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MATHEMATICS COMPETITIONS

JOURNAL OF THE

WORLD FEDERATION OF NATIONAL MATHEMATICS COMPETITIONS

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MATHEMATICS COMPETITIONS VOLUME 26 NUMBER 1 2013

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The aims of the Federation are:

1. ***to promote excellence in, and research associated with, mathematics education through the use of school mathematics competitions;***
2. ***to promote meetings and conferences where persons interested in mathematics contests can exchange and develop ideas for use in their countries;***
3. ***to provide opportunities for the exchanging of information for mathematics education through published material, notably through the Journal of the Federation;***
4. ***to recognize through the WFNMC Awards system persons who have made notable contributions to mathematics education through mathematical challenge around the world;***
5. ***to organize assistance provided by countries with developed systems for competitions for countries attempting to develop competitions;***
6. ***to promote mathematics and to encourage young mathematicians.***

From the President

Dear Fellow Federalists!

I hope you are enjoying happy and productive lives. The 7$^{\text{th}}$ Congress of our World Federation of National Mathematics Competitions is being organized by our Past President Dr Maria de Losada. It will take place on July 21–24 (arrival on July 20) 2014 at the Hotel El Prado in Barranquilla, Colombia, and will include a half-day excursion to the historic Cartagena. I hope to see you all there. Full details should appear in the next issue of the journal. So, plan your travel and start your creative engines!

During our previous congresses and our sections at ICME, the Program Committee was not always utilized. I know that first hand as a long term member of the committee and its past chair. I hope this situation will change, and Dr Kiril Bankov's Program Committee will be actively involved in creating academic structure of our 2014 Congress.

I believe that our 2010 Riga Congress had a sound basic structure, built around the following four sections:

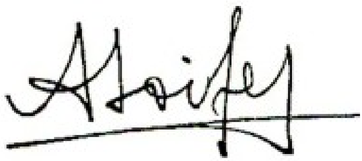
1. Competitions around the World
2. Creating Competition Problems and Problem Solving
3. Work with Students and Teachers
4. Building Bridges between Research and Competition Problems

This can serve as a foundation for the 2014 Congress and beyond. In addition, we should have a number of plenary lectures, workshops, and discussions on issues important to us. Another session was successful in Riga: a session where we presented our favorite problems and solutions, which resulted in a small book. I hope such a session will take place in 2014 and more participants will submit their problems, thus creating a larger book than the one published in Riga.

There is no time for complacency. We can and ought to pull our efforts together and sustain a vibrant World Federation. It means work, hard work, but work we love. As the brave American journalist Edward R. Murrow (the one who in 1954 challenged U.S. Senator Joseph McCarthy on live television) said on October 15, 1958,

> "Our history will be what we make it. If we go on as we are, then history will take its revenge, and retribution will not limp in catching up with us."

Best wishes,



Alexander Soifer
President of WFNMC

From the Editor

Welcome to *Mathematics Competitions* Vol. 26, No. 1.

First of all I would like to thank again the Australian Mathematics Trust for continued support, without which each issue (note the new cover) of the journal could not be published, and in particular Heather Sommariva, Bernadette Webster and Pavel Calábek for their assistance in the preparation of this issue.

In July 2012 at the WFNMC miniconference in Seoul the new MC's Editorial Board was formed. It comprises *Waldemar Pompe* (Poland), *Sergey Dorichenko* (Russia), *Alexander Soifer* and *Don Barry* (both USA).

Submission of articles:

The journal *Mathematics Competitions* is interested in receiving articles dealing with mathematics competitions, not only at national and international level, but also at regional and primary school level. There are many readers in different countries interested in these different levels of competitions.

- The journal traditionally contains many different kinds of articles, including reports, analyses of competition problems and the presentation of interesting mathematics arising from competition problems. Potential authors are encouraged to submit articles of all kinds.

- To maintain and improve the quality of the journal and its usefulness to those involved in mathematics competitions, all articles are subject to review and comment by one or more competent referees. The precise criteria used will depend on the type of article, but can be summarised by saying that an article accepted must be correct and appropriate, the content accurate and interesting, and, where the focus is mathematical, the mathematics fresh and well presented. This editorial and refereeing process is designed to help improve those articles which deserve to be published.

At the outset, the most important thing is that if you have anything to contribute on any aspect of mathematics competitions at any level, local, regional or national, we would welcome your contribution.

Articles should be submitted in English, with a black and white photograph and a short profile of the author. Alternatively, the article can be submitted on an IBM PC compatible disk or a Macintosh disk. We prefere LaTeX or TeX format of contributions, but any text file will be helpful.

Articles, and correspondence, can also be forwarded to the editor by mail to

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June 2013

What "Problem Solving" Ought to Mean and how Combinatorial Geometry Answers this Question: Divertismento in Nine Movements

Alexander Soifer

Born and educated in Moscow, Alexander Soifer has for 33 years been a Professor at the University of Colorado, teaching math, and art and film history. He has published over 200 articles, and a good number of books. In the past 3 years, 6 of his books have appeared in Springer: The Mathematical Coloring Book: Mathematics of Coloring and the Colorful Life of Its Creators*;* Mathematics as Problem Solving: How Does One Cut a Triangle?*;* Geometric Etudes in Combinatorial Mathematics; Ramsey Theory Yesterday, Today, and Tomorrow*; and* Colorado Mathematical Olympiad and Further Explorations*. He has founded and for 29 years ran the Colorado Mathematical Olympiad. Soifer has also served on the Soviet Union Math Olympiad (1970–1973) and USA Math Olympiad (1996–2005).*

> *The goal of mathematics education should be showing in a classroom what mathematics is and what mathematicians do. This can be achieved not by teaching but rather by creating an atmosphere in which students learn mathematics by doing it. As in "real" mathematics, this can be done by solving problems that require not just deductive reasoning, but also experiments, construction of examples, and synthesis in a single problem of ideas from various branches of mathematics. My recent five Springer books provide just right illustrations of these ideas, fragments of "live" mathematics.*

> *Give a man a fish, and you will feed him for a day.*
> *Teach a man how to fish, and you will feed him for a lifetime.*
>
> Lǎozǐ (VI century BC)

1 Lǎozǐ and a Problem Solving Approach to Life

The great Chinese sage *Lǎozǐ* proposes to teach a man to fish as a method of solving the problem of survival. This does go much further than giving a man a fish. However, is it good enough in today's rapidly changing world? We will come back to *Lǎozǐ* at the end of this essay.

2 The Principal Goal of Mathematics Education

What is the main goal of mathematics education?

Is it passing standardized three-letter tests, such as SAT, ACT, GRE, KGB, CIA (well, the two latter triples are from a different opera :-)).

Is it "teaching to the test," as USA President George W. Bush believed?

Most would agree that problem solving is the goal. Fair enough, but there is no agreement on the answer to a natural question: Just what is *problem solving*?

A typical secondary school problem looks like $A \Rightarrow B$, i.e., given A prove B by using theorem C. In real life, no one gives a research mathematician a B, it is discovered by intuition and is based on experimentation. And of course, no one has a C since nobody knows what would work to solve the problem which is not yet solved: a research mathematician is a pioneer, moving along an untraveled road!

And so, we ought to bring the school mathematics as close as possible to research mathematics. We ought to let our students experiment in our classroom-laboratory. We ought to let them develop intuition and use it to come up with conjectures B. And we ought to let our students find that tool, theorem C that does the job of deductive proving.

In my opinion, the real goal of mathematical education is to demonstrate in the classroom *what mathematics is*, and *what mathematicians do*.

3 Experiment in Mathematics

I have a good news and bad news for you. The bad news is, it is impossible to "teach" problem solving. The good news is, we can create an environment in which students learn problem solving by solving problems on their own, with our gentle guidance. And I do not single out mathematics—all fields of human endeavour are about problem solving. Alright, but what kind of problems should we offer our students?

First of all, we ought to set up a mathematical laboratory, where students conduct mathematical experiments, develop *inductive reasoning* and create conjectures. The following classic example comes from [2].

Partitioning the Plane. In how many regions do n straight lines in general position partition the plane?

We say that several straight lines on the plane are in general position if no two lines are parallel and no three lines have a point in common.

Solution. Let us denote by $S(n)$ the number of regions into which n straight lines in general position partition the plane. Now let us experiment: we draw one line on the plane and get $S(1) = 2$; we add another line to see that $S(2) = 4$; we add one more line to shows that $S(3) = 7$; and one more line demonstrates that $S(4) = 11$ (please see Figures 1 and 2).

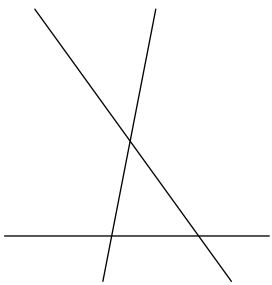

Figure 1. $S(3) = 7$

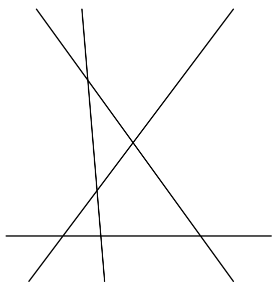

Figure 2. $S(4) = 11$

Let us put the data in a table:

<table>
<tr><th>Number of Lines N</th><th>$S(n)$</th><th>Difference $S(n)-S(n-1)$</th></tr>
<tr><td>1</td><td>2</td><td rowspan="2">2</td></tr>
<tr><td rowspan="2">2</td><td rowspan="2">4</td></tr>
<tr><td rowspan="2">3</td></tr>
<tr><td rowspan="2">3</td><td rowspan="2">7</td></tr>
<tr><td rowspan="2">4</td></tr>
<tr><td>4</td><td>11</td></tr>
</table>

We notice that

$$S(n) = S(n-1) + n.$$

This recursive formula strikingly resembles the growth in the well-known equality $1 + 2 + \cdots + n = \frac{1}{2}n(n+1)$, i.e., if $S_1(n) = 1 + 2 + \cdots + n$, we would get the same recursive relationship as in our original problem: $S_1(n) = S_1(n-1) + n$.

So let us check the hypothesis $S(n) = \frac{1}{2}n(n+1)$:

n	$S(n)$	$\frac{1}{2}n(n+1)$
1	2	1
2	4	3
3	7	6
4	11	10

Our hypothesis does not work, but we can now see from the table above that $S(n)$ and $\frac{1}{2}n(n+1)$ differ only by 1, and apparently always by 1! Thus, we can now conjecture:

$$S(n) = \frac{1}{2}n(n+1) + 1.$$

All there is left is to prove our conjecture, say, by mathematical induction.

As we already know, our conjecture holds for $n = 1$.

Assume that it is true for $n = k$, i.e., k straight lines in general position partition the plane into $S(k) = \frac{1}{2}k(k+1) + 1$ regions.

Let now $n = k + 1$, i.e., let $k + 1$ straight lines in general position be given in the plane. If we remove one of the lines L, then by the inductive assumption the remaining k lines would partition the plane into $S(k) = \frac{1}{2}k(k+1) + 1$ regions. Since we have $k+1$ lines in general position, the remaining k lines all intersect the line L; moreover, they intersect L in k different points $a_1, a_2, \ldots, a_k$ (see Figure 3).

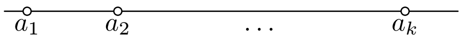

Figure 3

These k points split the line L into $k+1$ intervals. Each of these intervals splits one region of the partition of the plane by k lines into two new regions, i.e. instead of $k+1$ old regions we get $2(k+1)$ new regions, i.e.,

$$S(k+l) = S(k) + (k+l),$$

therefore,

$$S(k+l) = \frac{1}{2}k(k+1) + 1 + (k+l) = \frac{1}{2}(k+1)(k+2) + 1.$$

In other words, our conjecture holds for $n = k+1$. Thus, n straight lines in general position partition the plane into $\frac{1}{2}n(n+1) + 1$ regions.

4 Construction of Examples in Mathematics

Construction of examples and counterexamples plays a major role in mathematics, amounting to circa 50 % of its results. In fact, the great Russian mathematician Israel M. Gelfand once said,

Theories come and go; examples live forever.

Yet, practically the entire school mathematics consists of analytical proofs. In order to bring instruction closer to "the real mix" we ought to include construction of examples and counterexamples in education. Let me share one example, where a construction solves the problem [5].

Positive^2 (*A. Soifer*, 2001). Is there a way to fill a 2001×2001 square table T with pluses and minuses, one sign per cell of T, such that no series of interchanging all signs in any 1000×1000 or 1001×1001 square of the table can fill T with all pluses?

Solution. Having created this problem and a solution for the 2001 Colorado Mathematical Olympiad, I felt that another solution was possible using an invariant, but failed to find it. Two days after the Olympiad, on April 22, 2001, the past double-winner of the Olympiad Matt Kahle, now Professor at Ohio State University, found the solution that eluded me. It is concise and beautiful.

Define (please see Figure 4)

$$\Phi = \{\text{the set of all cells of } T\text{, except those in the middle row}\}.$$

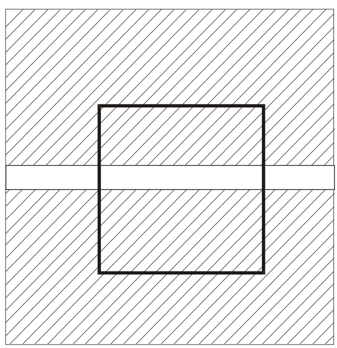

Figure 4

Observe that no matter where a 1000×1000 square S is placed in the table T, it intersects Φ in an even number of cells, because there are 1000 equal columns in S. Observe also that no matter where a 1001×1001 square S' is placed in T, it also intersects Φ in an even number of unit squares, because there are 1000 equal rows in S' (one row is always missing, since the middle row is omitted in S.)

