}{k+1} \geqslant |\sin(k+1)x|.$$

Hence when $n = k+1$, the original inequality holds.

Combining (1), (2) and the principle of mathematical induction we have proved that $|\sin x| + \frac{|\sin 2x|}{2} + \frac{|\sin 3x|}{3} + \cdots + \frac{|\sin nx|}{n} \geqslant |\sin nx|$.

**Solution to Exercise 6.**

**Proof:** For any fixed $x \in (0, \pi)$, let $m = \left[\frac{\sqrt{\pi}}{x}\right]$, then $m \leqslant \frac{\sqrt{\pi}}{x} < m+1$, and

$$\sum_{k=1}^{n} \frac{\sin kx}{k} = \sum_{k=1}^{m} \frac{\sin kx}{k} + \sum_{k=m+1}^{n} \frac{\sin kx}{k}$$

When $m = 0$, the first sum of right hand is 0; when $m \geqslant n$, the second sum of right hand is 0, and in the first sum $k$ is from 1 to $n$.

When $0 < x < \pi$, we have $\sin kx < kx$. Thus

$$\sum_{k=1}^{m} \frac{\sin kx}{k} \leqslant \sum_{k=1}^{m} \frac{kx}{k} = mx \leqslant \sqrt{\pi}$$

Let $S_i = \sum\limits_{k=m+1}^{i} \sin kx$, $i = m+1, m+2, \cdots, n$. Then

$$\begin{aligned}S_i \cdot \sin\frac{x}{2} &= \frac{1}{2}\sum_{k=m+1}^{i}\left[\cos\left(k-\frac{1}{2}\right)x - \cos\left(k+\frac{1}{2}\right)x\right]\\ &= \frac{1}{2}\left[\cos\left(m+\frac{1}{2}\right)x - \cos\left(i+\frac{1}{2}\right)x\right]\\ &\leqslant 1\end{aligned}$$

thus $S_i \leqslant \frac{1}{\sin\frac{x}{2}}$ for $i = m+1, m+2, \cdots, n$.

Let $a_k = \sin kx, b_k = \frac{1}{k}$, $k = m+1, m+2, \cdots, n$. Then $b_{m+1} \geqslant b_{m+2} \geqslant \cdots \geqslant b_n$, so by Abel inequality we know

$$\sum_{k=m+1}^{n} \frac{\sin kx}{k} \leqslant \left| \sum_{k=m+1}^{n} \frac{\sin kx}{k} \right| \leqslant \frac{1}{\sin \frac{x}{2}} \cdot \frac{1}{m+1},$$

Since $0 < \frac{x}{2} < \frac{\pi}{2}$, by Jordan's inequality we have $\sin \frac{x}{2} > \frac{2}{\pi} \cdot \frac{x}{2} = \frac{x}{\pi}$, thus

$$\frac{1}{\sin \frac{x}{2}} \cdot \frac{1}{m+1} \leqslant \frac{1}{\frac{x}{\pi} \cdot \frac{\sqrt{\pi}}{x}} = \sqrt{\pi}$$

Then

$$\sum_{k=1}^{n} \frac{\sin kx}{k} = \sum_{k=1}^{m} \frac{\sin kx}{k} + \sum_{k=m+1}^{n} \frac{\sin kx}{k} \leqslant 2\sqrt{\pi}$$

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# Concurrency, collinearity and other properties of a particular hexagon

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**Abstract**
This paper briefly describes a heuristic investigation using dynamic geometry of some concurrency, collinearity and other properties of a particular hexagon that would be of interest to mathematics olympiad enthusiasts. After managing to prove the initial result, further reflection on the proof led to an immediate generalisation, illustrating the so-called 'discovery' function of proof.

# Introduction

It is often said that theorems in mathematics are mostly discovered by means of intuition and/or experimental methods, before they are verified by the production of proofs. However, there are perhaps just as many examples in the history of mathematics where new results were discovered or invented in a purely deductive manner.

For example, it is completely unlikely that some results (like the non-Euclidean geometries) could ever have been chanced upon merely by intuition and/or only using experimental and/or inductive methods. Many similar historical examples can be given in regard to the development of abstract algebra, set theory, calculus, etc. Generally, to the working mathematician, proof is therefore not just a means of verifying an already-discovered result, but often also a means of exploring, analyzing, discovering and inventing new results. Indeed, quite frequently explaining (proving) why a result is true enables further generalisation (or specialisation). This valuable function of proof has been called the discovery function of proof by De Villiers (1990, 1997). Apart from presenting some new concurrency, collinearity and parallel results related to a special type of hexagon that should be accessible and of interest to talented mathematics olympiad students at college and high school level, this paper will also roughly describe the heuristic process by which these results were arrived at.

## Start of the Investigation

The first conjecture below was initially discovered and experimentally verified with a dynamic geometry program. However, as shown further on, proving it allowed us to immediately generalise it, and provides an instructive example of the discovery function of proof.

**Conjecture 1**. Given a hexagon $ABCDEF$ with $AB = BC, CD = DE, EF = FA$, and $\angle A = \angle C = \angle E = 120°$, then $AD, BE$, and $CF$ are concurrent at $P$.

Proof. The result follows directly from the following useful theorem by Anghel (2016): Given a hexagon $ABCDEF$, then the main diagonals $AD, BE$ and $CF$ are concurrent, if and only if
$\sin(\angle BCE) \cdot \sin(\angle DEA) \cdot \sin(\angle FAC) \cdot \sin(\angle DCE) \cdot \sin(\angle FEA) \cdot \sin(\angle BAC)$
$= \sin(\angle ACD) \cdot \sin(\angle CEF) \cdot \sin(\angle EAB) \cdot \sin(\angle BCA) \cdot \sin(\angle DEC) \cdot \sin(\angle FAE)$.
Since $\angle FEA = \angle FAE, \angle BAC = \angle BCA$ and $\angle DCE = \angle DEC$ from the three formed isosceles triangles, the above concurrency condition simplifies to showing that the fraction

$$\frac{\sin(\angle BCE) \cdot \sin(\angle DEA) \cdot \sin(\angle FAC)}{\sin(\angle ACD) \cdot \sin(\angle CEF) \cdot \sin(\angle EAB)} = 1.$$

If the angles are now labelled as shown in Figure 1, with $\angle CAE = a, \angle ECA = c$ and $\angle AEC = e$, and letting $\angle A = \angle C = \angle E = 120°$, we obtain the following fraction by substitution into the aforementioned condition:

$$\frac{\sin(x+c) \cdot \sin(120° - x - c + e) \cdot \sin(120° - x - a + a)}{\sin(120° - x - c + c) \cdot \sin(120° - x - a + e) \cdot \sin(x + a)} = \frac{\sin(x+c) \cdot \sin(120° - x - c + e)}{\sin(120° - x - a + e) \cdot \sin(x+a)}.$$

From the sum of the angles at say vertex $E$, we obtain the following identity $2x + a + c - e = 120°$. Respectively substituting the value of $120°$ from this identity into $\sin(120° - x - c + e)$ and $\sin(120° - x - a + e)$ in the fraction above, we obtain

$$\frac{\sin(x+c) \cdot \sin(x+a)}{\sin(x+c) \cdot \sin(x+a)} = 1.$$

This then completes the proof that $AD, BE$, and $CF$ are concurrent.

