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MATHEMATICS COMPETITIONS

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The aims of the Federation are:

1. ***to promote excellence in, and research associated with, mathematics education through the use of school mathematics competitions;***
2. ***to promote meetings and conferences where persons interested in mathematics contests can exchange and develop ideas for use in their countries;***
3. ***to provide opportunities for the exchanging of information for mathematics education through published material, notably through the Journal of the Federation;***
4. ***to recognize through the WFNMC Awards system persons who have made notable contributions to mathematics education through mathematical challenge around the world;***
5. ***to organize assistance provided by countries with developed systems for competitions for countries attempting to develop competitions;***
6. ***to promote mathematics and to encourage young mathematicians.***

From the President

VIII Congress of WFNMC, Summer 2018

Hosting Proposals

Dear Colleagues and Friends

As you know, our next Congress of the Federation will take place in Colombia in July 2014. It is now time to plan ahead, to the VIII Congress. On behalf of the Executive, I am inviting interested persons to submit to me proposals to host the 2018 Congress of the Federation.

We prefer the proposals to be backed by professional organizations and/or the host country. We expect the Congress to last for 5–7 days. We prefer the participants to stay together or in very close proximity to each other. The proposers will closely estimate the cost of lodging, meals and registration fee per delegate; include cultural attractions of the Congress's locale; offer an excursion for all delegates; and provide tentative dates.

We would like to receive all proposals by *May 1, 2014*, so that the Executive can choose the winning proposal during our July 2014 Congress in Colombia.

We would be happy to work with the proposers and address any questions they may have.

With warm regards to everyone,

Alexander Soifer
President of WFNMC

From the Editor

Welcome to *Mathematics Competitions* Vol. 26, No. 2.

First of all I would like to thank again the Australian Mathematics Trust for continued support, without which each issue of the journal could not be published, and in particular Heather Sommariva, Bernadette Webster and Pavel Calábek for their assistance in the preparation of this issue.

Submission of articles:

The journal *Mathematics Competitions* is interested in receiving articles dealing with mathematics competitions, not only at national and international level, but also at regional and primary school level. There are many readers in different countries interested in these different levels of competitions.

- The journal traditionally contains many different kinds of articles, including reports, analyses of competition problems and the presentation of interesting mathematics arising from competition problems. Potential authors are encouraged to submit articles of all kinds.
- To maintain and improve the quality of the journal and its usefulness to those involved in mathematics competitions, all articles are subject to review and comment by one or more competent referees. The precise criteria used will depend on the type of article, but can be summarised by saying that an article accepted must be correct and appropriate, the content accurate and interesting, and, where the focus is mathematical, the mathematics fresh and well presented. This editorial and refereeing process is designed to help improve those articles which deserve to be published.

At the outset, the most important thing is that if you have anything to contribute on any aspect of mathematics competitions at any level, local, regional or national, we would welcome your contribution.

Articles should be submitted in English, with a black and white photograph and a short profile of the author. Alternatively, the article can be submitted on an IBM PC compatible disk or a Macintosh disk. We prefer LaTeX or TeX format of contributions, but any text file will be helpful.

Articles, and correspondence, can also be forwarded to the editor by mail to

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Jaroslav Švrček
December 2013

VII Congress of the World Federation of Mathematics Competitions—WFNMC

Barranquilla, Colombia

July 20—July 25, 2014

Venue: Hotel El Prado

Activities

include plenary addresses, workshops and contributed talks.

Plenary speakers include

- Ron Graham, University of California at San Diego (Keynote address)
- John Jayne, University College London
- Radu Gologan, University "Politehnica" Bucharest
- Alexander Soifer, University of Colorado at Colorado Springs, President WFNMC (Closing address)

Registration

Early registration deadline: *March 31, 2014* Fee: US$ 900[1]

Registration deadline: *June 15, 2014* Fee: US$ 1050

[1] Based on double occupancy (includes accommodation and meals). Single occupancy early registration fee is $1100 and single occupancy late registration fee is $1250.

Call for papers

The VII Congress of the World Federation of National Mathematics Competitions, WFNMC, will be held in **Barranquilla, Colombia** from *July 21—24, 2014*.

The Programme Committee of the Congress hereby issues a call for papers in three formats.

- Long communications, talks 50 minutes in length.
- Short communications, talks 25 minutes in length.
- Workshops: workshops will be held for 50 minutes daily on *July 21, 22 and 24*.

Proposals should be submitted in English in complete text for long and short communications, and in partial text for workshops[1]. The deadline for submissions is *January 31, 2014*, and proponents will be notified of acceptance by *March 15, 2014*.

Send your submission to María de Losada (`mariadelosada@gmail.com`).

Barranquilla is located on the Caribbean coast of Colombia. It is a busy port city near the mouth of the Magdalena River.

[1] Times New Roman, 12 pts, 1.5 spaces and should not exceed 8 pages for short communications, 15 pages for long communications, 12 pages for workshops.

Call for nominations

Paul Erdös Award

The Paul Erdös Award was established to recognize the contributions of mathematicians who have played a significant role in the development of mathematical challenges at the national or international level and whose contributions have been a stimulus for the enrichment of mathematics learning.

Each recipient of the award is selected by the Executive and Advisory Committee of the World Federation of National Mathematics Competitions on the WFNMC Awards Subcommittee.

Paul Erdös Award 2014

The WFNMC Awards Committee hereby issues a call for nominations for the Erdös Award 2014.

Send nominations to Awards Committee Chair, María de Losada (`mariadelosada@gmail.com`) by *December 15, 2013*.

Awards will be given at the VII Congress of WFNMC to be held in Barranquilla, Colombia, July 21—24, 2014.

Technology and the Creation of Challenging Problems

Sergei Abramovich & Eun Kyeong Cho

Sergei Abramovich is Professor in the Department of Curriculum and Instruction at State University of New York at Potsdam, USA. He was born in St. Petersburg (then Leningrad), Russia, did all his studies at the University there and earned his PhD in mathematics in 1981. Throughout his career he has authored/co-authored over 150 publications, including two books and numerous articles on the use of technology in the teaching of mathematics.

Dr. Eun Kyeong Cho is an assistant professor of education at the University of New Hampshire in the United States. Dr. Cho has earned an Ed.D. in Curriculum and Teaching from Teachers College, Columbia University in 2005. Her research interests are pre-service teacher preparation, professional development, technology integration, and diversity.

Abstract. By revisiting existing mathematical problems and modifying them into more intellectually stimulating ones, educators can challenge students to be more actively engaged in the creative process of learning. The article provides examples of using technology for creating challenging problems to motivate different populations of students. Five avenues toward the creation of challenging problems in a technological paradigm are discussed.

1 Introduction

Teaching and learning with gifted and talented students as well as struggling students require educators to re-examine traditional ways in designing, implementing, and assessing curriculum materials. In mathematics education, such re-examination would involve revisiting currently available mathematical problems and reformulating them into more challenging ones, which are motivating for advanced students. Technology contributes to envision a world where one can pose and solve motivating mathematical problems appropriate to individual students.

The article concerns the use of computing technology in posing problems. It has been observed that in the presence of technology one can solve many (traditionally challenging) problems by using one or other type of mathematics software and, therefore, such problems may be considered somewhat outdated. Just like the use of calculators by students resulted in the algorithm of finding square roots (similar to that of long division) excluded from the school mathematics curriculum, the development of software, like *Wolfram Alpha* (available free online) with powerful computational and graphical capabilities has made many challenging problems for pre-college students interested in mathematics not as motivational as they used to be.

For example, if one is asked to solve the equation

$$x^3 - \lfloor x \rfloor = 2, \tag{1}$$

where $\lfloor x \rfloor$ is the largest integer smaller than or equal to x (a typical mathematical Olympiad problem of the past), plugging into *Wolfram Alpha* the text "solve $x^3 -$IntegerPart$\,(x) = 2$" yields the result, $x = \sqrt[3]{3}$. Whereas the software does not reveal the solution process, it appears to be difficult to motivate a modern student to find such (readily available) answer using pencil and paper. The question then arises as to how one can use technology as a medium within which traditionally challenging problems solvable nowadays at the click of a button, can be revised in order to motivate the student to do what technology is not capable of doing. To this end, note that after trying several other similar equations, one may conjecture that replacing the right-hand side of equation (1) by n yields $x = \sqrt[3]{n+1}$ as a solution. This, however, is not true as the case

$n = 8$ indicates. Thus, we can construct an equation with parameter,

$$x^3 - \lfloor x \rfloor = n, \tag{2}$$

the solution of which in terms of n cannot be easily found by using technology. At the same time, the use of *Wolfram Alpha* can facilitate the discussion of this problem with a larger population of students who are just interested in mathematics and want to become more "mathematically proficient" (Common Core State Standards Initiative, 2012). This shows a dual role of technology: it can assist in the creation of challenging problems and may help in "democratizing" (Kaput, 1994) access to such problems to a broader audience.

2 Different ways of using technology for the creation of problems

This article extends the authors' earlier work on using technology for posing problems (Abramovich & Cho, 2006, 2008) to include challenging problems that can emerge from the very use of technology. It has been more than twenty-five years since dynamic geometry programs have been in use in mathematics classrooms as cognitive tools for formulating geometric conjectures (Battista, 2008; Hoyles & Sutherland, 1986; Laborde, 1995; Lingefjärd & Holmquist, 2003; Yerushalmy, Chazan & Gordon, 1993). This paradigm shift in the teaching of mathematics, however, was not observed in the case of non-geometric software. In this article, the use of an electronic spreadsheet as well as *Wolfram Alpha* as tools for the creation of challenging problems informed by a computational experiment will be discussed. Such problems can be extensions of routine problems with a hidden challenging component, which can be revealed and utilized through the appropriate use of technology. They also may result from the appropriate revision of the well-known challenging problems carried out in a technological environment, which, sometimes, is specifically developed for the purpose of problem formulating.

The use of technology in creating challenging problems can take many different routes, five of which are discussed in this article. The first route is to create challenging, technology-immune problems for mathematically gifted students. An example of such a problem is equation (2) the solution of which can, nevertheless, be informed by a computational experiment. The second route, as noted above, is to take advantage of technology for introducing challenging problems to *all* students. The third route is to design challenging problems by modifying (or altering numeric data of) routine, sometimes well-known, problems with a hidden open-ended component (as shown in sections 3 and 4 below). The fourth route is to revise traditional problems for advanced students that are solvable by technology in order to make them technology-immune (section 5). The fifth route is to use computing technology as a medium for both solving problems through a computational experiment and posing new problems to enable computational efficiency of the experiment (section 6).

3 Using a spreadsheet for creating challenging problems at the elementary level

Teachers of mathematics need experience in the creation of problems the difficulty of which can somehow be parameterized so that starting from a simple situation one can move into more and more challenging contexts. In that way, mathematics can indeed be taught to all students, motivating everyone at their level of mathematical proficiency and responsiveness to a challenge. In other words, we talk about the joint use of technology and open-ended problems commonly found in a traditional curriculum as a springboard into the domain of grade-appropriate challenging problems.

It should be noted that beginning from the 1970, the effectiveness of using open-ended problems in developing one's higher order thinking skills in mathematics was emphasized by mathematics education researchers (Becker & Selter, 1996; Shimada, 1977). In many cases, an open-ended structure of a mundane problem may be hidden; yet it can be revealed through the use of technology. For example, in a test for grade four mathematics (New York State Testing Program, 1998) the following problem with a hidden open-ended structure can be found: *Michael has two quarters, two nickels and two pennies, while Tara has a quarter, a nickel and two dimes. Which coins could Michael give Tara so that they both have the same amount of money?*

While the problem itself is not a challenge (especially when images of the coins are provided), it can be explored in a specially constructed spreadsheet environment (Abramovich & Cho, 2006) allowing for the variation of coins and other conditions. Through such an exploration (by teacher as a problem creator), the following questions can be formulated: *For which combination of coins could Michael share money with Tara so that after sharing Michael has two (three, four, five, etc.) times as much money as Tara?* Note that using the spreadsheet makes it possible for the teacher to control the level of complexity of a problem and its solvability. This, in turn, leads to the notion of didactic coherence of a problem introduced by the authors elsewhere (Abramovich & Cho, 2008).