We can now create the required assignment of signs in T that cannot be converted into all pluses. Let Φ have any assignment with an *odd* number of + signs, and the missing in Φ middle row be assigned signs in any way. No series of operations can change the parity of the number of pluses in Φ, and thus no series of allowed operations can create all pluses in Φ.

5 Method & Anti-Method

Tiling with Dominoes. (Method). Can a chessboard with two diagonally opposite squares missing, be tiled by dominoes (Figure 5)?

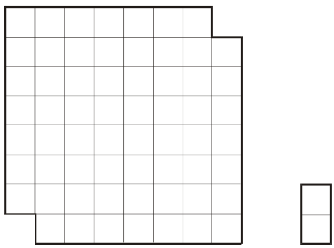

Figure 5

Solution. Color the board in a chessboard fashion (Figure 6). No matter where a domino is placed on the board, vertically or horizontally, it would cover one black and one white square.

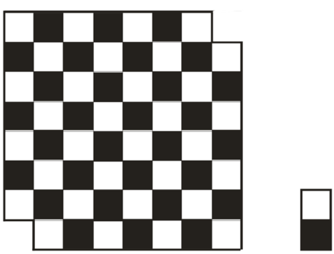

Figure 6

Thus, it is necessary for tileability to have equal numbers of black and white squares in the board—but they are not equal in our truncated board. Therefore, the required tiling does not exist.

It is certainly impressive and unforgettable to see for the first time how coloring can solve a mathematical problem, as we have witnessed here. However, I noticed that once a student learns this coloring idea, he always resorts to it when a chessboard and dominoes are present in the problem. This is why I created the following "Anti-Method" Problem and used it in the Colorado Mathematical Olympiad [5].

The Tiling Game (Anti-Method, *A. Soifer*, 1989). Mark and Julia are playing the following tiling game on a 1998×1999 board. They in turn are putting 1×1 square tiles on the board. After each of them made exactly 100 moves (and thus they covered 200 squares of the board) a winner is determined as follows: Julia wins if the tiling of the board can be completed with dominoes. Otherwise Mark wins. (Dominoes are 1×2 rectangles, which cover exactly two squares of the board.) Can you find a strategy for one of the players allowing him to win regardless of what the moves of the other player may be? You cannot? Let me help you: Mark goes first!

Solution. Julia (i.e., the second player) has a strategy that allows her to win regardless of what Mark's moves may be. All she needs is a bit of home preparation: Julia prepares a tiling template showing one particular way, call it T, of tiling the whole 1998×1999 chessboard. Figure 7 shows one such tiling template T for a 8×13 chessboard.

The strategy for Julia is now clear. As soon as Mark puts a 1×1 tile M on the board, Julia puts her template T on the board to determine which domino of the template T contains Mark's tile M. Then she puts her 1×1 tile J to cover the second square of the same domino (Figure 8).

6 Synthesis & Combinatorial Geometry

School mathematics consists predominantly of problems with single-idea solutions, found by analysis. We ought to introduce a sense of *mathemat-*

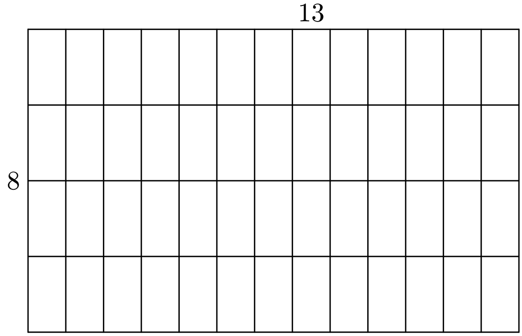

Figure 7. Tiling template for a 8×13 chessboard

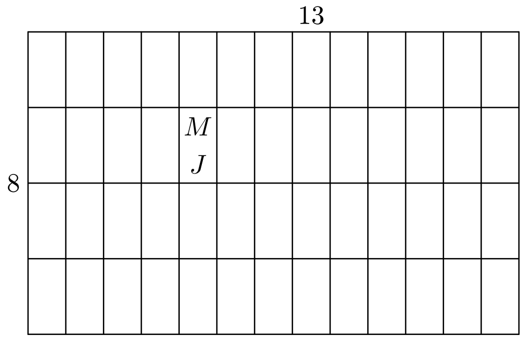

Figure 8. Winning Strategy

ical reality in the classroom by presenting *synthesis*, by offering problems that require for their solution ideas from a number of mathematical disciplines: geometry, algebra, number theory, trigonometry, linear algebra, etc.

And here comes *Combinatorial Geometry!* It offers an abundance of problems that sound like a "regular" secondary school geometry, but require for their solutions synthesis of ideas from geometry, algebra number theory, trigonometry, ideas of analysis, etc. See for example [3]; [4]; and [1].

Moreover, combinatorial geometry offers us open-ended problems. And it offers problems that any geometry student can understand, and yet no one has yet solved! Let us stop this discrimination of our students based on their young age, and allow them to touch and smell, and work on real mathematics and its unsolved problems. They may find a partial advance into solutions; they may settle some open problems completely. And they will then know the answer to what ought to become the fundamental questions of mathematical education:

What is Mathematics?
What do mathematicians do?

In fact, I would opine that every discipline is about problem solving. And so the main goal of every discipline ought to be to enable students to learn *how to think within the discipline*, *how to solve problems of the discipline*, and finally *what that discipline is*, and *what the professionals within the discipline do*!

And more generally, we can ask, what is life about? I believe that

Life itself is about overcoming difficulties, i.e., solving problems.

And mathematics to sciences does what gymnastics does to sports:

Mathematics is gymnastics of the mind.

Doing mathematics develops a universal approach to problem solving and intuition that go a long way in preparing our students for solving problems, no matter what kind of problems they will face in their lives.

7 Open Ended and Open Problems

As a junior at the university, I approached my supervisor Professor Leonid Yakovlevich Kulikov with an open problem I liked (he was my supervisor ever since my freshman year). He replied, “Learn first, the time will come later to enter research.” He meant well, but politically speaking, this was discrimination based on my age. Seeing my disappointment, Kulikov continued, “It does not look like I can stop you from doing research. Alright, whatever results you obtain on this open problem, I will count them as your course work." Soon I received first research results, and my life in mathematics began.

We ought to allow our students to learn what mathematicians do by offering them not just unrelated to each other exercises but rather series of problems leading to a deeper and deeper understanding. And we ought to let students "touch" unsolved problems of mathematics, give them a taste of the unknown, a taste of adventure and discovery. And combinatorial geometry serves these goals very well, providing us with easy-to-understand, hard-to-solve—or even unsolved—problems!

I will formulate here two examples. The volume of this essay does not allow including the solutions. You can find them in my five recent Springer books, listed in References.

Points in a Triangle [3] Out of any n points in or on the boundary of a triangle of area 1, there are 3 points that form a triangle of area at least $\frac{1}{4}$.

a) Prove this statement for $n = 9$.

b) Prove this statement for $n = 7$.

c) Prove this statement for $n = 5$.

d) Show that the statement is not true for $n = 4$, thus making $n = 5$ best possible.

Chromatic Number of the Plane [1] No matter how the plane is colored in n colors, there are two points of the same color distance 1 apart.

a) Prove this statement for $n = 2$.

b) Prove this statement for $n = 3$.

c) Disprove this statement for $n = 7$.

d) The answer for $n = 4$, 5, and 6 is unknown to man—this is a forefront of mathematics!

8 Beauty in Mathematics

I suspect you all have encountered examples of beauty of the mathematical kind. And so, I will share just one example, a problem that my high school friend and I created when we were fellow graduate students in Moscow. It was published in *Kvant* magazine, and we were even handsomely compensated!

Forty-One Rooks (*A. Soifer and S. Slobodnik*, 1973). Forty-one rooks are placed on a 10×10 chessboard. Prove that you can choose five of them that do not attack each other. (We say that two rooks "attack" each other if they are in the same row or column of the chessboard.)

Solution, first found during the 1984 Olympiad (!) by Russel Shaffer; the idea of using symmetry of all colors by gluing a cylinder, belongs to my university student Bob Wood.

Let us make a cylinder out of the chessboard by gluing together two opposite sides of the board, and color the cylinder diagonally in 10 colors (Figure 9).

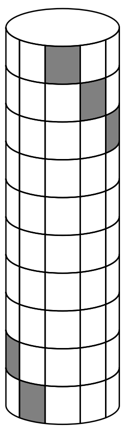

Figure 9. One out of the ten one-color diagonals is shown

Now we have $41 = 4 \times 10 + 1$ pigeons (rooks) in 10 pigeonholes (one-color diagonals), therefore, by the Pigeonhole Principle, there is at least one hole that contains at least 5 pigeons. But the 5 rooks located on the same one-color diagonal do not attack each other!

9 Returning to Lăozĭ 2600 Years Later

On September 5, 2011, I wrote a letter to the Editor of *The New York Times* precisely on the debate taking place between two most popular approaches to mathematical education. Let me present this letter here in its entirety:

> In "In Classroom of Future, Stagnant Scores" (September 4, 2011, p. 1), Matt Richtel discusses two approaches to mathematical education, "embrace of technology" vs. "back to the basics." Should we not first ask, what are the goals of education? To paraphrase the great Chinese thinker *Lăozĭ*, I would say: *Give student skills, and you will feed him in the short run. Let student learn ideas, solve problems, and you will feed him for a lifetime.*
>
> In the new "embrace of technology" approach I support taking teacher off the pedestal of a lecturer: one cannot teach mathematics, or anything for that matter. Students can learn math, and anything else, only by doing it, with a gentle guidance of the teacher. Technology in the classroom more often than not treats students like robots, and preprograms them with skills of today. But technology nowadays changes rapidly, as does the societal demands for particular skills. A student who has learned critical thinking and problem solving will easier adopt to the rapidly changing world.
>
> "Back to the basics" is also not the best solution, for it emphasizes mind numbing drill, and it also treats students as robots and preprograms them with skills.
>
> The goal in the mathematical classroom ought to be a practical demonstration of what mathematics is and what mathematicians do. Everything, life herself included, is about overcoming difficulties, solving problems.
>
> We ought to abandon standardized multiple choice testing of skills. There are more important things to test. Over the past 29 years, in the Colorado Mathematical Olympiad, we offered middle and high school students 5 problems and 4 hours to think and solve. We "test" not topics, not skills, but creativity and originality of thought.

Providing public education is not just an ethical thing to do—it is a profitable investment. Are there many jobs today for computer-illiterate persons? And yet just one generation ago, computers were a monopoly of researchers, and one generation before that did not exist at all.

Not every education is as good an investment as another. And we ought to go beyond *Lǎozǐ* and his famous lines:

Give a man a fish, and you will feed him for a day.
Teach a man how to fish, and you will feed him for a lifetime.

Not quite, dear Sage. Not in today's day and age. What if there is no more fish? What if the pond has dried out? And your man has only one skill, to fish?

Education ought to introduce students to ideas and how to solve problems no matter what field. A problem solver will not die if the fish disappears in a pond—he'll learn to hunt, to grow veggies, to solve whatever problems life puts in his path.

And so, we will go a long way by putting emphasis not on training skills but on creating atmosphere for developing problem solving abilities and attitudes:

Enable a man to learn how to solve problems,
And you will feed him for a lifetime!

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Mathematics Competitions and Gender Issues: a Case of the Virtual Marathon

Mark Applebaum & Margo Kondratieva & Viktor Freiman

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1 Introduction

In today's technologically enhanced world mathematics competitions become available to more students who are interested in challenging tasks. In this paper we look at gender-related issues pertinent to participation in the Virtual Mathematical Marathon over two years. Our study concentrates on the following questions: what were boys' and girls' participation patterns and how successful were they in online problem-solving competition.

The Virtual Mathematical Marathon (VMM, `http://www8.umoncton.ca/umcm-mmv/index.php`) is an online competition open to everybody who is interested in solving challenging problems over a long period of time. As an extension of the virtual interactive learning community CAMI (Chantier d'Apprentissages Mathématiques Interactifs; Freiman et al., 2009), the Marathon provides Grade 3–9 students with challenging mathematical problems.

Besides CAMI's regular *Problem of the week* activity that was conducted over the school year, we developed a long-term summer competition for young students who may have interest in solving more challenging tasks in a form of competition. A new section became available in summer 2008 and since that time, four summer rounds have been organized. Moreover, support received from the Canadian Natural Sciences and Engineering Research Council (Promoscience Grant, 2009–2011) helped us to develop a bilingual version of the Marathon (French and English) and introduce a winter round since 2010, thus making it a year-round competition. First results of the project based on 2008-2009 participation data were presented at the PME-36 Research Forum (Freiman & Applebaum, 2009) and in a journal article (Freiman &Applebaum, 2011). This paper is based on the work presented in the Topics Study Group 34 on mathematics competitions at the ICME-12 congress while extending our investigation of gender-related issues in virtual environments.