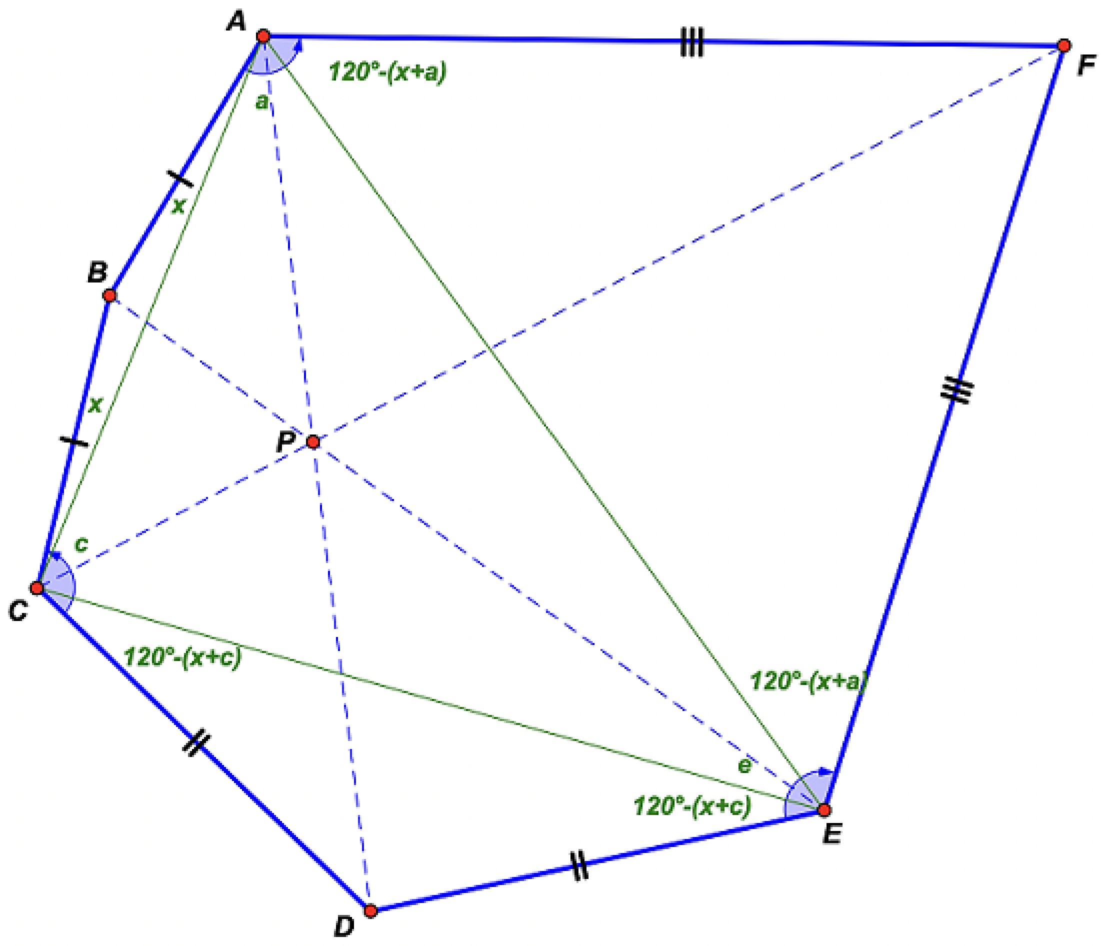


Figure 7

# Looking Back

However, reflecting on the above proof in the problem solving style of Pólya(1945), it immediately became clear that the preceding proof remains valid if the $120°$ angle was replaced by any angle $\theta$. Hence, the initial conjecture immediately generalises to the following theorem. This nicely demonstrates the discovery function of proof as proving it led to discovering this further generalisation.

**Theorem 1**. Given a hexagon $ABCDEF$ with $AB = BC, CD = DE, EF = FA$, and $\angle A = \angle C = \angle E = \theta$, then $AD, BE$, and $CF$ are concurrent at $P$.

In addition, the angle identity given in the proof is helpful in constructing a dynamic sketch as shown in the online example available for the reader at: `http://dynamicmathematicslearning.com/hung-generalization.html`

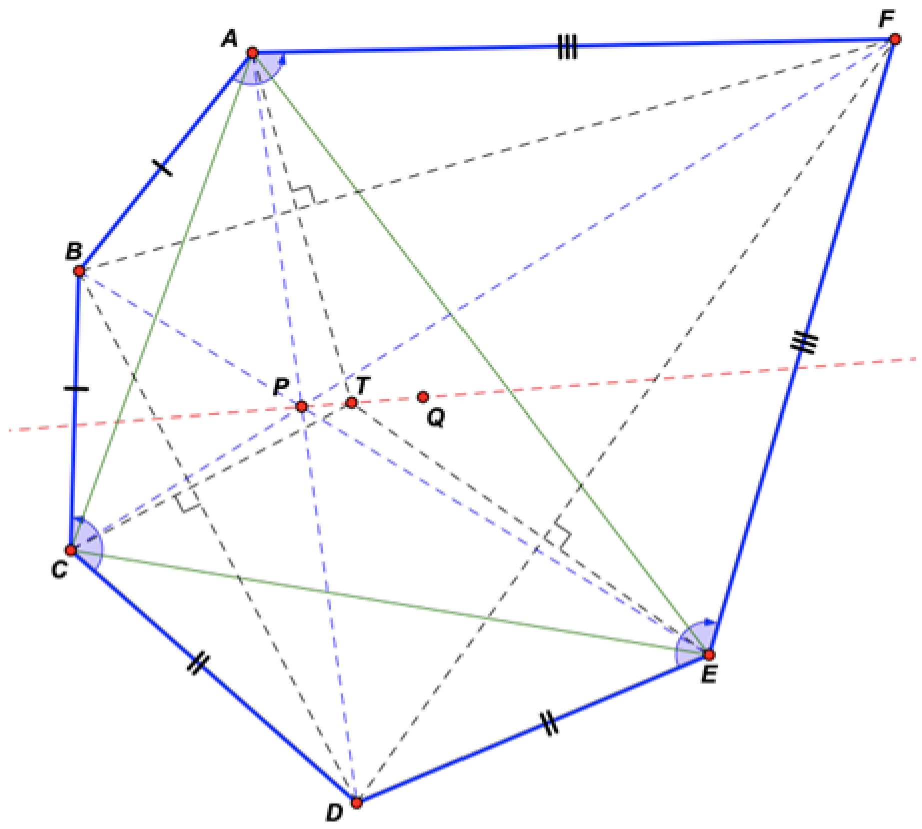


Figure 8

# Collinearity of concurrency points

Further exploration of the dynamic geometry sketch subsequently revealed the following theorem.

**Theorem 2**. The point $P$ defined as in Theorem 1, the powerpoint $T$[10] of $\triangle BDF$, and the circumcenter $Q$ of $\triangle ACE$ are collinear (see Figure 2).

Proof. The proof follows directly from the 1894 theorem of Sondat-Sollerstinsky on perspective orthologic triangles: If two nondegenerate orthologic triangles are also perspective, then the perspector and the two orthology centers are collinear (Yiu, 2015).

The two triangles $ACE$ and $DFB$ are orthologic to each other since the perpendiculars from vertices $A, C$ and $E$, respectively to the sides $FB, BD$ and $DF$ of $\triangle BDF$ are concurrent in $T$.

Likewise the perpendiculars from vertices $D, F$ and $B$, respectively to the sides $CE, EC$ and $AC$ of $\triangle ACE$ are concurrent at its circumcenter $Q$. But as shown in Theorem 1, the two triangles $ACE$ and $DFB$ are also in perspective to each other with the point of perspectivity (perspector) located at $P$. Hence, according to Sondat-Sollerstinsky's theorem, the point $P$, and

[10]Note that the powerpoint $T$ of $\triangle BDF$ is located at the point of concurrency of the perpendiculars from vertices $A, C$ and $E$, respectively to the sides $FB, BD$ and $DF$ of $\triangle BDF$. This is a well-known result and can be proved with the concept of the power of a point or Carnot's perpendicularity theorem.

the two orthology centers $T$ and $Q$ are collinear.

Another noteworthy aspect of the configuration, which also follows directly from the Sondat-Sollerstinsky theorem is that the line of perspective formed by the extension of the corresponding sides of the two perspective triangles $ACE$ and $DFB$ is perpendicular to the line $PTQ$ (Thebault, 1952).