Didactic coherence is composed of three mutually non-exclusive components (i.e., numerical, pedagogical, and contextual coherence). Didactic coherence of a problem is achieved when it is solvable (e.g., by choosing appropriate numeric data), pedagogically appropriate (e.g., by considering students' interests, attention span, mathematical level of understanding, number of possible solutions, etc.), and contextually appropriate (e.g., by considering sociocultural relevance) for the target group or individual students. Using technology, a challenging problem can be created to meet the three aspects of the notion of didactic coherence.

4 Using spreadsheets for generating similar didactically coherent problems

There is a well-known challenging problem about census taker and farmer's daughters (e.g., New York State Education Department, 1998, p. 77) the mathematical structure of which is based on the fact that there exist two (or more) sets of three integers with the same sum and the same product. The task for students is, given the product, to overcome the hurdle of the multiplicity of answers by using the context of the problem. It turns out that problems of that type are not easy to duplicate if a teacher wants to provide his or her students with rich problem-solving practice. Simply altering numeric data typically yields either numerically or contextually incoherent problems.

For example, the triples $(1, 9, 32)$ and $(2, 4, 36)$ do have the same product, 288, and the same sum, 42, thus manifesting numerical coherence of the corresponding problem about census taker and farmer's daughters. Yet contextually, as the ages of the farmer's daughters, these triples are hardly plausible. The use of the spreadsheet attached to the first author's website (Abramovich, 2012) makes it possible to generate both numerically and contextually coherent problems of that kind without much difficulty (though not immediately).

Consider another case. Given the product 144 of the ages of three daughters and the sum of ages a (not given) house number, there are two pairs of triples, $\{(3,6,8),(4,4,9)\}$ and $\{(3,4,12),(2,8,9)\}$, being appropriate as the ages of siblings; the first pair has the sum 17 and the second pair has the sum 19; all four triples have the same product, 144; however, only the triple $(4,4,9)$ would satisfy the problem situation in the context of having twins. This is just one example when technology (such as the spreadsheet shown in Figure 1) can be used as a generator of similar challenging problems satisfying the conditions of didactic coherence.

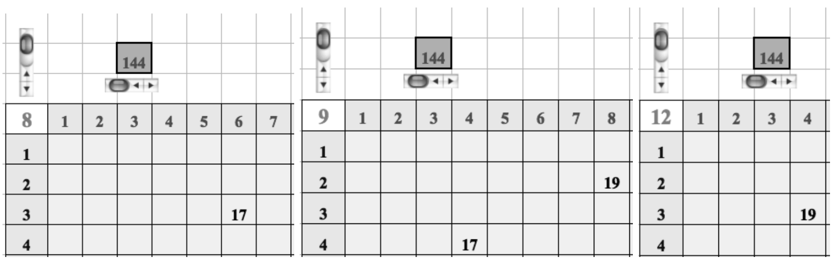

Figure 1: Generating two pairs of triples through a spreadsheet-based experiment

5 Revision of problems for advanced high school students

Consider the inequality

$$\log_x(16-6x-x^2) > 1, \tag{3}$$

another typical problem that may be offered in a number of contexts such as mathematical enrichment, a school with specialized curriculum emphasizing mathematics and science, or college entrance examination in mathematics. In the pre-computer age, solving inequality (3) required advanced skills in algebra. These skills consisted in reducing the inequality to simultaneous inequalities not involving logarithmic functions, solving the inequalities and deciding their solution set in common. However, nowadays one can enter the inequality in *Wolfram Alpha* (Figure 2) and immediately get the final result, $1 < x < \frac{1}{2}(-7 + \sqrt{13})$.

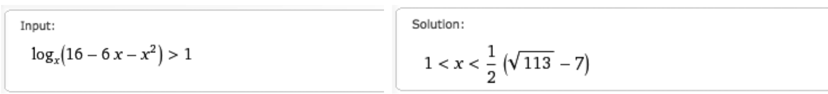

Figure 2: “Solving” inequality (3) with *Wolfram Alpha*

Thus, as was mentioned above, it becomes difficult to motivate an analytical solution of a complicated inequality when technology can do the job. That is why the traditional problems on solving equations or inequalities have to be revisited with the goal to make them technology-immune by adding new inquiries to traditional ones. Note that knowing the solution set of (3) does not make it easier to construct a non-trivial inequality with the identical solution set. That is, the construction of an inequality with a given solution set is a challenge in itself. For example, one can check to see that (3) and the inequality $x^{16-7x-x^2} > 1$ have identical solution sets. The latter inequality, though visually resembling (3), cannot be constructed without conceptual understanding of algebraic relationships structuring the behavior of logarithmic and exponential functions.

Such revised task of constructing an inequality with a given solution set is rich in formulation. First, it is an open-ended task allowing for multiple inequalities to be constructed and different reasoning used in the process of construction. Second, the task allows one to continue using technology, yet in a conceptually rich way. Third, the task still enables one to use *Wolfram Alpha* (or a similar application) but as a tool that verifies the correctness of the inequality constructed. Finally, one can still solve inequality (3) using paper and pencil and then select one of the equivalent intermediate inequalities as an answer to the task proposed.

6 Problems arising in the context of programming a computer

Consider the problem of finding the number of rectangles of integer side lengths x and y whose area and semi-perimeter are, numerically, in the ratio of n to 1. The problem's solution consists in solving the Diophantine equation $xy = n(x + y)$ whence

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{n}; \tag{4}$$

in other words, finding the number of decompositions of a unit fraction into the sum of two like fractions, a problem which appeared in Eggleton's (1996) work. Equation (4) can be explored with technology by modeling it within a spreadsheet (Abramovich, 2011). However, for sufficiently large values of n knowing the boundaries within which the variables x and y vary is important to provide the computational efficiency of modeling. Towards this end, the following problem, which requires skills in proof techniques, can be created: Assuming $x \le y$, prove that all solutions to equation (4) but one satisfy the inequalities $n + 2 \le x \le 2n \le y \le n(n + 2)/2$.

A similar problem can be created in the context of the Diophantine equation

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{1}{n}; \tag{5}$$

which, in particular, describes right rectangular prisms of integer side lengths x, y and z whose volume, numerically, is n times as much as the half of its surface area, that is, $xyz = n(xy + yz + zx)$. Equation (5) has another geometric interpretation: n is the radius of a circle inscribed in a triangle with the heights x, y and z, where $0 \le x \le y$ and $\frac{1}{x} < \frac{1}{y} + \frac{1}{z}$ (the triangle inequality). One can also use equation (5) as a model for finding all triples of workers to be hired to complete a certain job in n days when working together. In that way, challenging problems, when approached from a modeling perspective, can give rise to new types of problems the solutions of which can ensure the efficiency of computing applications.

7 Conclusion

Advanced students in every classroom often get less attention than students who struggle. While other students still are grappling with a newly introduced concept and solving a given set of accompanied problems, advanced students may have already grasped the concept and finished solving the set in a much shorter time and would be in need of a more intellectually stimulating environment than just waiting for other students to finish. Mathematics educators are faced with challenges to meet the vastly diverse needs of students in a group using a limited time and resources. If developmentally and pedagogically appropriate environments are provided, students at various levels can be motivated to think mathematically and creatively. In this article, a few ways of using technology for creating problems and reformulating a routine problem into a challenging one are shared, reflecting the potential of technology for promoting intellectual capacity of students and their teachers alike.

The use of technology in mathematics education has mostly been related to solving problems rather than posing problems. The use of *Wolfram Alpha* by a student to find the answer to a given problem is one of such examples. Yet, problem solving and problem posing are just two sides of the same coin, technology. By paying attention to such interrelated nature and function of technology, one can be encouraged to approach commonly available technology tools under a different angle; thus, using them not only for solving problems but also for creating problems. Revisiting a routine problem and how it is represented in a computer-based environment can help educators to recognize the hidden non-routine structure of a problem and be able to reformulate it into a challenging problem towards the end of helping students to advance their mathematical knowledge and skills. The authors encourage mathematics educators to engage in an intellectual journey with students through creative use of available technology.

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Counterexamples for Cevian Triangles

Valery Zhuravlev & Peter Samovol

Valery Zhuravlev is a Russian mathematician. He is living and working in Moscow. Much of his time is spent working with mathematically talented students of secondary schools in Russia.

Peter Samovol (Ph.D) is a Lecturer at the Department of Mathematics, Kaye Academic College of Education and at the Mathematical Club, Ben Gurion University, Be'er-Sheva, Israel. He has a high level of experience preparing students for regional and international Mathematics Olympiads. His area of interest includes Development of the Mathematical Thinking and Number Theory.

Ever since people started proving obvious statements, most of them turned out wrong.

Bertrand Russell

At first glance it seems that if we research an isosceles triangle, we most likely won't discover anything new. At the very beginning of geometry studies in school, it is proven that if a triangle is isosceles, then it has two equal angles, medians, heights and angle bisectors respectively. The opposite theorems are also true, meaning, that if a triangle has two equal angles, medians, heights or angle bisectors then the triangle will be necessarily isosceles. Even so, the proof of the opposite theorem in the case where the triangle has two equal angle bisectors turns out to be a bit more complicated than the case in which the triangle has two equal medians or heights. This theorem is called the *Steiner–Lehmus theorem.*

Now we will define a couple of terms:

a cevian – a segment which connects a vertex of a triangle with a point on the opposite side.

a cevian triangle – a triangle composed by the endpoints of the 3 cevians on all three sides.

We will take an arbitrary triangle and observe the cevian triangle that is formed by the endpoints of its three medians, heights or angle bisectors. It isn't hard to prove that in an isosceles triangle, all the cevian triangles we mentioned before will be also isosceles. Intuitively, we may assume that the opposite theorems will be true. An examination of the problem for the triangle composed by the heights and the medians strengthens our assumption, but in contradiction to that, the problem phrased by the mathematician Igor Sharygin, that deals with the existence of equilateral triangles where the cevian triangle constructed by the endpoints of their angle bisectors is isosceles surprises us.

In this article, we will concentrate mainly on the angle bisector case, and in addition examine a number of other famous cases of cevian triangles. We hope that the examples and the counter examples that we will present will convince our readers that even a simple geometric field such as isosceles triangles may conceal a significant number of surprises.

Exercise 1 Prove that a triangle is isosceles if and only if the cevian triangle is constructed by its:

a) heights;

b) medians;

is also isosceles.

Exercise 2 (I. Sharygin, [1, II.203]) In a given triangle, it is known that the cevian triangle constructed by the endpoints of its angle bisectors is isosceles. Will it be true to claim that the given triangle is isosceles?

The solution of our first problem we will leave to our readers as a simple "warm-up".

The second problem was published in 1982. Despite its simple wording, the attempt to solve it shows its depth. In order to emphasize its level of difficulty, we will quote the original article of I. Sharygin:

> Let A_1, B_1, C_1 be the endpoints of the angle bisectors in triangle ABC. If $A_1B_1 = A_1C_1$, and the triangle ABC isn't isosceles, then angle A is an obtuse angle, and $\cos A$ is in the interval $\left(-\frac{1}{4}; \frac{\sqrt{17}-5}{4}\right)$. Unfortunately, the writer hasn't succeeded in constructing a specific triangle with this exotic attribute (or in other words, stating precisely its angles or sides). Perhaps the readers will succeed where the writer failed?

The first known example for such a triangle, where all angles are stated, was discovered only at the beginning of this century.

Exercise 3 (S. Tokarev, the 27th International Tournament of Towns) Let ABC be a triangle, where the angle bisectors AA_1, BB_1 and CC_1 were constructed. It is known the ratio between the angles A, B, C is $4:2:1$. Prove that $A_1B_1 = A_1C_1$.