2 Gender-related data on maths competitions: is there an issue?

Several educators express a concern regarding gender difference in mathematics performance and the underrepresentation of women in science, technology, engineering and mathematics (STEM) careers (National Academy of Science, Beyond Bias and Barriers: Finding the potential of women in academic science and engineering, 2006; Hyde et al., 2008). Gender inequity is particularly evident in data related to the number of girls participating in the International Math Olympiad, or the number of female professors in university mathematics and engineering departments (Hyde & Mertz, 2009). There are several ways in which this problem may be addressed.

First, psychologists are looking for gender differences in brain structure, in hormones, in the use of brain hemispheres, nuances of cognitive or behavioural development and consequent spatial and numerical abilities that may predispose males to a greater aptitude and success in mathematics (Halpern, 1997; Moir & Jessel, 1989). However, several findings reported in the literature regarding this matter are not consistent (Spelke, 2005), partly due to the fact that experience alters brain structures and functioning (Halpern, et al., 2007).

Second, detailed measurements of students' achievements in mathematics are being performed by educators at different stages of schooling in an attempt to identify the moment of occurrence and further dynamics of gender gaps in mathematics. Many studies are consistent in their observation that the gender gap becomes more evident as students progress towards higher grades, especially if testing involves advanced topics in mathematics and higher cognitive level items. In contrast to earlier findings, some more current data provide no evidence of a gender difference favouring males emerging in the high school years (Hyde et al., 2008).

Yet another interesting observation is that "achievement gains are insufficient unless the self-beliefs of girls have changed correspondingly" (Lloyd, Walsh & Yailagh, 2005, p. 385). Research that views gender differences through the lenses of the attribution theory (see e.g. Bandura, 1997) suggests that girls tend to attribute their math successes to external factors and to effort and their failures to their own lack of ability (self defeating pattern), whereas boys tend to attribute the causes of their successes to internal factors and their failure to external factors (self-enhancing pattern). Since it is better for an individual to attribute success to ability, rather than to effort, because ability attributions are more strongly related to motivation and skill development (Schunk & Gunn, 1986), these patterns have explained in part girls' poorer achievement(Lloyd et al., 2005).

A report of the American Association of University Women *How Schools Shortchange Girls* (1992) focused on girls being discouraged from studying math and science. The report indicates that "girls receive less attention in the classroom than boys and less encouragement for their efforts. In addition, the study showed that many classrooms created the atmosphere of competition among students. Such an atmosphere played to the strength of boys, who were socialized to compete, but often intimidated girls, who were more often socialized to collaborate." (Williams, 2006, p. 301)

A third way of addressing the gender gap in mathematics is to investigate the influence of socio-cultural factors. According to Von Glaserfeld (1989), the context in which learners find themselves is important in the acquisition of knowledge. First, it was found that parents have greater expectations for sons regarding their mathematical performance that they have for daughters, and this has an influence on the students' results (Leder, 1993). It was also observed that even talented and motivated girls "are not immune to the ill effects of gender bias" (Leedy, LaLonde, & Runk, 2003, p. 290). In this respect it is unfortunate that stereotypes that girls and women lack mathematical ability persist and are widely held by parents and teachers (Hyde et al., 2008). Leedy et al. (2003) studied beliefs held by students participating in regional math competitions, as well as by their parents and teachers. They found that mathematics is still viewed as a male domain by men, while girls and women fail to acknowledge the existence of the bias. They argue that the task of the school is not to ignore or deny differences in learning styles, attitudes and performance but to acknowledge them and use them to develop strategies aimed at providing gender equitable education.

In conclusion, in all three perspectives in research on gender in mathematics—cognitive, instructional, and socio-cultural—care is needed in considering how the data are collected, examined and interpreted because within no approach is there a fully consistent theory that could explain the existing gender difference observed at the higher level of mathematical tasks. As Halpern et al. (2007) point out, "there are no single or simple answers to the complex question about sex difference in mathematics", and all "early experience, biological factors, educational policy, and cultural context" need to be considered when approaching this question.

3 Technology and gender: what patterns emerge in mathematics competitions?

While the previous section summarizes research related to gender issues in mathematics education showing no conclusive findings, similar observations can be drawn from technology-related studies that we will review very briefly. For instance Fogasz (2006) reports that when talking about classroom practices that involve computers as a learning tool, mathematics teachers held gender-based beliefs about their students that the incorporation of technology has more positive effects on males' classroom engagement and on their affective responses, and thus using a technological approach benefits boys' learning to a greater extent.

At the same time Wood, Viskic & Petocz (2003) found no gender differences in the students' use of computers or in their attitudes towards the use of computers. This agrees with ideas expressed by Willams (2006) quoted above, who reviews studies showing that girls are as confident and active as are boys in creating webpages, writing blogs, reading websites, and chatting online, among other activities.

As we mentioned in our earlier publications (Freiman et al., 2009; Freiman & Applebaim, 2009), the internet can be a suitable challenging environment for organizing mathematics competitions and problem solving activities, contributing potentially to the development of mathematical ability and giftedness. In a recent analysis of middle-school students participating in a web-based mathematics competition Carreira et al. (2012) argue that although it cannot be said that by solving problems online, students do better in mathematics, their data provide us with an evidence that the use of technology tends to involve more complex mathematical thinking.

Moreover, the use of technology can be considered as an inclusive form of mathematical enrichment, providing a tool, an inspiration, or a potentially challenging and motivating independent learning environment for any student. For the gifted ones, it is often a means to reach the appropriate depth and breadth of curriculum, to advance at the appropriate pace for each learner, as well as to achieve better engagement and task commitment (Johnson, 2000; Jones & Simons, 2000; Renninger & Shumar, 2004; Freiman & Lirette-Pitre, 2009; Sullenger & Freiman, 2011).

Being a part of a powerful set of out-of-regular-classroom activities such as mathematical clubs, mathematical camps, mathematics competitions (Olympiads), on-line mathematics competitions play a significant role in nurturing interest and motivating young learners of mathematics, as well as in identification and fostering the most able and talented (Skvortsov, 1978; Karnes & Riley, 1996; Robertson, 2007; Bicknell, 2008). The choice of appropriate challenging tasks is also an important condition of the success of mathematics competitions in developing students' learning potential. Leikin (2004, 2007) claims such tasks must be appropriate to students' abilities, neither too easy, nor too difficult. They should motivate students to persevere with task completion and develop mathematical curiosity and interest in the subject. As well, they must support and advance students' beliefs about the creative nature of mathematics, the constructive nature of the learning process, and the dynamic nature of mathematical problems as having different solution paths and supporting individual learning styles and knowledge construction.

While designing our Virtual Mathematical Marathon, we aimed to provide students with an opportunity to discover their talent, which they cannot normally demonstrate in the regular classroom (Taylor, Gourdeau & Kenderov, 2004) thus we considered the marathon as a stimulus for improving students' informal learning. Fomin, Genkin & Itenberg (2000) described that during the marathon that they conducted on a face-to-face basis, their students managed to increase the number of problems they solved, relatively to other non-competing frameworks in which the same students participated.

Regarding gender issues in virtual mathematics competitions, we found a lack of data that we aim to address in our paper. In the following section we describe the Virtual Mathematical Marathon's structure that allowed us to collect data about participants, including data according to their gender. The main question we asked in our study was: *What kind of differences have been observed in boys' and girls' behaviour during their participation in VMM?* We divided our investigation in two parts addressing the following two sub-questions:

- Was there a gender difference in the initial enrolment of student-participants of VMM? How did participation evolve during the competition, according to the gender?
- What were the gender-related patterns in participants' behaviour according to the difficulty levels for each year in terms of both, participation and success rate?

4 Structure of the virtual mathematical marathon

According to our model of the VMM, one set of 4 non-routine challenging problems was posted twice a week on the CAMI website (`www.umoncton.ca/cami`) over 10 weeks, from June to August in 2008 and 2009. In total, 20 sets of problems were offered to the participants of each round. Every registered member could login, choose a problem, solve it, and submit an answer by selecting it from a multiple-choice menu. The automatic scoring system immediately evaluated students' success producing a score for the problems and adjusting a total score that affected the overall standing.

According to the level of difficulty, scores per problem were determined as follows: level 1(easiest) was scored with 3 points, level 2 with 5 points, level 3 with 7 points, and level 4 (hardest) with 10 points. To support students' participation in the marathon, unsuccessful attempts were also rewarded with 1, 2, 3, and 4 points respectively. Participants could join the marathon, solve as many problems as they wished, withdraw, and come back at any time. The tasks were developed by a team of experts in mathematics and mathematics education.

Here are examples of such tasks coming from one set:

Level 1 problem: How many numbers from 10 to 200 have the property that reversing their digits does not change the number?

A) 17 B) 18 C) 19 D) 20

Comment: students can approach this problem by simply listing all numbers with the required property. These numbers are 11, 22, 33, 44, 55, 66, 77, 88, 99, 101, 111, 121, 131, 141, 151, 161, 171, 181, and 191. Thus the answer is 19.

Level 2 problem: Two dice are thrown at random. What is the probability that the two numbers shown are the digits of a two-digit perfect square?

A) 1/9 B) 5/36 C) 2/9 D) 5/18

Comment: students need to be familiar with the notion of probability, some counting techniques, and apply reasoning. They should notice that the only 2-digit perfect squares that can be constructed from digits 1, 2, 3, 4, 5, 6, are 16, 25, 36, and 64. This gives 4 possible squares, with two ways for each to occur. Since there are 36 possible outcomes, the probability is $8/36 = 2/9$.

Level 3 problem: The volume of a cube is 64 units cubed. What is the surface area of the cube?

A) 16 units squared B) 64 units squared
C) 96 units squared D) 256 units squared

Comment: students need to know basic facts about cubes. They could reason as follow. If the total volume is 64, then the side length is 4 units. This means each face has area $4 \times 4 = 16$. There are 6 faces, so the surface area is $6 \times 16 = 96$ units squared.

Level 4 problem: You have 100 tiles, numbered 0 to 99. Take any set of three tiles. If the number on one of the tiles is the sum of the other two numbers, call the set "good". Otherwise, call the set "bad". How many good sets of three tiles are there?

A) 160 B) 1600 C) 1225 D) 2401

Comment: students need to notice a pattern and invent some useful counting technique in order to solve this problem. For example, they may reason as follows. If 3 is the largest number, there is one good set, $\{1, 2, 3\}$. If 4 is the largest, there is one good set, $\{1, 3, 4\}$. For 5 and 6 there are two, $\{1, 4, 5\}$, $\{2, 3, 5\}$ and $\{1, 5, 6\}$, $\{2, 4, 6\}$, respectively. For 7 and 8 there are three each, for 9 and 10 there are four each, and so on. In this way, when we reach 97 there are 48 good sets, for 98 there are also 48, and for 99 there are 49. It remains to compute the sum

$$\begin{aligned} &1+1+2+2+3+3+\cdots+47+47+48+48+49 \\ &= 49+(1+48)+(1+48)+(2+47)+(2+47) \\ &+\cdots+(24+25)+(24+25)=49\times 49=2401. \end{aligned}$$

5 Proving and disproving conjectures: fostering exploration and questioning

The participants of the marathon were all members of the CAMI community. They received an invitation by email to take part in the marathon. Most of them were from New Brunswick, Canada. We also had a few participants from Quebec and from France. We have no reliable data on students' age, but the most frequent CAMI users are Grades 6–8 (ages 12–14) which is a good approximation.

In order to investigate the first sub-question, we have collected and analysed the data about boys' and girls' participation for each of 20 sets (in both years). We collected and compared the numbers of initial enrolment and on-going visits for boys and girls separately.

In order to address the second sub-question, we have analysed the data about boys' and girls' attempts to solve either all or some particular problems from each set. For example, some students could attempt only easier questions (levels 1–2). We were interested to see if the student was trying to stay in a 'safer' zone, or to take some greater 'risks' solving more challenging problems (levels 3–4). In this respect, we were curious whether a virtual problem-solving environment had allowed girls to exhibit risk-taking behaviour at a rate comparable to the one of boys. Moreover, we draw on our data from previous analyses that emphasized particular behaviour of students who were the most active and successful (the winners of each 20-round game), the group we called the 'most persistent'(Freiman & Applebaum, 2011). We have compared the number of girls and boys among this group. The next section presents our findings.

6 Preliminary results and discussion

There were 298 students (194 in the first year and 104 in the second year) who participated in at least one round (of the total of 20 rounds each year) of the marathon. In the first year, there were more boys ($n = 110$, or 56.7 %) than girls ($n = 84$, or 43.3 %). In the second year the number of girls was slightly higher than number of boys ($n = 56$, or 53.8 % against $n = 48$, or 46.2 %). Over two years, our data did not indicate any significant difference in participation according to the gender: girls seem to be as active as boys.

Further, Figures 1 and 2 below show how the number was changing over each competition. From Figure 1, we learn that in the first three sets of the Year 1, the number of boys was higher than number of girls, but starting from set 7, the numbers are nearly the same in each of remaining sets. We can see therefore that girls who decided to continue their participation were as persistent as boys. A similar trend can be observed in Year 2 data (Figure 2); while the number of participants is much lower than in the first year, there were still more boys than girls in the first set but starting from the set 7, the number of girls and boys was nearly the same until the end of the competition. It is remarkable that among the winners of the two 20-round games there were 6 boys and 5 girls, so nearly the same number of each gender.

The repartition of the number of attempts by gender, according to Table 1 (Year 1 and 2) shows that there was no significant difference in the number of attempts related to the difficulty levels between girls and boys. Usually, participants who tried a problem of level 1 (easiest) did attempt to solve problems of other levels; some difference is only between the levels one and two for both genders. This observation is particularly valuable in view of the fact that in a regular classroom setting "teachers perceived that girls... produced fewer exceptional, risk-taking [learners] than did boys." (Williams, 2006).