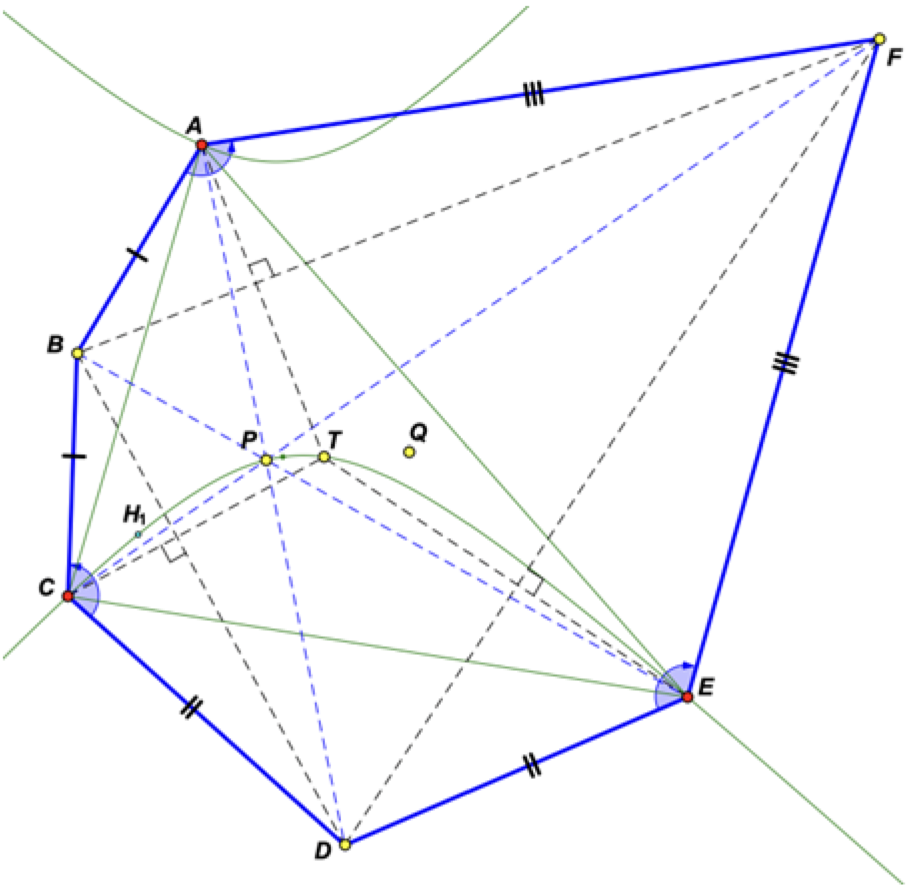


Figure 9

# Two six-point hyperbola

Though the isosceles triangles on the sides of $\triangle ACE$ are not in general similar to each other, our hexagon construction reminded us of the similar isosceles triangles on the sides of a triangle that produce the famed Kiepert hyperbola (Eddy & Fritsch, 1994). So we next explored whether some of the concurrency points lay on analogous conics passing through the vertices. Experimentation with the dynamic sketch of the configuration next revealed the following theorem involving two 6 point hyperbola. This is also illustrated in the dynamic sketch available at the URL given earlier.

**Theorem 3**. The points $A, B, C, P, T$ and the orthocenter $H_1$ of $\triangle ACE$ lie on a rectangular hyperbola[11] (see Figure 3). Likewise, the points $A, B, C, P, Q$ and the orthocenter $H_2$ of $\triangle DFB$

[11]Rectangular hyperbola are also sometimes called equilateral hyperbola.

lie on a rectangular hyperbola (not shown in Figure 3).

Proof. To prove the result, we first apply the following theorem, which is a corollary of the proof by Thebault (1952) of the Sondat-Sollerstinsky theorem, and is formulated by Yiu (2015) as follows: If $ABC$ and $A'B'C'$ are perspective at $P$ and the perpendiculars from $A$ to $B'C'$, $B$ to $C'A'$, and $C$ to $A'B'$ intersect at $Q'$, then $A, B, C, P, Q'$ lie on a rectangular hyperbola.

Applied to our configuration in Figure 3, this theorem immediately implies that the points $A, B, C, P$, and $T$ related to $\triangle ACE$ lie on a rectangular hyperbola (and likewise for the points $A, B, C, P$, and $Q$ related to $\triangle DFB$).

Next we use the Brianchon-Poncelet (1822) theorem which states that if the vertices of a triangle lie on a rectangular hyperbola, then the orthocentre of the triangle also lies on the hyperbola. Since both these hyperbola are rectangular, it now follows from this theorem that the orthocenters, $H_1$ of $\triangle ACE$, and $H_2$ of $\triangle DFB$, respectively lie on the rectangular hyperbolas $ABCPT$ and $ABCPQ$. This concludes the proof.

The Brianchon-Poncelet theorem appears on several websites, and can be easily proven synthetically (Besant, 1895:129) or analytically (Margetson & Buckingham, 1989). It provides a novel, but quite accessible challenge for mathematically talented high school or college students. An additional interesting property proved by Brianchon & Poncelet is that the centres of the rectangular hyperbolas $ABCPT$ and $ABCPQ$, respectively, lie on the nine-point circles of triangles $ACE$ and $DFB$ inscribed on each of the hyperbola. This is also an accessible challenge to talented mathematical students at different levels (Margetson & Buckingham, 1989).

It should also be noted that for the hyperbola $ABCPT$ above to coincide precisely with the corresponding Kiepert hyperbola of $\triangle ACE$, the isosceles triangles on its sides need to be similar. But since for our hexagon $ABCDEF$ it is required that $\angle A = \angle C = \angle E$, it follows that the hyperbola $ABCPT$ will only coincide with the Kiepert hyperbola when $\triangle ACE$ is equilateral.

# Interesting Special Case

Further dynamically exploring the special case of the hexagon $ABCDEF$, when $\angle A = \angle C = \angle E = 120°$, next revealed the following theorem. This is also available for the reader to explore at the earlier provided URL.

**Theorem 4**. Given a hexagon $ABCDEF$ with $AB = BC, CD = DE, EF = FA$ and $\angle A = \angle C = \angle E = 120°$, then the line $PTQ$ is parallel to the Euler line of $\triangle BDF$.

However, in order to prove this theorem, we first need to prove the following Lemma. This useful Lemma appears as a theorem in Fettis (1946), but the proof given below is somewhat different.

**Lemma**. Let $ABC$ be a triangle with the first Fermat point $T$. Let $S$ be the isogonal conjugate of $T$. Then $ST$ is parallel to the Euler line of triangle $ABC$.

Proof. Let $DEF$ be the pedal triangle of $S$ as shown in Figure 4. Then sides $EF, FD, DE$ are perpendicular to $TA, TB, TC$. If $XYZ$ is the Napoleon triangle of $ABC$, then $YZ, ZX, XY$ are also perpendicular to $TA, TB, TC$ Thus, triangles $DEF$ and $XYZ$ have parallel sides, this means $DX, EY, FZ$ are concurrent at $P$ . Consider the homothety with center $P$ which swaps

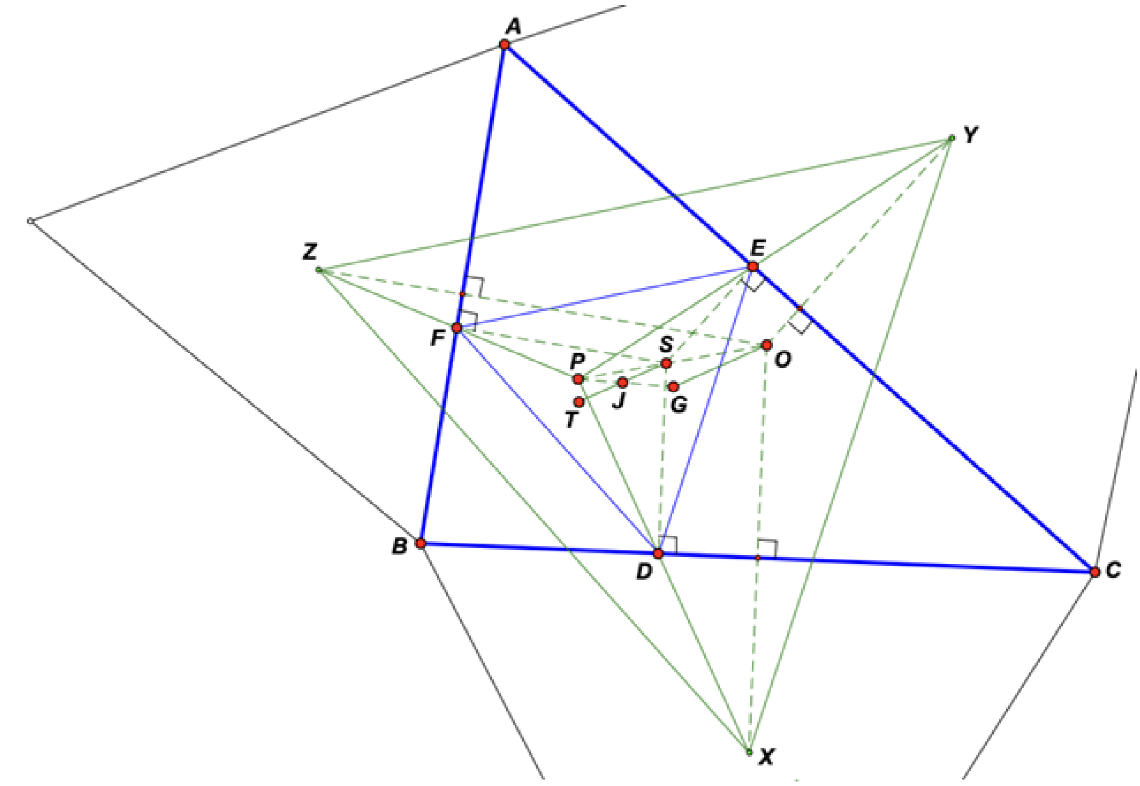


Figure 10

triangle $DEF$ to $XYZ$. The circumcenter of $DEF$ is the midpoint $J$ of $ST$ which swaps to the circumcenter of $XYZ$, which in turn is the centroid $G$ of $ABC$. (1)