The proof of the exercise (that we won't describe here) is based on the existence of a triangle that isn't isosceles (since the ratio is $4:2:1$) however the cevian triangle built by the endpoints of its angle bisectors is $(A_1B_1 = A_1C_1)$. Our readers will probably evaluate the esthetics of this example. So far two different proofs were published to this problem.

In order to find more examples the writers of this article used algebraic tools. We managed to point out all the possible lengths of scalene triangles, that the cevian triangle, constructed by the endpoints of their angle bisectors is isosceles.

Constructing a counterexample like in exercise 2
Let a, b, c be the lengths of the triangle's sides, $\angle B = \beta$, $\angle C = \gamma$ (figure 1).

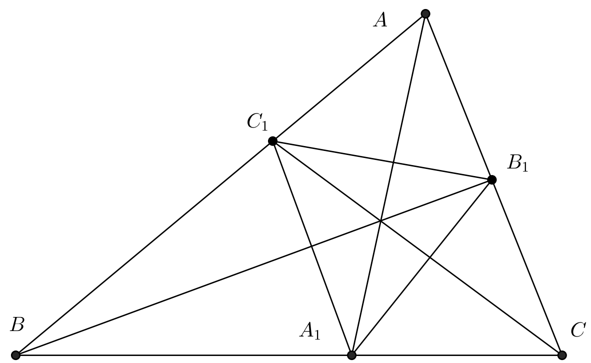

Fig. 1

According to known attributes of angle bisectors we can conclude that

$$BA_1 = \frac{ac}{b+c}, \qquad CA_1 = \frac{ab}{b+c}, \qquad BC_1 = \frac{ac}{a+b}, \qquad CB_1 = \frac{ab}{a+c}.$$

According to the law of cosines in triangles we can deduce that $2ac\cos\beta = a^2+c^2-b^2$ and that $2ab\cos\gamma = a^2+b^2-c^2$. We will use the law of cosines again in triangles A_1BC_1 and A_1B_1C (noticing that $A_1C_1 = A_1B_1$) and have that

$$\left(\frac{ac}{b+c}\right)^2 + \left(\frac{ac}{a+b}\right)^2 - \frac{2a^2c^2\cos\beta}{(b+c)(a+b)} = \left(\frac{ab}{b+c}\right)^2 + \left(\frac{ab}{a+c}\right)^2 - \frac{2a^2b^2\cos\gamma}{(b+c)(a+c)}.$$

And then we will get that

$$\frac{abc(b-c)(b+c)\left(b^3+c^3-a^3+b^2c+b^2a+c^2b+c^2a-a^2b-a^2c+abc\right)}{(a+b)^2(b+c)^2(a+c)^2} = 0.$$

In order for the given triangle not to be isosceles it must have that $b \neq c$ then we will necessarily have to get that

$$b^3+c^3-a^3+b^2c+b^2a+c^2b+c^2a-a^2b-a^2c+abc = 0.$$

We will divide the equality we received by a^2, substitute $x = \frac{b}{a}$, $y = \frac{c}{a}$ and get that

$$x^3 + x^2(y+1) + x(y^2+y-1) + y^3 + y^2 - y - 1 = 0.$$

We will relate to this equality as a cubic equation for x where y is a parameter. In order to construct an example we will have to find one solution to this equation. We will multiply the equation by 27 and substitute $x = t - \frac{y+1}{3}$ and in this way get that

$$27t^3 + t(18y^2+9y-36) + 20y^3 + 15y^2 - 21y - 16 = 0.$$

Then $y = \frac{1}{2}$ will fit, and after substitution we will have that $t^3 - t - \frac{3}{4} = 0$. Using Cardano's formula for solving cubic equations we will get that

$$t = \sqrt[3]{\frac{3}{8} + \sqrt{-\frac{1}{27} + \frac{9}{64}}} + \sqrt[3]{\frac{3}{8} - \sqrt{-\frac{1}{27} + \frac{9}{64}}}.$$

And that's why

$$x = -\frac{1}{2} + \sqrt[3]{\frac{3}{8} + \frac{1}{8}\sqrt{\frac{179}{27}}} + \sqrt[3]{\frac{3}{8} - \frac{1}{8}\sqrt{\frac{179}{27}}} \approx 0.763.$$

Thus, we find a scalene triangle with sides

$$a = 2, \qquad b = -1 + \sqrt[3]{3 + \sqrt{\frac{179}{27}}} + \sqrt[3]{3 - \sqrt{\frac{179}{27}}} \qquad \text{and} \qquad c = 1.$$

The cevian triangle constructed by the endpoints of the angle bisectors will be isosceles.

It is reasonable to assume that our readers will think that this counterexample seems terrifying compared to the one described in Exercise 3. Esthetically it will be hard to argue with, but our interest is in mathematics and not in esthetics. In mathematical manners, both solutions are equal and worthy. Furthermore, the algebraic tools allow us to construct an infinite number of examples of triangles such as that. In order to do so, we will have to change a bit the value of y, in a way that x, y and 1 will continue maintaining the triangle inequality.

We will observe the following problem.

Exercise 4 (A. Zaslavsky, V. Senderov [5] M1862) The angle bisectors AD, BE and CF in triangle ABC intersect in point I. Prove that:

a) If $ID = IF = IE$ then triangle ABC is an equilateral triangle.

b) If DFE is an equilateral triangle then ABC is an equilateral triangle.

In this problem we can see that there is no alternative—if the cevian triangle constructed by the endpoints of the angle bisectors is equilateral so the given triangle is an equilateral triangle. The proof of this problem we will leave to our readers.

We will summarize what is known to us by now. There are two options:

a) The cevian triangle is isosceles; therefore the given triangle is isosceles.

b) The cevian triangle is isosceles; nevertheless the given triangle isn't necessarily isosceles.

The first option is true for medians and heights and the second option is true for angle bisectors.

In the frame of this short article we will look in detail at three more famous examples of cevian triangles—the cevian triangles for the Gergonne point, the Nagel point and the Lemoine point.

The cevian triangle for the Gergonne point is the triangle constructed by the three touch points of the incircle on the three sides, where the Gergonne point itself is defined as the intersection point of the three segments that connect the triangle's vertices and their opposite touch points (figure 2). The examination of this case is relatively simple.

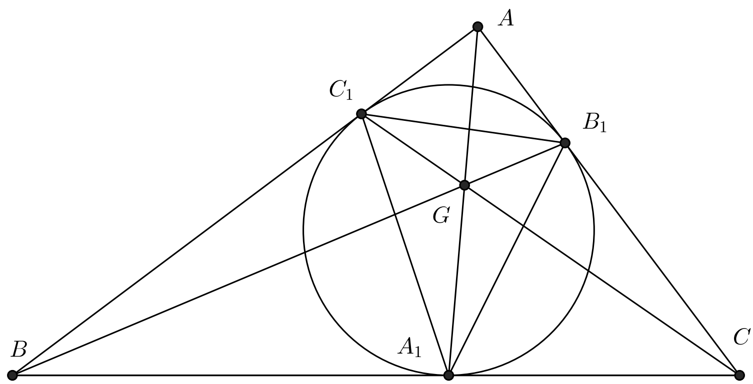

Fig. 2

Exercise 6 Prove that a given triangle ABC is isosceles if and only if the cevian triangle constructed by the three touch points of the incircle on the three sides is also isosceles.

Let's assume that the given triangle ABC is isosceles and that $AB = AC$. It is also known that $AB_1 = AC_1$ since they are two tangent lines that are tangent from the same point. Nonetheless AA_1 is an angle bisector and that's why $\triangle AA_1B_1 \cong \triangle AA_1C_1$ (according to SAS). Therefore $A_1B_1 = A_1C_1$.

On the other hand, if $A_1B_1 = A_1C_1$ then since $AB_1 = AC_1$ we will get that $\triangle AA_1B_1 \cong \triangle AA_1C_1$ (SSS). Following by that $\angle AB_1A_1 = \angle AC_1A_1$. Since the triangles A_1C_1B and A_1B_1C are also isosceles we will get that $\angle ABC = \angle ACB$, or in other words ABC is an isosceles triangle.

The cevian triangle for the Nagel point is the triangle constructed by touch points of the excircles on the sides of the triangle, where the Nagel point itself is defined as the intersection point of the three segments connecting the triangle's vertices and their opposite touch points (figure 3).

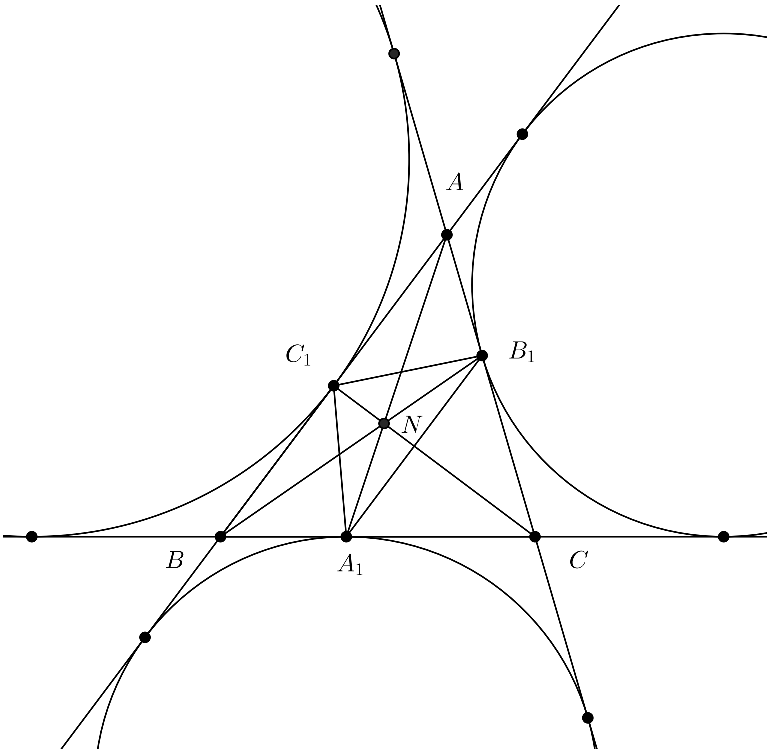

Fig. 3

Exercise 7 Let A_1, B_1, C_1 be the touch points of the excircles on the triangle sides BC, CA and AB. Prove that if $A_1B_1 = A_1C_1$ then the triangle ABC is isosceles.

Let $p = (a + b + c)/2$. It isn't complicated to express the length of the segments $BA_1 = p - c$, $CA_1 = p - b$, $BC_1 = p - a$, $CB_1 = p - a$. According to the law of cosines in triangle ABC we will get that $2ac \cos \beta = a^2 + c^2 - b^2$ and that $2ab \cos \gamma = a^2 + b^2 - c^2$. We will use the law of cosines again in triangles A_1BC_1 and A_1B_1C (noticing that $A_1C_1 = A_1B_1$) and have that

$$(p-c)^2 + (p-a)^2 - 2(p-a)(p-c)\cos\beta = (p-b)^2 + (p-a)^2 - 2(p-a)(p-b)\cos\gamma.$$

And after basic substitution we will have that

$$(p-c)^2 - 2(p-a)(p-c)\frac{a^2+c^2-b^2}{2ac} = (p-b)^2 - 2(p-a)(p-b)\frac{a^2+b^2-c^2}{2ab}.$$

Or

$$\frac{(b-c)(a^4+b^4+c^4+4ab^2c+4abc^2-2a^2b^2-2a^2c^2-2b^2c^2)}{4abc} = 0.$$

We will examine the case where $b \geq c$. We will get that

$$\begin{aligned} a^4+b^4+c^4+4ab^2c+4abc^2-2a^2b^2-2a^2c^2-2b^2c^2 \\ = (a^2-b^2+c^2)^2 + 4ac(-ac+b^2+bc) \geq 4ac(-ac+b(b+c)) \\ \geq 4ac(-ac+c(b+c)) = 4ac^2(b+c-a) > 0. \end{aligned}$$

This means that an equality exists only if $b = c$.