The dynamics of success rates is similar between the girls and the boys in the first year; also, both genders were more successful on easier levels (1 and 2) and less in more difficult levels (3 and 4). In the Year 2, however, the boys have clearly outperformed girls at all levels; with the same trends between levels 1–2 (easier—better solved) and 3–4 (harder—less success).

7 Conclusive remarks

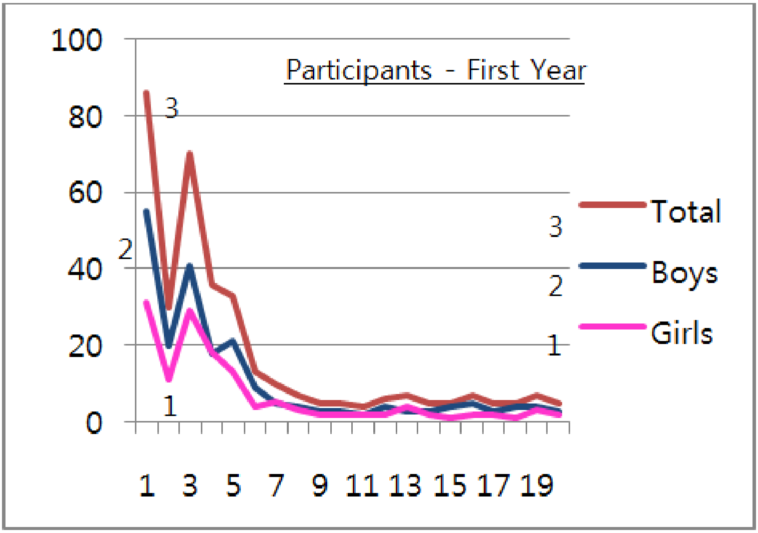
The paper explores gender-related data of students' participation in a virtual mathematics competition, a marathon. Based on studies that indicate virtual environments have the potential to attract as many girls as boys to take part in solving challenging problems, we analysed participation patterns and success rate at the VMM, according to the gender. While observing participation over a long period of time during the first two years of the competition, we found that for both years girls and boys showed similar patterns when, after first few weeks, a number of participants who decided to stay remains relatively stable, independently of success or failure on certain tasks, thus demonstrating risk-taking behaviour and persistence for both genders. This similarity between both genders is consistent with other researchers' findings (Lloyd et al., 2005; Williams, 2006) indicating non-significant gender difference at the junior high level in mathematics as well as equal abilities and interest of both boys and girls to participate in online activities.

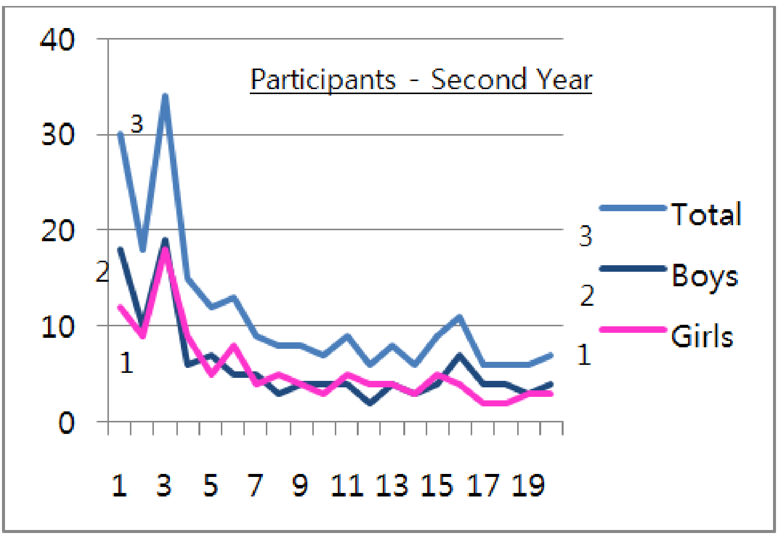
Our preliminary data analysis has several limitations: the major being that we neither analysed students' solutions nor conducted interviews asking them about their level of satisfaction with the game. However, from the data of the online survey, we learn that both, girls and boys, seem to express their interest for more challenging mathematics and to appreciate the experience.

When asked "What motivates you to take part in Marathon?", students answered: "desire to improve my mathematical abilities", "I love math", "...it [VMM] allows me to practice my mathematics", "I like the fact that lot of students compete and the questions come with solutions with detailed explanations in case of a mistake", "I like to see how high I can rank with everybody else. I especially like to getting top 10 and getting name on the home page", "I love to solve math problems", "I like math and solving problems. Plus I like to compete", "Challenge by new, interesting, unusual problems".

For our question "What do you like about Virtual Mathematical Marathon?", students' answers were: "I get to do challenging problems. The explanations are good.", "It's fun and helps me learn more about math", "Short problems, ranking system, solutions are available immediately", "It's possible to do it anytime during the week, so I can easily work around homework", "That it makes me think and sweat", "It challenges me and I love it all. I like seeing the problems every week". Such students' responses are very encouraging and reassuring that the goal of building a virtual community of junior mathematics problem solvers set by our team can be achieved by continuing our collaborative work on the VVM project.

Our future work will use more data and look at more detailed data analysis including students' interviews that could allow for a deeper insight into students' behaviour and better understanding of their thoughts and attitudes about the online problem-solving activity.





Figures 1 and 2: Dynamics of students' participation: total (3), boys (2), and girls (1), X - Set number; Y - Number of participants

			Level 1	Level 2	Level 3	Level 4	Total
First year	Boys	Number of successful solutions / Number of attempts	$\frac{124}{211}$	$\frac{115}{202}$	$\frac{89}{201}$	$\frac{72}{200}$	$\frac{400}{814}$
		Per cents	58.77%	56.93%	44.28%	36%	49.14%
	Girls	Number of successful solutions / Number of attempts	$\frac{81}{137}$	$\frac{70}{121}$	$\frac{60}{122}$	$\frac{39}{122}$	$\frac{250}{502}$
		Per cents	59.12%	57.85%	49.18%	31.97%	49.80%
Second year	Boys	Number of successful solutions / Number of attempts	$\frac{78}{119}$	$\frac{65}{108}$	$\frac{51}{105}$	$\frac{50}{104}$	$\frac{244}{436}$
		Per cents	65.55%	60.19%	48.57%	48.08%	55.96%
	Girls	Number of successful solutions / Number of attempts	$\frac{52}{106}$	$\frac{45}{94}$	$\frac{29}{98}$	$\frac{34}{100}$	$\frac{160}{398}$
		Per cents	49.06%	47.87%	29.59%	34%	40.20%

Table 1: The boys' and girls' success rates at each difficulty level

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The Good, the Bad and the Pleasure (not Pressure!) of Mathematics Competitions

Siu Man Keung

Siu Man Keung obtained a BSc as a double major in mathematics and physics from University of Hong Kong and went on to earn a PhD in mathematics from Columbia University. Like the Oxford cleric in Chaucer's The Canterbury Tales, *"and gladly would he learn, and gladly teach". He had been doing that for more than three decades until he retired in 2005. He has published some research papers in mathematics and computer science, some more papers in history of mathematics and mathematics education, and several books in popularizing mathematics. In particular he is most interested in integrating history of mathematics with the learning and teaching of mathematics, and has been participating actively in an international community of History and Pedagogy of Mathematics since the mid 1980s.*

1 Introduction

A mathematics competition, as but one among many different kinds of extracurricular activity, should enhance the teaching and learning of mathematics in a positive way rather than present a controversy in a negative way. But why does it sometimes become a controversial issue for some people as to its negative effect? What are the pros and cons of this extracurricular activity known as a mathematics competition?

I cannot claim myself to be actively involved in mathematics competitions, but will attempt to share some of my views of this activity gleaned from the limited experience gained in the past years. The intention is to invite more discussion of the topic from those who are much more experienced and know much more about mathematics competitions[1].

To begin with I'd like to state clearly which aspect of this activity I will *not* touch on. In the recent decade mathematics competitions have mushroomed into an industry, some of which are connected with profit-making or fame-gaining intention. Even if these activities may have indirect benefit to the learning of mathematics, which I seriously doubt, an academic discussion of the phenomenon is, mathematically speaking, irrelevant. Rather, it is more a topic of discussion for its social and cultural aspect, namely, what makes parents push their children to these competitions and to training centres which are set up to prepare the children for these competitions, sometimes even against the liking of the children? With such a disclaimer let me get back to issues on mathematics competitions that have to do with mathematics and mathematics education.

The "good" of mathematics competitions

The only experiences I had of an international mathematics competition occurred in 1988 and in 1994. I helped as a coach when Hong Kong first entered the 29th IMO (International Mathematical Olympiad) in 1988 held in Canberra, and I worked as a coordinator to grade the answer scripts of contestants in the 35th IMO in 1994 held in Hong Kong. Through working in these two instances I began to see how the IMO can exert good influence on the educational side, which I had overlooked before. I wrote up my reflections on the IMO in an article, from which I now extract the three points on "the good" of mathematics competitions [4, pp. 74–76].

1. All contestants know that clear and logical presentation is a necessary condition for a high score. When I read their answers I had a markedly different feeling from that I have in reading the answer scripts of many of my students. I felt cozy. Even for incomplete or

[1]This paper is an extended text of an invited talk given at a mathematics education forum held in conjunction with the International Mathematics Competition scheduled on July 24–27, 2012 in Taipei. I thank the organizers, particularly SUN Wen-Hsien of the Chiu Chang Mathematics Education Foundation, for inviting me to give a talk so that I can take the opportunity to organize my thoughts and share them with those who are interested in mathematics competitions.

incorrect answers I could still see where the writing is leading me. But in the case of many of my students, despite advice, coaxing, plea, protest (anything short of threat!) on my part, they write down anything that comes to their minds, disconnected, disorganized and perhaps irrelevant pieces. One possible reason for this bad habit is the examination strategy they have adopted since their school days—write down everything you can remember, for you will score certain marks for certain key points (even if these key points are not necessarily presented in a correct logical order!) and the kind-hearted examiner will take the trouble to sift the wheat from the chaff! Many undergraduates still follow this strategy. Correct or incorrect answer aside, the least we can ask of our students should be clear communication in mathematics (but sadly we cannot).

2. All contestants know that one can afford to spend up to one and a half hour on each problem on average, and hence nobody expects to solve a problem in a matter of minutes. As a result, most contestants possess the tenacity and the assiduity required for problem solving. They will not give up easily, but will try all ways and means to probe the problem, to view it from different angles, and to explore through particular examples or experimental data. On the contrary, many of my students, again too much conditioned by examination techniques since their school days, would abandon a problem once they discover that it cannot be disposed of readily by routine means. In an examination when one races against time, this technique may have its excuse. Unfortunately, many students bring the same habit into their daily study. Any problem that cannot be disposed of in 3 minutes is a difficult problem and is beyond one's capacity, hence no time should be wasted in thinking about it! This kind of "instant learning" is detrimental to the acquirement of true understanding and it kills curiosity, thence along with it the pleasure of study.

3. Some contestants have the commendable habit of writing down not only the mathematics, but remark on their progress as well. Some would write down that they could not go from that point on, or what they did so far seemed to lead nowhere, or that they decided to try a new approach. I really appreciate the manifestation of this kind of "academic sincerity". (It is ironic to note that some leaders or deputy leaders tried to argue that those contestants were

almost near to the solution and should therefore be credited with a higher score. They might be, but they did not, and the good intention of the leaders or deputy leaders is like filling in between the lines for the contestants.) On the contrary, some of my students behave in the opposite way. They write down the given in the first line (amounting to copying the first part of the question), and write down the conclusion at the end (amounting to copying the final part of the question), then fill in between with disconnected pieces of information which may be relevant or irrelevant, ending with an unfounded assertion "hence we conclude ..."! I am deeply disappointed at this kind of insincerity, passing off gibberish as an answer. I would have felt less disappointed if the student did not know the answer at all.

I should add one more point about the "good" of mathematics competitions. A young friend of mine and a member of the Hong Kong 2012 IMO team, Andy Loo, by recounting his own experience since primary school days with mathematics competitions, highlights the essential "good" of mathematics competitions as lying in arousing a passion in the youngster and piquing his or her interest in the subject. The experience of participating in a mathematics competition can exert strong influence on the future career of a youngster, whether he or she chooses to become a research mathematician or not. For those who finally do not benefit from this experience for one reason or another, perhaps it is just an indication that they lack a genuine and sustained passion for the subject of mathematics itself.

2 The "bad" of mathematics competitions

Despite what has been said in the previous section I have one worry about mathematics competitions which has to do with the way of studying mathematics and doing mathematics, even more so for those who are doing well in mathematics competitions. Those who do well in mathematics competitions tend to develop a liking for solving problems by very clever but sometimes quite *ad hoc* means, but lack the patience to do things in a systematic but hard way or view things in a more global manner. They tend to look for problems that are already well-posed for them and they are not accustomed to dealing with vague situations.

Pursuing mathematical research is not just to obtain a prescribed answer but to explore a situation in order to understand it as much as one can. It is far more important to be able to raise a good question than to be able to solve a problem set by somebody else who knows the answer already. One may even change the problem (by imposing more conditions or relaxing the demand) in order to make progress. This is, unfortunately, not what a contestant is allowed to do in a mathematics competition!