The perpendicular lines from $Y$ and $Z$ respectively to $CA$ and $AB$, meet at the circumcenter $O$ of $ABC$. The perpendicular lines from $E$ and $F$ respectively to $CA$ and $AB$, meet at $S$. From this, we see that the homothety swaps $E \to Y, F \to Z$; so $S \to O$. (2)

From (1) and (2), we deduce that the homothety swaps the line $JS$ (or $TS$) to Euler line $GO$, thus $ST \parallel OG$. This completes the proof of the Lemma, and now we are ready to prove Theorem 4.

Proof of Theorem 4. Consider Figure 5. Let $S$ be the reflection of $E$ in $DF$. We see that $FE = FS = FA$ and $DE = DS = DC$. We get

$$\begin{aligned}\angle ASC &= 360^\circ - \angle ASE + \angle CSE \\ &= 360^\circ - (180^\circ - (\angle AFE/2) + 180^\circ - (\angle CDE/2)) \\ &= (\angle AFE/2) + (\angle CDE/2)\end{aligned}$$

But $\angle A + \angle B + \angle C + \angle D + \angle E + \angle F = 720^\circ$.
Therefore,

$$\begin{aligned}\angle ASC &= (\angle AFE/2) + (\angle CDE/2) \\ &= 360^\circ - (60^\circ + (\angle ABC/2) + 60^\circ + 60^\circ) \\ &= 180^\circ - (\angle ABC/2).\end{aligned}$$

Draw a circle with center $B$ and radius $BA = BC$, then since the angle subtended by chord $AC$ at the center of a circle is twice the angle subtended by the chord on the circumference, any angle on

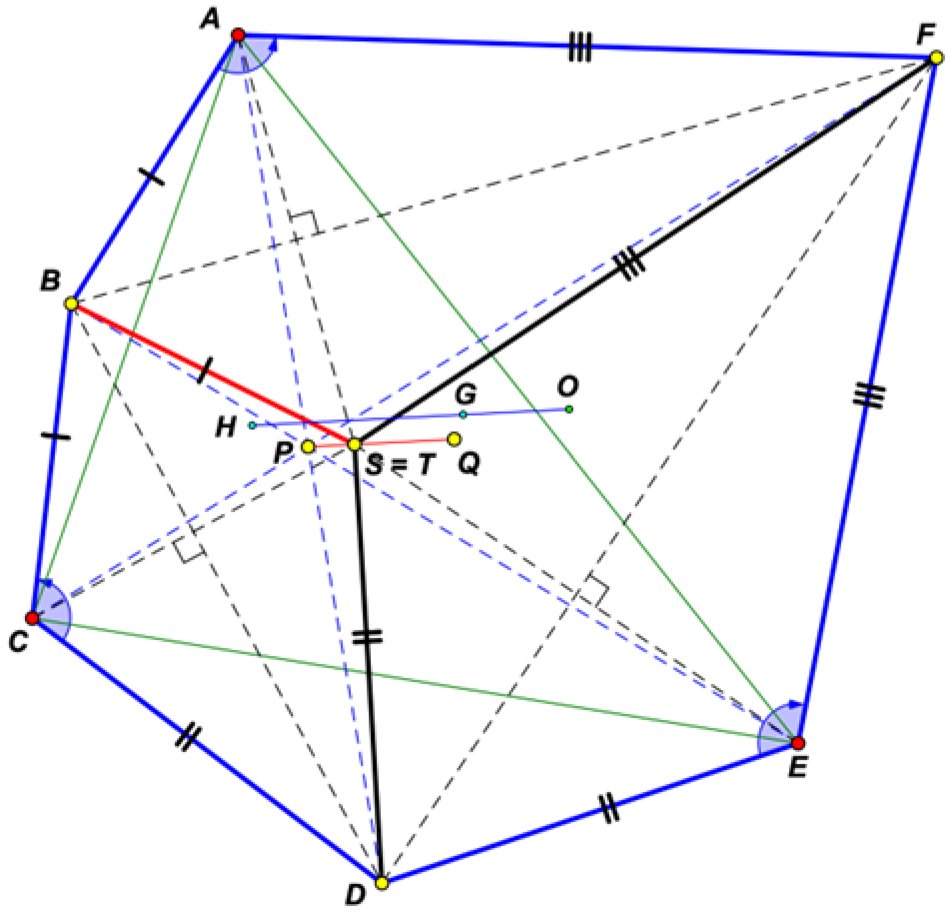


Figure 11

the circumference subtended by chord $AC$ (on the appropriate side) will be $180° - (\angle ABC/2)$. Since $\angle ASC = 180° - (\angle ABC/2)$, the point S must lie on the circumference of the circle. Hence, $AB = BC = BS$. So $A$ and $S$ are reflections of each other in the line $BF$, and we also see that S coincides with the powerpoint T (since the perpendiculars from $A, C$ and $E$ respectively to $BF$, $BD$ and $DF$ are unique). More-over, from the reflections around $BF$, $BD$ and $DF$, we note that all three angles surrounding $T$ are equal to $120°$ ; hence $T$ is the Fermat point of $\triangle BDF$. It is now not hard to further observe that $Q$ is the isogonal conjugate of T, and hence, from the Lemma, it follows that the line $PTQ$ is parallel to the Euler line of $\triangle BDF$.

## Further Observations

From results proven in Beluhov (2009), it's also interesting to note that for the special case above when $\theta = 120°$, since the point $T$ is the Fermat point of $\triangle BDF$, that the Euler lines of triangles $TBD, TDF$ and $TBF$ are concurrent at the centroid $G$ of $\triangle BDF$. The particular hexagon explored here is a close 'cousin' of the so-called Haag hexagon, which is a hexagon $ABCDEF$, also with $AB = BC, CD = DE, EF = FA$ but with $\angle B = \angle D = \angle F = 120°$ (see Schattschneider, 1990, p. 90; De Villiers, 2014). Like the hexagon discussed here, the Haag hexagon also has its main diagonals $AD, BE$, and $CF$ concurrent. (This can easily be proved from Jacobi's generalisation of the Fermat-Torricelli point of a triangle). However, the Haag hexagon also tessellates, which is not generally the case for the particular hexagon discussed in this paper.

## Concluding Remarks

This paper has briefly described the fruitful interplay between inductive and deductive processes in the creation of some new mathematics, as well demonstrating the discovery function of proof. It is hoped that it will provide some enrichment ideas to mathematics teachers at college and high school for challenging their talented mathematics students students beyond the narrow confines of the prescribed curriculum. Apart from the four interesting theorems themselves, students can also learn a lot from constructing their own dynamic sketches of the results, and exploring their properties further.

Web Supplement.
http://dynamicmathematicslearning.com/hung-generalization.html

Disclaimer. No potential competing interest was reported by the authors.

Acknowledgement. The authors wish to thank Waldemar Pompe from the University of Warsaw, Poland for bringing the Sondat-Sollerstinsky theorem to our attention.

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# The 62nd International Mathematical Olympiad

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He enjoys composing Olympiad problems for mathematics contests.

The 62nd International Mathematical Olympiad (IMO) was held 14–24 July 2021. Due to COVID-19, this was a distributed IMO administered from St Petersburg, Russian Federation. This was the third time that Russia has hosted the IMO.

A total of 619 high school students from 107 countries participated. Of these, 64 were female.

To ensure the integrity of the contest, students sat the contest papers at Exam Centres in their own countries. The exams were invigilated from St Petersburg using the *Zoom* video conferencing platform. Moreover, the IMO Board appointed an *IMO Commissioner* for each Exam Centre. The Commissioner was a resident of the country, but generally not a citizen. They were to be trusted individuals at each Exam Centre who would ensure fair play in the administering of the contest and in the scanning and uploading of students' scripts.

