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MATHEMATICS COMPETITIONS
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World Federation of National Mathematics Competitions

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The aims of the Federation are:

1. ***to promote excellence in, and research associated with, mathematics education through the use of school mathematics competitions;***
2. ***to promote meetings and conferences where persons interested in mathematics contests can exchange and develop ideas for use in their countries;***
3. ***to provide opportunities for the exchanging of information for mathematics education through published material, notably through the Journal of the Federation;***
4. ***to recognize through the WFNMC Awards system persons who have made notable contributions to mathematics education through mathematical challenge around the world;***
5. ***to organize assistance provided by countries with developed systems for competitions for countries attempting to develop competitions;***
6. ***to promote mathematics and to encourage young mathematicians.***

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From the President

Dear Fellow Federalists!

I am writing these lines in August 2018 for the second issue of 2017 and first issue of 2018. We fell a bit behind, but will catch up, I am advised by our long term excellent Editor and my friend Jarek Švrček.

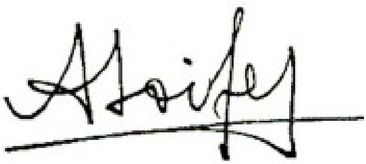
Working for 20 years on the book *The Scholar and the State: In Search of Van der Waerden* (Birkhäuser, 2015) changed my priorities in life, from aesthetics to ethics, from accepting the Ivory Tower to assuming my share of responsibility for my country and my world.

The prominent Russian poet Evgeny Evtushenko wrote, "Poet in Russia is more than a Poet." He meant, the Poet is a Prophet and also a conscience of the people. I would not expect this much from a Mathematician. But I certainly hope that every Mathematician will become more than a Mathematician—will also become a Citizen, will assume his or her share of responsibility for what is done by our leaders on our behalf, will become a Citizen of the world.

This is all I have to say today. And I do not think it is too little or too late.

With warm regards,

Yours as ever,



Alexander Soifer
President of WFNMC
August 2018

From the Editor

Welcome to *Mathematics Competitions* Vol 30, No 2 and Vol 31, No 1.

First of all I would like to thank again the Australian Mathematics Trust for continued support, without which each issue of the journal could not be published, and in particular Bernadette Webster, Alexandra Carvajal and Pavel Calábek for their assistance in the preparation of this issue.

Submission of articles:

The journal *Mathematics Competitions* is interested in receiving articles dealing with mathematics competitions, not only at national and international level, but also at regional and primary school level. There are many readers in different countries interested in these different levels of competitions.

- The journal traditionally contains many different kinds of articles, including reports, analyses of competition problems and the presentation of interesting mathematics arising from competition problems. Potential authors are encouraged to submit articles of all kinds.
- To maintain and improve the quality of the journal and its usefulness to those involved in mathematics competitions, all articles are subject to review and comment by one or more competent referees. The precise criteria used will depend on the type of article, but can be summarised by saying that an article accepted must be correct and appropriate, the content accurate and interesting, and, where the focus is mathematical, the mathematics fresh and well presented. This editorial and refereeing process is designed to help improve those articles which deserve to be published.

At the outset, the most important thing is that if you have anything to contribute on any aspect of mathematics competitions at any level, local, regional or national, we would welcome your contribution.

Articles should be submitted in English, with a black and white photograph and a short profile of the author. Alternatively, the article can be submitted on an IBM PC compatible disk or a Macintosh disk. We prefer LaTeX or TeX format of contributions, but any text file will be helpful.

Articles, and correspondence, can also be forwarded to the editor by mail to

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Jaroslav Švrček
June 2018

A functional equation arising from compatibility of means

Marcin E. Kuczma

Marcin E. Kuczma (born 1944) teaches advanced calculus and various branches of analysis at the University of Warsaw. His research areas are real variable functions and measure theory. Engaged in mathematical competitions activities for decades, he is also strongly active at the Polish MO and the Austro-Polish MC. He worked at eight IMOs as Problem Selection Committee member and at twelve IMOs as coordinator. He was chair of the Jury at Baltic Way Competition in 1998. He has been involved with WFNMC since its beginning: awarded Hilbert Prize (1992), chair of the Problems Section at the 3rd WFNMC Congress.

The following functional equation is the object of this presentation:

$$x + g\big(y + f(x)\big) = y + g\big(x + f(y)\big); \qquad f, g\colon \mathbb{R} \to \mathbb{R}. \tag{1}$$

Before telling how it has emerged, let us try to get some feeling of the equation. It involves two unknown functions; a *solution* is a pair (f, g). Here are some examples:

$$f(x) = -\lfloor x \rfloor, \qquad g(x) = \begin{cases} x - \lfloor x \rfloor, & x \geq 0, \\ x, & x \leq 0; \end{cases} \tag{2}$$

$$f(x) = -\lfloor x \rfloor, \qquad g(x) = \begin{cases} x - \lfloor x \rfloor, & x \leq 0, \\ x, & x \geq 0 \end{cases} \tag{3}$$

(note the lack of uniqueness in g, given f); or

$$f\colon \mathbb{R} \to \{0,1\} \ \text{ arbitrary with } \ f(x+1) = f(x); \quad g(x) = x + f(x); \tag{4}$$

and many others. Verification that these pairs indeed satisfy (1) is straightforward. The function f in (4) is in fact the characteristic function of some set $E \subset [0,1)$, extended by periodicity to the whole real line. Since E can be *any* set in the unit interval, we already get a family of solutions, equipotent with the class of all functions $\mathbb{R} \to \mathbb{R}$. Taking E to be the empty set, we find the (trivial) pair

$$f(x) = 0, \qquad g(x) = x. \tag{5}$$

These examples can be further modified. E.g., in (4) one might take, instead of a two-valued function f, any integer-valued, 1-periodic function f.

Another useful observation is the following. Let $f, g\colon \mathbb{R} \to \mathbb{R}$ be any pair of functions and $b, c \in \mathbb{R}$ be any pair of real numbers, and define

$$\overline{f}(x) = f(x) + b, \qquad \overline{g}(x) = g(x-b) + c. \tag{6}$$

Then

$$f, g \text{ satisfy (1)} \qquad \text{iff} \qquad \overline{f}, \overline{g} \text{ satisfy (1);} \tag{7}$$

this is so because the value (at x, y) of LHS(1) for f, g, and the value of LHS(1) for $\overline{f}, \overline{g}$ differ by a constant:

$$x + \overline{g}\big(y + \overline{f}(x)\big) = x + g\big(y + \overline{f}(x) - b\big) + c = x + g\big(y + f(x)\big) + c;$$

and the same applies to RHS(1).

In view of (6), (7) there is no loss of generality in restricting attention to the case where

$$f(0) = g(0) = 0; \tag{8}$$

any solution of the equation (pair f, g) can be brought down to a solution with property (8), via transformation (6).

A general feature of functional equations in two (or more) independent variables is that symmetry impairs information. Here, equation (1) is pure symmetry; the expression on the right is the mirror image of that on the left, upon interchanging x and y. Vast variety of solutions is therefore no surprise. Note, however, that the examples (2), (3), (4) are discontinuous functions—all of them except (5).

Solutions showing much more regularity can also be given. We exhibit three important families:

$$f(x) = \frac{1}{r}\,\ln\bigl(ae^{-rx}+1-a\bigr), \quad g(x) = \frac{1}{r}\,\ln\frac{e^{rx}+a}{1+a} \quad (r \neq 0;\;\; 0 \le a \le 1); \tag{9}$$

$$f(x) = -ax, \qquad g(x) = \frac{x}{1+a} \qquad (a \neq -1); \tag{10}$$

$$f(x) = -x; \quad g(x) = \frac{x}{2} + h(x); \quad h\colon \mathbb{R} \to \mathbb{R} \text{ any even function.} \tag{11}$$

Again, verification that these actually are solutions is a matter of routine calculation.

Now, (9) is a two-parameter family (of pairs of functions); (10) is a one-parameter family; and they satisfy $f(0) = g(0) = 0$. Dropping this last condition and applying transformation (6), we enlarge these families by two more degrees of freedom. Family (11) is more puzzling—it shows a lot of freedom, involving an *arbitrary* even function h, which of course can be as regular or irregular as we please.

These three families only scarcely overlap: (9) coincides with (10) only for $a = 0$ (becoming (5)); and each of (9), (10) coincides with an instant of (11) for $a = 1$ (and a suitable even function h).

Means are mentioned in the title of this note. Everybody knows that if we consider, say, the arithmetic mean of N numbers, and if we pick some n of them ($n < N$) and replace them by *their* arithmetic mean (repeated n times), the overall average shall not change. The same happens with geometric mean of N positive numbers, or any other power mean. I call this feature *compatibility.* To be precise:

A *mean* is defined here as any function

$$\mu\colon \mathbb{R}_+^n \to \mathbb{R}_+\,; \qquad \mathbb{R}_+ = (0,\infty); \qquad n \in \mathbb{N}, \quad n \ge 2, \tag{12}$$

which satisfies following assumptions: is symmetric, i.e. $\mu(\mathbf{x}) = \mu(\mathbf{x}')$ for $\mathbf{x}'$ a permutation of $\mathbf{x} = (x_1, \ldots, x_n)$, nondecreasing in each x_i, and normalized: $\mu(t,\ldots,t) = t$ for $t \in \mathbb{R}_+$; it is *homogeneous* iff $\mu(t\mathbf{x}) = t\mu(\mathbf{x})$ for $t \in \mathbb{R}_+$, $\mathbf{x} \in \mathbb{R}_+^n$.

The *power means* with exponent $r \in \mathbb{R}$, defined by

$$\left(\frac{1}{n}\sum_{i=1}^{n} x_i^r\right)^{1/r} \quad \text{for } r \neq 0, \qquad \text{and} \quad \left(\prod_{i=1}^{n} x_i\right)^{1/n} \quad \text{for } r = 0, \tag{13}$$

are a model example.

Now suppose that μ is some homogeneous mean in $\mathbb{R}_+^n$ and $\widetilde{\mu}$ is some homogeneous mean in $\mathbb{R}_+^{n+1}$. We say that μ and $\widetilde{\mu}$ are *compatible* if, for any $(x_1, \ldots, x_{n+1}) \in \mathbb{R}_+^{n+1}$

$$\widetilde{\mu}(x_1, \ldots, x_n, x_{n+1}) = \widetilde{\mu}(\underbrace{z, \ldots, z}_{n}, x_{n+1}) \quad \text{where} \quad z = \mu(x_1, \ldots, x_n). \tag{14}$$

The power means (in $\mathbb{R}_+^n$, $\mathbb{R}_+^{n+1}$, with the same exponent r) satisfy this property. This is basically due to the fact that their defining formula (13) has the same shape and is equally meaningful, both in n variables and in $n+1$ variables. What about other examples (with μ, $\widetilde{\mu}$ not necessarily given by any algebraic formula)? A tool to address that question is the following observation.

Let μ and $\widetilde{\mu}$ be compatible homogeneous means, in $\mathbb{R}_+^n$ and $\mathbb{R}_+^{n+1}$. Define:

$$f(x) = \ln \mu(\underbrace{1, \ldots, 1}_{n-1}, e^{-x}), \quad g(x) = \ln \widetilde{\mu}(\underbrace{e^x, \ldots, e^x}_{n}, 1), \quad x \in \mathbb{R} \tag{15}$$

and call $f, g\colon \mathbb{R} \to \mathbb{R}$ the functions *induced* by $\mu, \widetilde{\mu}$. Clearly, $f(0) = g(0) = 0$; f is nonincreasing and g is nondecreasing. We now show that f and g satisfy equation (1).

Choose $x, y \in \mathbb{R}$ and consider the number

$$M(x, y) = \widetilde{\mu}(\underbrace{1, \ldots, 1}_{n-1}, e^{-x}, e^{-y}).$$

Writing

$$z = e^{f(x)} = \mu(\underbrace{1, \ldots, 1}_{n-1}, e^{-x}) \quad \text{and} \quad w = ze^y = e^{y+f(x)}$$

we obtain, by compatibility and homogeneity,

$$M(x,y)=\widetilde{\mu}(\underbrace{z,\dots,z}_{n},e^{-y})=e^{-y}\widetilde{\mu}(\underbrace{w,\dots,w}_{n},1).$$

Taking logarithms, we get by (15) and by $w=e^{y+f(x)}$,

$$\ln M(x,y)=-y+g\big(y+f(x)\big).$$

In the last equation, the left side is symmetric in x,y. Hence

$$-y+g\big(y+f(x)\big)=-x+g\big(x+f(y)\big),$$

and this is equation (1).