The case where $b \leq c$ can be checked in a similar way and we can reach the same conclusion. Thus, we concluded that necessarily $b = c$ and that's why ABC is an isosceles triangle. In order to explain what the Lemoine point is we will have to define the term symmedian: a segment constructed by taking a median and reflecting it over the corresponding angle bisector. The Lemoine point is the intersection point of all three symmedians in the triangle. Therefore, the cevian triangle for the Lemoine point is the triangle constructed by the endpoints of the three symmedians in the triangle. We hope that proving the following problem won't be a challenge for our readers.

Exercise 8 Prove that if a given triangle is isosceles, then the triangle constructed by the endpoints of its symmedians is also isosceles.

The opposite theorem is surprising.

Exercise 9 In a given triangle it is known that the triangle constructed by the endpoints of its three symmedians is isosceles. Will it be right to claim that the given triangle is isosceles?

In order to investigate this problem we will have to use a known theorem. We will phrase it as a lemma.

Lemma Let K be the intersection point of the symmedian from vertex B (in triangle ABC) with the side AC. Then

$$\frac{AK}{KC} = \frac{AB^2}{BC^2}.$$

The proof of the lemma can be found in different sources (see [1, II.171]).

Examination of exercise 9

We will operate the same way we did in the angle bisector case, and use the same signs from figure 1. Using the lemma it isn't hard to express the following segments

$$BA_1 = \frac{ac^2}{b^2+c^2}, \quad CA_1 = \frac{ab^2}{b^2+c^2}, \quad BC_1 = \frac{a^2c}{a^2+b^2}, \quad CB_1 = \frac{a^2b}{a^2+c^2}.$$

In the same way we will use the law of cosines in triangle ABC and get that $2ac\cos\beta = a^2+c^2-b^2$ and that $2ab\cos\gamma = a^2+b^2-c^2$. We will use the law of cosines again in triangles A_1BC_1 and A_1B_1C (noticing that $A_1C_1 = A_1B_1$) and have that

$$\left(\frac{ac^2}{b^2+c^2}\right)^2 + \left(\frac{ca^2}{a^2+b^2}\right)^2 - \frac{2a^3c^3\cos\beta}{(b^2+c^2)(a^2+b^2)}$$
$$= \left(\frac{ab^2}{b^2+c^2}\right)^2 + \left(\frac{ba^2}{a^2+c^2}\right)^2 - \frac{2a^3b^3\cos\gamma}{(b^2+c^2)(a^2+c^2)}.$$

After substitution and division by a^2 we will have that

$$\frac{c^4}{(b^2+c^2)^2} + \frac{c^2a^2}{(a^2+b^2)^2} - \frac{c^2\left(a^2+c^2-b^2\right)}{(a^2+b^2)(b^2+c^2)}$$
$$= \frac{b^4}{(b^2+c^2)^2} + \frac{b^2a^2}{(a^2+c^2)^2} - \frac{b^2\left(a^2+b^2-c^2\right)}{(a^2+c^2)(b^2+c^2)}.$$

We will substitute $x = \frac{c^2}{a^2}$, $y = \frac{b^2}{a^2}$ and have that

$$\frac{x^2}{(x+y)^2} + \frac{x}{(1+y)^2} - \frac{x(1+x-y)}{(1+y)(x+y)} - \frac{y^2}{(x+y)^2} - \frac{y}{(1+x)^2} + \frac{y(1+y-x)}{(1+x)(x+y)} = 0.$$

And that's why

$$\frac{xy(y-x)\left(x^2+y^2-xy-x-y-3\right)}{(1+x)(1+y)(x+y)} = 0.$$

If $x = y$ then $b = c$ and ABC is an isosceles triangle. However if the equality $x^2+y^2-xy-x-y-3 = 0$ exists then for $x \neq y$ we can construct a counterexample. We will notice that the equality is equivalent to the quartic equation for the triangle sides

$$c^4 + b^4 - c^2b^2 - a^2c^2 - a^2b^2 - 3a^4 = 0.$$

We will relate to this equation as a quadratic equation for x where y is a parameter and have that $x^2 - x(1+y) + y^2 - y - 3 = 0$. This equation has a solution when $\Delta \geq 0$, which means that $\Delta = (1+y)^2 - 4\left(y^2 - y - 3\right) = -3y^2 + 6y + 13 \geq 0$.

From that we can conclude that $0 < y \leq \frac{3+4\sqrt{3}}{3}$. Nonetheless for every y we get two different solutions since $x = \frac{1+y\pm\sqrt{-3y^2+6y+13}}{2}$. In addition one must remember that the triangle inequality must exist, meaning that

$$\begin{cases} a < b+c \\ b < a+c \\ c < a+b \end{cases} \Leftrightarrow \begin{cases} 1 < \sqrt{y} + \sqrt{x} \\ \sqrt{y} < 1 + \sqrt{x} \\ \sqrt{x} < 1 + \sqrt{y} \end{cases}.$$

We will examine each condition separately (and skip the technical part of solving the system of inequalities). We will have that

1. If $x = \frac{1+y+\sqrt{-3y^2+6y+13}}{2}$ then $\left(\frac{\sqrt{5}-1}{2}\right)^2 < y \leq \frac{3+4\sqrt{3}}{3}$.

2. If $x = \frac{1+y-\sqrt{-3y^2+6y+13}}{2}$ then $\left(\frac{1+\sqrt{5}}{2}\right)^2 < y \leq \frac{3+4\sqrt{3}}{3}$.

Since what really interests us is the ratio between the triangle sides and not their lengths (if we will find one example we will be able to construct an infinite number of similar triangles to the one in the example), we can assume without the loss of generality that $a = 1$.

We will summarize our conclusions:

- If $\left(\frac{1+\sqrt{5}}{2}\right)^2 < y < \frac{3+4\sqrt{3}}{3}$ then both of the solutions are suitable and we find two groups of similar triangles with sides
$$a = 1, \qquad b = \sqrt{y}, \qquad c = \sqrt{\frac{1 + y \pm \sqrt{-3y^2 + 6y + 13}}{2}}.$$
- If $\left(\frac{\sqrt{5}-1}{2}\right)^2 < y \leqslant \left(\frac{1+\sqrt{5}}{2}\right)^2$ then only one solution is suitable and we find a group of similar triangles with sides
$$a = 1, \qquad b = \sqrt{y}, \qquad c = \sqrt{\frac{1 + y \pm \sqrt{-3y^2 + 6y + 13}}{2}}.$$
- If $\Delta = 0$ then $y = \frac{3+4\sqrt{3}}{3}$ and we find a triangle with sides $a = 1$, $b = \sqrt{\frac{3+4\sqrt{3}}{3}}$, $c = \sqrt{\frac{3+2\sqrt{3}}{3}}$ which is also suitable as a counterexample.

It is important to state that $y = 1$ isn't a counter example because then the triangle has sides with the lengths $a = b = 1$, $c = \sqrt{3}$ meaning that it's isosceles. Nonetheless, this case is interesting because even though the original triangle isn't equilateral the cevian triangle constructed by the endpoints of the three symmedians is!

As we can see the counter examples for the symmedian case are simpler because we had to solve a quadratic equation. Even though the equation that connected between the side lengths was a quartic equation it was a biquadratic equation so the solution was simpler, compared to the angle bisector case where we couldn't avoid solving a cubic equation.

In the symmedian case we found the alternative for the case which in the cevian triangle is equilateral but the original triangle isn't as we've seen in the triangle with side lengths $a = b = 1$, $c = \sqrt{3}$. In the angle bisector such an example can't be constructed.

Readers may continue the research if they wish to—all they have to do is observe cevian triangles for other special known points in a triangle.

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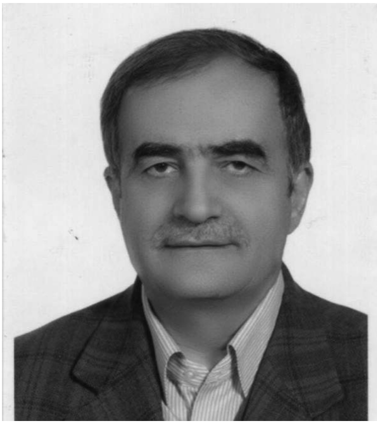
Challenging Mathematics through the Improvement of Education

Ali Rejali & Neda Hematipour



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1 Introduction

As mentioned in the introductory chapter of [1], "Mathematics is praised for its utility and regarded as a foundation of our modern technological and information society." But the results of different tests and international comparative exams show that not many students are comfortable with using mathematical ideas and concepts in their daily lives and many of them are disdainful of mathematics.

Mathematics awareness in many societies works well but popularizing mathematics is not so successful among many students. Many years after the World Mathematical Year 2000, the image of mathematics has not been changed in the eyes of the general public. Many activities such as mathematics competitions, mathematics clubs and mathematics museums have had effects on some groups of students, but the aims of popularizing mathematics for all is not fulfilled by these activities.

2 Iran's experience

In Iran, we have a long experience of mathematics competitions since 1982 [2], and the establishment of mathematics houses since 1997 [3, 4], but we are still not satisfied with our achievements in attracting all the students. We have proceeded to attract some gifted students but mathematics for all is not well achieved in our society. Iranian Mathematical Society [5], Iranian Statistical Society [6], Mathematics Houses [7], Mathematics Teachers' societies, and even the ministry of education performed many programs for improvement of mathematical sciences awareness and popularization of mathematics, but the audience for most of these programs are not all the students. Only some gifted ones, mostly due to encouragement by their teachers or parents, are being attracted to these programs.

Most mathematicians remember the effect of good teachers encouraging them to learn mathematics. Challenging problems in classes raised by teachers at different levels of schooling were the main reason for most of us to continue our studies in mathematical sciences. These days in many countries, we do not get such teachers to be able to encourage their students for learning mathematics. Some general publications such as popular mathematics books and journals attract some students as well, but they have not the same effect as teacher's activities in the class. In Iran, they used to attract motivated students to go into teaching by giving them a scholarship based on their achievements and recommendation by their teachers, while they were high school students. These well-qualified teachers, who refused to go to engineering schools, medical schools or law schools for their university studies, encouraged many of us to go to mathematics, physics or other basic sciences. I remember the time when they have changed our curriculum from old mathematical subjects to the so-called "modern mathematics". The mathematics teachers worked together to learn the new ideas of modern mathematics without enough help from outside in 1960.

But later on the situation changed, and the prestige for the teaching position in Iran declined. In the late 1970 we had problems with a lack of interest in students for studying mathematics. Therefore the Iranian Mathematical Society with the help of some authorities in the ministry of education started a study on the lack of interest in students for studying mathematics [8]. As a result, mathematics competitions started in Iran and turned to Iranian Mathematics Olympiad and other Scientific Olympiads, as well as the participation of Iranian teams in IMO[1] and other international competitions. But as we mentioned in our paper [9], not many teachers and students got involved in mathematics competitions. It became an isolated program which attracted some so-called gifted students and does not have a real impact on all the students, mathematics teachers and the school system, in general.

Then on the occasion of the WMY-2000, we started some new projects and as a result mathematics houses [10] and Mathematics Teachers' Societies have been established or developed in many cities and provinces in Iran.

Different competitions, exhibitions, lectures, workshops and other programs of the houses have attracted some students and teachers. They had some impacts on the system of education in Iran, but not enough. These attempts did not fulfill the goals of popularizing mathematics.

What we and other countries are missing are the lack of prestige for teaching and the lack of support for school system. In most countries, education is considered to be a service to citizens not a productive tool for the development of the country. As a result of this attitude, the schools are small and the students are not free to play in schools and

[1] International Mathematics Olympiad

enjoy as well as learning everyday life skills, and the teachers are not being paid enough [11]. The programs are not well designed and the textbooks are not well written, because we do not give the appropriate value to education.