As per normal IMO rules, each participating country may enter a team of up to six students, a Team Leader and a Deputy Team Leader.[1]

Participating countries also submit problem proposals for the IMO. This year there were 175 problem proposals from 51 countries. The local Problem Selection Committee shortlisted 32 of these for the contest and then went on to set the two contest exam papers.

At the IMO the Team Leaders, as an international collective, form what is called the *Jury*. The Jury normally makes various decisions such as approving marking schemes, and setting medal boundaries. However, this year, decisions such as these were made on behalf of the Jury by the Jury Chair taking advice from the IMO Board and, if necessary, the IMO Ethics Committee.

The six problems that ultimately appeared on the IMO exam papers may be described as follows.

1. An easy number theory problem proposed by Australia.
2. An algebraic inequality proposed by Canada. Although intended to be a medium level problem, it turned out to be much more difficult than many would have anticipated.

[1]The IMO regulations also permit countries to enter a small number of additional staff as Observers. These may fulfil various roles such as meeting child safety obligations, assisting with marking and coordination or learning about how to host an IMO.

3. A difficult classical geometry problem proposed by Ukraine.
4. An easy classical geometry problem proposed by Poland.
5. A medium combinatorics problem proposed by Spain.
6. A difficult problem proposed by Austria. Although formally classified as an algebra problem, it also contains flavours of number theory and combinatorics.

These six problems were posed in two exam papers held on Monday 19 July and Tuesday 20 July for 4.5 hours each day starting at a time between 07:30 and 12:00 Universal Coordinated Time (UTC). This helped ensure the integrity of the contest as no student would finish the contest before another had started. Each paper had three problems. The contestants worked individually. Each problem was scored out of a maximum of seven points.

After the exams, the Leaders and their Deputies spent about two to three days assessing the work of the students from their own countries, guided by marking schemes. A local team of markers called *Coordinators* also assessed the papers. They too were guided by the marking schemes but were allowed some flexibility if, for example, a Leader brought something to their attention in a contestant's exam script that was not covered by the marking scheme. The Team Leader and Coordinators must agree on scores for each student of the Leader's country in order to finalise scores.

The contestants found Problem 1 to be the easiest with an average score of 4.39. Problems 2 and 3 were the hardest, each averaging just 0.37. The score distributions by problem number were as follows.

| **Mark** | **P1** | **P2** | **P3** | **P4** | **P5** | **P6** |
|---|---|---|---|---|---|---|
| 0 | 131 | 522 | 488 | 218 | 404 | 562 |
| 1 | 36 | 61 | 110 | 33 | 12 | 12 |
| 2 | 41 | 12 | 4 | 39 | 13 | 2 |
| 3 | 10 | 2 | 1 | 2 | 4 | 3 |
| 4 | 41 | 3 | 1 | 12 | 2 | 1 |
| 5 | 38 | 1 | 0 | 1 | 5 | 2 |
| 6 | 36 | 2 | 0 | 5 | 4 | 0 |
| 7 | 286 | 16 | 15 | 309 | 175 | 37 |
| Mean | 4.39 | 0.37 | 0.37 | 3.82 | 2.15 | 0.48 |

The medal cuts were set at 24 points for Gold[2], 19 for Silver and 12 for Bronze. The medal distributions[3] were as follows.

| | **Gold** | **Silver** | **Bronze** | **Total** |
|---|---|---|---|---|
| Number | 52 | 103 | 148 | 303 |
| Proportion | 8.4% | 16.6% | 23.9% | 48.9% |

[2]This is the lowest cut for Gold in the history of the IMO.

[3]The total number of medals must (normally) be approved by the Jury and should not normally exceed half the total number of contestants. The numbers of Gold, Silver and Bronze medals should be approximately in the ratio 1:2:3.

These awards were announced during the online closing ceremony. Of those who did not get a medal, a further 98 contestants received an Honourable Mention for scoring full marks on at least one problem.

Yichuan Wang of the People's Republic of China was the sole contestant who achieved the most excellent feat of a perfect score of 42.

The 2021 IMO was organised by the Ministry of Education of the Russian Federation, the Government of St. Petersburg, Herzen University, and the Presidential Physics and Mathematics Lyceum No. 239, Talent Academy.

Hosts for future IMOs have been secured as follows.

| | |
|---|---|
| 6–16 July, 2022 | Oslo, Norway |
| 2–13 July, 2023 | Chiba, Japan |
| 2024 | Kyiv, Ukraine |
| 2025 | Melbourne, Australia |

Much of the statistical information found in this report can also be found on the official website of the IMO.

www.imo-official.org

English (eng), day 1

*Monday, 19. July 2021*

**Problem 1.** Let $n \geq 100$ be an integer. Ivan writes the numbers $n, n+1, \ldots, 2n$ each on different cards. He then shuffles these $n+1$ cards, and divides them into two piles. Prove that at least one of the piles contains two cards such that the sum of their numbers is a perfect square.

**Problem 2.** Show that the inequality

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\sqrt{|x_i - x_j|} \leq \sum_{i=1}^{n}\sum_{j=1}^{n}\sqrt{|x_i + x_j|}$$

holds for all real numbers $x_1, \ldots, x_n$.

**Problem 3.** Let $D$ be an interior point of the acute triangle $ABC$ with $AB > AC$ so that $\angle DAB = \angle CAD$. The point $E$ on the segment $AC$ satisfies $\angle ADE = \angle BCD$, the point $F$ on the segment $AB$ satisfies $\angle FDA = \angle DBC$, and the point $X$ on the line $AC$ satisfies $CX = BX$. Let $O_1$ and $O_2$ be the circumcentres of the triangles $ADC$ and $EXD$, respectively. Prove that the lines $BC$, $EF$, and $O_1O_2$ are concurrent.

*Language: English*

*Time: 4 hours and 30 minutes.*

*Each problem is worth 7 points.*

English (eng), day 2

*Tuesday, 20. July 2021*

**Problem 4.** Let $\Gamma$ be a circle with centre $I$, and $ABCD$ a convex quadrilateral such that each of the segments $AB$, $BC$, $CD$ and $DA$ is tangent to $\Gamma$. Let $\Omega$ be the circumcircle of the triangle $AIC$. The extension of $BA$ beyond $A$ meets $\Omega$ at $X$, and the extension of $BC$ beyond $C$ meets $\Omega$ at $Z$. The extensions of $AD$ and $CD$ beyond $D$ meet $\Omega$ at $Y$ and $T$, respectively. Prove that

$$AD + DT + TX + XA = CD + DY + YZ + ZC.$$

**Problem 5.** Two squirrels, Bushy and Jumpy, have collected 2021 walnuts for the winter. Jumpy numbers the walnuts from 1 through 2021, and digs 2021 little holes in a circular pattern in the ground around their favourite tree. The next morning Jumpy notices that Bushy had placed one walnut into each hole, but had paid no attention to the numbering. Unhappy, Jumpy decides to reorder the walnuts by performing a sequence of 2021 moves. In the $k$-th move, Jumpy swaps the positions of the two walnuts adjacent to walnut $k$.

Prove that there exists a value of $k$ such that, on the $k$-th move, Jumpy swaps some walnuts $a$ and $b$ such that $a < k < b$.

**Problem 6.** Let $m \geq 2$ be an integer, $A$ be a finite set of (not necessarily positive) integers, and $B_1, B_2, B_3, \ldots, B_m$ be subsets of $A$. Assume that for each $k = 1, 2, \ldots, m$ the sum of the elements of $B_k$ is $m^k$. Prove that $A$ contains at least $m/2$ elements.