That's how equation (1) came into being. It can be easily checked that the r-th power means (13), with the same exponent $r\neq 0$, induce via (15) the pairs f,g of type (9) with $a=1/n$; and the geometric means (13), $r=0$, induce the pair (10), also with $a=1/n$.

Less evident is that if *some* pair $\mu,\widetilde{\mu}$ (means in $\mathbb{R}^n_+,\mathbb{R}^{n+1}_+$) induce via (15) a pair f,g of type (9) or (10), with parameters a,r, then automatically $a=1/n$; a proof is given in [2].

On the other hand, the third family—pairs f,g of type (11)—never occur as functions induced by compatible homogeneous means. This is true because (see (11)), if $f(x)=-x$ then the formula in (15) would imply $e^{-x}=\mu(1,\dots,1,e^{-x})$ for all $x\in\mathbb{R}$; whence by setting $x=-1$ and $x=1$ (and by coordinatewise monotonicity) we would get

$$e=\mu(1,1,\dots,1,e)\le\mu(1,e,\dots,e,e)=e\mu(e^{-1},1,\dots,1,1)=e\cdot e^{-1}=1,$$

a contradiction.

All of this taken into account, we see that if, *in some function class $\mathcal{F}$*, equation (1) with condition (8) has no other solutions than those of types (9), (10), (11), then for each $n\in\mathbb{N}$, $n\geq 2$, there are no pairs of homogeneous compatible means (one in $\mathbb{R}^n_+$, the other in $\mathbb{R}^{n+1}_+$), other than power means (13), that would coordinatewise be functions of class $\mathcal{F}$.

So—what regularity conditions on functions f, g satisfying (1) with (8) guarantee that they must belong to type (9), (10) or (11)?

My first step in this direction was to show that analyticity of both f and g suffices (or analyticity of one of them plus some technical condition on the other one); basically, that was work on power series [2]. Closer inspection of their low-order coefficients led to the hypothesis that second-order differentiability of f, g should also suffice. This was turned into theorem by Justyna Sikorska [3]. Shortly later Nicole Brillouët–Belluot [1] showed that first-order differentiability is enough.

A further significant push was made again by J. Sikorska [4]: if f, g, satisfying (1) with (8), are continuous, strictly monotone, and convex or concave, then they are of type (9), (10) or (11). Accordingly, if $\mu, \widetilde{\mu}$ are compatible homogeneous means, of coordinatewise regularity as above, they must be power means.

In the same paper [4], J. Sikorska obtained yet another interesting result, which is more easily stated assuming that f, g satisfy equation (1) and dropping condition (8): if f, g are continuous, monotone, but not strictly monotone, and f is not identically zero, then they are of one of the two following types (for some $a, b, c \in \mathbb{R}$):

$$f(x) = \max\{b,\, a + b - x\}, \qquad g(x) = \max\{c,\, x + c - a - b\} \qquad (16)$$

or

$$f(x) = \min\{b,\, a + b - x\}, \qquad g(x) = \min\{c,\, x + c - a - b\}. \qquad (17)$$

It is worth noticing that (16) with $a = b = c = 0$ represents the pair of functions induced by the means $\max_{i\le n} x_i$, $\max_{i\le n+1} x_i$; and analogous is the claim about the pair (17), with min in place of max. (Of course, these are the limit cases of the power means (13) as $r \to \infty$, resp. $r \to -\infty$).

Although the equation has arisen merely as a tool in research on means and their compatibility, it apparently attracted more attention than the parent problem and started a life of its own. Note, however, that the results just quoted have been obtained some considerable time ago; I have not heard about any progress ever since then. The major problem still waiting for an answer is:

Find the general solution of equation (1) *in the class of continuous functions.*

In other words:

Are there any continuous functions satisfying equation (1) *with condition* (8) *other than those pertaining to families* (9), (10), (11), (16), (17)*?*

If not, that shall mean that there are no pairs of compatible homogeneous means, one in $\mathbb{R}_+^n$, the other in $\mathbb{R}_+^{n+1}$, other than the power means (13) or their limit $\max / \min$ forms. This holds because the conditions adopted as definition of a homogeneous mean (see the text around (12)) imply its continuity (a rather standard exercise in multivariate calculus); consequently, pairs of functions f, g induced by pairs of means as above, are automatically continuous.

Note that also the question of general solution of (1) in the class of *monotone* functions has so far found no satisfactory answer.

What is the rationale for a talk like this at a WFNMC conference, merely reporting a piece of rather specialized research? My intention was of course to give some publicity to the equation (1)—a nice functional equation, after all. But not only that. The presentation was primarily addressed to listeners who are teachers, instructors, coaches of olympic youth. The open problem of finding a full solution of (1) in continuous functions (or monotone functions) can be attractive to the young. It can be hard, but is not likely to require advanced knowledge or techniques. I believe that it might be solved by some young student, smart rather than experienced.

The word *problem* has in our community different meanings. An *open* problem is one of them. But we are also using this word when preparing a class-test, an exam, a competition, as a substitute for *exercise* (with solution known to the proposer); perhaps more challenging as compared to everyday classroom work. So, to end up, I wish to propose, as a Math-Comp-bonus, an olympiad-style "problem" related to the topics discussed above. The function F which appears in it obviously imitates the $\widetilde{\mu}$ of the foregoing considerations (and is visibly compatible with root mean square in $\mathbb{R}_+^2$):

Let $F\colon \mathbb{R}_+^3 \to \mathbb{R}_+$ satisfy

$$F(x,y,z) = F\left(\sqrt{\frac{x^2+y^2}{2}}\,, \sqrt{\frac{x^2+y^2}{2}}\,, z\right) = F(y,z,x) = F(z,x,y).$$

Show that if $a^2+b^2+c^2 = u^2+v^2+w^2$ then $F(a,b,c) = F(u,v,w)$.

(Note that the conditions say nothing about continuity.)

The exercise admits several possible variations: $\mathbb{R}_+^n$ in place of $\mathbb{R}_+^3$; averaging not in pairs but rather in triples or other k-tuples; the root mean square (of x, y or, in an extended version, of a k-tuple) replaced by some other power mean. If one starts to seek far reaching generalizations, one will probably sooner or later arrive at the actual problems that have made the subject of this talk.

(The talk presented at the 8th WFNMC Congress in Austria.)

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Inequalities on excursion: From China to Romania and then US

Robert Bosch

Robert Bosch is Coach and Coordinator of Mathematics Competitions at Archimedean Academy, USA. Involved in problem solving since 2000, he won a Bronze Medal in VIII Iberoamerican Mathematical Olympiad for University Students (2005).

In this article we study a group of related inequalities. The first of them was proposed in China Girls Math Olympiad 2007, for three variables. Later, a version for four variables appear in Romania Math Olympiad 2015 for ninth grade. Automatically a generalization to n variables was proposed in a paper in *Gazeta Matematica* (Seria B). Finally the Olympiad Problem O347, which is a very strong refinement to four variables inequality is proposed in the journal *Mathematical Reflections*. We present a new solution to this problem.

1 China

The problem 6 from China Girls Math Olympiad 2007 is:

For nonnegative real numbers a, b, c with $a+b+c=1$, prove that

$$\sqrt{a+\frac{(b-c)^2}{4}}+\sqrt{b}+\sqrt{c} \leq \sqrt{3}. \qquad (1)$$

We present three solutions that can be found in [1] and [2].

Solution 1. Without loss of generality, we may assume that $b \geq c$. We set $\sqrt{b} = x + y$ and $\sqrt{c} = x - y$ for some nonnegative real numbers x and y. Hence $b - c = 4xy$ and $a = 1 - 2x^2 - 2y^2$. It follows that

$$\sqrt{a + \frac{(b-c)^2}{4}} + \sqrt{b} + \sqrt{c} = \sqrt{1 - 2x^2 - 2y^2 + 4x^2y^2} + 2x. \qquad (2)$$

Note that $2x = \sqrt{b} + \sqrt{c}$, implying that

$$4x^2 = \left(\sqrt{b} + \sqrt{c}\right)^2 \leq 2b + 2c \leq 2,$$

by the AM–GM inequality. Thus $4x^2y^2 \leq 2y^2$ and

$$1 - 2x^2 - 2y^2 + 4x^2y^2 \leq 1 - 2x^2.$$

Substituting the last inequality into (2) yields

$$\sqrt{a + \frac{(b-c)^2}{4}} + \sqrt{b} + \sqrt{c} \leq \sqrt{1 - 2x^2} + 2x,$$
$$\sqrt{1 - 2x^2} + x + x, \leq \sqrt{3},$$

by the Cauchy–Schwarz inequality.

Solution 2. Let $a = u^2$, $b = v^2$, and $c = w^2$. Then $u^2 + v^2 + w^2 = 1$ and the desired inequality becomes

$$\sqrt{u^2 + \frac{(v^2 - w^2)^2}{4}} + v + w \leq \sqrt{3}. \qquad (3)$$

Note that

$$\begin{aligned} u^2 + \frac{(v^2-w^2)^2}{4} &= 1-(v^2+w^2)+\frac{(v^2-w^2)^2}{4}, \\ &= \frac{4-4(v^2+w^2)+(v^2-w^2)^2}{4}, \\ &= \frac{4-4(v^2+w^2)+(v^2+w^2)^2-4v^2w^2}{4}, \\ &= \frac{(2-v^2-w^2)^2-4v^2w^2}{4}, \\ &= \frac{(2-v^2-w^2-2vw)(2-v^2-w^2+2vw)}{4}, \\ &= \frac{\left[2-(v+w)^2\right]\left[2-(v-w)^2\right]}{4}, \\ &\le 1-\frac{(v+w)^2}{2}. \end{aligned}$$

(Note that $(v+w)^2 \le 2(v^2+w^2) \le 2$). Substitute the above inequality into (3) and it gives

$$\sqrt{1-\frac{(v+w)^2}{2}}+v+w \le \sqrt{3}.$$

Set $\frac{v+w}{2} = x$. We can rewrite the above inequality as

$$\sqrt{1-2x^2}+2x \le \sqrt{3},$$

and we can complete the proof as we did in the first solution. *Note*: The

second proof reveals the motivation of the substitution used in the first proof.

Solution 3. Set $x = \sqrt{bc}$, then we have

$$\sqrt{b}+\sqrt{c} = \sqrt{b+c+2\sqrt{bc}} = \sqrt{1-a+2\sqrt{bc}} = \sqrt{1-a+2x},$$

and

$$\sqrt{a+\frac{(b-c)^2}{4}} = \sqrt{a+\frac{(b+c)^2-4bc}{4}} = \frac{1}{2}\sqrt{(1+a)^2-4x^2}.$$

Therefore, the inequality in question now rewrites as

$$\sqrt{(1+a)^2-4x^2}+2\sqrt{1-a+2x}\le 2\sqrt{3}.$$

By the Cauchy–Schwarz inequality we have

$$\sqrt{(1+a)^2-4x^2}+2\sqrt{1-a+2x}\le \sqrt{3\left[(1+a)^2-4x^2+2(1-a+2x)\right]},$$

and hence it suffices to prove that

$$(1+a)^2-4x^2+2(1-a+2x)\le 4,$$

or equivalently

$$a^2-4x^2+4x-1\le 0.$$

This can be also rewritten as

$$(1-2x-a)(1+a-2x)\ge 0,$$

which is obviously true since

$$1+a-2x\ge 1-2x-a=1-2\sqrt{bc}-a\ge 1-(b+c)-a=0.$$

2 Romania

The following problem was proposed by *Costel Anghel*, Negreni, Olt to Romanian Mathematical Olympiad 2015 for ninth grade.