3 Worldwide Experience

Looking at the results from TIMSS[2] [12] and Pisa[3] [13], (although Iran has not participated in the second one), shows that the countries with high priorities on education achieved better results in these two international studies. (We should not be misleaded by IMO results [14]. These days mathematics competitions leading to IMO are mostly assigned for mathematically gifted students and they are isolated programs for special students [9]. We studied the relation between the results of IMO and TIMSS for the countries which participated in both. The figures show the existence of a zero correlation between these two scores. That is, *high achievement in IMO does not mean good performance of a country in mathematics*. This is one of reasons for the WFNMC[4] to encourage the participation of the countries in other mathematics competitions as well.)

Park [15] mentions that some of the factors contributing to high achievement in mathematics for Korean students are the national enthusiasm for education, the eagerness for study, and the ethic of hard work which are the characteristic of East Asian Countries. He also mentions that the focus of primary and secondary education in Korea is on subjects required in the national college entrance examination (known as the College Scholastic Ability Test[5]), and since mathematics is one of the four areas that are assessed in CSAT, it results in a relatively high importance on this subject.

Moreover, there are many private institutions and tutoring courses dedicated to preparing students for the subjects of CSAT. Similar practices and attitudes exist in Iran for our national university entrance examination (Konkour), but the Iranian students are not achieving the high

[2]Trends in International Mathematics and Science Study
[3]Program for International Student Assessment
[4]World Federation of National Mathematics Competitions
[5]CSAT

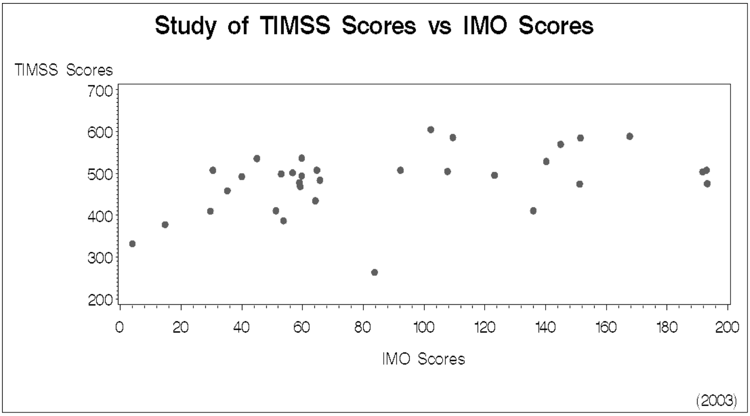

Figure 1: Study of TIMSS Scores versus IMO Scores in 2003

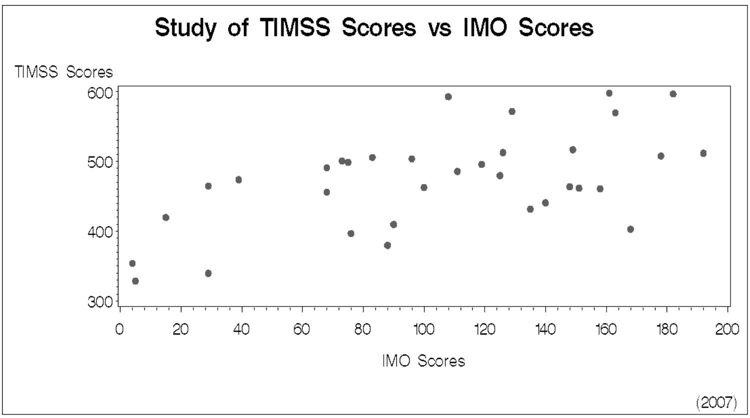

Figure 2: Study of TIMSS Scores versus IMO Scores in 2007

standards in international comparison tests on mathematics [12]. The reasons for that could be the prestige of teaching as a job, low enthusiasm among teachers for their job, and lack of interest in well-qualified and talented students to continue their studies toward teaching. See also the importance of teacher education in Korea [15], which does not exist in Iran and many other countries of the world. Similar results have been shown in Ferrucci, Kaur and Carter's study [16] on exploration of differences in mathematics achievement in Singapore and the United States.

Many suggestions have been made by Ahuja [17] for all K-12 American students to have a world-class high quality mathematics education, which are all useful tools for improvement, but what we are missing is a lack of talented people in the business of teaching mathematics. To argue about this claim, I would like to refer to the recent report by Commonwealth Foundation for Public Policy Alternatives, February 2012, regarding Pennsylvania K-12 education spending which shows that public schools revenue per student increased by 44 percent, while performance on National Assessment of Education Progress using the national exam has changed a little. According to this report, there is only a little or no correlation between student achievement and class size, teacher salaries or per student expenditure.

Also a 2010 study by 21st century Partnership for Science, Technology, Engineering and Mathematics education (21PSTEM) comparing 11th grade mathematics, reading and science scores on Pennsylvania State test with distinct per-student spending found low-spending districts often outperformed high-spending ones. This evidence shows that spending on education without preparation, well-designed programs, and good teachers does not improve the performance of students.

4 Proposal

Our proposal involves using the Chinese, Korean, Japanese and Singaporean attitudes toward education, and the Iranian tradition in attracting gifted students for teaching positions.

In Iran, parents are willing to pay for their children's education, but since the system is not prepared, this spendings are not useful and the students do not get the benefit.

Multiple-choice questions for the first round of mathematics competitions, emphasis on individualism in the school system and the university entrance examinations mean learning of mathematics in schools, although it very important subject for students and their parents, is not working properly.

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PitGame

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Abstract. PitGame is a new model for a problem-solving contest which is based on teamwork and it is imposed by a process of role playing game too. PitGame minimizes the downsides and obstacles of contests and through a double creativity would cause a fresh environment toward the joy of problem solving.

Contests are mainly a competitive learning activity and it is usually extra curriculum which develops creativity and problem-solving skills. Competitive learning could assist educators in discovering students' abilities and creativity; it also improves students' skills and would support improvement of educational system too.

PitGame is mainly a problem-solving contest based on: Competitive learning, Game theory, and Role playing game, it will be discussed in this paper.

1 Introduction

PitGame is a model for a problem-solving contest which is based on game theory and a role playing game as the main theoretical aspects. Main ideas of developing game are considered as: teamwork, fun, a multitasking game which all the team members would have a feeling to be in system and would play their own role. PitGame minimizes the downsides and obstacles of contests and through a double creativity would cause a fresh environment toward joy of problem solving in a group contest. PitGame model and rules and regulations are simplified such that it would easily be deployed.

PitGame model and rules and regulations will be introduced in the following sections.

2 PitGame Model

Earning the most money is the main object in the game, the currency which is used is called "Pit". Each team receives some initial Pits and they should increase this fund as best as possible based upon the rules and regulations of the game such as solving the problems, buying and selling problems etc.

Teams with initial Pits may buy problems in auctions: in each auction some problems will be offered and each team will give their bid separately. The winner will receive the problem and they should submit the solution in a certain period of time, for correct solution they will receive more Pits based on the difficulty of the problems.

Earning more Pits are the main object which will happen through solving the problems. Teams may also sell and buy unsolved problems to and/or from other teams, sell and buy hints to and/or from other teams as a possibility for earning or losing Pits.

Besides problem solving, role playing games are in the heart of PitGame which allows, Time and Resource Management, Strategic Planning, Communication and Negotiation, and Cooperation.

3 Rules and Regulations

A jury of 5 to 7 experts will run the game. Each team will receive 250 Pits as initial funds but they also will receive a success factor too. Success factor is zero for all teams at the beginning but it may increase or decrease during the competition and would affect as a factor on earning benefits for solved problems.

Rules and regulations are given to teams in an instruction manual and extracting the rules and applying them is part of the game. At the beginning of the game there will be an auction on an amount of problems as the number of participating teams.

As the game proceeds up to the decision of jury new problems will come to auction as well. In any auction one or more problems will be on auction and there will be a closed Vickery auction. A Vickery auction is a type of sealed-bid auction where bidders submit written bids without knowing the bid of the other people in the auction, and in which the highest bidder wins, but the price paid is the second-highest bid.

To enter a new auction each team should deposit a minimum charge, problems will be on bid just in closed form and teams may decide up to three factors the bid to enter. Factors as time which is estimated for solving problem, a factor of 1 to 5 which will effect on final benefit of each problem based on the difficulty of the problem and the extra bonus factor which will be added to the success factor if the problem is solved in the period of estimated time. The winner of a problem will be the team(s) which has suggested the highest price but they have to pay the suggested price of the second group. In case of more than one winner, all the winners will receive the problem but the first group to solve the problem will receive all the bonuses.

After passing half of the estimated time for solving a problem, the jury may put some hints also again in the bid. Teams also may sell and/or buy problems with each other too.

4 Solution Submission

Solution submissions also will be considered under rules and regulations which in brief are:

- If the correct solution is submitted in the timeline they receive the amount they paid for buying the problem and more Pits as benefit, extra bonuses and success factor also will be considered.
- If any team succeeds in solving the problem within the half of the suggested time they will receive 25 % extra bonuses.
- Any team in the duration of solving a problem may ask the jury to check their solution but they will be charged 50 % of the problem cost.
- In 3 minutes after receiving a not easy problem the team may cut the deal and give the problem back, and they receive all the money back but they lose success factor.
- Within half of the suggested period for solving a problem also they may cut the deal and give back the problem but they will receive back only 30 % of Pits which they have paid and also lose success factor too.
- Any team may ask for extra time for solving a problem but they will be charged with extra Pits.
- The team may cut the deal in the end of deadline upon unsuccessful attempt of solving a problem and give it back, they lose the cost of problem and also will lose success factor.

5 Trade

Selling and buying problems and hints are also another way of gaining more Pits, but any such business should be registered by the jury. Teams are free to negotiate and make agreement on the cost of the deal but they should pay 5 % of the deal as registration fee. Rules of the trade in brief are:

- Problems: Any team which does not like a problem may sell it to another team but there will be no extra time for solving the problem and the next team should work on the problem just within the remaining period of problem time.
- Hints: Any team may receive hints from another team under certain deal but they should register the deal and pay the registration fee.

6 Epilogue

In brief, PitGame is a problem-solving contest which is played by teams in a certain business model. Contestants form teams of 3 to 5 and initially each team will receive an amount of credit in “Pit” currency. Teams can participate in auctions and buy some problems using their own Pits. The easiest job is to solve the problem and earn some Pits, but they also can sell their unsolved problems to other teams, and even sell hints about other teams’ problems. At the end each team with the earned Pits their abilities in negotiation and business and also problem-solving ability will be ranked. Team ranking would not be only in problem solving ability but it will show also other abilities such as:

- Time and Resource Management
- Strategic Planning
- Communication and Negotiation
- Cooperation.

Developing such a contest virtually on the Internet also may be considered which also brings youth around the world together to experience more than problem solving but role playing games as well.

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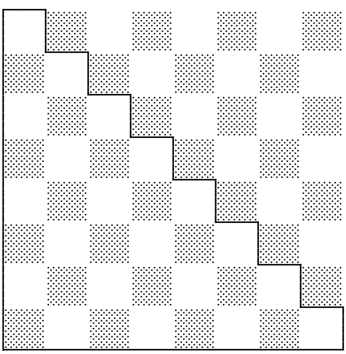
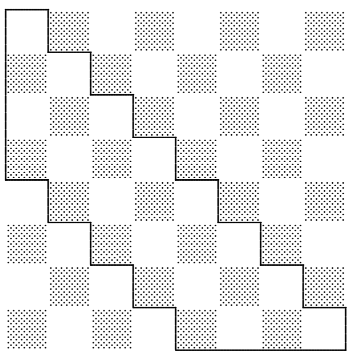
Black or White

Yunhao Fu & Ryan Morrill

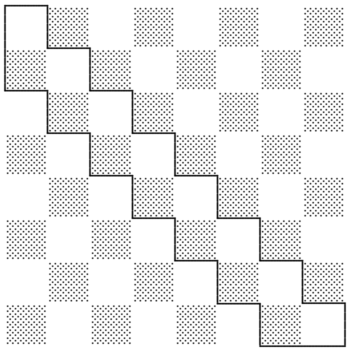
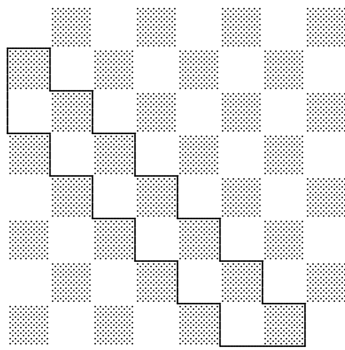
Anna chooses a cell of a standard 8×8 chessboard. She challenges Boris to deduce whether the chosen cell is black or white. He may choose a subboard consisting of one or more cells, bounded by a closed polygonal line with no self-intersections. She will then announce whether her cell is inside or outside this subboard. Naturally, Boris wants to minimize the number of subboards needed for accomplishing his objective.