*Language: English*

*Time: 4 hours and 30 minutes.*

*Each problem is worth 7 points.*

## Some Country Totals

| Rank | Country | Total |
|---|---|---|
| 1 | People's Republic of China | 208 |
| 2 | Russian Federation | 183 |
| 3 | Republic of Korea | 172 |
| 4 | United States of America | 165 |
| 5 | Canada | 151 |
| 6 | Ukraine | 149 |
| 7 | Israel | 139 |
| 7 | Italy | 139 |
| 9 | Taiwan | 131 |
| 9 | United Kingdom | 131 |
| 11 | Mongolia | 130 |
| 12 | Germany | 129 |
| 13 | Poland | 126 |
| 14 | Vietnam | 125 |
| 15 | Singapore | 123 |
| 16 | Czech Republic | 121 |
| 16 | Thailand | 121 |
| 18 | Australia | 120 |
| 18 | Bulgaria | 120 |
| 20 | Kazakhstan | 117 |
| 21 | Croatia | 113 |
| 21 | Hong Kong | 113 |
| 23 | Philippines | 111 |
| 24 | Belarus | 109 |
| 25 | Japan | 108 |
| 26 | India | 106 |
| 27 | France | 105 |
| 27 | Romania | 105 |
| 29 | Islamic Republic of Iran | 104 |
| 30 | Peru | 103 |

## Distribution of Awards at the 2021 IMO

| Country | Total | Gold | Silver | Bronze | HM |
|---|---|---|---|---|---|
| Albania | 11 | 0 | 0 | 0 | 1 |
| Algeria | 16 | 0 | 0 | 0 | 0 |
| Argentina | 66 | 0 | 2 | 0 | 2 |
| Armenia | 91 | 0 | 2 | 3 | 0 |
| Australia | 120 | 2 | 2 | 1 | 1 |
| Austria | 47 | 0 | 0 | 2 | 1 |
| Azerbaijan | 62 | 0 | 0 | 2 | 3 |
| Bangladesh | 68 | 0 | 0 | 3 | 2 |
| Belarus | 109 | 0 | 4 | 1 | 1 |
| Belgium | 68 | 0 | 0 | 3 | 3 |
| Bolivia | 28 | 0 | 0 | 0 | 1 |
| Bosnia and Herzegovina | 81 | 0 | 0 | 5 | 1 |
| Botswana | 0 | 0 | 0 | 0 | 0 |
| Brazil | 96 | 0 | 2 | 3 | 1 |
| Bulgaria | 120 | 1 | 3 | 2 | 0 |
| Canada | 151 | 3 | 3 | 0 | 0 |
| Chile | 23 | 0 | 0 | 0 | 2 |
| Colombia | 51 | 0 | 1 | 1 | 1 |
| Costa Rica | 23 | 0 | 0 | 1 | 0 |
| Croatia | 113 | 1 | 2 | 3 | 0 |
| Cyprus | 28 | 0 | 0 | 0 | 2 |
| Czech Republic | 121 | 1 | 3 | 1 | 0 |
| Denmark | 46 | 0 | 0 | 1 | 3 |
| Ecuador | 34 | 0 | 0 | 2 | 0 |
| Egypt | 2 | 0 | 0 | 0 | 0 |
| El Salvador | 37 | 0 | 0 | 1 | 1 |
| Estonia | 63 | 0 | 1 | 1 | 3 |
| Finland | 41 | 0 | 0 | 1 | 3 |
| France | 105 | 1 | 1 | 3 | 0 |
| Georgia | 74 | 0 | 1 | 3 | 1 |
| Germany | 129 | 2 | 2 | 1 | 1 |
| Ghana | 11 | 0 | 0 | 0 | 1 |
| Greece | 47 | 0 | 0 | 2 | 1 |
| Honduras | 13 | 0 | 0 | 0 | 1 |
| Hong Kong | 113 | 1 | 3 | 1 | 0 |
| Hungary | 101 | 0 | 1 | 5 | 0 |
| Iceland | 11 | 0 | 0 | 0 | 1 |
| India | 106 | 1 | 1 | 3 | 0 |
| Indonesia | 99 | 0 | 2 | 4 | 0 |

| Country | Total | Gold | Silver | Bronze | HM |
|---|---|---|---|---|---|
| Iraq | 16 | 0 | 0 | 1 | 0 |
| Ireland | 12 | 0 | 0 | 0 | 1 |
| Islamic Republic of Iran | 104 | 0 | 3 | 3 | 0 |
| Israel | 139 | 3 | 2 | 1 | 0 |
| Italy | 139 | 1 | 4 | 1 | 0 |
| Japan | 108 | 1 | 2 | 3 | 0 |
| Kazakhstan | 117 | 1 | 3 | 2 | 0 |
| Kenya | 2 | 0 | 0 | 0 | 0 |
| Kosovo | 23 | 0 | 0 | 0 | 1 |
| Kyrgyzstan | 34 | 0 | 0 | 0 | 2 |
| Latvia | 64 | 0 | 0 | 3 | 3 |
| Lithuania | 31 | 0 | 0 | 1 | 0 |
| Luxembourg | 7 | 0 | 0 | 0 | 1 |
| Macau | 60 | 0 | 0 | 3 | 2 |
| Malaysia | 74 | 0 | 2 | 0 | 3 |
| Mauritania | 7 | 0 | 0 | 0 | 0 |
| Mexico | 98 | 0 | 2 | 4 | 0 |
| Mongolia | 130 | 2 | 2 | 2 | 0 |
| Montenegro | 27 | 0 | 1 | 0 | 1 |
| Morocco | 33 | 0 | 0 | 0 | 3 |
| Nepal | 15 | 0 | 0 | 0 | 1 |
| Netherlands | 65 | 0 | 0 | 2 | 3 |
| New Zealand | 41 | 0 | 0 | 2 | 0 |
| Nicaragua | 25 | 0 | 0 | 1 | 1 |
| Nigeria | 6 | 0 | 0 | 0 | 0 |
| North Macedonia | 67 | 0 | 1 | 2 | 2 |
| Norway | 62 | 0 | 1 | 1 | 3 |
| Oman | 2 | 0 | 0 | 0 | 0 |
| Pakistan | 2 | 0 | 0 | 0 | 0 |
| Panama | 36 | 0 | 0 | 1 | 1 |
| Paraguay | 18 | 0 | 0 | 1 | 0 |
| People's Republic of China | 208 | 6 | 0 | 0 | 0 |
| Peru | 103 | 0 | 2 | 4 | 0 |
| Philippines | 111 | 0 | 4 | 2 | 0 |
| Poland | 126 | 1 | 5 | 0 | 0 |
| Portugal | 60 | 0 | 1 | 2 | 1 |
| Puerto Rico | 27 | 0 | 0 | 1 | 1 |
| Republic of Korea | 172 | 5 | 1 | 0 | 0 |
| Republic of Moldova | 62 | 0 | 0 | 3 | 2 |
| Romania | 105 | 0 | 3 | 2 | 1 |
| Russian Federation | 183 | 5 | 1 | 0 | 0 |
| Rwanda | 3 | 0 | 0 | 0 | 0 |

| Country | Total | Gold | Silver | Bronze | HM |
|---|---|---|---|---|---|
| Saudi Arabia | 90 | 0 | 1 | 3 | 2 |
| Serbia | 102 | 1 | 2 | 1 | 1 |
| Singapore | 123 | 1 | 3 | 2 | 0 |
| Slovakia | 82 | 0 | 2 | 2 | 1 |
| Slovenia | 47 | 0 | 0 | 2 | 2 |
| South Africa | 53 | 0 | 0 | 3 | 1 |
| Spain | 50 | 0 | 0 | 1 | 3 |
| Sri Lanka | 25 | 0 | 0 | 0 | 2 |
| Sweden | 56 | 0 | 1 | 1 | 1 |
| Switzerland | 64 | 0 | 0 | 3 | 1 |
| Syria | 44 | 0 | 0 | 2 | 2 |
| Taiwan | 131 | 1 | 3 | 2 | 0 |
| Tajikistan | 54 | 0 | 0 | 3 | 1 |
| Thailand | 121 | 1 | 3 | 2 | 0 |
| Trinidad and Tobago | 10 | 0 | 0 | 0 | 1 |
| Tunisia | 57 | 0 | 1 | 1 | 2 |
| Turkey | 96 | 0 | 1 | 5 | 0 |
| Turkmenistan | 55 | 0 | 0 | 3 | 2 |
| Uganda | 3 | 0 | 0 | 0 | 0 |
| Ukraine | 149 | 3 | 2 | 1 | 0 |
| United Kingdom | 131 | 2 | 3 | 0 | 1 |
| United States of America | 165 | 4 | 2 | 0 | 0 |
| Uruguay | 17 | 0 | 0 | 0 | 1 |
| Uzbekistan | 51 | 0 | 1 | 1 | 1 |
| Venezuela | 24 | 0 | 0 | 0 | 2 |
| Vietnam | 125 | 1 | 2 | 3 | 0 |
| **Total** (107 teams, 619 contestants) | | **52** | **103** | **148** | **98** |

N.B. Not all countries entered a full team of six students.