Let a,b,c,d be nonnegative real numbers such that $a+b+c+d=1$. Prove that

$$\sqrt{a+\frac{(b-c)^2}{6}+\frac{(c-d)^2}{6}+\frac{(d-b)^2}{6}}+\sqrt{b}+\sqrt{c}+\sqrt{d}\le 2.$$

Solution.

$$\left(\sqrt{b}+\sqrt{c}\right)^2 = b+c+2\sqrt{bc} \le 2b+2c \le 2,$$

by AM–GM inequality and the given condition. Thus

$$(b-c)^2 = \left(\sqrt{b}+\sqrt{c}\right)^2\left(\sqrt{b}-\sqrt{c}\right)^2 \le 2\left(\sqrt{b}-\sqrt{c}\right)^2.$$

It follows that

$$a+\frac{(b-c)^2}{6}+\frac{(c-d)^2}{6}+\frac{(d-b)^2}{6}$$

is less than

$$\begin{aligned}
& a+\frac{1}{3}\left[\left(\sqrt{b}-\sqrt{c}\right)^2+\left(\sqrt{c}-\sqrt{d}\right)^2+\left(\sqrt{d}-\sqrt{b}\right)^2\right],\\
&= a+\frac{1}{3}\left[2(b+c+d)-2\left(\sqrt{bc}+\sqrt{cd}+\sqrt{db}\right)\right],\\
&= a+\frac{1}{3}\left[2(b+c+d)+(b+c+d)-S^2\right],\\
&= a+b+c+d-\frac{S^2}{3},\\
&= 1-\frac{S^2}{3},
\end{aligned}$$

where

$$S=\sqrt{b}+\sqrt{c}+\sqrt{d}.$$

Therefore the left side of the original inequality is less than

$$\sqrt{1-\frac{S^2}{3}}+S.$$

To completely prove the inequality we consider the next chain of equivalences

$$\begin{aligned}
\sqrt{1-\frac{S^2}{3}} &\le 2-S,\\
1-\frac{S^2}{3} &\le 4-4S+S^2,\\
4S^2-12S+9 &\ge 0,\\
(2S-3)^2 &\ge 0.
\end{aligned}$$

Notice that $S \leq 2$ because

$$\left(1 \cdot \sqrt{a} + 1 \cdot \sqrt{b} + 1 \cdot \sqrt{c} + 1 \cdot \sqrt{d}\right)^2 \leq (1^2+1^2+1^2+1^2)(a+b+c+d) = 4,$$

by Cauchy–Schwarz inequality and then $S \leq 2 - \sqrt{a} \leq 2$.

A generalization to n variables can be found in [3]. Say, let $a_1, a_2, ..., a_n$ be nonnegative real numbers ($n \geq 3$) such that $a_1 + a_2 + \cdots + a_n = 1$. The following inequality is true

$$\sqrt{a_1 + \frac{1}{2(n-1)} \sum_{2 \leq i < j \leq n} (a_i - a_j)^2} + \sum_{i=2}^{n} \sqrt{a_i} \leq \sqrt{n}. \qquad (4)$$

Equality holds if and only if $a_1 = a_2 = \cdots = a_n = \frac{1}{n}$.

The proof is similar to the previous one.

3 USA

The problem O347 was proposed by Marius Stanean, Zalau, Romania, to the journal *Mathematical Reflections*. Exactly issue 4, 2015. Here we show his solution first and later a joint proof by the author Robert Bosch and Arkady Alt, San Jose, California, USA.

O347. Let $a, b, c, d \geq 0$ be real numbers such that $a + b + c + d = 1$. Prove that

$$\sqrt{a + \frac{2(b-c)^2}{9} + \frac{2(c-d)^2}{9} + \frac{2(d-b)^2}{9}} + \sqrt{b} + \sqrt{c} + \sqrt{d} \leq 2.$$

Solution 1. First we prove the following Lemma.

Lemma: Let x, y, z be nonnegative real numbers.

$$2\sum_{cyc}(y^2-z^2)^2 \le 3(x^2+y^2+z^2)\sum_{cyc}(y-z)^2.$$

Proof: Since the inequality is symmetric and homogeneous, without loss of generality we assume that $x+y+z=3$ and for an easy computing, denote $q = xy+yz+zx$. Rewrite the inequality to the following forms

$$2\left(\sum_{cyc}x^4-\sum_{cyc}x^2y^2\right) \le 3\left(\sum_{cyc}x^2\right)\left(\sum_{cyc}x^2-\sum_{cyc}xy\right),$$
$$\Leftrightarrow 2\left[(x^2+y^2+z^2)^2-3(x^2y^2+y^2z^2+z^2x^2)\right] \le 3(9-2q)(9-3q),$$
$$\Leftrightarrow 2\left[(9-2q)^2-3(q^2-6xyz)\right] \le 3(9-2q)(9-3q),$$
$$\Leftrightarrow 36xyz \le 81-63q+16q^2.$$

Now, since $3q \le (x+y+z)^2 = 9 \Leftrightarrow q \le 3$, there is a real number $t \in [0,1]$ such that $q = 3(1-t^2)$. Therefore, the inequality becomes

$$4xyz \le 16t^4-11t^2+4.$$

Now we have by ABC-Method (see the next section) $xyz \le (1-t)^2(1+2t) = 2t^3-3t^2+1$. Returning back to the previous inequality we only need to prove that

$$\begin{aligned} 4-12t^2+8t^3 &\le 16t^4-11t^2+4, \\ t^2(16t^2-8t+1) &\ge 0, \\ t^2(4t-1)^2 &\ge 0, \end{aligned}$$

which is obviously true. Equality holds for $t=0$ which means $x=y=z$ or for $t=\frac{1}{4}$ which means $2x=2y=z$.

Let us now return to our inequality. According to the Lemma we have

$$2\sum_{cyc}(b-c)^2 \le 3(b+c+d)\sum_{cyc}(\sqrt{b}-\sqrt{c})^2,$$

but $b+c+d \le 1$, so

$$\frac{2}{9}\sum_{cyc}(b-c)^2 \le \frac{1}{3}\sum_{cyc}(\sqrt{b}-\sqrt{c})^2 = b+c+d-\frac{(\sqrt{b}+\sqrt{c}+\sqrt{d})^2}{3}.$$

Applying the Cauchy–Schwarz inequality we have

$$\left(\sqrt{a+\frac{2(b-c)^2}{9}+\frac{2(c-d)^2}{9}+\frac{2(d-b)^2}{9}}+\sqrt{b}+\sqrt{c}+\sqrt{d}\right)^2$$

less than

$$\left(a+\frac{2(b-c)^2}{9}+\frac{2(c-d)^2}{9}+\frac{2(d-b)^2}{9}+\frac{(\sqrt{b}+\sqrt{c}+\sqrt{d})^2}{3}\right)$$
$$(1+3) \leq 4.$$

Equality holds when $a=b=c=d=\frac{1}{4}$.

ABC−Method

This theorem can be found in [4]. It's very powerful to solve hard inequalities since the bounds provided by it are optimal.

Let

$$\begin{aligned}
p &= \frac{x+y+z}{3},\\
q &= \frac{xy+yz+zx}{3},\\
r &= xyz,\\
s &= \frac{1}{3}\sqrt{\frac{(x-y)^2+(y-z)^2+(z-x)^2}{2}} = \sqrt{p^2-q}.
\end{aligned}$$

The following theorem is true:

$$\max\left\{0,(p+s)^2(p-2s)\right\} \leq r \leq (p-s)^2(p+2s),$$

the proof is considering the cubic polynomial

$$P(t)=(t-x)(t-y)(t-z)=t^3-3pt^2+3(p^2-s^2)t-r,$$

with three real roots, meaning that $P(t_1) \geq 0$ and $P(t_2) \leq 0$ if $t_1 \leq t_2$ are the roots of $P'(t)$. (Notice we used Rolle's theorem). But

$$P'(t) = 3(t-p+s)(t-p-s),$$

therefore

$$\begin{aligned}(p-s)^2(p+2s) - r \geq 0,\\ (p+s)^2(p-2s) - r \leq 0.\end{aligned}$$

This new solution to problem O347 was elaborated by the author Robert Bosch and Arkady Alt, San Jose, California, USA.

Proof 2. $a = 1-t^2,\ b = x^2t^2,\ c = y^2t^2,\ d = z^2t^2$ with $0 \leq t \leq 1$, and $x, y, z \geq 0$, and $x^2+y^2+z^2 = 1$. Then the original inequality becomes

$$\sqrt{1-t^2+Ht^4} + (x+y+z)t \leq 2,$$

where

$$H = \frac{2}{9}\left[(x^2-y^2)^2 + (y^2-z^2)^2 + (z^2-x^2)^2\right].$$

Clearly $(x^2-y^2)^2 \leq (x^2+y^2)^2 \leq 1$, so $H < 1$. Using $t^2 \leq 1$

$$\sqrt{1-(t^2-Ht^4)} = \sqrt{1-t^2\left(1-Ht^2\right)} \leq \sqrt{1-(1-H)t^2}.$$

Now by Cauchy–Schwarz inequality

$$\begin{aligned}\left[\sqrt{1-t^2+Ht^4} + (x+y+z)t\right]^2 &\leq \left[\sqrt{1-(1-H)t^2} + (x+y+z)t\right]^2,\\ &= \left[\sqrt{1-(1-H)t^2} + \frac{x+y+z}{\sqrt{1-H}} \cdot \sqrt{1-H}t\right]^2,\\ &\leq \left(1 + \frac{(x+y+z)^2}{1-H}\right)\left(1-(1-H)t^2 + (1-H)t^2\right),\\ &= 1 + \frac{(x+y+z)^2}{1-H}.\end{aligned}$$

Let us prove

$$(x+y+z)^2 \leq 3(1-H).$$

Notice that

$$3H = \frac{4}{3}\left(x^4 + y^4 + z^4 - (x^2y^2 + y^2z^2 + z^2x^2)\right).$$

Also,

$$\begin{aligned}(x+y+z)^2 &= x^2 + y^2 + z^2 + 2(xy + yz + zx) = 1 + 2(xy + yz + zx),\\ x^4 + y^4 + z^4 &= 1 - 2(x^2y^2 + y^2z^2 + z^2x^2),\end{aligned}$$

so the inequality to be proved becomes

$$6(x^2y^2 + y^2z^2 + z^2x^2) - 3(xy + yz + zx) + 1 \geq 0,$$

with $x^2 + y^2 + z^2 = 1$. The homogeneous form is

$$6\sum_{cyc} x^2y^2 - 3\sum_{cyc} xy \cdot \sum_{cyc} x^2 + \left(\sum_{cyc} x^2\right)^2 \geq 0. \tag{5}$$

Without loss of generality we can assume $x + y + z = 1$. Let

$$\begin{aligned} q &= xy + yz + zx,\\ r &= xyz. \end{aligned}$$

With this new notation (5) is

$$16q^2 - 7q + 1 - 12r \geq 0.$$

By $ABC-$Method the following upper bound holds

$$r \leq \frac{9q - 2 + 2(1-3q)\sqrt{1-3q}}{27}. \tag{6}$$

Here $q \leq \frac{1}{3}$ because

$$\begin{aligned} x^2 + y^2 + z^2 &\geq xy + yz + zx,\\ (x+y+z)^2 &\geq 3(xy + yz + zx),\\ 1 &\geq 3(xy + yz + zx),\\ 1 &\geq 3q. \end{aligned}$$

Writing $q = \frac{1-\lambda^2}{3}$ with $\lambda \in [0, 1)$ give opportunity to avoid radicals and we obtain the following equivalent rational form of (6)

$$r \leq \frac{2\lambda^3 - 3\lambda^2 + 1}{27}.$$

The inequality to be proved is

$$16q^2 - 7q + 1 - \frac{12}{27}\left(2\lambda^3 - 3\lambda^2 + 1\right) \geq 0,$$

and after multiply by 9 and expand

$$\begin{aligned} 16\lambda^4 - 8\lambda^3 + \lambda^2 &\geq 0, \\ \lambda^2 (4\lambda - 1)^2 &\geq 0. \end{aligned}$$

Done.

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[1] Xiong Bin, Lee Peng Yee. *Mathematical Olympiad in China (2007–2008). Problems and Solutions.* East China Normal University Press and World Scientific Publishing, (2009).

[2] Vo Quoc Ba Can, Cosmin Pohoata. *Old and New Inequalities.* Editura Gil, (2008).

[3] Marian Haiducu, Mircea Fianu. O Generalizare a unei Probleme de Concurs. *Gazeta Matematica*, (Seria B), (2015).

[4] Tran Phuong. *Diamonds in Mathematical Inequalities.*

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Addendum

In my article "Regular Lattice Polygons", in this journal *Mathematics Competitions*, Vol. 29, No 1, 2016, I showed a new proof of the following result:

The square is the only regular lattice polygon.

In this one is suggested the first proof was by Davor Klobucar in

"On nonexistence of an integer regular polygon" *Mathematical Communications* (3) 1998, 75–80.