His first approach takes advantage of the fact that all cells along any diagonal are of the same colour. He comes up with the following method which requires four subboards. The first subboard is shown in the diagram below on the left. We may assume that Anna's announcement is "Inside". This narrows down the chosen cell to eight diagonals. Then the second subboard is shown in the diagram below on the right.



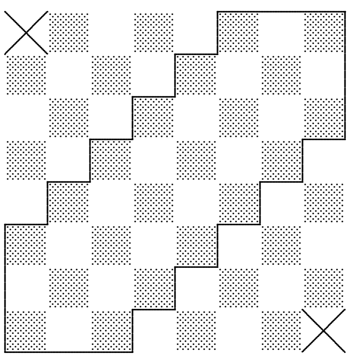
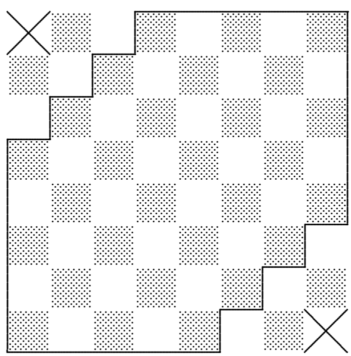


Again, we may assume that Anna's announcement is "Inside". This narrows down the chosen cell to four diagonals. Now the third subboard is shown in the diagram below on the left. We may assume that Anna's announcement is still "Inside". This narrows down the chosen cell to two diagonals. Finally, the fourth subboard is shown in the diagram below on the right.

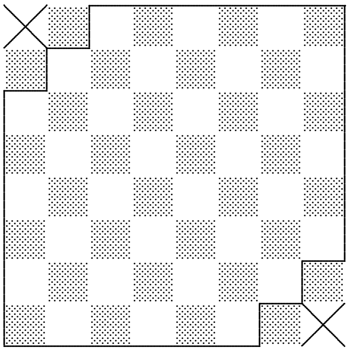
 

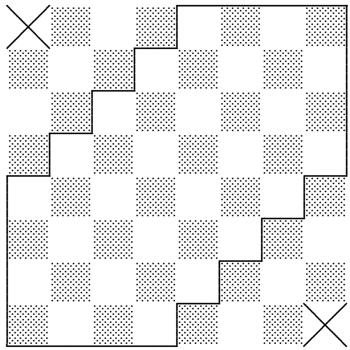
If Anna's announcement is "Inside", then her cell is black. Otherwise, it is white. In any of the preceding subboards, had Anna's announcement been "Outside", a similar and simpler situation arises.

Rather pleased with what he has accomplished so far, Boris tries to simplify his approach and reduce the number of subboards necessary down to three. To his chagrin, he is unable to do so. His method only works if two cells in opposite corners are deleted from the chessboard. Then the first subboard is as shown in the diagram below on the left. If Anna's first announcement is "Outside", then the second subboard is as shown in the diagram below on the right.

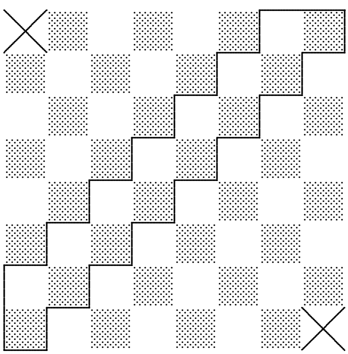
 

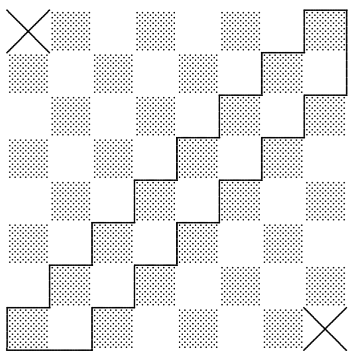
If Anna's second announcement is still "Outside", the third subboard is as shown in the diagram below on the left. If Anna's second announcement is "Inside", the third subboard is as shown in the diagram below on the right. In either case, "Outside" means black and "Inside" means white.



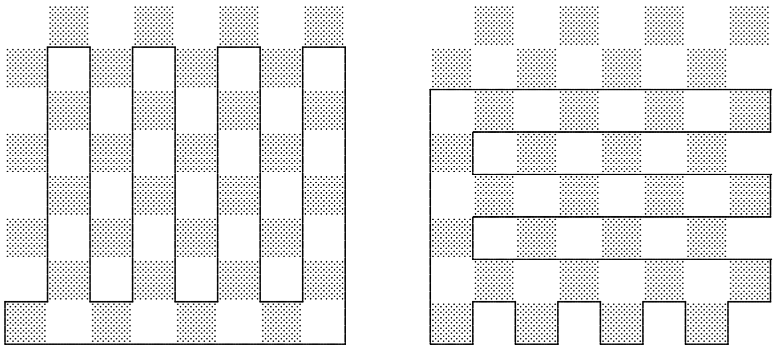


Suppose Anna's first announcement is "Inside". Then the second and the third subboards are as shown in the diagram below. If Anna's second and third announcements are the same, the cell is black. If they are different, the cell is white.



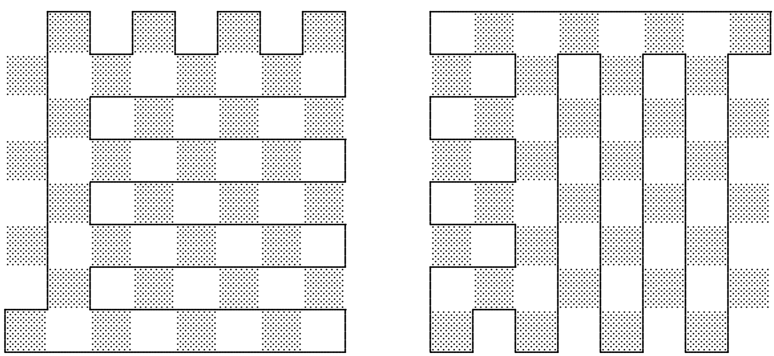


This is not entirely satisfactory. So Boris thinks of taking advantage of the fact that the black cells and white cells are symmetrically situated. In his new approach, the first subboard is shown in the diagram below on the left. By symmetry, we may assume that Anna's announcement is "Inside". Then the second subboard is shown in the diagram below on the right. If Anna's announcement is "Inside", then her cell is black. Otherwise, it is white.

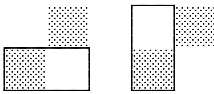


Clearly, one subboard will not be sufficient, since such a subboard must separate all the white cells from all the black cells. So the best possible result has been achieved. However, Boris is still not satisfied. In both approaches so far, he must wait for Anna's announcement before he can choose his next subboard. This is known as an *adaptive solution.*

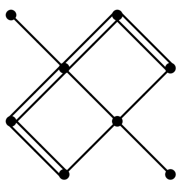
What Boris would like is a non-adaptive solution, in which he would present all subboards to Anna at the same time, and make the deduction upon receiving all the announcements simultaneously. After a while, he comes up with one. The two subboards are as shown in the diagram below. If Anna's announcements are the same, the cell is black. If they are different, the cell is white.



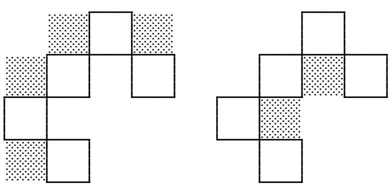
When Boris finally gets around to answering Anna's challenge, Anna is duly impressed. She has not thought that the task can be done using less than four subboards. So she tries to see if she can duplicate the accomplishment of Boris. She starts with the 2×2 chessboard, and there is indeed a non-adaptive solution using only two subboards. They are shown in the diagram below. If the two announcements are the same, the cell must be black. Otherwise, it is white.



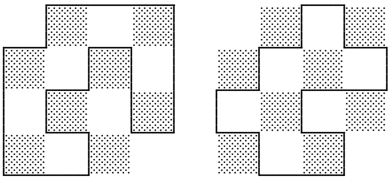
Anna now moves onto the 4×4 chessboard. There are eight white cells. She constructs a graph where each vertex represents a white cell, and two vertices are joined by an edge if a Bishop can move directly between the two white cells they represent. The longest Bishop path without including four vertices forming a square is indicated by a doubled line with three segments. The five white cells represented by the vertices on this path will be inside both subboards. The reason why a square is forbidden is because the four white cells represented by the vertices forming a square will enclose a black square, which must then appear inside both subboards also.



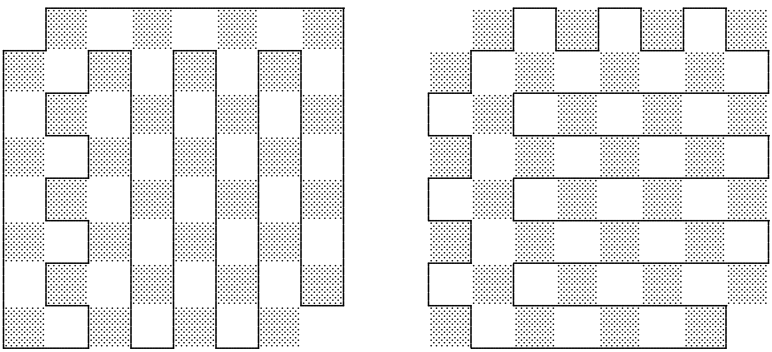
These five white cells may be connected by two disjoint sets of black squares, as shown in the diagram below.



The two subboards are shown in the diagram below. They have been extended so that each includes one of two black squares near the bottom right corner. If the two announcements are the same, then the chosen cell must be white. If the announcements are different, the cell must be black.



This scheme may be generalized to work for any $2n \times 2n$ chessboard. The diagram below shows the solution for the 8×8 chessboard.



The Fall 2013 A-Level paper in the International Mathematics Tournament of the Towns is the source of the challenge from Anna.

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The 54th International Mathematical Olympiad, Santa Marta, Colombia, 2013

The 54th International Mathematical Olympiad (IMO) was held July 18–28 in Santa Marta, Colombia. This is the second country in South America to have hosted an IMO.[1] A total of 527 high school students from 97 countries participated.

Each country sends a team of up to six students, a Team Leader and a Deputy Team Leader. At the IMO the Team Leaders, as an international collective, form what is called the *Jury*. This Jury was chaired by María Falk de Losada.[2]

The first major task facing the Jury is to set the two competition papers. During this period the Leaders and their observers are trusted to keep all information about the contest problems completely confidential. The local Problem Selection Committee had already shortlisted 27 problems from 149 problem proposals submitted by 50 of the participating countries from around the world. During the Jury meetings four of the shortlisted problems had to be discarded from consideration due to being too similar to material already in the public domain. A proposal by UK Leader Geoff Smith was tried this year. The proposal stipulated that all four major areas of algebra, combinatorics, geometry and number theory be represented among the two easy and two medium problems. The idea was that since the two difficult problems are usually quite inaccessible to most contestants, then at least the four more accessible problems would provide a balanced contest. In hindsight this worked quite well, and may well be tried again next year. Eventually, the Jury finalised the exam questions and then made translations into all the more than 50 languages required by the contestants.

[1] Argentina hosted the IMO in 1997 and in 2012.

[2] It is noteworthy that three generations of a single family were involved in the successful running of this year's IMO. As mentioned María Falk de Losada, amongst other things, chaired the Jury meetings. Her daughter, Maria Elizabeth Losada, played key roles on the Organising Committee. And Maria Elizabeth's twelve year old daughter, Isabella Mijares, helped out by being a microphone runner for some of the Jury meetings.