Of course, many strong contestants in various mathematics competitions go on to become outstanding mathematicians, but many stay at the level of being good competition problem solvers even if they go on to pursue mathematics. Many leave the field altogether. That is not a problem in itself, because everybody has his or her own aspiration and interest, and there is no need for everybody to become a research mathematician. On the other hand, it would be a pity if they leave the field because they get tired of the subject or acquire a lopsided view of the subject as a result of over-training during the youthful years they spent on mathematics competitions.

Looking at the history of several famed mathematics competitions we see a host of winners in the Eötvös Mathematics Competition of Hungary, started in 1894[2], went on to become eminent mathematicians [5]; we see many medalists in the IMO's, since the event started in 1959, received in their subsequent career various awards for their important contribution to the field of mathematics, including the Fields Medal, Navanlinna Prize, Wolf Prize, ... [6]; we see the same happens for many Putnam Fellows in the William Lowell Putnam Mathematical Competition in the USA for undergraduates [8]. On the other hand, the Fields Medalist, Crafoord Prize and Wolf Prize recipient, YAU Shing-Tung, is noted for his public view against mathematics competitions. The eminent Russian mathematician of the last century, Pavel Sergeevich Aleksandrov (1896–1982), was reported to have once said that he would not have become the mathematician he was had he joined the Mathematical Olympiad! An explanation of this polarity in opinions is to be sought in the way

[2]The role played by the journal *Középiskolai Matematikai és Fizikai Lapok* on mathematics and physics for secondary school founded in that same year merits special attention. For more detailed information readers are invited to visit the website of the journal [7].

how one regards this activity known as a mathematics competition from the impression one gets in witnessing how it is run.

When I worked as a coordinator in the 35^{th} IMO in 1994 I noticed that some teams scored rather high marks, but all six contestants in the team worked out the problem in the same way, indicating solid training on the team's part. However, there were some teams, not all of whose members scored as high marks, but each of whom approached the same problem in a different way, indicating a free and active mind that works independently and imaginatively. It made me wonder: will such qualities like independence and imagination be hampered by over-training, and if so, does that mean over-training for mathematics competitions defeats the purpose of this otherwise meaningful activity? Rather than over-training would an extended follow-up investigation of a competition problem enable the youngsters to better appreciate what mathematical exploration is about? I am sure many contestants who go on to become outstanding mathematicians followed this practice of follow-up investigation during the youthful years they spent on mathematics competitions.

I will now illustrate with two examples. The first example is a rather well-known problem in one IMO. We will see how one can view it as more than just a competition problem begging for just an answer. The other example is on a research topic where the main problem is still open to this date (as far as I am aware of). We will see how a research problem differs from a problem viewed in the context of a mathematics competition problem.

It was natural that I paid some special attention to the questions set in the 29^{th} IMO, although I did not take part in the actual event in July of 1988. Question 6 of the 29th International Mathematical Olympiad reads:

> Let a and b be positive integers such that $ab+1$ divides a^2+b^2. Show that $\dfrac{a^2+b^2}{ab+1}$ is the square of an integer.

A slick solution to this problem, offered by a Bulgarian youngster (Emanouil Atanassov) who received a special prize for it, starts by supposing that $k = \frac{a^2+b^2}{ab+1}$ is *not* a perfect square and rewriting the expression in the form

$$a^2 - kab + b^2 = k, \quad \text{where } k \text{ is a given positive integer.} \qquad (*)$$

Note that for any integral pair (a, b) satisfying $(*)$ we have $ab \geq 0$, or else $ab \leq -1$, and $a^2 + b^2 = k(ab + 1) \leq 0$, implying that $a = b = 0$ so that $k = 0$! Furthermore, since k is not a perfect square, we have $ab > 0$, that is, none of a or b is 0. Let (a, b) be an integral pair satisfying $(*)$ with $a > 0$ (and hence $b > 0$) and $a + b$ *smallest*. We may assume $a > b$. (By symmetry we may assume $a \geq b$. Note that $a \neq b$ or else k is a number lying strictly between 1 and 2.) Regarding $(*)$ as a quadratic equation with a positive root a and another root a', we see that $a + a' = kb$ and $aa' = b^2 - k$. Hence a' is also an integer and (a', b) is an integral pair satisfying (*). Since $a'b > 0$ and $b > 0$, we have $a' > 0$. But

$$a' = \frac{b^2 - k}{a} \leq \frac{b^2 - 1}{a} \leq \frac{a^2 - 1}{a} < a,$$

so that $a' + b < a + b$, contradicting the choice of (a, b)! This proves that $\frac{a^2+b^2}{ab+1}$ must be the square of an integer. (Having no access to the original answer script I have tried to reconstruct the proof based on the information provided by a secondary source [1, p. 505]. The underlying key ideas are (i) choice of a minimal solution (a, b), and (ii) the expression $(*)$ viewed in the context of a quadratic equation.)

Slick as the proof is, it also invites a couple of queries.

1. What makes one suspect that $\frac{a^2+b^2}{ab+1}$ is the square of an integer?
2. The argument by *reductio ad absurdum* should hinge crucially upon the condition that k is not a perfect square. In the proof this condition seems to have slipped in casually so that one does not see what really goes wrong if k is *not* a perfect square. More pertinently, this proof *by contradiction* has *not explained* why $\frac{a^2+b^2}{ab+1}$ must be a perfect square, even though it *confirms* that it is so.

In contrast let us look at a less elegant solution, which is my own attempt. When I first heard of the problem, I was on a trip in Europe and had a "false insight" by putting $a = N^3$ and $b = N$ so that

$$a^2 + b^2 = N^2(N^4 + 1) = N^2(ab + 1).$$

Under the impression that any integral solution (a, b, k) of $k = \frac{a^2+b^2}{ab+1}$ is of the form (N^3, N, N^2) I formulated a strategy of trying to deduce from $a^2 + b^2 = k(ab + 1)$ the equality

$$\big(a-(3b^2-3b+1)\big)^2+(b-1)^2 = \Big(k-(2b-1)\Big)\Big(\big(a-(3b^2-3b+1)\big)(b-1)+1\Big).$$

Were I able to achieve that, then I could have reduced b in steps of one until I got down to the equation $a^2+b^2 = k(ab+1)$ for which $a = k = 1$. By reversing steps I would have solved the problem. I tried to carry out this strategy while I was travelling on a train, but to no avail. Upon returning home I could resort to systematic brute-force checking and look for some actual solutions, resulting in a (partial) list shown below.

a	**1**	**8**	**27**	30	**64**	112	**125**	**216**	240	**343**	418	**512**	...
b	**1**	**2**	**3**	8	**4**	30	**5**	**6**	27	**7**	112	**8**	...
k	**1**	**4**	**9**	4	**16**	4	**25**	**36**	9	**49**	4	**64**	...

Then I saw that my ill-fated strategy was doomed to failure, because there are solutions other than those of the form (N^3, N, N^2). However, not all was lost. When I stared at the pattern, I noticed that for a fixed k, the solutions could be obtained recursively as (a_i, b_i, k_i) with

$$a_{i+1} = a_i k_i - b_i, \qquad b_{i+1} = a_i, \qquad k_{i+1} = k_i = k.$$

It remained to carry out the verification. Once that was done, all became clear. There is a set of "basic solutions" of the form (N^3, N, N^2) where $N \in \{1, 2, 3, \dots\}$. All other solutions are obtained from a "basic solution" recursively as described above. In particular,

$$k = \frac{a^2 + b^2}{ab + 1}$$

is the square of an integer. I feel that I understand the phenomenon much more than if I just learn from reading the slick proof.

[Thanks to Peter Shiu we can turn the indirect proof into a more transparent direct proof based on the same key ideas. Proceed as before and set $c = \min(a, b)$ and $d = \max(a, b)$. Look at the quadratic equation

$$x^2 - kxc + (c^2 - k) = 0,$$

for which d is a positive root with another root d'. Since $d+d' = kc$ and $dd' = c^2 - k < c^2 \leq dc$, we know that d' is an integer less than c so that $d' + c < 2c \leq a + b$. By the choice of (a, b), d' cannot be positive. On the other hand,

$$\begin{aligned}(d+1)(d'+1) &= dd' + (d+d') + 1 = (c^2 - k) + kc + 1 \\ &= c^2 + (c-1)k + 1 \geq c^2 + 1 > 0.\end{aligned}$$

Therefore, $d' + 1 > 0$, implying that $d' = 0$. Hence, $k = c^2 - dd' = c^2$ is the square of an integer.]

The next example is a research problem on the so-called Barker sequence, which is a binary sequence of length S for which the sequence and each off-phase shift of itself differ by at most one place of coinciding entries and non-coinciding entries in their overlapping part. Technically speaking we say that the sequence has its aperiodic autocorrelation function having absolute value 0 or 1 at all off-phase values. For instance, 11101 is a Barker sequence of length 5, while 111010 of length 6 is not a Barker sequence. Neither is 11101011 of length 8 a Barker sequence, but 11100010010 is a Barker sequence of length 11. For application in group synchronization digital systems in communication science R. H. Barker first introduced the notion in 1953. Such sequences for S equal to $2, 3, 4, 5, 7, 11, 13$ were soon discovered and in 1961 R. Turyn and J. Storer proved that there is no Barker sequence of *odd* length S larger than 13. A well-known conjecture in combinatorial designs says that there is no Barker sequence of length S larger than 13, which has withstood the effort of many able mathematicians for more than half a century. Although the conjecture itself remains open, it has stimulated much research in combinatorial designs and in the design of sequences and arrays in communication science. In order to better understand the original problem researchers change the problem and look at the 2-dimensional analogue of arrays or even analogues in higher dimensions, or other variations such as non-binary sequences and arrays over an alphabet set with more than two elements, or instead of a single sequence a pair of sequences (known as Golay complementary sequence pairs) satisfying some suitably formulated modification on their aperiodic autocorrelation functions. In this sense the problem, instead of looking like an interesting piece of curio, opens up new fields and generates new methods and techniques which prove useful elsewhere. (I would recommend an excellent survey paper to those who wish to know more about this topic [2]).

3 School mathematics and "Olympiad mathematics"

Since many mathematics competitions aim at testing the contestants' ability in problem solving rather than their acquaintance with specific subject content knowledge, the problems are set in some general areas which can be made comprehensible to youngsters of that age group, independent of different school syllabi in different countries and regions. That would cover topics in elementary number theory, algebra, combinatorics, sequences, inequalities, functional equations, plane and solid geometry and the like. Gradually the term "Olympiad mathematics" is coined to refer to this conglomeration of topics. One question that I usually ponder over is this: why can't this type of so-called "Olympiad mathematics" be made good use of in the classroom of school mathematics as well? If one aim of mathematics education is to let students know what the subject is about and to arouse their interest in it, then interesting non-routine problems should be able to play their part well when used to supplement the day-to-day teaching and learning. In the preface to *Alice in Numberland: A Students' Guide to the Enjoyment of Higher Mathematics* (1988) the authors, John Baylis and Rod Haggarty remark, "The professional mathematician will be familiar with the idea that entertainment and serious intent are not incompatible: the problem for us is to ensure that our readers will enjoy the entertainment but not miss the mathematical point, ..."

By making use of "Olympiad mathematics" in the classroom I do not mean transplanting the competition problems directly there. Rather, I mean making use of the kind of topics, the spirit and the way the question is designed and formulated, even if the confine is to be within the official syllabus. The so-called "higher-order thinking" is (and should be) one of the objectives in school mathematics as well. Sometimes we may have underestimated the capability and the interest of our students in the classroom. It is not true that they only like routine (and hence usually regarded as "easy"?) material. Perhaps they lack the motivation to learn because they find the diluted content dull and are tired of it.

Besides, good questions not just benefit the learners in the classroom; it is also a challenging task for the teachers to design good questions, thereby upgrading themselves in the process. In this respect mathematics competitions can benefit teachers as well, if they try to make use of the competition problems to enrich the learning experience of their students. To be able to better reap such benefit carefully designed seminars for teachers and suitably prepared didactical material will be helpful [3, pp. 1596–1597].

There is a well-known anecdote about the famous mathematician John von Neumann (1903–1957). A friend of von Neumann once gave him a problem to solve. Two cyclists A and B at a distance 20 miles apart were approaching each other, each going at a speed of 10 miles per hour. A bee flew back and forth between A and B at a speed of 15 miles per hour, starting with A and back to A after meeting B, then back to B after meeting A, and so on. By the time the two cyclists met, how far had the bee travelled? In a flash von Neumann gave the answer—15 miles. His friend responded by saying that von Neumann must have already known the trick so that he gave the answer so fast. His friend had in mind the slick solution to this quickie, namely, that the cyclists met after one hour so that within that one hour the bee had travelled 15 miles. To his friend's astonishment von Neumann said that he knew no trick but simply summed an infinite series! (I leave it as an exercise for you to find the answer by summing an infinite series.)

For me this anecdote is very instructive. For one thing, it tells me that different people may have different ways to go about solving a mathematical problem. There is no point in forcing everybody to solve it in just the same way you solve it. Likewise, there is no point in forcing everybody to learn mathematics in just the same way you learn it. This point dawned on me quite late in my teaching career. For a long time I thought a geometric explanation would make my class understand linear algebra in the easiest way, so I emphasized the geometric viewpoint along with an analytic explanation. I still continue to do that in class to this date, but it did occur to me one day that some students may prefer an analytic explanation because they have difficulty with geometric visualization. To von Neumann, who could carry out mental calculation with lightning speed, maybe an infinite series was the first thing that came up in his mind rather than the time spent by the cyclists meeting each other!