The above statement is not completely correct, reading the book *Mathematical Diamonds*, by Ross Honsberger, from the series Dolciani Mathematical Expositions, MAA, 2003, exactly Section 13, *Semi-regular Lattice Polygons* you can read this is proved in the book *Combinatorial Geometry in the Plane* by Hadwiger, Debrunner, and Klee, **1964**.

In the above Section 13, the following two theorems are proved:

Theorem 1 There exists an *equilateral* lattice polygon if and only if n is an even number and $n \geq 4$.

Theorem 2 There exists an *equiangular* lattice polygon if and only if $n = 4$ or $n = 8$.

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The Flavor of the Colorado Mathematical Olympiad: A Concerto in Four Movements[1]

Alexander Soifer

Born and educated in Moscow, Alexander Soifer has for over 38 years been a Professor at the University of Colorado, teaching math, and art and film history. He has published over 300 articles, and a good number of books. In the past several years, 7 of his books have appeared in Springer: The Scholar and the State: In the search of Van der Waerden*;* The Mathematical Coloring Book: Mathematics of Coloring and the Colorful Life of Its Creators*;* Mathematics as Problem Solving*;* How Does One Cut a Triangle?*;* Geometric Etudes in Combinatorial Mathematics*;* Ramsey Theory Yesterday, Today, and Tomorrow*; and* Colorado Mathematical Olympiad and Further Explorations: From the Mountains of Colorado to the Peaks of Mathematics*. He has founded and for 32 years ran the Colorado Mathematical Olympiad. Soifer has also served on the Soviet Union Math Olympiad (1970–1973) and USA Math Olympiad (1996–2005). He has been Secretary of WFNMC (1996–2008), and Senior Vice President of the World Federation of National Mathematics Competitions (2008–2012); from 2012 he has been the president of the WFNMC. He is a recipient of the Federation's Paul Erdős Award (2006). Soifer's Erdős number is 1.*

Dedicated to the Members of Problem Committee and Jury of the Colorado Mathematical Olympiad

Note: CMO stands below for the Colorado Mathematical Olympiad; problem number $m.n$ stands for problem n of CMO-m. At CMO, we offer 5 problems of increasing difficulty and 4 hours to solve and write them up.

[1]This is a full text of the talk presented at the 8th International Congress of the World Federation of National Mathematics Competitions in Semriach (Graz), Austria, July 2018.

Figure 1: Judges at work, Sixth Colorado Mathematical Olympiad, 1989 (left to right): Robert Ewell, Keith Mann, Gary Miller, and Alexander Soifer. *Photograph by Mary Kelley, The Gazette*

Movement 1: ARCHEOLOGY IN SERVICE OF MATH

17.3. MORE STONE AGE ENTERTAINMENT (*F. Harary and A. Soifer*, 2000)
(A) Fred Flintstone and Barney Rubble in turn color unit squares of a 2000×2000 square grid, one unit square per move. Fred uses red, and Barney uses blue. Fred wins if he gets a red 5-unit-square cross (that is, a unit square with its adjacent squares above, below, right and left). Otherwise Barney wins. Find a strategy that allows Fred or Barney to win regardless of how the other one may play. Fred goes first.

(B) What is the winning strategy if the 5-unit-square cross is replaced by a 2×2 square?

Solution of 17.3 (A). Barney tiles the given 2000×2000 square grid by dominoes and comes to play with this home-prepared template (Figure 17.1). Every time Fred colors one square of a domino red, Barney colors the second square of the same domino blue. This guarantees that there are no completely red dominoes in Barney's tiling. On the other hand, no matter where you place the 5-unit-square cross on the grid, it would cover completely at least one domino in Barney's tiling. Thus, Barney wins.

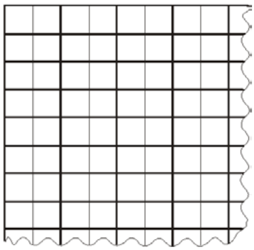


Figure 17.1

Solution of 17.3 (B). Observe that the same strategy would not work, because it is possible to place a 2×2 square on the grid so that it does not cover completely any domino in Barney's tiling. However, we can change the template a little to make it work: just translate consecutive rows of the template through the width of one unit square (Figure 17.2), and let Barney use this template in following the same strategy of coloring the second square of the domino. In this template we have some monominoes (single unit squares) along the boundary. If Fred colors one of monominoes, let Barney color another monomino.

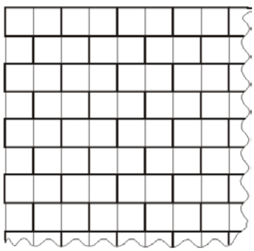


Figure 17.2

Second Solution of 17.3 (B). Something amazing happened with me during the week between the Olympiad and the Award Presentation Ceremonies. I found a new solution of this problem on... an ancient

6000-year old seal! You can see it in Figure 17.3. The Tuesday, April 25, 2000, auction described it as follows.

Ancient Persian Seal Stamp Sialk 4000 BC
A very rare and large Chalcolithic period[2], of Shalk II/III, Persian pottery impressed seal, or seal stamp. Probably used as both an official pass, and a design impression marker. A rare type found at Sialk, Rey, Susu and, or Tepe Giyan, generally north of Tehran or near Kashan, and dating from 4200–3400 B.C. Very fine, large and in perfect condition, with buff orange body, deep design, and untouched burial surface. A rather esoteric museum type item, or for a serious collector of the formative period of seals. 4” long × 2 1/4” wide × 1 1/16” deep.

Figure 17.3

As you can see, the ancient seal presents a “zigzag” tiling of the plane with dominoes. This tiling, taken as a template (please see Figure 17.4), allows us to use exactly the same strategy as we used in the first solution of this problem.

In order to win, Barney uses this template and colors the second square of the domino the first square of which was colored by Fred. If Fred colors one of monominoes, Barney colors another monomino.

You may be wondering, what happened to the ancient seal at the auction? I had to buy it as a memento of such an unusual way of discovering a mathematical solution!

[2]The Chalcolithic is the name given to the period in the Near East and Europe after the Neolithic period and before the Bronze Age, between approximately 4500 and 3500 BCE.

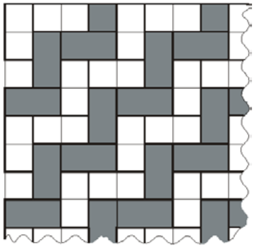


Figure 17.4

Movement 2: TILING & COLORING

34.4. The Four-Color Game (*A. Soifer*, 2017)
Two players in turn color one unit edge of a 2017×2017 square grid in one of four available colors. The first player wins if a circuit is created with no adjacent edges of the same color; otherwise, the second player wins. Find a strategy for one of the players, guaranteeing a win regardless of the moves the other player makes.

A *circuit* is a continuous line along distinct edges of the grid that starts and ends at the same point.

Solution. Name the first player A and the second player B. Since a circuit, desired by A, must have at least four points at which the circuit makes a 90-degree turn, a winning strategy is readily available for player B. Before the game starts, B prepares a tiling of the unit lines of the grid by L-shaped pairs of unit lines (Fig. 34.1)

Every time A colors a unit edge, B colors the second unit edge of the same L, using the same color, thus eliminating any L-shaped 90 degree turn that the first player needs to complete the required circuit. There is a set S consisting of two lines of the grid, not covered by the L-tiles: the top and the right boundary lines of the grid, a total of 2×2017 unit lines. Every time A colors a unit line from S, B colors any other unit line of S.

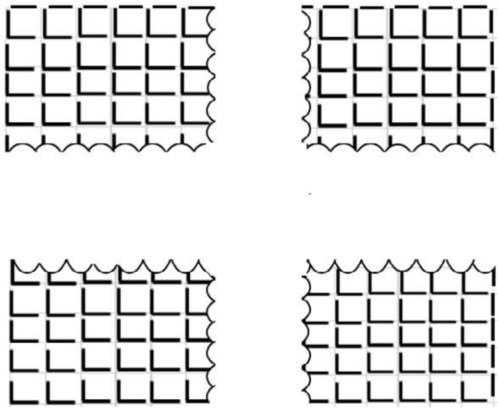


Figure 34.1

34.5. The Game of Tetris (*A. Soifer, 2017*)

A *tromino* is a figure made of three equal-sized squares connected edge-to-edge. A tromino in the shape of an L is called an L-tromino.

(A). In the game of Tetris, L-trominoes fall on a 3×2017 square grid G, each covering completely three cells of G. Can all cells of G end up covered equally many times?

(B). Solve the same problem for a 5×2025 square grid G.

Solution of 34.5 (A). The answer is *no*. Draw coordinate axes with the origin in a corner of the grid G, and place number 2 in all squares with coordinates (*odd*, *odd*) and number (-1) in all other squares of G (see Fig. 34.2).

Figure 34.2

There are 2×1009 squares with 2, and $(3\times2017-2\times1009) = 4\times1009-3$ squares with (-1). Now we can easily calculate the sum S of all numbers in G:

$$S = 4 \times 1009 - 4 \times 1009 + 3 = 3 > 0. \tag{1}$$

Assume that each square of G is covered by n L-trominoes. Observe that no matter where and how an L-tromino covers 3 squares of G, the sum of numbers in the L-tromino will be *non-positive*: 0 or (-3); see Fig. 34.3.

2	-1	2	-1	2	-1	2	-1	2	-1	2
-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
2	-1	2	-1	2	-1	2	-1	2	-1	2

Figure 34.3

This implies that $nS \leq 0$, which contradicts the inequality (1).

Solution of 34.5 (B). Here the answer is *yes*. Fig. 34.4 shows a tiling of a 5×9 grid by L-trominoes.

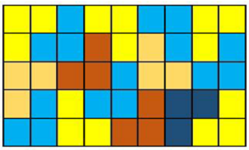


Figure 34.4

Since 2025 is divisible by 9, the 5×2025 grid can be partitioned into 5×9 parts, which can be tiled by L-trominoes—moreover, tiled in one layer.

Movement 3: MATHS FOR MEDS

32.4. Stopping the Ebola Epidemic (*A. Soifer*, 2015)
A square region 2016×2016 miles is divided into 2016^2 cells each of which is a square of side 1 mile. Some cells are contaminated by the Ebola virus. Every month the virus spreads to those cells which have at least two sides in common with the contaminated cells. Find the maximum number of contaminated cells, such that no matter where they are located, the Ebola epidemic will not spread to cover the entire region.

Solution. As the epidemic spreads, the perimeter of the contaminated region cannot increase, for with each newly contaminated cell the perimeter loses at least two sides (shared with previously contaminated cells), and gains at most two new sides. If at most 2015 cells are contaminated initially, the starting perimeter is at most 2015×4, and thus the perimeter will never reach 2016×4, which is the perimeter of the entire region.

The contaminated main diagonal of the region (Figure 32.1) spreads to cover the entire region, thus showing that 2016 contaminated cells can *possibly* cause the spread of the Ebola on the entire region. The answer is thus 2015.

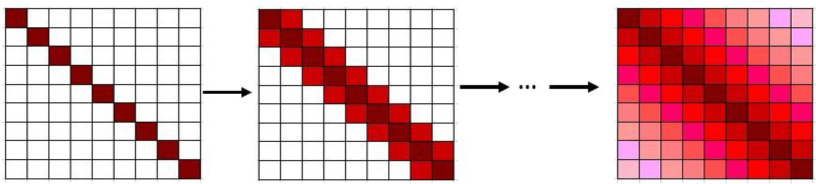


Figure 32.1

Movement 4: MATH OF POLITICS

31.3. A Dream for a Peaceful Ukraine (*A. Soifer*, 2014)
Each Ukrainian city flies one flag, Ukrainian or Russian, and connects by roads directly to 11 or 19 other Ukrainian cities, its *neighbors*. A city *lives in peace* if it flies the same flag as the majority of its neighbors, and *at war* otherwise. Each morning one city at war, if there is one, changes its flag. Will the day come when all Ukrainian cities will live in peace?

Solution. Create a graph with the Ukrainian cities as vertices and roads connecting them as edges. Denote by x the number of edges that connect cities flying opposite flags. With each change of a city flag, x reduces by at least 1 while remaining non-negative. Therefore, after finitely many steps there will be no flags to change, and peace will come to Ukraine.