The six questions can be described as follows.

1. An easy but novel number theory problem proposed by Japan.
2. A medium combinatorial geometry problem proposed by Australia. This problem required no technical background whatsoever, only a couple of simple original ideas.
3. A difficult classical geometry problem proposed by Russia.
4. A very easy classical geometry problem proposed by South Africa.
5. A medium functional inequality proposed by Bulgaria. One is required to investigate a function that is simultaneously super-additive and sub-multiplicative.
6. This very difficult but most beautiful combinatorial number theory problem was proposed by Russia. It asked one to count the number of permutations of the remainders modulo n in a circle which satisfy a certain arithmetical property. The various solutions to this problem basically amount to a short professional mathematical paper.

These six questions were posed in two exam papers held on Tuesday July 23 and Wednesday July 24. Each paper had three problems. The contestants worked individually. They were allowed $4\frac{1}{2}$ hours per paper to write their attempted proofs. Each problem was scored out of a maximum of seven points.

For many years now there has been an Opening Ceremony prior to the first day of competition. Following the formal speeches there was the parade of the Teams, flanked by Colombian dancers. At the conclusion of the Opening Ceremony the 2013 IMO was declared open.

After the exams the Leaders and their Deputies spent about two days assessing the work of the students from their own countries, guided by marking schemes discussed earlier. A local team of markers called *Coordinators* also assessed the papers. They too were guided by the marking schemes but are allowed some flexibility if, for example, a Leader brought something to their attention in a contestant's exam script not covered by the marking scheme. The Team Leader and Coordinators have to agree on scores for each student of the Leader's country in order to finalise scores.

Question 4 turned out to be very easy as expected. It averaged 5.4 points. Being hard to train for, question 2 mixed things up somewhat. No country achieved a team perfect score for this question. Yet for some students from traditionally weaker countries, this was the only question they could solve. As expected, question 6 was very difficult, averaging just 0.3 points. Only seven students scored full marks on this question.

There were 278 (=52.8%) medals awarded, a little more generous than usual. The distributions[3] being 141 (=26.8%) Bronze, 92 (=17.5%) Silver and 45 (=8.5%) Gold. No student achieved the perfect score of 42. However, two students, Yutao Liu of China and Eunsoo Jee of South Korea, jointly topped the IMO with outstanding scores of 41 points each. The medal cuts were set at 31 for Gold, 24 for Silver and 15 for Bronze. These awards were presented at the Closing Ceremony. Of those who did not get a medal, a further 141 contestants received an Honourable Mention for solving at least one question perfectly.

The 2013 IMO was organized by the Colombian Mathematics Olympiad, along with the generous support of the University Antonio Nariño.

Venues for future IMOs have been secured up to 2018 as follows:

2014	South Africa (Cape Town July, 3–13)		
2015	Thailand	2017	Brazil
2016	Hong Kong	2018	Romania

Much of the statistical information found in this report can also be found at the official website of the IMO. `www.imo-official.org`

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Australian IMO Team Leader
AUSTRALIA

[3]The total number of medals must be approved by the Jury and should not normally exceed half the total number of contestants. The numbers of gold, silver and bronze medals must be approximately in the ratio 1:2:3.

1 IMO Papers

First Day

Problem 1. Prove that for any pair of positive integers k and n, there exist k positive integers $m_1, m_2, \ldots, m_k$ (not necessarily different) such that

$$1 + \frac{2^k - 1}{n} = \left(1 + \frac{1}{m_1}\right)\left(1 + \frac{1}{m_2}\right) \ldots \left(1 + \frac{1}{m_k}\right).$$

Problem 2. A configuration of 4027 points in the plane is called *Colombian* if it consists of 2013 red points and 2014 blue points, and no three of the points of the configuration are collinear. By drawing some lines, the plane is divided into several regions. An arrangement of lines is *good* for a Colombian configuration if the following two conditions are satisfied:

- no line passes through any point of the configuration;
- no region contains points of both colours.

Find the least value of k such that for any Colombian configuration of 4027 points, there is a good arrangement of k lines.

Problem 3. Let the excircle of triangle ABC opposite the vertex A be tangent to the side BC at the point A_1. Define the points B_1 on CA and C_1 on AB analogously, using the excircles opposite B and C, respectively. Suppose that the circumcentre of triangle $A_1B_1C_1$ lies on the circumcircle of triangle ABC. Prove that triangle ABC is right-angled.

The excircle *of triangle ABC opposite the vertex A is the circle that is tangent to the line segment BC, to the ray AB beyond B, and to the ray AC beyond C. The excircles opposite B and C are similarly defined.*

Second Day

Problem 4. Let ABC be an acute-angled triangle with orthocentre H, and let W be a point on the side BC, lying strictly between B and C. The points M and N are the feet of the altitudes from B and C, respectively. Denote by ω_1 the circumcircle of BWN, and let X be the point on ω_1 such that WX is a diameter of ω_1. Analogously, denote by ω_2 the circumcircle of CWM, and let Y be the point on ω_2 such that WY is a diameter of ω_2. Prove that X, Y and H are collinear.

Problem 5. Let $\mathbb{Q}_{>0}$ be the set of positive rational numbers. Let $f\colon \mathbb{Q}_{>0} \to \mathbb{R}$ be a function satisfying the following three conditions:

(i) for all $x, y \in \mathbb{Q}_{>0}$, we have $f(x)f(y) \geq f(xy)$;

(ii) for all $x, y \in \mathbb{Q}_{>0}$, we have $f(x+y) \geq f(x) + f(y)$;

(iii) there exists a rational number $a > 1$ such that $f(a) = a$.

Prove that $f(x) = x$ for all $x \in \mathbb{Q}_{>0}$.

Problem 6. Let $n \geq 3$ be an integer, and consider a circle with $n+1$ equally spaced points marked on it. Consider all labellings of these points with the numbers $0, 1, \ldots, n$ such that each label is used exactly once; two such labellings are considered to be the same if one can be obtained from the other by a rotation of the circle. A labelling is called *beautiful* if, for any four labels $a < b < c < d$ with $a + d = b + c$, the chord joining the points labelled a and d does not intersect the chord joining the points labelled b and c.

Let M be the number of beautiful labellings, and let N be the number of ordered pairs (x, y) of positive integers such that $x + y \leq n$ and $\gcd(x, y) = 1$. Prove that

$$M = N + 1.$$

2 Results

Mark Distribution by Question						
Mark	Q1	Q2	Q3	Q4	Q5	Q6
0	118	229	438	82	235	481
1	96	32	10	16	84	15
2	9	65	15	14	33	6
3	6	33	16	14	11	6
4	14	22	0	2	0	2
5	3	12	3	5	10	6
6	5	16	4	9	19	4
7	276	118	41	385	135	7
Total	527	527	527	527	527	527
Mean	4.1	2.5	0.8	5.4	2.5	0.3

Some Country Scores		
Rank	Country	Score
1	China	208
2	South Korea	204
3	U.S.A	190
4	Russia	187
5	North Korea	184
6	Singapore	182
7	Vietnam	180
8	Taiwan	176
9	U.K.	171
10	Iran	168
11	Canada	163
11	Japan	163
13	Israel	161
13	Thailand	161
15	Australia	148

Some Country Scores		
Rank	Country	Score
16	Ukraine	146
17	Mexico	139
17	Turkey	139
19	Indonesia	138
20	Italy	137
21	France	136
22	Belarus	134
22	Hungary	134
22	Romania	134
25	Netherlands	133
26	Peru	132
27	Germany	127
28	Brazil	124
29	India	122
30	Croatia	119

The medal cut-offs were set at 31 for gold, 24 for silver and 15 for bronze.

Distribution of Awards at the 2013 IMO					
Country	Total	Gold	Silver	Bronze	H.M.
Argentina	46	0	0	1	1
Armenia	88	0	1	1	4
Australia	148	1	2	3	0
Austria	77	0	1	1	2
Azerbaijan	73	0	0	2	3
Bangladesh	60	0	0	3	1
Belarus	134	1	2	3	0
Belgium	82	0	1	2	2
Bolivia	5	0	0	0	0
Bosnia & Herzegovina	56	0	0	1	4
Brazil	124	0	3	1	2
Bulgaria	101	0	1	2	3
Canada	163	2	2	2	0
Chile	35	0	0	1	2
China	208	5	1	0	0
Colombia	77	0	0	2	2
Costa Rica	59	0	0	1	5
Croatia	119	2	0	2	2
Cuba	11	0	0	0	1
Cyprus	52	0	0	1	3
Czech Republic	108	1	0	3	2
Denmark	31	0	0	0	3
Ecuador	45	0	0	1	2
El Salvador	14	0	0	0	2
Estonia	67	0	0	2	3
Finland	46	0	1	0	0
France	136	0	2	4	0
Georgia	75	0	0	2	4
Germany	127	0	2	4	0
Greece	101	0	2	1	3
Honduras	0	0	0	0	0
Hong Kong	117	0	1	5	0

Distribution of Awards at the 2013 IMO

Country	Total	Gold	Silver	Bronze	H.M.
Hungary	134	0	2	4	0
Iceland	27	0	0	0	2
India	122	0	2	3	0
Indonesia	138	1	1	4	0
Iran	168	2	3	1	0
Ireland	33	0	0	0	4
Israel	161	1	3	2	0
Italy	137	1	2	1	2
Japan	163	0	6	0	0
Kazakhstan	116	0	1	4	1
Kosovo	25	0	0	0	3
Kyrgyzstan	36	0	0	1	2
Latvia	47	0	0	1	3
Liechtenstein	15	0	0	1	0
Lithuania	78	0	0	3	3
Luxembourg	25	0	0	1	0
Macedonia (FYR)	34	0	0	1	1
Malaysia	117	0	2	3	0
Mexico	139	0	3	3	0
Moldova	71	0	0	2	4
Mongolia	84	0	0	3	3
Montenegro	1	0	0	0	0
Morocco	17	0	0	0	1
Netherlands	133	0	2	3	1
New Zealand	77	0	0	2	3
Nicaragua	22	0	0	0	3
Nigeria	18	0	0	1	0
North Korea	184	2	4	0	0
Norway	36	0	0	1	1
Pakistan	25	0	0	0	3
Panama	19	0	0	0	2
Paraguay	38	0	0	2	1
Peru	132	0	3	2	1

Distribution of Awards at the 2013 IMO					
Country	Total	Gold	Silver	Bronze	H.M.
Philippines	72	0	0	3	2
Poland	79	0	1	1	3
Portugal	111	1	0	4	1
Puerto Rico	14	0	0	0	1
Romania	134	0	3	3	0
Russia	187	4	2	0	0
Saudi Arabia	84	0	0	4	0
Serbia	112	1	1	2	2
Singapore	182	1	5	0	0
Slovakia	112	0	1	3	2
Slovenia	34	0	0	0	4
South Africa	64	0	0	2	3
South Korea	204	5	1	0	0
Spain	63	0	0	2	3
Sri Lanka	65	0	0	1	4
Sweden	62	0	1	1	2
Switzerland	88	0	0	3	2
Syria	14	0	0	0	1
Taiwan	176	2	4	0	0
Tajikistan	65	0	0	1	4
Thailand	161	1	4	1	0
Trinidad & Tobago	16	0	0	0	1
Tunisia	49	0	0	1	3
Turkey	139	1	2	3	0
Turkmenistan	78	0	0	4	1
Uganda	1	0	0	0	0
Ukraine	146	1	3	1	1
United Kingdom	171	2	3	1	0
United States of America	190	4	2	0	0
Uruguay	7	0	0	0	0
Venezuela	9	0	0	0	1
Vietnam	180	3	3	0	0
Total (97 teams, 527 contestants)		**45**	**92**	**141**	**141**

N.B. Not all countries sent a full team of six students.

Tournament of Towns
Selected Problems, Spring 2013

Andy Liu

Remark:
The heading in my last column should have been "Selected Problems from Fall 2012" rather than "Selected Problems from Spring 2012".