Secondly, both methods of solution have their separate merits. The method of first calculating when the cyclists met is slick and captures the key point of the problem, killing it in one quick and direct shot. The other method of summing an infinite series, which is slower (but not for von Neumann!) and is seemingly more cumbersome and not as clever, goes about solving the problem in a systematic manner, resorting even to brute force. It indicates patience, determination, down-to-earth approach and meticulous care. Besides, it can help to consolidate some basic skills and nurture in a student a good working habit.

It makes me think that there are two approaches in doing mathematics. To give a military analogue one is like positional warfare and the other guerrilla warfare. The first approach, which has been going on in the classrooms of most schools and universities, is to present the subject in a systematically organized and carefully designed format supplemented with exercises and problems. The other approach, which goes on more predominantly in the training for mathematics competitions, is to confront students with various kinds of problems and train them to look for points of attack, thereby accumulating a host of tricks and strategies. Each approach has its separate merit and they supplement and complement each other. Just as in positional warfare flexibility and spontaneity are called for, while in guerrilla warfare careful prior preparation and groundwork are needed, in the teaching and learning of mathematics we should not just teach tricks and strategies to solve special types of problems or just spend time on explaining the general theory and working on problems that are amenable to routine means. We should let the two approaches supplement and complement each other in our classrooms. In the biography of the famous Chinese general and national hero of the Southern Song Dynasty, Yue Fei (1103–1142) we find the description: "Setting up the battle formation is the routine of art of war. Manoeuvring the battle formation skillfully rest solely with the mind."

Sometimes the first approach may look quite plain and dull, compared with the excitement acquired from solving competition problems by the second approach. However, we should not overlook the significance of this seemingly bland approach, which can cover more general situations and turns out to be much more powerful than an *ad hoc* method which, slick as it is, solves only a special case. Of course, it is true that frequently a clever *ad hoc* method can develop into a powerful general method or can become a part of a larger picture. A classic case in point is the development of calculus in history. In ancient time, only masters in mathematics could calculate the area and volume of certain geometric figures, to name just a couple of them, Archimedes (c. 287 B.C.E. – c. 212 B.C. E.) and Liu Hui (3$^{\text{rd}}$ century). Today we admire their ingenuity when we look at their clever solutions, but at the same time feel that it is rather beyond the capacity of an average student to do so. With the development of calculus since the seventeenth and eighteenth century, today even an average school pupil who has learnt the subject will have no problem in calculating the area of many geometric figures.

Let me further illustrate with one example, which is a competition problem posed to me by the father of a contestant. In isosceles $\triangle ABC$, where $AB = AC$ and the measure of $\angle BAC$ is $20°$, D is taken on the side AC such that $AD = BC$. Find θ, the measure of $\angle ADB$ (see Figure 1). Clearly, if one is to employ the law of sines, then the answer can be

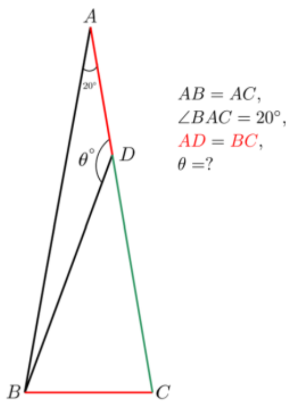

Figure 1

readily obtained in a routine manner, namely,

$$\frac{AD}{\sin(\alpha+\theta)} = \frac{AB}{\sin\theta}, \qquad AD = BC = 2AB\sin\frac{\alpha}{2},$$

where α is the measure of $\angle BAC$, thereby arriving at

$$\tan\alpha = \frac{\sin\alpha}{2\sin\frac{\alpha}{2} - \cos\alpha}.$$

When $\alpha = 20°$, $\theta = 150°$. However, the problem appeared in a primary school mathematics competition in which the contestant was not expected to possess the knowledge of the law of sines! Is there a way to avoid the use of this heavy machinery (for a primary school pupil)? I hit upon a solution by constructing an equilateral $\triangle FBC$ with F inside the given $\triangle ABC$. Pick a point E on AB such that $AE = CD$ (see Figure 2).

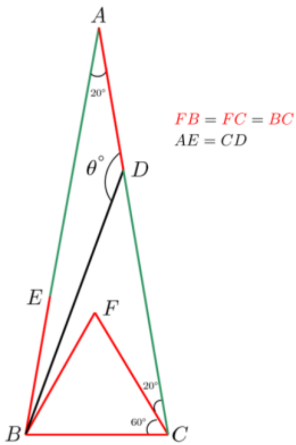

Figure 2

Then it is not hard (by constructing DE, DF) to find out that the measure of $\angle DBE$ is $10°$ (Exercise) so that $\theta = 180° - 20° - 10° = 150°$. Why would I throw in the equilateral $\triangle FBC$ as if by magic? It is because I had come across a similar-looking problem before: In an isosceles $\triangle ABC$, where $AB = AC$ and the measure of $\angle BAC$ being $20°$, points D and E are taken on AC, AB respectively such that the measure of $\angle DBC$ is $70°$ and that of $\angle ECB$ is $150°$; find ϕ, the measure of $\angle BDE$ (see Figure 3). By constructing an equilateral $\triangle FBC$ with F inside the

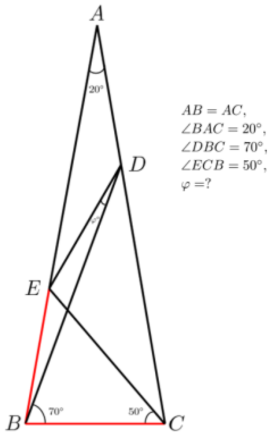

Figure 3

given $\triangle ABC$ we can arrive at the answer $\phi = 10°$ (Exercise). These two versions are indeed the description of the same situation, because it can be proved that $AD = BC$ in the second problem. Only knowledge of congruence triangles suffices. No knowledge of trigonometry is required. However, if the measure of $\angle DBC$ and that of $\angle ECB$ are not 70° and 50° respectively, then the geometric proof completely breaks down! But we can still compute the measure of $\angle BDE$ by employing the law of sines, which is within what an average pupil learns in school. It has to be admitted that the method is routine and not as elegant, but it covers the general case and can be handled by an average pupil who has acquired that piece of knowledge.

4 The pleasure (not pressure!) of mathematics competitions

Before working as a coordinator for the 35$^{\text{th}}$ IMO I harboured a distrust of the value of mathematics competitions. I still harbour this distrust to some extent, all the more when I witnessed during coordination of the IMO in 1994 how some leaders or deputy leaders over-reacted out of too much concern for winning high scores. Putting strong emphasis on winning/losing will inculcate in the youngsters an unhealthy attitude towards the whole activity. Attaching undue importance to the competition by organizers, teachers, parents, students, is one main source that may cause distortion of the good intention of mathematics competitions, not to mention the more "commercial" consequences that take advantage of this misplaced emphasis. Not only does it fail to bring about the ideal outcome of fostering genuine intrinsic interest and enthusiasm in the subject, it takes the fun and meaning out of a truly extracurricular activity as well. Instead of pleasure we are imposing pressure on the youngsters.

Furthermore, the unilateral strengthening of ability to attain high score on these so-called "Olympiad mathematics" problems may have adverse effect on the overall growth of a youngster, not just in terms of academic pursuit in other disciplines (or in mathematics itself!) but even in terms of personal development. In particular, I am disappointed at not finding how mathematics competitions breathe life into a general mathematics culture in the local scene. On the contrary, many people may be misled into believing that those difficult "Olympiad mathematics" problems present the high point in mathematics, and that mathematics is therefore too difficult to lie within reach of an average person.

5 Concluding remark

On the whole I have great admiration for the talent of those youngsters who take part in a mathematics competition. What little I accomplish in trying out those competition problems with all my might they accomplish at a stroke, and explain it in a clear and lucid manner. I also have great respect for the dedication and enthusiasm of those organizers who believe in the value of a healthy mathematics competition. They are serving the mathematical community in many ways.

One predominant objection to mathematics competitions is the requirement to work out the problems within a fixed time span, say three to four hours. Some regard this as an act to undermine the intellectual and intrinsic pleasure of doing mathematics. In a comprehensive paper on mathematics competitions and mathematics education Petar Kenderov points out how this requirement disadvantages those creative youngsters who are "slow workers". Along with it he points out some important features which are not encouraged in a traditional mathematics competition but which are essential for doing good work in mathematics. These include "the ability to formulate questions and pose problems, to generate, evaluate, and reject conjectures, to come up with new and non-standard ideas". Moreover, he points out that all such activities "require ample thinking time, access to information sources in libraries or the internet, communication with peers and experts working on similar problems, none of which are allowed in traditional competitions." [3, p. 1592]

My good friend, Tony Gardiner, who is known for his rich experience in mathematics competitions and had served as the leader of the British IMO team four times, after reading my article in 1995 [4] commented that I should not blame the negative aspects on the mathematics competition itself. He went on to enlighten me on one point, namely, a mathematics competition should be seen as just the tip of a very large, more interesting, iceberg, for it should provide an incentive for each country to establish a pyramid of activities for masses of interested students. It would be to the benefit of all to think about what other activities besides mathematics competitions can be organized to go along with it. These may include the setting up of a mathematics club or publishing a magazine to let interested youngsters share their enthusiasm and their ideas, organizing a problem session, holding contests in doing projects at various levels and to various depth, writing book reports and essays, producing cartoons, videos, softwares, toys, games, puzzles, ... I wish more people will see mathematics competitions in this light, in which case the negative impression, which I might have conveyed in this paper, will no longer linger on!

The good, the bad and the pleasure of mathematics competitions
  Are to which we should pay our attention.
Benefit from the good; avoid the bad;
  And soak in the pleasure.
Then we will find for ourselves satisfaction!

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The Twin Towers of Hanoi

Jack Chen & Richard Mah & Steven Xia

In a famous puzzle known as the Tower of Hanoi, there are three pegs in the playing board. There are n disks of different sizes, all stacked on the first peg, in ascending order of size from the top. The objective is to transfer this tower to the third peg. The rule is that we may only move one disk on top of a peg to the top of another peg, and a disk may not be placed on top of a smaller disk. Figure 1 shows the starting position when $n = 3$.

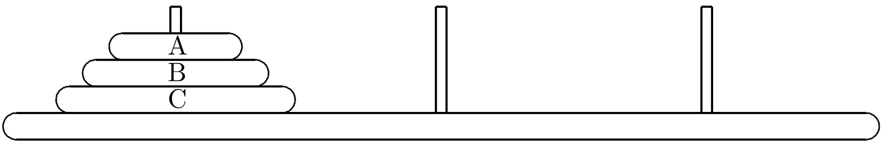


Figure 1

If $n = 1$, the task can be accomplished in 1 move: A_3. If $n = 2$, the task can be accomplished in 3 moves: A_2BA_3. If $n = 3$, the task can be accomplished in 7 moves:: $A_3BA_2CA_1BA_3$. The subscript for A shows the number of the peg to which it is moving. B and C never have choices.

Based on these three simple cases, we conjecture that the minimum number of moves required to transfer a tower with n disks is $2^n - 1$. Let us prove this by mathematical induction. We assume that the minimum number of moves required to transfer a tower with n disks is $2^n - 1$. We now try to transfer a tower with $n + 1$ disks.

A critical moment occurs when the bottom disk is moving, from the first peg to the third. In order for this to be possible, the n smaller disks must form a tower on the second peg. Thus before the move of the bottom disk, we must transfer a tower with n disks from the first peg to the second. By the induction hypothesis, this takes $2^n - 1$ moves. After the move of the bottom disk, we must complete the task by transferring the tower on the second peg, consisting of the n smaller disks, to the third peg. Once again, this also takes $2^n - 1$ moves. Together with the move of the bottom disk, the minimum number is $(2^n-1)+1+(2^n-1) = 2^{n+1}-1$. This completes the inductive argument.

We can express the solution to the problem of the Tower of Hanoi using a different terminology. Let a_n be the minimum number of moves to transfer a tower of height n from one peg to another. From our analysis above, we see that $a_n = 2a_{n-1}+1$. This result, which defines a_n in terms of a_{n-1}, is an example of what is called a **recurrence relation**. Along with $a_1 = 1$, which is called an **initial value**, they define the sequence $\{a_n\}$ uniquely.

Using the recurrence relation and the initial value, we can generate additional terms of the sequence, as shown in the chart below.

$a_1 = 1$	$a_2 = 3$	$a_3 = 7$
$a_4 = 15$	$a_5 = 31$	$a_6 = 63$
$a_7 = 127$	$a_8 = 255$	$a_9 = 511$

We now consider a variant which we call the Twin Towers of Hanoi. As before, there are three pegs in the playing board. There are n disks of sizes 1, 2, 3, …, n. Those of odd sizes are stacked on the first peg, and those of even sizes are stacked on the third peg. On both pegs, the disks are in ascending order of size from the top. The rule is the same, in that we may only move a disk on top of a peg to the top of another peg, and a disk may not be placed on top of a smaller disk. Figure 2 illustrates the starting position of the case $n = 6$.