Ukrainian Flag

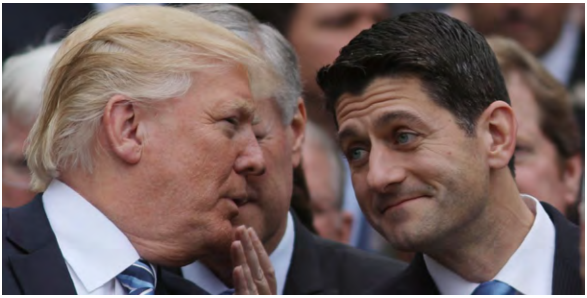
35.3. AfterMath of Tax Reform: A Game for Two Players (*A. Soifer*, 2018)

(A). The U. S. Congress Joint Committee on Taxation estimated that the December 20, 2017, Tax Reform will increase the national debt by $1,460,000,000 over 10 years. In trying to offset this debt increase, President Donald J. Trump and Speaker of the House of Representatives Paul D. Ryan, Jr. are playing the game of *AfterMath of Tax Reform*. On the table between them there are three piles of money: Welfare consisting of 2018 billion dollars, Medicare consisting of 2019 billion dollars, and Medicaid consisting of 2020 billion dollars. Mr. Trump brought with him 25 billion dollars he got from Mexico for the Great Trump Wall; Mr. Ryan arrived with 1 million dollars of his family savings. In turn, the players take any non-zero integral amount of money from one or two piles or add any non-zero integral amount from their money to one or two piles. The player wins who takes the last dollar off the table. Find the strategy for the President or the Speaker allowing him to win regardless of the moves of his opponent. As always, President Trump goes first.

(B). In fact, each of these two players desires to blame the other for playing the *AfterMath of Tax Reform*. Find the strategy for the President or the Speaker that allows him to make the opponent take the last dollar regardless of the opponent's moves. As always, President Trump goes first.

Note: The inspiration for this problem comes from President Trump's public statements and the facts reported by The Washington Post: `https://www.washingtonpost.com/news/wonk/wp/2017/12/01/gop-`

eyes-post-tax-cut-changes-to-welfare-medicare-and-social-security/?utm_term=.af45d572a3e7



President Donald J. Trump (left) and
Speaker of the House of Representatives Paul D. Ryan, Jr.

Solution of 35.3 (A). President Trump wins. In his first move, Trump takes 1 billion dollars from Medicare and 2 billion dollars from Medicaid, thus leaving on the table equal amounts in each of the three piles. No matter what Speaker Ryan takes, President Trump can take money from one or two piles and leaves again equal amounts of money in the three piles. This process will end with the President and the Speaker taking all the money from Welfare, Medicare, and Medicaid, with President Trump taking the last dollar. If at any point the Speaker adds his own money to the piles, the President immediately takes to himself all the money added by the Speaker.

Solution of 35.3 (B). Once again, President Trump wins—he likes winning he often says.:-) In his first move, Trump takes 1 billion dollars from Medicare and 2 billion dollars from Medicaid, thus leaving on the table equal amounts in each of the three piles. No matter what Speaker Ryan takes, President Trump can take money from one or two piles and leave again equal amounts of money in the three piles. However, the President must not equalize the amounts in all three piles if he gets the lowest pile consisting of $1 or none at all. Let us take a closer look at these positions.

Let (x, y, z) be the position in the game, i.e., dollar amounts in the three piles, $x \leq y \leq z$. If the Speaker leaves after his move the position $(1, y, z)$, the President takes everything from the piles with the amounts

y, z, and forces the Speaker to pocket the last dollar. If the Speaker leaves after his move the position $(0, y, z)$ with $y + z > 1$, the President takes everything except 1 dollar from the piles with the amounts y, z, and forces the Speaker to take the last dollar. Finally, the Speaker cannot leave after his move to the position $(0, 0, 1)$ because he gets from the President a position (x, x, x) with $x \geq 2$, and there is no way to get from there to $(0, 0, 1)$ as this would require taking money from all three piles.

If at any point the Speaker adds his own money to the piles, the President takes to himself all the money added by the Speaker.

You will find CMO's detailed history and all problems and solutions of the first 30 years of our Olympiad in [1] and [2]. The Fourth Decade book has to wait for the decade to run its course.

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Solving mathematical competition problems with triangle equalizers

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1 Introduction

An interesting issue arising from classical Euclidean geometry concerns the existence of lines called "equalizers" that bisect both the area and the perimeter of a surface. This paper examines the possibility of a straight line bisecting a right triangle, its position on the triangle and the possible number of such lines. In this case, such a line is known as a *triangle equalizer*[1]. The search for such lines can be seen as a trivial process, but this abstains from the real picture. The complete study concerning the special case of a triangle was conducted by Kontokostas (2010). The possibility of the existence of an equalizer that can be applied to an arbitrary planar shape is an important parameter. However, a general method may not exist in order to solve this problem.

In general, an equalizer can be applied to any body and that is a fact that came up from a useful topology theorem: the *Ham-Sandwich Theorem*, also called the *Stone-Tukey Theorem* (after Arthur H. Stone and John W. Tukey). The theorem states that, given $d \geq 2$ measurable solids in $\mathbb{R}^d$, it is possible to bisect all of them in half with a single $(d-1)$-dimensional hyperplane. In other words, the Ham-Sandwich Theorem provides the following paraphrased statement: *Take a sandwich made of a slice of ham and two slices of bread. No matter where one places the pieces of the sandwich in the kitchen, or house, or universe, so long as one's knife is long enough one can cut all three pieces in half in only one pass.* Proving the theorem for $d = 2$ (known as the *Pancake Theorem*) is simple and can be found in Courant and Robbins (1996, p. 267).

In the present paper, we'll give a general overview to various international competitions, where their core is the equalizer of a body. The reason for conducting such a mathematical search was an article written by George Berzsenyi (1997, p. 51), in which he assumed that in each triangle there are either only one or three equalizers. Meanwhile, no effort is made in order to solve the problem, but the author encourages readers to explore it. Besides, H. Bailey also states that there are no triangles with exactly two equalizers. Furthermore, the article of Kodokostas (2010) provides a detailed account of the location of the equalizers in a

[1]Many writers use several alternative names for the equalizer. For example, Berele and Catoiu (2016, p. 20) used the term *AP-bisecting line*, Todd (1999, p. 31) used the term *B-line*, Yiu (2016, p. 7) used the term *perimeter-area bisector.*

triangle. An interesting lemma in that article states that any equalizer of a triangle goes through the incenter of the triangle, a line through that incenter is an area divider if and only if it is a perimeter splitter and the equalizers of a triangle are the area dividers through its center (Kodokostas, 2010, p. 142).

2 Brief history about equalizer problems

There is a great difference between the knowledge of the existence of a problem's solution and the ability to construct that solution. Mathematical learning cannot be considered a passive learning process and the only way to learn mathematics is by solving problems (Chen, 2016, xi). The problem of bisecting a shape into equal areas and equal perimeters has been studied in several papers. It is desirable to know which lines (or planes) bisect a given figure. Such lines always exist and a triangle could have one, two or three equalizers, while no triangle can have more than three equalizers according to Anthony Todd (1999, p. 37).

In 1994, Alexander Shen[2], professor at the Independent University of Moscow, published in *The Mathematical Intelligencer* a selection of problems, known as "coffin problems", which were offered to "undesirable" applicants at the entrance examinations at the Department of Mechanics and Mathematics (Mekh-mat) of Moscow University at 1970s and 1980s. Four examinations were held at the Mekh-Mat: written math, oral math, literature essay composition, and oral physics (Frenkel, 2013, p. 28). These problems appear to resemble greatly with the Olympiad problems. It should be noted that these problems also differ from the Olympiad problems by being, in many cases, either false or poorly stated. Their solution does not require knowledge of a higher level of mathematics, but require, however, ingenuity, creativity and unorthodox attitudes. Solutions to these problems were thoroughly analyzed by Ilan Vardi (2005a, 2005b, 2005c).

The Mathematics Department of Moscow State University, the most prestigious mathematics school in Russia, had at that time been actively trying to keep Jewish students (and other "undesirables") from

[2] Shen, Alexander (1994). Entrance examinations to the Mekh-mat. *The Mathematical Intelligencer*, 16 (4): 6–10.

enrolling in the department (Vershik, 1994, p. 5). One of the methods they used for doing this was to give the unwanted students a different set of problems on their oral exam. These problems were carefully designed to have elementary solutions (so that the Department could avoid scandals) that were nearly impossible to find. Any student who failed to answer could be easily rejected, so this system was an effective method of controlling admissions. These kinds of math problems were informally referred to as "Jewish" problems or "coffins". Coffins is the literal translation from Russian (Khovanova and Radul, 2012, p. 815). These problems along with their solutions were, of course, kept as a secret, but Valera Senderov and his friends had managed to collect a list. In 1975, they approached us to solve these problems, so that they could train the Jewish students following these mathematical ideas. *Problem 5* of Shen's catalogue, which had been proposed by Podkolzin in 1978, states *Draw a straight line that halves the area and perimeter of a triangle*. A solution was included in the first chapter of Mikhail Shifman's book (2005, pp. 50–51).

Expanding on the question of existence and focusing on a triangle's equalizer, George Berzsenyi in the *Quantum* magazine conjectured that no triangle has more than three equalizers. A companion conjecture by the Emeritus Professor H. Bailey states that no triangle has exactly two equalizers. Berzsenyi stated that the present investigation was prompted by a problem posed in the 1988–89 Scottish Mathematical Challenge (Berzsenyi, 1997, p. 51).

In 1999, seventeen year-old Rio G. Bennin[3], a home-schooled senior from Berkeley (California), won a $20000 scholarship for his project *Triangular equalizers, Pythagorean quadrilaterals, equalizers of n-dimensional convex sets and perfect equalizers*. Bennin earned the fifth place in Intel *Science Talent Search*[4], the oldest and most prestigious science and math competition for high school seniors in U.S. Finalists were selected based on their scientific rigor and the world-changing potential of their research projects. Bennin's paper dealt with dividing two-dimensional

[3] Additional information could be found at the website of the Mathematical Association of America: www.maa.com/mathland/mathtrek_3_22_99.html.

[4] Additional information could be found at: http://www.societyforscience.org/STS

figures, such as triangles, into two pieces of equal area and perimeter. He managed to extend those ideas and results to higher-dimensional shapes.

As mentioned in the introduction, the ham-sandwich theorem took its name from the case when $n = 3$ and the three objects of any shape can all be simultaneously bisected with a single cut. The early history backround of this result seems not to be well known. Stone and Tukey attribute this theorem to Ulam. They mention that they got the information from a referee. The problem appears in *The Scottish Book* as *Problem 123*. The problem is posed by Hugo Steinhaus (Mauldrin, 1981). A reference is made to the pre-World War II journal *Mathesis Polska* (Latin for "Polish Mathematics")[5]. Steinhaus wrote a paper in 1945[6], that represents the work that Steinhaus did in Poland on the ham-sandwich problem in World War II while hiding out with a Polish farm family. According to Beyer and Zardecki (2004), the earliest known paper about the ham-sandwich theorem, specifically the $n = 3$ case of bisecting three solids with a plane, is proposed by Hugo Steinhaus (1938). Beyer and Zardecki's paper includes a translation of the 1938 paper. It attributes the posing of the problem to Hugo Steinhaus, and credits Stefan Banach as the first to solve the problem, by a reduction to the Borsuk-Ulam theorem. Ham-Sandwich theorem appeared at *American High School Mathematics Examinations* (AHSME) destined for middle and high school students (Berzsenyi and Maouro, 1997, p. 31).

Problem 27 (AHSME, 1987)
A cube of cheese $C = \{(x, y, z) \mid 0 \le x, y, z \le 1\}$ is cut along the planes $x = y$, $y = z$ and $z = x$. How many pieces are there? (No cheese is moved until all three cuts are made)
(A) 5 (B) 6 (C) 7 (D) 8 (E) 9

The solution to the above problem is given in detail in Berzsenyi and Maouro (1997, p. 124). At the first round of 29th British Mathematical Olympiad, which was held in Wednesday 13th January 1993, a topic related to the Ham-Sandwich theorem and the triangle's equalizer was given:

[5]Steinhaus, Hugo (1938). A note on the ham-sandwich theorem. *Mathesis Polska*, 9: 26–28.

[6]Steinhaus, Hugo (1945). Sur la division des ensembles de l'espace par les plans et des ensembles plans par les cercles. *Fundamenta Mathematicae*, 33 (1): 245–263.