1. The weight of each of 11 objects is a distinct integral number of grams. Whenever two subsets of these objects of unequal sizes are weighed against each other, the subset with the larger number of objects is always the heavier one. Prove that the weight of one of the objects exceeds 35 grams.

   Solution:
   Let S be the total weight of the lightest 6 objects, and let T be the total weight of the other 5 objects. Let k be the weight of the heaviest object among the lightest 6. Then

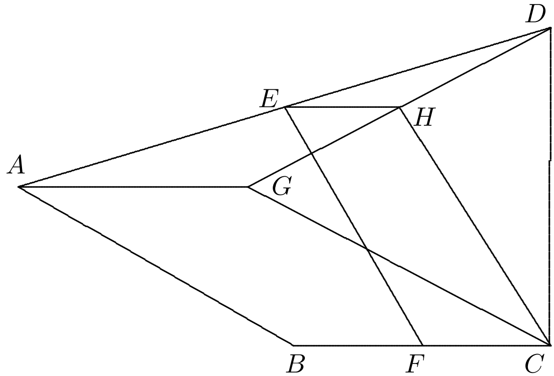
$$\begin{aligned}&(k-5)+(k-4)+(k-3)+(k-2)+(k-1)+k\\ \geq\ & S\\ >\ & T\\ \geq\ & (k+1)+(k+2)+(k+3)+(k+4)+(k+5).\end{aligned}$$

   Hence $k > 30$ and the weight of the heaviest object is at least $k + 5 > 35$.

2. In the quadrilateral $ABCD$, $AB = CD$, $\angle B = 150°$ and $\angle C = 90°$. Determine the angle between BC and the line joining the midpoints of AD and BC.

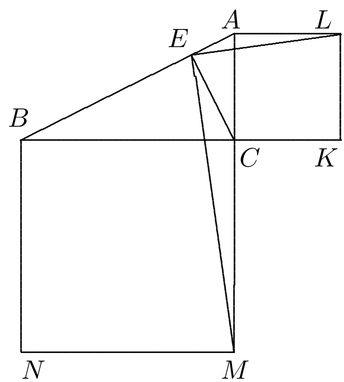
   Solution:
   Construct the parallelogram $ABCG$. CDG is an equilateral triangle since $CG = BA = CD$ and $\angle GCD = 90° - \angle GCB = 90° - (180° - 150°) = 60°$. Let E, F and H be the respective midpoints of AD, BC and DG. Then EH is parallel to AG and thus

to BC. Moreover, $EH = \frac{AG}{2} = \frac{BC}{2} = FC$. It follows that $EHFC$ is also a parallelogram. The acute angle between EF and BC is equal to $\angle BCH = \angle BCG + \angle GCH = (180° - 150°) + \frac{60°}{2} = 60°$.



3. In triangle ABC, $\angle C = 90°$ and CE is an altitude. Squares $ACKL$ and $BCMN$ are constructed outside ABC. Prove that $\angle LEM = 90°$.

Solution:
We have $\angle CAE = 90° - \angle ACE = \angle BCE$. Hence triangles CAE and BCE are similar, so that $\frac{AE}{CE} = \frac{AC}{BC} = \frac{AL}{CM}$. Since $\angle LAE = 90° + \angle CAE = 90° + \angle BCE = \angle MCE$, triangles LAE and MCE are also similar, so that $\angle LEA = \angle MEC$. Hence $\angle LEM = \angle MEA - \angle LEA = \angle MEA - \angle MEC = \angle CEA = 90°$.



4. Each of 100 stones with distinct weights has a label showing its weight. Is it always possible to rearrange the labels so that the total weight of any group of less than 100 stones is not equal to the sum of the corresponding labels?

Solution:
Let the correct weight of the i-th object be a_i with $a_1 < a_2 < \cdots < a_{100}$. Let the false label of the i-th object indicate that its weight is b_i. A possible rearrangement of the labels is defined by $b_i = a_{i+1}$ for $1 \leq i \leq 99$ and $b_{100} = a_1$. Any subset of objects not including the heaviest one will have total weight less than the sum of their labels. Any subset of objects including the heaviest one but not all 100 will have total weight greater than the sum of their labels.

5. Ten boys and ten girls are standing in a row. Each boy counts the number of girls to his left, and each girl counts the number of boys to his right. Prove that the total number counted by the boys is equal to the total number counted by the girls.

Solution:
The only countings arise from a girl sitting to the right of a boy, in which case they count each other. Hence the total number counted by the girls is equal to the total number counted by the boys. The number of girls does not have to be equal to the number of boys.

6. On a circle are 1000 non-zero numbers painted alternately black and white. Each black number is the sum of its two neighbours while each white number is the product of its two neighbours. What are the possible values of the sum of these 1000 numbers?

Solution:
Let a, b, c, d, e, f, g, h, i and j be ten numbers in succession along the circle, with a black. Then $c = \frac{b}{a}$, $d = c - b = \frac{b(1-a)}{a}$, $e = \frac{d}{c} = 1 - a$, $f = e - d = \frac{(1-a)(a-b)}{a}$, $g = \frac{f}{e} = \frac{a-b}{a}$, $h = g - f = a - b$, $i = \frac{h}{g} = a$ and $j = i - h = b$. It follows that the pattern will repeat in a cycle of length 8. Now $a + b + c + d + e + f + g + h = a + b + (1-a) + (a-b) + \frac{b+b(1-a)+(1-a)(a-b)+(a-b)}{a}$, which simplifies to 3. Since $1000 = 8 \times 125$, the sum of the 1000 numbers is $3 \times 125 = 375$.

7. In a school with more girls than boys, there is only one ping-pong table. A boy is playing against a girl while all remaining students form a single line. At the end of a game, the student at the head of the line replaces the student of the same gender who has just finished playing, and the replaced student goes to the end of the line. Prove that eventually, every boy has played every girl.

Solution by Central Jury:
Let there be m girls and n boys with $m > n$. Number the girls $G_1, G_2, \ldots, G_m$ and the boys $B_1, B_2, \ldots, B_n$ so that initially G_1 is playing B_1, G_i is ahead of G_{i+1} and B_j is ahead of B_{j+1} in the line. Note that the players of the same gender, including the one who is playing, are always in the same cyclic order. Two players of opposite gender may trade places, but only just after they have played each other. Group the games into cycles each consisting of $m+n-2$ games. During the first cycle, $m+n-2$ players will go to the end of the line, all except G_m and B_n. Therefore these players start the second cycle. Similarly G_{m-1} and B_{n-1} start the third cycle, and so on. Thus each girl goes to the end of the line exactly $(m-1)$ times during m cycles and exactly $(m-1)n$ times during mn cycles. Similarly each boy goes to the end of the line exactly $m(n-1)$ times during mn cycles. Now $n(m-1) - m(n-1) = m - n \geq 1$. Assume that some girl G has never played against some boy B. During $2mn$ cycles G goes to the end of the line at least 2 more times than B. Therefore there are two consecutive exits of G such that between them there were no exits of B. Between these to exits of G, B remained in the line. However after the first exit of G, B is in front of G in the line, but when G enters the table again, B is behind her. This is a contradiction.

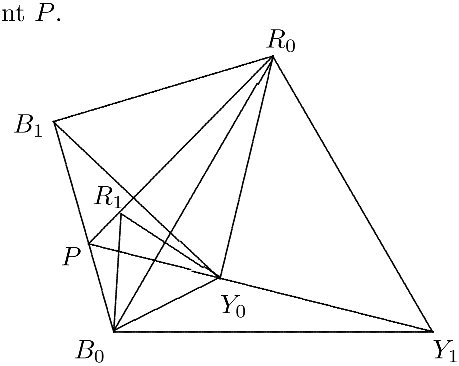
8. A boy and a girl are sitting at opposite ends of a long bench. One at a time, twenty other children take seats on the bench. If a boy takes a seat between two girls or if a girl takes a seat between two boys, he or she is said to be brave. At the end, the boys and girls were sitting alternately. What is the number of brave children?

Solution:
A space between two children of the same gender is called a brave spot. If a boy sits between two girls, he is brave. If he sits between

a boy and a girl, he creates a brave spot. If he sits between two boys, he changes one brave spot into two. Hence each boy is either brave or creates a brave spot. By symmetry, the same can be said about each girl. At the beginning, there are no brave spots, which is also the case at the end. Hence of the twenty children who come later, half of them create a brave spot and the other half, namely ten of them, are brave.

9. Initially, only three points on the plane are painted: one red, one yellow and one blue. In each step, we choose two points of different colours. A point is painted in the third colour so that an equilateral triangle with vertices painted red, yellow and blue in clockwise order is formed with the two chosen points. Note that a painted point may be painted again, and it retains all its colours. Prove that after any number of moves, all points of the same colour lie on a straight line.

Solution:
An equilateral triangle with one vertex in each of red, yellow and blue and in the correct orientation is said to be *good*. Let R_0, Y_0 and B_0 be the initial red, yellow and blue points. We may assume that they do not form a good triangle. Then $R_0Y_0B_1, Y_0B_0R_1$ and $B_0R_0Y_1$ are distinct good triangles. We claim that R_0R_1, Y_0Y_1 and B_0B_1 make $60°$ angles with one another, and are concurrent at some point P.



A $60°$-rotation about B_0 brings R_0 to Y_1 and R_1 to Y_0. Hence the angle between R_0R_1 and Y_0Y_1 is $60°$. Similarly, the angle between B_0B_1 and either R_0R_1 or Y_0Y_1 is also $60°$. Let P be the point of R_0R_1 and Y_0Y_1. Then $\angle R_0PY_1 = 60° = \angle R_0B_0Y_1$. Hence $B_0PR_0Y_1$ is a cyclic quadrilateral, so that $\angle B_0PY_1 = \angle B_0R_0Y_1 = 60°$. Since $\angle R_0PY_0 = 60° = \angle R_0B_1Y_0$, $R_0Y_0PB_1$ is also a cyclic quadrilateral. Hence $\angle B_1PR_0 = 60°$. Now $\angle B_1PR_0 + \angle R_0PY_1 + \angle Y_1PB_0 = 180°$. It follows that P also lies on B_0B_1.

We now prove that all subsequent red points lie on R_0R_1, all subsequent yellow points lie on Y_0Y_1, and all subsequent blue points lie on B_0B_1. Without loss of generality, we may take R as a point on R_0R_1 and Y as a point on Y_0Y_1. Let B be the point such that RYB is a good triangle. We claim that B must lie on B_0B_1. Note that $\angle RPY$ is either $60°$ or $120°$. In the former case, $RPBY$ is a cyclic quadrilateral. In the latter case, $RPYB$ is a cyclic quadrilateral. In either case, $\angle BPY = 60°$, and the claim is justified.

10. One thousand wizards are standing in a column. Each is wearing one of the hats numbered from 1 to 1001 in some order, one hat not being used. Each wizard can see the number of the hat of any wizard in front of him, but not that of any wizard behind. Starting from the back, each wizard in turn calls out a number from 1 to 1001 so that every other wizard can hear it. Each number can be called out at most once. At the end, a wizard who fails to call out the number on his hat is removed from the Council of Wizards. This procedure is known to the wizards in advance, and they have a chance to discuss strategy. Is there a strategy which can keep in the Council of Wizards

(a) more than 500 of these wizards;

(b) at least 999 of these wizards?

Solution:
We solve only (b). Let the wizards be $W_1, W_2, \ldots, W_{1000}$, and let n_i be the number on the hat of wizard W_i for $1 \le i \le 999$. Starting from the back, each wizard in turn will write down a permutation of $\{1, 2, \ldots, 1001\}$. W_{1000} can see $n_1, n_2, \ldots, n_{999}$, and write them down in order as the first 999 terms. He places the remaining two numbers in the last two places in the order which results in an even permutation. W_{999} can see $n_1, n_2, \ldots, n_{998}$ and can hear

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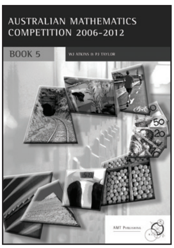

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