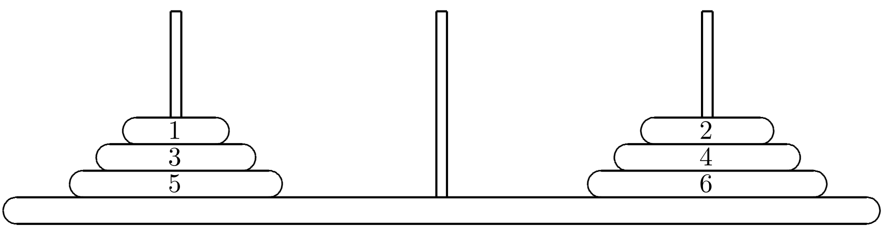


Figure 2

The objective is to have the two towers trade places. As in the Tower of Hanoi, a critical moment occurs when disk 6 is moving, from the third peg to the first. In order for this move to be possible, the 5 smaller disks must form a tower on the second peg, as illustrated in Figure 3. Thus before the move of disk 6, we must merge the disks 1 to 5 into a tower on the second peg.

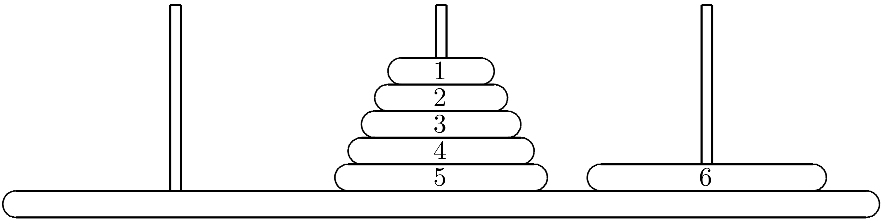
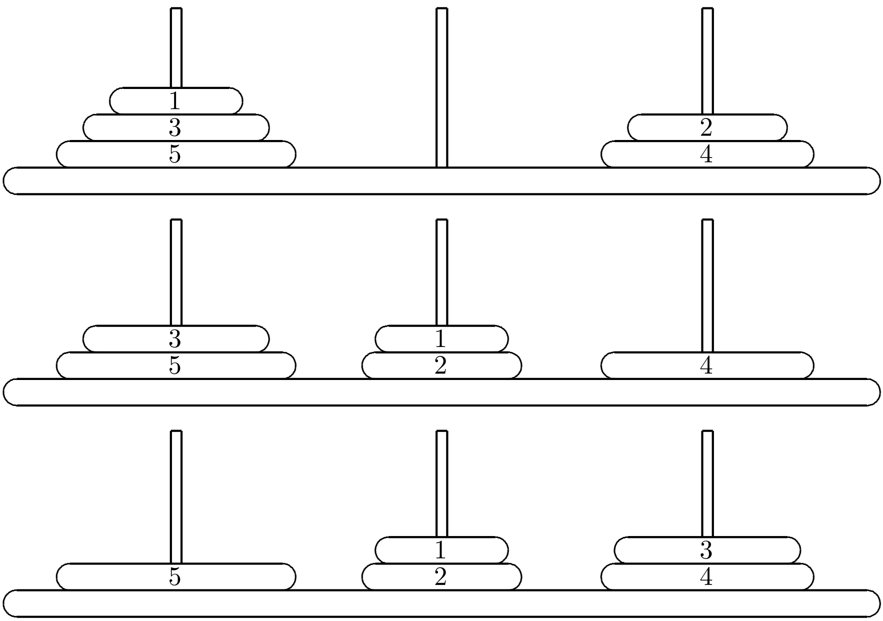


Figure 3

We have identified an intermediate objective for the Twin Towers of Hanoi, that of merging the n disks on the second peg. Figure 4 illustrates the case $n = 5$.



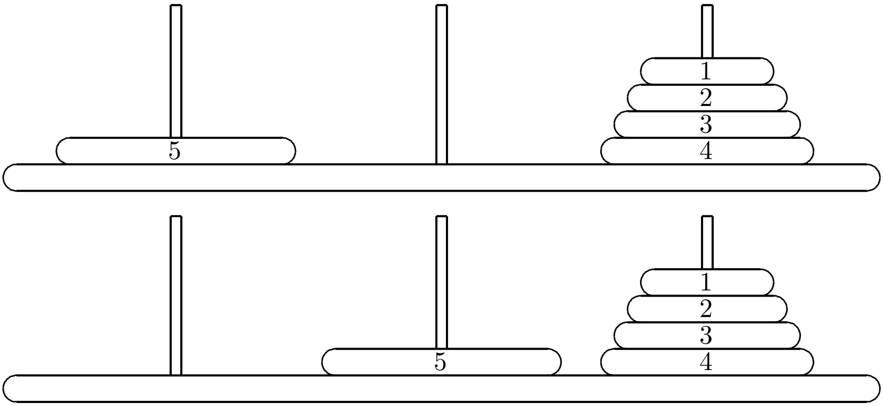


Figure 4

We first merge disks 1 and 2 on the second peg to pave the way for the move of disk 3. After disk 3 moves on top of disk 4, we transfer disks 1 and 2 on top of disk 3 to pave the way for the move of disk 5. After disk 5 moves to the second peg, we transfer disks 1, 2, 3 and 4 on top of it to complete the task.

In general, we merge disks 1, 2, 3, ..., $n-3$ on the second peg, move disk $n-2$ on top of disk $n-1$, transfer disks 1, 2, 3, ..., $n-3$ on top of disk $n-2$, move disk n to the second peg and transfer disks 1, 2, 3, ..., $n-1$ on top of disk n.

This means that if we let b_n be the minimum number of moves for merging n disks, then we have $b_n = b_{n-3} + 1 + a_{n-3} + 1 + a_{n-1}$. Since $a_n = 2^n - 1$, we have $b_n = b_{n-3} + 5 \cdot 2^{n-3}$.

This is a three-step recurrence relation since b_n is not defined in terms of b_{n-1} but in terms of b_{n-3}. Hence we need three initial values. It is not hard to determine b_1, b_2 and b_3, and generate the chart of values below.

$b_1 = 1$	$b_2 = 2$	$b_3 = 5$
$b_4 = 11$	$b_5 = 22$	$b_6 = 45$
$b_7 = 91$	$b_8 = 182$	$b_9 = 365$

We notice that b_n is either equal to $2b_{n-1}$ or $2b_{n-1}+1$, but it is not immediately clear what the general formula for b_n is.

We also notice that the numbers in the last column so far are all multiples of 5. Dividing them by 5 yields the quotients 1, 9 and 73. In the table of values of a_n given earlier, we notice that the numbers in the last column there are all multiples of 7. Dividing them by 7 yields the quotients 1, 9 and 73 also. Thus we suspect that b_n is roughly $\frac{5}{7}a_n$.

$\frac{5}{7}a_1 = 1 - \frac{2}{7}$	$\frac{5}{7}a_2 = 2 + \frac{1}{7}$	$\frac{5}{7}a_3 = 5$
$\frac{5}{7}a_4 = 11 - \frac{2}{7}$	$\frac{5}{7}a_5 = 22 + \frac{1}{7}$	$\frac{5}{7}a_6 = 45$
$\frac{5}{7}a_7 = 91 - \frac{2}{7}$	$\frac{5}{7}a_8 = 182 + \frac{1}{7}$	$\frac{5}{7}a_9 = 365$

Thus we conjecture that b_n is equal to $\frac{5}{7}a_n$ rounded off to the nearest integer. More specifically, we conjecture that

$$b_n = \begin{cases} \frac{5}{7}(2^n-1) & \text{when } n \equiv 0 \pmod 3, \\ \frac{5}{7}(2^n-2)+1 & \text{when } n \equiv 1 \pmod 3, \\ \frac{5}{7}(2^n-4)+2 & \text{when } n \equiv 2 \pmod 3. \end{cases}$$

We now use mathematical induction to prove these three formulae. First, let $n \equiv 0 \pmod 3$. We have $\frac{5}{7}(2^3-1) = 5 = b_3$. Suppose $b_n = \frac{5}{7}(2^n-1)$ for some $n \geq 3$. Then

$$\begin{aligned} b_{n+3} &= b_n - 5 \cdot 2^n \\ &= \tfrac{5}{7}(2^n-1) + 5 \cdot 2^n \\ &= \tfrac{5}{7}(2^n - 1 + 7 \cdot 2^n) \\ &= \tfrac{5}{7}(2^{n+3}-1). \end{aligned}$$

The other two formulae can be proved in an analogous manner.

Returning to the main task of having the two towers trade places. Let the minimum number of moves required be c_n where n is the total number of disks. We have already identified the critical moment when disk n moves, either from the first peg to the third or vice versa. In order for this move to be possible, the $n-1$ smaller disks must form a tower on the second peg. This merger requires b_{n-1} moves. After the move of disk n, we must disperse the merged tower. This also takes b_{n-1} moves since the process is reversible. It follows that $c_n = 2b_{n-1} + 1$. We have the chart of values below.

$c_1 = 1$	$c_2 = 3$	$c_3 = 5$
$c_4 = 11$	$c_5 = 23$	$c_6 = 45$
$c_7 = 91$	$c_8 = 183$	$c_9 = 365$

It follows that c_n and b_n are identical, except for $n \equiv 2 \pmod 3$, when they differ by 1. It may be interesting to discover what may have caused this discrepancy.

The method of mathematical induction allows us to prove that the specified minimum numbers of moves are indeed correct, but it does not explain why they should have those values. Moreover, we have to have an idea what these values are before we can apply the inductive argument. We have resorted to observing patterns and making inspired guesses. We would like to have a more formal and systematic approach to solving recurrence relations.

Let us revisit $a_n = 2a_{n-1} + 1$ with initial value $a_1 = 1$. Actually, $a_0 = 0$ will serve just as well. Note that the recurrence relation is true not just for one value of n, but for all values of n. In other words, all the statements below are true. This is called an iteration.

$$\begin{aligned} a_n &= 2a_{n-1} + 1, \\ a_{n-1} &= 2a_{n-2} + 1, \\ a_{n-2} &= 2a_{n-3} + 1, \\ &\cdots \\ a_2 &= 2a_1 + 1, \\ a_1 &= 2a_0 + 1. \end{aligned}$$

Multiplying each by a power of 2 one higher than the preceding one, we obtain the following statements.

$$\begin{aligned} a_n &= 2a_{n-1} + 1, \\ 2a_{n-1} &= 2^2 a_{n-2} + 2, \\ 2^2 a_{n-2} &= 2^3 a_{n-3} + 2^2, \\ &\cdots \\ 2^{n-2} a_2 &= 2^{n-1} a_1 + 2^{n-2}, \\ 2^{n-1} a_1 &= 2^n a_0 + 2^{n-1}. \end{aligned}$$

Addition results in massive cancellations, leading to

$$a_n = 2^n a_0 + 1 + 2 + 2^2 + \cdots + 2^{n-2} + 2^{n-1} = 2^n - 1.$$

We now turn to the recurrence relation $b_n = b_{n-3} + 5 \cdot 2^{n-3}$ with initial values $b_0 = 0$, $b_1 = 1$ and $b_2 = 2$. We consider only $n \equiv 0 \pmod 3$ as the other two cases are analogous.

$$\begin{aligned} b_n &= b_{n-3} + 5 \cdot 2^{n-3}, \\ b_{n-3} &= b_{n-6} + 5 \cdot 2^{n-6}, \\ b_{n-6} &= b_{n-9} + 5 \cdot 2^{n-9}, \\ \cdots &= \cdots, \\ b_6 &= b_3 + 5 \cdot 2^3, \\ b_3 &= b_0 + 5 \cdot 2^0. \end{aligned}$$

Addition yields

$$\begin{aligned} b_n &= b_0 + 5(1 + 2^3 + 2^6 + \cdots + 2^{n-3}) \\ &= \frac{5(2^n - 1)}{2^3 - 1} = \frac{5}{7}(2^n - 1). \end{aligned}$$

There is an alternative approach which is equally formal and systematic. Rewriting this recurrence relation as $a_n - 2a_{n-1} = 1$, we consider the associated recurrence relation $a_n - 2a_{n-1} = 0$. The simpler one is said to be homogeneous because all terms involve the sequence $\{a_n\}$. The original recurrence relation is said to be non-homogeneous since it has a non-zero term not involving the sequence $\{a_n\}$.

We may solve the homogeneous recurrence relation by setting $a_n = x^n$ for some non-zero number x. Then $x^n - 2x^{n-1} = 0$. Since $x \neq 0$, we can simplify to $x - 2 = 0$ so that $a_n = 2^n$ is a solution to the homogeneous recurrence relation. The general solution is $a_n = K2^n$ for some constant K. This value will eventually be determined by the initial value.

All we need now is a particular solution to the non-homogeneous recurrence relation. Since the non-homogeneous term is 1, it is reasonable to guess that $a_n = A$ for some constant A. It is called an (as yet) undetermined coefficient. Now $0 = a_n - 2a_{n-1} = A - 2A = -A$. Hence $A = -1$ and the particular solution is $a_n = -1$.

Combining the two partial solutions, we now have the general solution to the non-homogeneous recurrence relation, namely, $a_n = K2^n - 1$. Setting $n = 0$, we have $0 = a_0 = K2^0 - 1 = K - 1$. It follows that $K = 1$, so that $a_n = 2^n - 1$.