29th British Mathematical Olympiad, Problem 2
A square piece of toast $ABCD$ of side length 1 and centre O is cut in half to form two equal pieces ABC and CDA. If the triangle ABC has to be cut into two parts of equal area, one would usually cut along the line of symmetry BO. However, there are other ways of doing this. Find, with justification, the length and location of the shortest straight cut which divides the triangle ABC into two parts of equal area.

The solution to the particular problem is given in detail in Gardiner (1997, pp. 104–105) and Dinh (2012, pp. 685–686).

3 The equalizers of a right-angled triangle in mathematical contests

The paragraph is a collection of problems of the major mathematical competitions regarding the triangle's or right-angled triangle's equalizers. The Canadian Mathematical Olympiad is an annual premier national advanced mathematics competition sponsored by the Canadian Mathematical Society. In 1985, 17th Canadian Mathematical Olympiad was held, and the first problem was:

17th Canadian Mathematical Olympiad 1985, Problem 1
A triangle has sides 6, 8, 10. Show that there is a unique line which bisects the area and the perimeter.

The solution to the above problem is given in detail by Doob (1993, p. 169). It is important to point out that the person who won the particular competition was Minh Tue Vo coming from the city of Montréal, Canada and also a student at the École Secondaire St-Luc. The same person was declared a winner at the Canadian Mathematical Olympiad in 1984. *Problem 1* of the 17th Canadian Mathematical Olympiad (1985) investigates the total number of the equalizers of a special right-angled triangle. The same subject seems to appear as *Problem 9* at the Canadian mathematical magazine *Crux Mathematicorum* destined for students. Readers are invited to search for the number of equalizers included on a right triangle whose sides differ from those presented in *Problem 1* (Woodrow, 1991, p. 72):

Problem 9, *Crux Mathematicorum* 1991
The lengths of the sides of a triangle are 3, 4 and 5. Determine the number of straight lines which simultaneously halve the area and the perimeter of the triangle.

A solution to the magazine's *Problem 9* was given by Michael Selby from the University of Windsor. A solution was also already given to *Problem 1* of the Canadian Mathematical Olympiad stating that the questioned right triangle contains only one equalizer. The solution of the particular problem doesn't abstain from *Problem 1.* A relative problem was also proposed by the Flemish Mathematical Olympiad in 2004 in Belgium. It states:

Flanders Mathematics Olympiad 2004, Problem 1
Consider a triangle with side lengths 501 m, 668 m, 835 m. How many lines can be drawn with the property that such a line halves both area and perimeter?[7]

A relevant subject concerning the area and perimeter of a triangle was also proposed at the qualifying process (Team Selection Test)[8] of the Balkan Mathematical Olympiad in 2011 at Albania (Balkan Mathematical Olympiad, BMO). It states that:

Balkan Mathematical Olympiad 2011, Albanian Team Selection Test, Problem 2
The area and the perimeter of the triangle with sides 10, 8, 6 are equal. Find all the triangles with integral sides whose area and perimeter are equal.[9]

[7] Additional information could be found at: https://www.vwo.be/vwo/vorige-edities/alle-vragen

[8] A team selection test (TST), or team selection exam (TSE), is a test given to top finishers in a national Olympiad to determine the nation's team for an international Olympiad such as the International Mathematical Olympiad (IMO) or Balkan Mathematical Olympiad (BMO).

[9] This problem concerns a special case of triangles. A triangle whose sides and area are rational numbers is called a *rational triangle.* If the sides of a rational triangle is of integer length, it is called an *integer triangle.* If further these sides have no common factor greater than unity, the triangle is called *primitive integer triangle.* A *Heronian triangle* (named after Heron of Alexandria) is an integer triangle with the additional property that its area is also an integer. Perfect triangles are the triangles the sum of whose integer-valued sides equals the area of the triangle. There

A solution to this problem is proposed in the article of Starke (1969, p. 47), while various problems of the Balkan Mathematical Olympiad can be easily discovered in a publication of the Greek Mathematical Society (Felouris et al., 2002). It can be easily observed that in all four problems, the lengths of the sides correspond to real triangles since they satisfy the Pythagorean theorem. The integer length sides $(3, 4, 5)$ constitute a primitive Pythagorean triple, so a triangle with side lengths $(3\lambda, 4\lambda, 5\lambda)$, with $\lambda \in \mathbb{N}$ and $\lambda > 1$ can be considered a right triangle according to the Pythagorean theorem. If $\lambda = 2$, we have the following sides $(6, 8, 10)$ [CMO 1985 and BMO 2011], while for $\lambda = 167$ we have $(501, 668, 835)$ [Flanders Mathematics Olympiad 2004]. Another problem has also been presented at the Finnish National High School Mathematics Competition in 2011:

> **Finnish National High School Mathematics Competition 2011, Problem 1**
> An equilateral triangle has been drawn inside the circle. Split the triangle to two parts with equal area by a line segment parallel to the triangle side. Draw an inscribed circle inside this smaller triangle. What is the ratio of the area of this circle compared to the area of original circle?[10]

A relevant problem concerning the equalizer has been proposed by the III Olimpíada Iberoamericana de Matemática, OIM which was held in Lima, Perú on the 24th of April, 1988. In this problem, the candidates are asked to prove how an equalizer can pass through the center of a triangle as shown by Kodokostas (2010, p. 142).

> **III Ibero-american Mathematical Olympiad 1988, Problem 3**
> Prove that among all possible triangles whose vertices are 3, 5 and

exist only five perfect triangles: $T_1 = (6, 8, 10)$, $T_2 = (5, 12, 13)$, $T_3 = (9, 10, 17)$, $T_4 = (7, 15, 20)$, $T_5 = (6, 25, 29)$ (Starke, 1969, p. 47). Notice that each of the pairs (T_1, T_3) and (T_1, T_5) have a common side. These pairs can be placed along their common sides to form a large triangle in each case (Rabinowitz, 1992, p. 243). The triangles T_1 and T_2 are right-angled triangles (Markowitz, 1981, p. 222). Additional information could be found at Bonsangue et al. (1999) and at: `https://artofproblemsolving.com/community/c3908\_2011\_bmo\_tst`

[10] Additional information at: `http://matematiikkakilpailut.fi/valmennus/English/loppu2011eng.pdf`

7 apart from a given point P, the ones with the largest perimeter have P as incentre.[11]

In honour of I. F. Sharygin, a problem of the same interest (approaching equalizers) had been equally assigned to students on the occasion of the 5th Geometrical Olympiad in 2009 (V Geometrical Olympiad 2009 in honour of I.F. Sharygin, The Corresponding Round). This particular problem states:

V Geometrical Olympiad 2009 in honour of I. F. Sharygin, The Corresponding Round, Problem 2
Given non-isosceles triangle ABC. Consider three segments passing through different vertices of this triangle and bisecting its perimeter. Are the lengths of these segments certainly different? (proposed by B. Frenkin)[12]

A similar problem is presented at the 41st Austrian Mathematical Olympiad held in 2010 (41st Austrian Mathematical Olympiad 2010, Federal Competition for Advanced Students, part 1, Problem 4).

Austrian Mathematical Olympiad 2010, Federal Competition for Advanced Students, part 1, Problem 4
In a triangle ABC, if one draws through some interior point P the three parallels to the sides, then the triangle ABC is decomposed into three quadrangles (in the corners) and three triangles which rest on the sides. (a) Show that if $P = I$ is the center of the incircle, then the circumference of each of the new small triangles is equal to the length of the side on which it rests. (b) Determine for a given triangle ABC all interior points P for which the circumference of each of the new small triangles is equal to the length of the side on which it rests. (c) For which interior point P is the sum of the areas of the three triangles minimal?[13]

In conclusion, a relevant problem focusing on issues evoking equalizers is mentioned at the VII Mathematical Olympiad of Central America and

[11] Additional information could be found at the website of *Organización de Estados Iberomaricanos* (OEI) and at: `http://www.oei.es/historico/oim/iiioim.htm.`

[12] Additional information could be found at: `http://geometry.ru/olimp/2009/zaochsol-e.pdf`

[13] Additional information could be found at: `http://www.oemo.at/problems/bwb/b2010-en.pdf`

the Caribbean which took place at the city of San Salvador, capital of El Savador between the 17–26th of June 2005 (VII Olimpiada Matemática de CentroAmérica y El Caribe).

VII Mathematical Olympiad of Central America and the Caribbean 2005, Problem 5
Let ABC be an acute-angled triangle, H the orthocenter and M the midpoint of AC. Let ℓ be the parallel through M to the bisector of $\angle AHC$. Prove that ℓ divides the triangle in two parts of equal perimeters (Pedro Marrone, Panamá)[14].

Therefore, we have to point out the significance and importance of the international mathematical competitions concerning the cultivation and formation of a mathematical conscience on various issues such as the subject of equalizers. This is easily demonstrated throughout the number of books being published on the subject of this particular field (e.g. Chen, 2016; Louridas and Rassias, 2013; Andreescu and Feng, 2002; Klamkin, 1989).

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Community Outreach: Annual Mathematics Competitions Bootcamp at Morehouse College

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1 Introduction

A way to ensure that youth receive a quality education is to teach them at home. However, many parents and guardians do not have the needed resources to do this. Therefore, schools are solely responsible for education. Education is not only the responsibility of schools, community is responsible, also. The phrase "It takes a village to raise a child", holds true for education, too:

> It takes a community to educate a child.

One way community can assist with education is to provide Out-of-School Time (OST) learning opportunities.

2 Literature Review

OST learning opportunities is a way to enhance the in-school learning of students. These can be fun, rigorous activities for students to engage in without the requirement of earning a grade. Such opportunities as before/after school programs, museums, discovery learning themed playgrounds, nature hikes, local library programs, pre-college programs, and academic competitions are just a few. Baker, Larson and Krehbiel (2014) give two examples of programs that bridge OST and in-school learning. Locklear and Mustian (1998) found that before and after school programs "produce changes in perceived youth behavior". Patton and Capse (2014) state that "learners across all developmental periods ... gain knowledge and skills in multiple places and across all hours of the day." According to Lauver, Little and Weiss (2004), students who participate in OST learning:

> receive personal attention from caring adults, explore new interests, receive academic support, develop a sense of belonging to a group, develop new friendships with their peers, take on challenging leadership roles, and build a sense of self-esteem independent of their academic talent.

Ferreira (2001) found that "participants new sense of community transferred into the math classroom." Bouffard, Little and Weiss (2006) noted that "a rich network of supports—one that includes OST programs along with families and community-based organizations—can make a difference."

3 Math Competitions Bootcamp Description

The Department of Mathematics at Morehouse College takes part in community outreach by hosting an Annual Mathematics Competition Bootcamp, a four-hour workshop, on campus, devoted to disseminating information about Math competitions (Appendix A). Boys and girls, in grades 5–12 are invited to attend with the goals to:

- Increase participation in math competitions by members of underrepresented groups.
- Introduce participants to local, state, regional and national math competitions.
- Provide ongoing communication via social media to facilitate and encourage participants throughout the school year as they progress through all levels of math competitions.

– Provide an opportunity for participants to compete in a Mini-Math Competition.

4 Before the Day of the Math Competitions Bootcamp

At the end of the school year, Mathletes are encouraged to study all summer long to prepare for the first competition of the new upcoming academic year, the Math Competitions Bootcamp. The Mini-Math Contest, given at the bootcamp, consists of problems from all levels of Math. The design of the Mini-Math Contest is as follows:

Math Level	**Number of Problems**
Basic/General Math/Pre-Algebra	20
Algebra I/Algebra II	20
Geometry/Trigonometry	15
Georgia/Other States Grade Level Minimal Basic Math Skills Tests Grades 5–12	10
Georgia High School Graduation Test Math	10
MathCOUNTS/Other State High School Math Competition	10
SAT/ACT Math	10
GRE/GMAT/MCAT Math	5

Figure 1: Mini-Math Contest Design

The goal in having Math problems from the GRE, GMAT and MCAT is to show students that the math they are learning now will be used way into the future and to show students in higher grades that they already have the math skills required for some graduate/professional school entrance exams.

5 The Day of the Math Competitions Bootcamp

Early arrivers are challenged, by the Math faculty and volunteers (Math Majors, Math Club, Math Honor Society and other college students), with fun mental math activities. The program begins promptly at 10:00 a.m. with a welcome, introductions and instructions to let everyone know their room assignment for the Mini-Math Contest.

Beginning at 10:30 a.m., Mathletes take the one-hour Mini-Math Contest consisting of one hundred multiple choice problems from all levels of Math (Appendix B). Mathletes are instructed to do their best and do as much as they can.