*Angelo Di Pasquale*

*IMO Team Leader, Australia*

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# International Mathematics Tournament of the Towns

*Andy Liu*

In 1976, Andy Liu received a Doctor of Philosophy in mathematics and a Professional Diploma in elementary education, making him one of very few people officially qualified to teach from kindergarten to graduate school. He was heavily involved in the International Mathematical Olympiad. He served as the deputy leader of the USA team from 1981 to 1984, and as the leader of the Canadian team in 2000 and 2003. He chaired the Problem Committee in 1995, and was on the Problem Committee in 1994, 1998 and 2016. He had given lectures to school children in Canada, the United States, Colombia, Hungary, Latvia, Sweden, Tunisia, South Africa, Sri Lanka, Nepal, Thailand, Laos, Malaysia, Indonesia, the Philippines, Hong Kong, Macau, Taiwan, China and Australia. He ran a mathematics circle in Edmonton for thirty-two years, and continued his book-publishing after his retirement from the University of Alberta in 2013. He is currently writing his twentieth mathematics book, which is based on Greek Mythology.

**Selected Problems and Solutions from the Fall 2021 Papers**

1. In each of five bags are 30 coins. One bag contains only gold coins, another one contains only silver coins, the third one contains only bronze coins, and each of the remaining two contains 10 gold, 10 silver and 10 bronze coins. What is the minimum number of coins that must be drawn, from any combination of bags, in order to determine the content of at least one bag?

2. The convex $n$-gon $A_1A_2\dots A_n$, with $n > 4$, is such that $A_{n-1}A_nA_1$ and $A_nA_1A_2$ are isosceles triangles, as are $A_{i-1}A_iA_{i+1}$ for $2 \le i \le n-1$. Prove that there are at least two equal sides among any four sides of the $n$-gon.

3. There were 20 participants in a chess tournament. Each of them played every other participant twice, once as white and once as black. We say that participant X is no weaker than participant Y if X has won at least the same number of games playing white as Y and also has won at least the same number of games playing black as Y. Do there always exist two participants such that one is not weaker than the other?

4. In the first move, a point $x$ is chosen in the segment [0,1], dividing it into two segments $[0, x]$ and $[x, 1]$. The product $x(1-x)$ is recorded. In each subsequent move, a point $x$ is chosen in a segment $[a, b]$ with no other chosen points inside, dividing into two segments $[a, x]$ and $x, b]$. The product $(x-a)(b-x)$ is recorded. Prove that the sum of all recorded numbers never exceeds $\frac{1}{2}$.

5. Eight $1 \times 3$ cards are placed on a flat surface in the same orientation. Each is divided into three $1 \times 1$ squares, each of which is either black or white. The cards may not be rotated or reflected, and no two are identical. They may be moved in any direction by any distance which needs not be integral. Is it possible to move the cards so that they do not overlap, all the white squares form a connected region bounded by a closed polygonal line which does not intersect itself, and the same is true for all the black squares?

6. In triangle $ABC$, $\angle C = 90°$ and $AB = 1$. What are the possible values of the length of the chord of the circumcircle of $ABC$ determined by the points on $BC$ and $CA$ tangent to the incircle of $ABC$?

7. Anna and Boris play a game which starts with the number 7 on a board. Anna goes first and turns alternate thereafter. In each turn, the player adds a digit to the existing number, at the beginning, between any two digits or at the end. However, a 0 may not be added at the beginning. The moving player wins if the resulting number is the squares of an integer. Does either player has a winning strategy?

8. A parallelogram $ABCD$ is divided by the diagonal $BD$ into two equal triangles. A regular hexagon is inscribed into triangle $ABD$ so that two of its adjacent sides lie on $DA$ and $AB$ and one of its vertices lies on $BD$. Another regular hexagon is inscribed into triangle $CDB$ so that two of its adjacent vertices lie on $BC$ and $CD$ and one of its sides lies on $BD$. Which of the hexagons is bigger?

9. The wizards A, B, C and D know that the integers $1, 2, \ldots, 12$ are written on 12 cards, one integer on each card. Each wizard gets three cards, and sees only the numbers on his cards. The following true statements are made.
A: One of the numbers on my cards is 8.
B: The numbers on my cards are all primes.
C: The numbers on my cards are all multiples of the same prime.
Having heard these, D declares truthfully that he knows what numbers are written on the cards of each wizard. What numbers are on the other two cards of A?

10. There is a rook on some square of a $10 \times 10$ chessboard. At each turn it moves to an adjacent square in the same row or column. It visits each square exactly once. Consider each of the two diagonals joining two corner squares of the chessboard,. Prove that the rook has made two consecutive moves, first leaving the diagonal and then returning to it immediately.

11. Prove that for any positive integers $a_1, a_2, \ldots, a_n$,

$$\left\lfloor \frac{a_1^2}{a_2} \right\rfloor + \left\lfloor \frac{a_2^2}{a_3} \right\rfloor + \cdots + \left\lfloor \frac{a_n^2}{a_1} \right\rfloor \geq a_1 + a_2 + \cdots + a_n.$$

12. Anna and Boris play a game which starts with 20 gold coins and 20 silver coins arranged in a row at random. Anna goes first and turns alternate thereafter. In each turn, the player takes a coin from either end of the row. When all the coins have been taken, Alice wins if and only if she has obtained 10 coins of each type. Is it true for that any initial arrangement of the 40 cons, Anna always has a winning strategy?

**Solutions**

1. The minimum number of coins drawn is five. We first show sufficiency. Draw one coin from each bag. There must be at least one gold coin, at least one silver coin and at least one bronze coin. At least one kind of coin can appear only once, say a lone bronze coin. Then the bag from which it is drawn contains 30 bronze coins. We now show necessity. Suppose four coins are drawn. If they are from at most three bags, let them all be gold coins. Then one of these three contains 30 gold coins while each of the other two contains 10 coins of each kind, but we cannot tell which is which. One of the bags from which no coins have been drawn contains 30 silver coins while the other one contains 30 bronze coins. Again

we cannot tell which is which. Finally, suppose we draw one coin from each of four bags. In the following scenario, two cases are possible, and we cannot determine the content of any bag.

| Bag | First | Second | Third | Fourth | Fifth |
|---|---|---|---|---|---|
| Draw | Gold | Gold | Silver | Bronze | (none) |
| Case 1 | Gold | Mixed | Silver | Mixed | Bronze |
| Case 2 | Mixed | Gold | Mixed | Bronze | Silver |

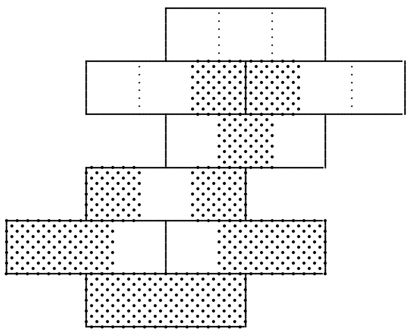
2. Divide the perimeter of the polygon into blocks of consecutive sides of equal length. If $AB$ and $BC$ belong to different blocks, then $AB \neq BC$. Since $ABC$ is isosceles, we must have $AB = AC$ or $AC = BC$. In either case, $\angle ABC$ is one of two equal angles in $ABC$, so that it must be acute. Since the polygon is convex, it has at most three acute angles and therefore at most three blocks. Thus all of its edges are of one of three lengths, and the desired conclusion follows immediately.

3. Suppose the answer is negative. If two players have the same number of wins as white, one of them is bound to be no weaker than the other. This is also true if they have the same number of wins as black. It follows that the numbers of wins as white for the twenty players are 19, 18, …, 2, 1 and 0, and the same can be said about the numbers of wins as black. Suppose the 19-game winners are two different players, say A as white and B as black. Consider the game with A as white and B as black. Since they cannot both win, this situation is impossible. It follows that the same player wins 19 games as white and as black. This player will not be weaker than any others. We have a contradiction.