We now turn to the recurrence relation $b_n - b_{n-3} = 5 \cdot 2^n$. The associated homogeneous recurrence relation is $b_n - b_{n-3} = 0$. Setting $b_n = x^n$, we have $x^n - x^{n-3} = 0$ so that

$$0 = x^3 - 1 = (x - 1)(x^2 + x + 1).$$

The first factor yields the root $x = 1$ while the second factor yields complex roots $\frac{1 \pm \sqrt{3}\mathrm{i}}{2}$, where i is a root of the quadratic equation $x^2 + 1 = 0$. We denote $\frac{1+\sqrt{3}\mathrm{i}}{2}$ by ω. It satisfies $\omega^2 = \frac{1-\sqrt{3}\mathrm{i}}{2}$, $\omega^3 = 1$ and $\omega^2 + \omega + 1 = 0$. Thus the general solution of the aassociated homogeneous recurrence relation is $b_n = K_1 + K_2\omega^n + K_3\omega^{2n}$.

We need a particular solution to the non-homogeneous recurrence relation. Since the non-homogeneous term is $5 \cdot 2^n$, it is reasonable to guess that $b_n = A2^n$ for some constant A. Then $A2^n = A2^{n-3} + 5 \cdot 2^{n-3}$. Canceling the factor 2^{n-3} yields $8A = A + 5$. Hence $A = \frac{5}{7}$ so that $b_n = \frac{5}{7}2^n$.

It follows that the general solution to the non-homogeneous recurrence relation is $b_n = \frac{5}{7}2^n + K_1 + K_2\omega^n + K_3\omega^{2n}$. Using the initial values

$b_0 = 0,\ b_1 = 1$ and $b_2 = 2$, we have

$$\begin{aligned} 0 = b_0 &= \frac{5}{7} + K_1 + K_2 + K_3, \\ 1 = b_1 &= \frac{10}{7} + K_1 + K_2\omega + K_3\omega^2, \\ 2 = b_2 &= \frac{20}{7} + K_1 + K_2\omega^2 + K_3\omega. \end{aligned}$$

It follows that

$$\begin{aligned} K_1 + \quad K_2 + \quad K_3 &= -\frac{5}{7}, \\ K_1 + \ \omega K_2 + \omega^2 K_3 &= -\frac{3}{7}, \\ K_1 + \omega^2 K_2 + \ \omega K_3 &= -\frac{6}{7}. \end{aligned}$$

We can solve this system of three equations in three unknowns and obtain $K_1 = -\frac{2}{3}$, $K_2 = -\frac{2}{21} - \frac{1}{7}\omega$ and $K_3 = -\frac{2}{21} - \frac{1}{7}\omega^2$. However, we do not really need these values.

For $n \equiv 0 \pmod 3$, we have

$$\begin{aligned} b_n &= \frac{5}{7}2^n + K_1 + K_2 + K_3 \\ &= \frac{5}{7}2^n - \frac{5}{7} \\ &= \frac{5}{7}(2^n - 1). \end{aligned}$$

For $n \equiv 1 \pmod 3$, we have

$$\begin{aligned} b_n &= \frac{5}{7}2^n + K_1 + K_2\omega + K_3\omega^2 \\ &= \frac{5}{7}2^n - \frac{3}{7} \\ &= \frac{5}{7}(2^n - 2) + 1. \end{aligned}$$

For $n \equiv 2 \pmod 3$, we have

$$\begin{aligned} b_n &= \frac{5}{7}2^n + K_1 + K_2\omega^2 + K_3\omega \\ &= \frac{5}{7}2^n - \frac{6}{7} \\ &= \frac{5}{7}(2^n - 4) + 2. \end{aligned}$$

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Tournament of Towns Selected Problems, Spring 2012

Andy Liu

1. Five students have the first names: Clark, Donald, Jack, Robin and Steven, and have the last names, in a different order: Clarkson, Donaldson, Jackson, Robinson and Stevenson. Clark is 1 year older than Clarkson, Donald is 2 years older than Donaldson, Jack is 3 years older than Jackson and Robin is 4 years older than Robinson. Who is older, Steven or Stevenson and what is the difference in their ages?

Solution:
Let Steven be n years older than Stevenson. The total age of Clark, Donald, Jack, Robin and Steven must be the same as the total age of Clarkson, Donaldson, Jackson, Robinson and Stevenson, because these are the same five people. Hence $1+2+3+4+n=0$ so that $n=-10$. It means that Steven is 10 years younger than Stevenson.

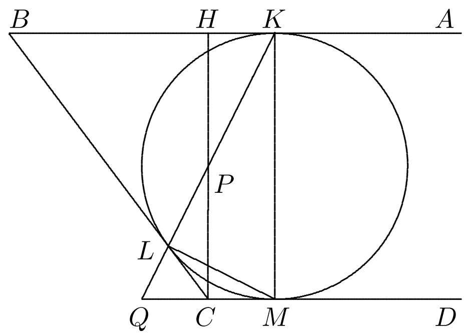
2. The game Minesweeper is played on a 10×10 board. Each cell either contains a bomb or is vacant. On each vacant cell is recorded the number of bombs in the neighbouring cells along a row, a column or a diagonal. Then all the bombs are removed, and new bombs are placed in all cells which were previously vacant, and the numbers of neighbours are recorded as before. Can the sum of all numbers on the board now be greater than the sum of all numbers on the board before?

Solution:
Represent each cell by a vertex and join two neighbouring cells by an edge. An edge is called a scoring edge if it joins a bomb cell and vacant cell, since it will contribute 1 to the number recorded on the vacant cell. An edge which joins two bomb cells or two vacant cells is non-scoring. Hence the sum of the recorded numbers is equal to the number of scoring edges. After the transformation of the board, a scoring edge remains a scoring edge, and a non-scoring

edge remains a non-scoring edge. Hence the sum of all the numbers on the board before must be equal to the sum of all the numbers on the board after.

3. A circle touches sides AB, BC and CD of a parallelogram $ABCD$ at points K, L and M respectively. Prove that the line KL bisects the perpendicular dropped from C to AB.

Solution:
Let KL intersect the perpendicular CH dropped from C to AB at P. Let the extensions of KL and DC intersect at Q. Now $\angle KLM = 90°$ since KM is a diameter of the circle. Hence QLM is a right triangle, so that its circumcentre lies on QM as well as on the perpendicular bisector of LM. Now $CL = CM$ since both are tangents from C to the circle. Hence C lies on the perpendicular bisector of LM. Being on QM, C must be the circumcentre of triangle QLM. This means that $QC = CM$. Now PC is parallel to KM. By the Midpoint Theorem, $PC = \frac{1}{2}KM = \frac{1}{2}HC$, which is equivalent to the desired result.



4. Among 239 coins which look the same, there are two counterfeit coins of the same weight, and 237 real coins of the same weight but different from that of the counterfeit coins. Determine in three weighings on a balance whether the counterfeit coins are heavier or lighter than the real coins. It is not necessary to identify the counterfeit coins.

Solution:
Put 80 coins into group A, 79 coins into each of groups B and C, and we have one coin left. If we add this coin to group B, we call

the expanded group B^+. C^+ is similarly obtained from C. In the first two weighings, we try to balance group A against group B^+ and group A against group C^+. We consider three cases.
Case 1. We have equilibrium both times.
This means that each group has one counterfeit coin, so that the extra coin must be counterfeit. The coins in both groups B and C are real. Weigh one of them against the known counterfeit coin, and that will tell us the desired answer.
Case 2. We have equilibrium only once, say between A and B^+.
Either each has a counterfeit or neither has a counterfeit coin. Divide the coins in A into two subgroups of 40 and weigh them against each other. If we have equilibrium, A does not have any counterfeit coins, and neither does B, which means C has both of them. If we do not have equilibrium, A has a counterfeit coin, and so does B^+. This means that the extra coin is real and C^+ does not have any counterfeit coins. In either situation, the weighing between A and C^+ tells us the desired answer.
Case 3. We have no equilibrium.
We claim that either A is heavy both times or A is light both times. Suppose to the contrary that B^+ is heavier than A and A is heavier than C^+. Then they must have two, one and zero counterfeit coins in either order. However, if A has one counterfeit coin, it is impossible for either B or C to have two of them. This justifies our claim. So either both B^+ and C^+ have no counterfeit coins or both have one. Divide the coins in B^+ into two subgroups of 40 and weigh them against each other. This will tell us the desired answer.

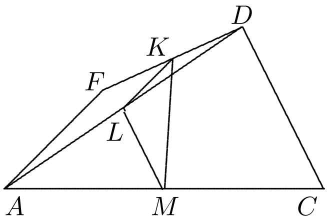
5. An infinite sequence of numbers $a_1, a_2, a_3, \ldots$ is given. For any positive integer k, there exists a positive integer $t = t(k)$ such that $a_k = a_{k+t} = a_{k+2t} = \ldots$ Is this sequence necessarily periodic?

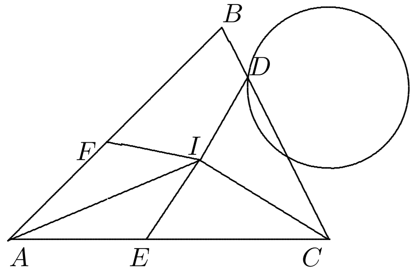
Solution:
The result is not true, and here is a counter-example. For any positive integer k, let a_k be the highest power of 2 which divides k. Thus the sequence is $0, 1, 0, 2, 0, 1, 0, 3, 0, 1, 0, 2, 0, 1, 0, 4, \ldots$ The choice $t(k) = 2^{a_k+1}$ satisfies the given condition and yet the sequence is not periodic as the maximum size of the terms increases without bound.

6. I is the incentre of triangle ABC. A circle through B and I intersects AB at F and BC at D. K is the midpoint of DF. Prove that $\angle AKC$ is obtuse.

Solution:

Let $ACDF$ be a convex quadrilateral. Let K and M be the respective midpoints of DF and AC. We claim that $KM < \frac{AF+CD}{2}$. Let L be the midpoint of AD, as shown in the diagram below on the left. By the Midpoint Theorem, $KL = \frac{AF}{2}$ and $LM = \frac{CD}{2}$. By the Triangle Inequality, $KM < KL + LM = \frac{AF+CD}{2}$, justifying our claim.





Let E be the point on AC such that $AE = AF$, as shown in the diagram above on the right. Let $\angle AEI = \alpha$ and $\angle CEI = \beta$, so that $\alpha + \beta = 180°$. Since AI bisects $\angle CAB$, triangles AEI and AFI are congruent. Hence $\angle AFI = \alpha$ and $\angle BFI = \beta$. Since $BDIF$ is cyclic, $\angle BDI = \alpha$ and $\angle CDI = \beta = \angle CEI$. Hence triangles CDI and CEI are also congruent, so that we have $CD = CE$. Let M be the midpoint of AC. By our earlier claim, $KM < \frac{AF+CD}{2} = \frac{AE+CE}{2} = \frac{AC}{2}$. Hence K lies within the semicircle with diameter AC, which implies that $\angle AKC$ is obtuse.

7. Peter chooses some positive integer a and Paul wants to determine it. Paul only knows that the sum of the digits of Peter's number is 2012. In each moves, Paul chooses a positive integer x and Peter tells him the sum of the digits of $|x - a|$. What is the minimal number of moves Paul needs to determine Peter's number for sure?

Solution:

We claim that Paul needs at most 2012 questions. Let Peter's number have n non-zero digits $a_1, a_2, \ldots, a_n$ from right to left, with $n > 1$ and $a_1 + a_2 + \cdots + a_n = 2012$. For $1 \le k \le n$, let b_k be the number of adjacent 0s to the right of a_k. Paul will carry out the following steps. In Step 1, Paul determines b_1 using one question. He chooses 1, and Peter's answer must be $2011 + 9b_1$. In step k, $2 \leq k \leq n$, Paul will determine a_{k-1} and b_k using a_{k-1} questions. Paul chooses a number which has a 2 in front of the known digits of Peter's number. If Peter's answer is 2010, Paul replaces the 2 by the 3. Continuing this way, when the first digit of Paul's choice is a_{k-1}, Peter will answer $2013 - a_{k-1}$. However, when Paul replaces a_{k-1} by $a_{k-1}+1$, Peter's answer will be $2012 - a_{k-1} + 9b_k$. Now only a_n is left, but Paul does not know that unless $a_n = 1$. However, by the time he asks the $(a_n - 1)$-st questions here, he will get 0 as a response and knows Peter's number. The total number of questions needed is $1 + a_1 + a_2 + \cdots + (a_n - 1) = 2012$. This justifies our claim. We now prove that Peter can force Paul to use 2012 questions, by constructing his number one step at a time. Let b_1 be the number of digits in Paul's first choice. Peter chooses the last b_1 digits of his number to be 0 and put a 1 in front, referred to as a_1. His response will be $2011 + 9b_1$. Paul can only tell that Peter's number ends in exactly b_1 zeros. Paul's next choice must be a number with more digits. Peter adds b_2 zeros in front of a_1 to match the length of Paul's choice, and put another 1 in front, referred to as a_2. Paul can only determine the last b_2+1+b_1 digits of Peter's number, plus the fact that the digit in front is non-zero. Continuing this way, we have $a_1 = a_2 = \cdots = a_{2012} = 1$, and Paul needs one question to determine each of $b_1, b_2, \ldots, b_{2012}$.

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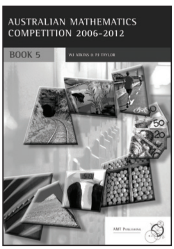

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