After the Mini-Math contest ends at 11:30 a.m., Mathletes are greeted by their support team. The purpose is to ensure that each is made to feel that they won the contest because they competed. Everyone is then provided a delicious lunch of pizza, chips, cookies and assorted drinks until 12:45 p.m. Over lunch faculty and college students chat with young Mathletes. Chalk and whiteboard markers are provided as lunch time is more time to do Math. During lunch, also, the scoring team grade the contest.

Following lunch, a panel discussion is held until 1:30 p.m. The panel consists of Alumni Mathletes, a current top Mathlete, a Math Competitions Coach, parents and college students who participated in other academic competitions. Each panelist shares their experiences, insight and wisdom in training and competing in Math competitions. Members of the audience share their experiences, too. Thus, the young Mathletes learn that it's okay to excel in Math and be a member of a Math Club.

Next, Mathletes are provided a short presentation with information about other Math competitions for the upcoming academic year (Appendix C) such as:

- University of Georgia Athens Math Tournament
- MathCOUNTS
- Online Math League
- Georgia Tech High School Math Competition

The workshop ends with the top scorers for each grade level and the overall top scorer awarded a trophy. Second and third place winners received a silver and bronze medal, respectively. Our philosophy is that every Mathlete who competes is a winner and each one receives an inexpensive intellectually stimulating consolation prize (Appendix D).

6 Conclusion

Students come to Morehouse Math Competitions Bootcamp from all over the city and other states. They come with their school team, church groups or as individuals. To see an article from one school that epitomizes the goals of the bootcamp, google this link: `http://fayette-news.com/sandy-creek-math-team-shares-love-for-their-favorite-subject/`

Because of students' participation and if they train for other competitions, this Out-of-School Time (OST) learning opportunity should help Mathletes in their studies to earn good grades on their Math assignments in school. That's Winning!

Enjoy The Math Challenge!

References

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[2] Bouffard, S., Little, P., & Weiss, H. Building and evaluating out-of-school time connections. *The Evaluation Exchange*, 12 (1&2), 2006, 2–6.

[3] Ferreira, M., Building communities through role models, mentors, and hands-on-science. *The School Community Journal*, 11 (2), 2001, 27–37.

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[5] Locklear, E. & Mustain, R., Extension-Supported School-Age Care Programs Benefit Youth. *Journal of Extension*, 36 (3) Article 3RIB4, 1998. Available at: `https://joe.org/joe/1998june/rb4.php`

[6] Patton, C. and Caspe, M., *Finding Time Together: Families, Schools, and Communities Supporting Anywhere, Anytime Learning.* Cambridge, MA: Harvard Family Research Project, 2014.

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Appendix

A. Sample Registration Form

Third Annual
Math Competitions Bootcamp

Morehouse College
August 23, 2014
10:00 a.m. – 2:00 p.m.

Registration Form

Name: ______________________________________ Grade: ___________ Age:__________
Mailing Address: ____
Phone Number: _____________________ Email: __________________________________
School Name: __________________________ County: _____________________________

Math Coach/Teacher/Club Advisor: ____
Parents: ____
Cell phone: ___________________________________ Email: _______________________
Are you in the Math Club or on a Math Team? ____
List previous Math Competitions: ____
____

Parents/Guardians/Coaches may attend with you. How many will attend? ______

Return to:
Morehouse College
Department of Mathematics
Attn: Math Competitions Bootcamp
830 Westview Drive, S.W.
Atlanta, GA 30314-3773

Contact:
(404)215-2614, Fax: (404)572-3645
Registration Deadline: August 8, 2014

Registration Fee: None

Donations to support this event are accepted online at https://giving.morehouse.edu/sslpage.aspx?pid=344
Select "Mathematic Restricted Fund" in the "Designation" dropdown menu.
Put "Math Competitions Bootcamp" in the "Comments" section.

B. Sample Mini-Math Contest Questions

Third Annual Math Competitions Bootcamp

Mini-Math Contest

1. Subtract: 8.02 – 3.696

(A) 4.498
(B) 2.894
(C) 4.324
(D) 11.716
(E) 6.932

10. If $3^{x+5} = \frac{1}{27}$, then $x =$

(A) -10
(B) -8
(C) -6
(D) -2
(E) 1

28. A bookstore determines the retail price by marking up the wholesale price 25%. The retail price of a physics book is \$111.60. What was the wholesale price?

(A) \$83.70
(B) \$89.28
(C) \$139.50
(D) \$64.28
(E) \$99.50

52. What are the horizontal and vertical asymptotes of the function $f(x) = \frac{2x-1}{x^2-4}$

(A) $y = 0, x = -2, x = 3$
(B) $y = 2, x = 0$
(C) $y = 2, x = -2, x = 2$
(D) $y = 0, x = -4, x = 4$
(E) $y = 0, x = 2, x = -3$

56. Which sign makes the statement true?

$$0.86 \; ? \; 0.83$$

(A) $\leq$
(B) $<$
(C) $=$
(D) $>$
(E) $\geq$

100. You are given the acid ionization constant

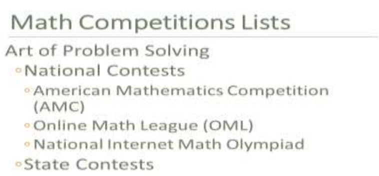
$$Ka = 3.75 \text{ x } 10^{-4}.$$

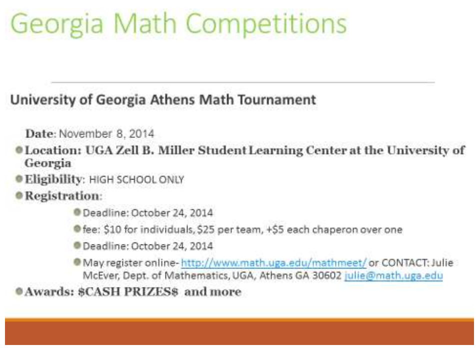
Find the pH level using the formula

$$pH = -\log(Ka).$$

(A) 5.4
(B) 4.3
(C) 3.7
(D) 3.4
(E) 7.5

C. Sample Presentation of other Math Competitions




Georgia Math Competitions
University of Georgia Athens Math Tournament
Date: November 8, 2014
Location: UGA Zell B. Miller Student Learning Center at the University of Georgia
Eligibility: HIGH SCHOOL ONLY
Registration:
Deadline: October 24, 2014
fee: $10 for individuals, $25 per team, +$5 each chaperon over one
Deadline: October 24, 2014
May register online- http://www.math.uga.edu/mathmeet/ or CONTACT: Julie McEver, Dept. of Mathematics, UGA, Athens GA 30602 julie@math.uga.edu
Awards: $CASH PRIZES$ and more


Florida Math Competitions
http://www.fgcu.edu/events/mathcompetition/
Florida Gulf Coast University Invitational Mathematics Competition
Date: December 15, 2014
Location: Ft. Myers, FL
Eligibility: Grades 6 – 12, Individuals and Teams
Registration:
Deadline: November 26, 2014
Fee: $12
CONTACT: Prof. Jerry Ellis, Contest Director
Florida Gulf Coast University
College of Arts & Sciences
10501 FGCU Blvd South
Ft. Myers, FL 33965-6565
jkellis@fgcu.edu

D. Sample Trophy and Inexpensive Intellectually Stimulating Consolation Prizes

Source: www.dollartree.com

E. Sample Ongoing Communication

On Sun, Jun 5, 2016 at 8:32 PM, Lamar, Tuwaner <Tuwaner.Lamar@morehouse.edu> wrote:

Hello Mathletes, Parents/Relatives/Guardians, Teachers, Coaches, Mentors, Community Facilitators, Faculty, Staff, Students and Volunteers,

Schools out for Summer! Oh yeah! It's time to have fun in the sun, get involved in summer enrichment programs and get ready for the first Math competition of the upcoming school year:

The Fifth Annual Morehouse Mathematics Competition Bootcamp
August 27, 2016
10:00 a.m. – 2:00 p.m.

The registration form is attached along with some problems for you to practice on. Also, attached is a description of the competition.

We just finished a historic year here at Morehouse. This year's graduating class had not 1, not 2, but 3, yes 3, valedictorians! You can witness the historical Baccalaureate and Commencement ceremonies on these links:

https://www.youtube.com/watch?v=AhxLEbvY-Yg

https://www.youtube.com/watch?v=bh6ITgNr1Ao

https://www.youtube.com/watch?v=p_i-9sMR748

Also, this year, the winner of Morehouse annual Vulcan Teaching Excellence Award is a Mathematics Professor, Dr. Ulrica Wilson.

We are so very proud of all our Mathletes of all ability levels. One point that has been repeatedly shared, in the panel discussion at past bootcamps is, how training for Math competitions helps you do good in other subjects and areas of your life. I had the pleasure of chatting with a former Mathlete who is now a world renowned English Professor. So, I humbly extended an invitation for her to be a participant on this year's panel discussion.

I receive reports of awards and recognitions that Mathletes receive. Let me just mention one bootcamp participant. He wasn't that interested in doing a Math competition the first year he came. His mama, grandma and aunt got him to the bootcamp. Well, he didn't take home a grade level trophy from the bootcamp. But, he did just what we encourage all bootcampers to do on the Mini-Math Contest: Do the best you can, do as much as you can and that makes you a winner! I have you to know that this middle school bootcamper, not only graduated middle school on the honor roll, but, he won the Principal Scholarship Award, which is given to the Scholar-Athlete with the highest GPA. Congratulations! Congratulations to all of our Mathletes for honors, awards and recognitions that you received this year.

Our Math majors have once again been accepted in to competitive Summer Research Experiences for Undergraduates (REUs) around the country. Congratulations to all of them. I hope everyone was able to get in their first choice summer enrichment program. We have a number of enrichment programs here at Morehouse for youth. See attached list of programs here and around the city. Many deadlines have passed. Some may still have openings. You can search other college/university websites for youth Pre-College programs. You may, also, contact the college/university Continuing Education Department to help you with your search.

Here's a new one for everyone. Have you ever wondered if you started college today, what Math class would you be placed in and if you could work to get placed in a higher Math class? Well, we are, in partnership with the ALEKS Corporation, offering an online opportunity to the community, for anyone of any age, who want to sharpen their Math skills for a one-time registration fee of $25. Go to www.aleks.com to register and enter class code VP39G-AV939. Please see attached registration instructions. You may use this class to challenge the Math you already know, reinforce your Math skills or cover topics that you may have missed during the school year. Since the class is online, you have 24/7 access to it and you can work at your own pace.

Notes:

1. This course is **not** for the Morehouse incoming Freshman class.
2. Freshman participating in the PSEP program should contact the PSEP office to get their class code.
3. All other incoming Freshman should contact the Student Success Center to get their class code.
4. Other Morehouse students, upper classmen still needing to meet Math requirements, may use the class code provided in the attached registration instructions.

Now, every year there is Joint Mathematics Meetings of the American Mathematical Society and the Mathematical Association of America. The next one is January 4-7 (Wed.-Sat.), 2017 and ATLANTA is the host city. So, mark your calendar. There is usually a session for High School Students and Teachers to present research. The deadline to submit your abstract for acceptance is September 20th. The registration fee for high school students is $7 and for high school teachers $71, if paid by December 20th. Everyone is invited and can find out more information at: http://jointmathematicsmeetings.org/meetings/national/jmm2017/2180_intro

Morehouse College Fifth Annual Mathematics Competitions Bootcamp accepts donations online to support this event. If you or you know of individuals who would like to support our effort, go to https://giving.morehouse.edu/sslpage.aspx?pid=309
and select the Designation: "Mathematics Restricted Fund" from the drop down menu. Put "Math Competitions Bootcamp" in the "Comments" section.

All donations are tax deductible.

Please feel free to forward this email, share it on facebook, twitter, instagram, snapchat, groupme, periscope, etc.

Enjoy the Math Challenge!
Dr. Lamar

The 59th International Mathematical Olympiad

The 59th International Mathematical Olympiad (IMO) was held 3–14 July 2018 in the city of Cluj-Napoca, Romania. This year was the sixth time that Romania[1] has hosted the IMO. A total of 594 high school students from 107 countries participated. Of these, 60 were girls.

Each participating country may send a team of up to six students, a Team Leader and a Deputy Team Leader. At the IMO the Team Leaders, as an international collective, form what is called the *Jury*. This Jury was ably chaired by Mihai Bălună.