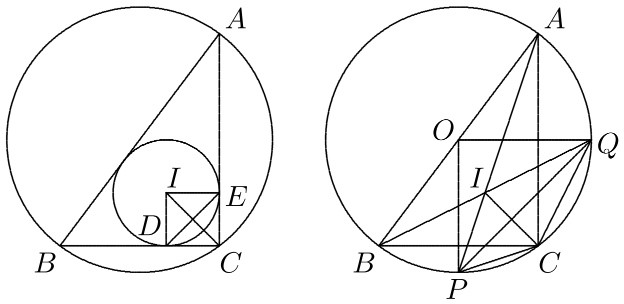
4. Suppose that after $n-1$ moves, the interval [0,1] has been divided into $n$ subintervals of lengths $a_1,\ a_2,\ \dots,\ a_n$ from left to right. We have $\sum_{i=1}^{n} a_i = 1$. For $1 \le i < j \le n$, the product $a_ia_j$ is recorded exactly once, when the intervals of respective lengths $a_i$ and $a_j$ are separated. No other numbers are recorded. Hence the sum of the recorded numbers is $S = \sum_{1\le i<j\le n} a_ia_j$. Now at least one $a_i$ is positive. Hence we have $S < \frac{1}{2}$ since

$$1 = \left(\sum_{i=1}^{n} a_i\right)^2 = \sum_{i=1}^{n} a_n^2 + 2\left(\sum_{1\le i<j\le n} a_ia_j\right) > 2S.$$

5. The task can be accomplished as shown in the diagram below.



6. Let $I$ be the incentre of triangle $ABC$ and let the incircle be tangent to $BC$ at $D$ and $CA$ at $E$. Since $\angle C = 90°$, $IDCE$ is a square so that $DE$ is the perpendicular bisector of $IC$.



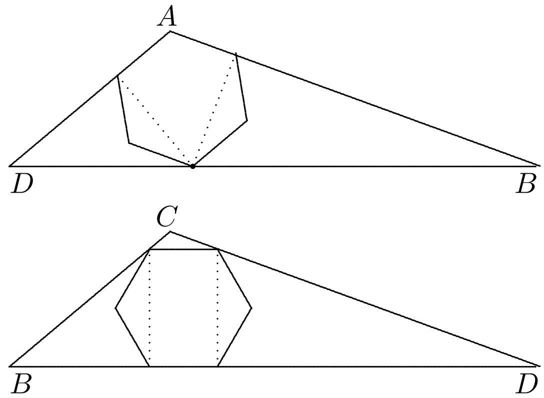

Let $O$ be the circumcentre of triangle $ABC$. Drop perpendiculars from $O$ to $BC$ and $CA$, intersecting the circumcircle at $P$ and $Q$ respectively. Then $AP$ and $BQ$ are the respective bisectors of $\angle A$ and $\angle B$, so that they intersect at $I$. Now

$$\angle PCI = \angle PCB + \angle BCI = \angle PAB + \angle 45^\circ = \angle PAC + \angle ICA = \angle PIC.$$

Hence $PI = PC$. Similarly, $QI = QC$ so that $PQ$ is the perpendicular bisector of $IC$. It follows that $PQ$ is the unique chord of the circumcircle determined by $DE$. Since we have $\angle POQ = 90^\circ$, $PQ = \frac{1}{\sqrt{2}}$.

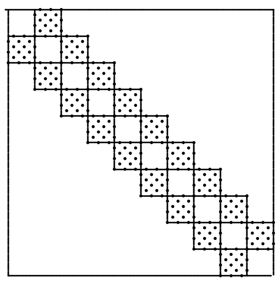
7. We claim that each player can prevent the other from winning. Anna cannot win on her opening move. In any of her subsequent turns, she appends 3 or 7 at the end. Since the square of an integer cannot end in 2, 3, 7 or 8, Boris must append (always at the end) 0, 1, 4, 5, 6 or 9 in order to win on the move. Now a number ending in 30, 70, 34 or 74 is congruent modulo 4 to 2, and a number ending in 31, 71, 35, 75, 39 or 79 is congruent modulo 4 to 3. None of them can be the square of an integer. It follows that Boris can only win by appending 6. We consider three cases.
**Case 1.** The last digit of the current number is 0, 4 or 8.
Anna appends 7 while Boris appends 6. A number ending in 076, 476 or 876 is divisible by 4. The quotient obtained when dividing by 4 ends in 19, which is congruent modulo 4 to 3. Hence the quotient is not the square of an integer, and neither is 4 times the quotient.
**Case 2.** The last digit of the current number is 2 or 6.
Anna appends 3 while Boris appends 6. A number ending in 276 or 676 is divisible by 4. The quotient obtained when dividing by 4 ends in 59, which is congruent modulo 4 to 3. Hence the quotient is not the square of an integer, and neither is 4 times the quotient.
**Case 3.** The last digit of the current number is odd.
Anna appends 3 or 7 while Boris appends 6. In either case, the number is divisible by 8. Since they differ by 40, only one of them is a multiple of 16. Anna chooses 3 or 7 so that this number is not divisible by 16. Hence it cannot be the square of an integer.
This proves that Anna can prevent Boris from winning. Exactly the same strategy can be used by Boris to prevent Anna from winning.

8. Triangles $BAD$ and $DCB$ are congruent and have the same area. The former is dissected into a regular hexagon, a small non-convex quadrilateral and a large non-convex quadrilateral. The latter is likewise dissected, with an extra triangle. The two small quadrilaterals are obtained from right triangles with an acute angle equal to the larger acute angle of $BAD$ and $DCB$, by cutting off a $30^\circ - 30^\circ - 120^\circ$ triangle whose base coincides with the side opposite this angle. It follows that the two quadrilateral are similar. The same argument shows that the two large quadrilaterals are similar too. The coefficient of similarity is the ratio of the side lengths of the two regular hexagons. Since there is an extra triangle inside $DCB$, the hexagon inscribed in it is smaller than the one inscribed in $ABD$.




9. C may hold three of 2, 4, 6, 10 and 12, or three of 3, 6, 9 and 12. In order for D to cut the number of possibilities down to one, he must hold at least one of 6 and 12. We consider two cases.
**Case 1.** D holds both 6 and 12.
C must hold (2,4,10). With the prime 2 out of the picture, B may hold three of 3, 5, 7 and 11. In order for D to cut the number of possibilities down to one, he must hold the odd prime not held by B. It follows that A holds (1,8,9).
**Case 2.** D holds only one of 6 and 12.
By symmetry, we assume that D holds 6 but not 12. Then C may hold any of (3,9,12), (2,4,10), (2,4,12), (2,10,12) and (4,10,12). In the last scenario, C does not hold any primes. Hence D must hold (2,3,6), forcing B to hold (5,7,11). Again, A must hold (1,8,9).

10. The diagram below shows the ten white squares on the diagonal $D$ from the northwest corner to the southeast corner, along with the eighteen black squares on the adjacent diagonals. Since the starting square and the finishing square of the rook tour have opposite colours, at most one of them is on $D$. From this square, only one move is made to an adjacent black square. From every other squares on $D$, two moves are made to adjacent black squares. Thus the total number of such moves is at least 19. However, the number of black squares adjacent to $D$ is only 18. By the Pigeonhole Principle, two moves are made to at least one of them. It follows that when the rook exits $D$ to this square, it must immediately return to $D$. The same argument applies to the black diagonal from the southwest corner to the northeast corner.



11. We have $0 \leq (a_n - a_1)^2 = a_n^2 - 2a_na_1 + a_1^2$ so that $\frac{a_n^2}{a_1} \geq 2a_n - a_1$. Since $2a_n - a_1$ is an integer, $\lfloor \frac{a_n^2}{a_1} \rfloor \geq 2a_n - a_1$. Similarly, $\lfloor \frac{a_k^2}{a_{k+1}} \rfloor \geq 2a_k - a_{k+1}$ for $1 \leq k \leq n-1$. Summation and cancellation yield the desired inequality.

12. We say that we are in equilibrium if before a move by Anna, she has the same number of gold coins as Boris. We are in equilibrium at the beginning, and if we are in equilibrium at the end, then Anna has won. Let the coins be placed on a $1 \times 40$ chessboard. Anna makes

an arbitrary move whenever we are in equilibrium. To break the equilibrium, Boris must take a coin different from the one Anna has just taken. We may assume that Boris takes a silver coin after Anna has taken a gold coin. An odd number of gold coins and an odd number of silver coins remain on the chessboard. Half of these coins are on black squares and the other half on white squares. We may assume that there are more gold coins on black squares than on white squares. Anna takes the coin on the end-square which is white. Now both end-squares are black, and Boris must take one of them, exposing a white square at that end. If equlibrium has not been restored, Anna takes the coin on this white square, forcing Boris to take a coin from a black square. Since there are more silver coins on white squares than on black squares, equilibrium must be restored at some point. Anna can start all over again by making an arbitrary move. It follows that Boris cannot prevent her from winning.

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