The first major task facing the Jury is to set the two competition papers. During this period the Leaders and their observers are trusted to keep all information about the contest problems completely confidential. The local Problem Selection Committee had already shortlisted 28 problems from the 168 problem proposals submitted by 49 of the participating countries from around the world. During the Jury meetings four of the shortlisted problems had to be discarded from consideration due to being too similar to material already in the public domain. Eventually, the Jury finalised the exam problems and then made translations into the 57 languages required by the contestants.

The six problems that ultimately appeared on the IMO contest papers may be described as follows.

1. A relatively easy classical geometry problem proposed by Greece. It is remarkable that new easy problems in geometry are still being composed that have a relatively uncluttered diagram. This is a fine example of such a problem!

2. A medium sequence problem proposed by Slovakia.

3. A deceptively difficult combinatorics problem concerning the existence of a triangular array of positive integers. It was proposed by Iran.

[1] Romania hosted the very first IMO in 1959, where just seven countries participated.

4. An easy combinatorial problem in the form of a two-player game that may be played on a 20×20 chessboard. It was proposed by Armenia.

5. A medium number theory problem proposed by Mongolia.

6. A difficult geometry problem, whose diagram consists merely of five points and eight line segments. It was proposed by Poland.

These six problems were posed in two exam papers held on Monday 9 July and Tuesday 10 July. Each paper had three problems. The contestants worked individually. They were allowed four and a half hours per paper to write their attempted proofs. Each problem was scored out of a maximum of seven points.

For many years now there has been an opening ceremony prior to the first day of competition. The President of Romania attended this in person and addressed the audience. Following this and other formal speeches there was the parade of the teams, a dance show, and the 2018 IMO was declared open.

After the exams the Leaders and their Deputies spent about two days assessing the work of the students from their own countries, guided by marking schemes, which had been agreed to earlier. A local team of markers called *Coordinators* also assessed the papers. They too were guided by the marking schemes but are allowed some flexibility if, for example, a Leader brought something to their attention in a contestant's exam script that is not covered by the marking scheme. The Team Leader and Coordinators have to agree on scores for each student of the Leader's country in order to finalise scores. Any disagreements that cannot be resolved in this way are ultimately referred to the Jury. No such referrals occurred this year.

The contestants found Problem 1 to be the easiest with an average score of 4.93. Problem 3 was the hardest, averaging just 0.28. Only 11 contestants scored full marks on it. The score distributions by problem number were as follows.

Mark	**P1**	**P2**	**P3**	**P4**	**P5**	**P6**
0	96	158	548	148	175	419
1	54	85	7	13	184	108
2	24	87	9	106	31	26
3	15	66	14	18	7	11
4	10	18	4	18	6	5
5	7	16	1	15	8	2
6	7	7	0	5	11	5
7	381	157	11	271	172	18
Mean	4.93	2.95	0.28	3.96	2.70	0.64

The medal cuts were set at 31 points for gold, 25 for silver and 16 for bronze. The medal distributions[2] were as follows.

	Gold	**Silver**	**Bronze**	**Total**
Number	48	98	143	289
Proportion	8.1%	16.5%	24.1%	48.7%

These awards were presented at the closing ceremony. Of those who did not get a medal, a further 138 contestants received an honourable mention for scoring full marks on at least one problem.

The following two contestants achieved the most excellent feat of a perfect score of 42.

- Agnijo Banerjee, United Kingdom
- James Lin, United States

They were given a standing ovation during the presentation of medals at the closing ceremony. Fittingly their gold medals were awarded by Ciprian Manolescu.[3]

[2]The total number of medals must be approved by the Jury and should not normally exceed half the total number of contestants. The numbers of gold, silver, and bronze medals should be approximately in the ratio 1:2:3.

[3]Ciprian Manolescu, of Romania, has the singular distinction of achieving three perfect scores at the IMO. He competed at the IMO in 1995, 1996, and 1997.

The 2018 IMO was organised by the Romanian Mathematical Society with support from the Romanian government.

The 2019 IMO is scheduled to be held 10–22 July in Bath, United Kingdom. Hosts for future IMOs have been secured up to 2023 as follows.

2020	Russia
2021	United States
2022	Norway
2023	Japan

Much of the statistical information found in this report can also be found on the official website of the IMO.

www.imo-official.org

Angelo Di Pasquale
IMO Team Leader, Australia
School of Mathematics and Statistics
University of Melbourne
AUSTRALIA
email: adp@unimelb.edu.au

Monday, July 9, 2018

Problem 1. Let Γ be the circumcircle of acute-angled triangle ABC. Points D and E lie on segments AB and AC, respectively, such that $AD = AE$. The perpendicular bisectors of BD and CE intersect the minor arcs AB and AC of Γ at points F and G, respectively. Prove that the lines DE and FG are parallel (or are the same line).

Problem 2. Find all integers $n \geq 3$ for which there exist real numbers $a_1, a_2, \ldots, a_{n+2}$, such that $a_{n+1} = a_1$ and $a_{n+2} = a_2$, and

$$a_i a_{i+1} + 1 = a_{i+2}$$

for $i = 1, 2, \ldots, n$.

Problem 3. An *anti-Pascal triangle* is an equilateral triangular array of numbers such that, except for the numbers in the bottom row, each number is the absolute value of the difference of the two numbers immediately below it. For example, the following array is an anti-Pascal triangle with four rows which contains every integer from 1 to 10.

$$\begin{array}{ccccccc} & & & 4 & & & \\ & & 2 & & 6 & & \\ & 5 & & 7 & & 1 & \\ 8 & & 3 & & 10 & & 9 \end{array}$$

Does there exist an anti-Pascal triangle with 2018 rows which contains every integer from 1 to $1 + 2 + \cdots + 2018$?

Language: English

Time: 4 hours and 30 minutes
Each problem is worth 7 points

Tuesday, July 10, 2018

Problem 4. A *site* is any point (x, y) in the plane such that x and y are both positive integers less than or equal to 20.

Initially, each of the 400 sites is unoccupied. Amy and Ben take turns placing stones with Amy going first. On her turn, Amy places a new red stone on an unoccupied site such that the distance between any two sites occupied by red stones is not equal to $\sqrt{5}$. On his turn, Ben places a new blue stone on any unoccupied site. (A site occupied by a blue stone is allowed to be at any distance from any other occupied site.) They stop as soon as a player cannot place a stone.

Find the greatest K such that Amy can ensure that she places at least K red stones, no matter how Ben places his blue stones.

Problem 5. Let $a_1, a_2, \ldots$ be an infinite sequence of positive integers. Suppose that there is an integer $N > 1$ such that, for each $n \geq N$, the number

$$\frac{a_1}{a_2} + \frac{a_2}{a_3} + \cdots + \frac{a_{n-1}}{a_n} + \frac{a_n}{a_1}$$

is an integer. Prove that there is a positive integer M such that $a_m = a_{m+1}$ for all $m \geq M$.

Problem 6. A convex quadrilateral $ABCD$ satisfies $AB \cdot CD = BC \cdot DA$. Point X lies inside $ABCD$ so that

$$\angle XAB = \angle XCD \quad \text{and} \quad \angle XBC = \angle XDA.$$

Prove that $\angle BXA + \angle DXC = 180^\circ$.

Language: English

Time: 4 hours and 30 minutes
Each problem is worth 7 points

Some Country Totals

Rank	Country	Total
1	United States of America	212
2	Russia	201
3	China	199
4	Ukraine	186
5	Thailand	183
6	Taiwan	179
7	South Korea	177
8	Singapore	175
9	Poland	174
10	Indonesia	171
11	Australia	169
12	United Kingdom	161
13	Japan	158
13	Serbia	158
15	Hungary	157
16	Canada	156
17	Italy	154
18	Kazakhstan	151
19	Iran	150
20	Vietnam	148
21	Bulgaria	146
22	Croatia	145
23	Slovakia	140
24	Sweden	138
24	Turkey	138
26	Israel	136
27	Georgia	133
28	Brazil	132
28	India	132
28	Mongolia	132
31	Germany	131
32	Armenia	130
33	France	129
33	Romania	129
35	Peru	125

Distribution of Awards at the 2018 IMO

Country	Total	Gold	Silver	Bronze	HM
Albania	37	0	0	0	1
Algeria	18	0	0	0	0
Argentina	115	0	1	4	0
Armenia	130	0	2	4	0
Australia	169	2	3	1	0
Austria	72	0	0	3	1
Azerbaijan	50	0	0	0	5
Bangladesh	114	1	0	3	2
Belarus	102	0	0	4	1
Belgium	92	0	0	4	1
Bolivia	33	0	0	0	3
Bosnia and Herzegovina	103	0	0	4	2
Botswana	12	0	0	0	1
Brazil	132	1	0	4	1
Bulgaria	146	1	3	1	1
Cambodia	11	0	0	0	0
Canada	156	0	5	1	0
Chile	19	0	0	0	2
China	199	4	2	0	0
Colombia	59	0	0	1	2
Costa Rica	65	0	0	2	2
Croatia	145	0	4	1	0
Cyprus	45	0	0	1	1
Czech Republic	115	0	2	2	2
Denmark	71	0	0	3	0
Ecuador	48	0	0	0	3
Egypt	10	0	0	0	1
El Salvador	20	0	0	0	2
Estonia	80	0	1	0	3
Finland	70	0	0	2	2
France	129	1	1	4	0
Georgia	133	0	1	5	0
Germany	131	1	2	1	2
Ghana	13	0	0	0	0
Greece	74	0	0	2	2
Guatemala	11	0	0	0	1

Country	Total	Gold	Silver	Bronze	HM
Honduras	6	0	0	0	0
Hong Kong	89	0	0	2	4
Hungary	157	0	4	2	0
Iceland	56	0	0	1	3
India	132	0	3	2	1
Indonesia	171	1	5	0	0
Iran	150	1	3	1	1
Iraq	9	0	0	0	0
Ireland	43	0	0	1	1
Israel	136	0	2	4	0
Italy	154	0	4	2	0
Ivory Coast	8	0	0	0	1
Japan	158	1	3	2	0
Kazakhstan	151	0	4	2	0
Kosovo	21	0	0	0	2
Kyrgyzstan	41	0	0	0	4
Latvia	40	0	0	0	2
Lithuania	77	0	0	2	3
Luxembourg	14	0	0	0	1
Macau	61	0	0	1	3
Macedonia (FYR)	27	0	0	0	2
Malaysia	90	0	0	2	3
Mexico	123	0	1	4	1
Moldova	86	0	0	3	3
Mongolia	132	0	1	5	0
Montenegro	20	0	0	0	1
Morocco	46	0	0	0	3
Myanmar	23	0	0	0	2
Nepal	5	0	0	0	0
Netherlands	123	0	1	4	1
New Zealand	102	0	1	2	3
Nigeria	26	0	0	0	2
Norway	73	0	0	2	1
Pakistan	35	0	0	0	3
Panama	21	0	0	0	2
Paraguay	12	0	0	0	0

Country	**Total**	**Gold**	**Silver**	**Bronze**	**HM**
Peru	125	0	2	3	1
Philippines	121	1	1	2	2
Poland	174	1	5	0	0
Portugal	77	0	0	2	3
Puerto Rico	46	0	0	1	1
Romania	129	1	1	2	2
Russia	201	5	1	0	0
Saudi Arabia	69	0	1	1	1
Serbia	158	2	2	2	0
Singapore	175	2	3	1	0
Slovakia	140	0	3	3	0
Slovenia	104	0	1	1	4
South Africa	66	0	0	1	4
South Korea	177	3	3	0	0
Spain	74	0	0	2	4
Sri Lanka	47	0	0	1	3
Sweden	138	1	2	2	1
Switzerland	52	0	0	1	1
Syria	69	0	0	2	2
Taiwan	179	3	1	2	0
Tajikistan	103	0	0	5	1
Tanzania	1	0	0	0	0
Thailand	183	3	3	0	0
Trinidad and Tobago	26	0	0	0	1
Tunisia	49	0	0	0	3
Turkey	138	1	1	4	0
Turkmenistan	65	0	0	1	4
Uganda	9	0	0	0	0
Ukraine	186	4	2	0	0
United Kingdom	161	1	4	0	1
United States of America	212	5	1	0	0
Uruguay	7	0	0	0	0
Uzbekistan	21	0	0	0	2
Venezuela	2	0	0	0	0
Vietnam	148	1	2	3	0
Total (107 teams, 594 contestants)		**48**	**98**	**143**	**138**

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