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MATHEMATICS COMPETITIONS

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MATHEMATICS COMPETITIONS VOLUME 25 NUMBER 2 2012

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The aims of the Federation are:

1. ***to promote excellence in, and research associated with, mathematics education through the use of school mathematics competitions;***
2. ***to promote meetings and conferences where persons interested in mathematics contests can exchange and develop ideas for use in their countries;***
3. ***to provide opportunities for the exchanging of information for mathematics education through published material, notably through the Journal of the Federation;***
4. ***to recognize through the WFNMC Awards system persons who have made notable contributions to mathematics education through mathematical challenge around the world;***
5. ***to organize assistance provided by countries with developed systems for competitions for countries attempting to develop competitions;***
6. ***to promote mathematics and to encourage young mathematicians.***

From the President

Dear Fellow Federalists!

Our Federation was born in 1984 in Adelaide, Australia. In 1986, the founder Peter O'Halloran invited me to join it based on the recommendation of our common friend George Berzsenyi. Peter felt that the *Mathematics Competitions* journal published worthy reports about competitions around the world, but asked me to write on the topic he wanted to be represented better: creating and solving Olympiad problems. Hence from the early years of the Federation we have had two directions: *Competitions around the World* and *Creating Competition Problems and Problem Solving.*

Later there came a realization that we ought to have a third direction, *Work with Students and Teachers.* This direction has manifested itself in an ICMI study, in which several of our members participated as authors. The result was the publication of a volume edited by our Treasurer and Past President Peter J. Taylor and Ed Barbeau.

As many of you, I have spent a lifetime working in research mathematics and in mathematical Olympiads. Already in the late 1960s, as an undergraduate student, I realized that in fact these two activities have deep connections and each influences the other. Mathematical research is a treasure trove of beautiful ideas that could inspire new generations of math Olympiads problems. On the other hand, once a good Olympiad problem is solved, its solution often inspires further problems that only research can hope to settle. As a mathematical endeavour, our Federation needed a fourth direction dedicated to these links. This direction was formally added to our congresses, for the first time, at our Riga-2010 Congress and named *Building Bridges between Research and Competition Problems.*

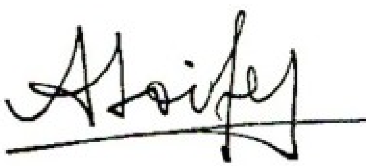
With these four directions, well represented in our journal *Mathematics Competitions*, edited by Jaroslav Švrček (who now has Editorial Board at his disposal), and in our congresses, the Federation has reached a certain level of maturity. Ever since Budapest-1988, we have held the Federation's sections at each of the quadrennial ICME congresses. Our

Immediate Past President Maria Falk de Losada has organized mini-conferences of the Federation at each of the two last ICME sites. We have held six congresses of the Federation: Waterloo-1990, Pravets-1994, Zhong Shan-1998, Melbourne-2002, Canbridge-2006, and Riga-2010. The 2014 Federation's Congress will take place in Colombia, organized by Maria Falk de Losada with the assistance of our Program Committee chaired by Senior Vice President Kiril Bankov.

Everything in our Federation seems to be working well. And yet, I see, with sadness, how some of the active participants of the years past have disappeared, leaving their countries unrepresented. The Executive is presently making an effort, led by Vice President Robert Geretschläger, to attract a greater participation. At the 2014 General Membership Meeting in Colombia, the Executive will propose to move the Federation elections of the officers from taking place at ICMEs to our own congresses, where general meetings are much better attended, and thus are more representative of our membership.

The future of the Federation is in our hands (or is it our minds?). So, roll up your sleeves and contribute fine papers to our journal and exciting talks to our congresses. Assemble project groups to share your—and thus our Federation's—valuable knowledge and experiences, in the form of books and events, with the world that is hungry for innovations of a mathematical kind!

Happy New 2013$^{\text{th}}$ Year, my friends!



Alexander Soifer
President of WFNMC

From the Editor

Welcome to *Mathematics Competitions* Vol. 25, No. 2.

First of all I would like to thank again the Australian Mathematics Trust for continued support, without which each issue (note the new cover) of the journal could not be published, and in particular Heather Sommariva, Bernadette Webster and Pavel Calábek for their assistance in the preparation of this issue.

In July 2012 at the WFNMC miniconference in Seoul the new MC's Editorial Board was formed. It comprises *Waldemar Pompe* (Poland), *Sergey Dorichenko* (Russia), *Alexander Soifer* and *Don Barry* (both USA).

Submission of articles:

The journal *Mathematics Competitions* is interested in receiving articles dealing with mathematics competitions, not only at national and international level, but also at regional and primary school level. There are many readers in different countries interested in these different levels of competitions.

- The journal traditionally contains many different kinds of articles, including reports, analyses of competition problems and the presentation of interesting mathematics arising from competition problems. Potential authors are encouraged to submit articles of all kinds.
- To maintain and improve the quality of the journal and its usefulness to those involved in mathematics competitions, all articles are subject to review and comment by one or more competent referees. The precise criteria used will depend on the type of article, but can be summarised by saying that an article accepted must be correct and appropriate, the content accurate and interesting,

> and, where the focus is mathematical, the mathematics fresh and well presented. This editorial and refereeing process is designed to help improve those articles which deserve to be published.

At the outset, the most important thing is that if you have anything to contribute on any aspect of mathematics competitions at any level, local, regional or national, we would welcome your contribution.

Articles should be submitted in English, with a black and white photograph and a short profile of the author. Alternatively, the article can be submitted on an IBM PC compatible disk or a Macintosh disk. We prefere LaTeX or TeX format of contributions, but any text file will be helpful.

Articles, and correspondence, can also be forwarded to the editor by mail to

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December 2012

Heaven and Earth

Andy Liu

Andy Liu is a professor of mathematics at the University of Alberta in Canada. His research interests span discrete mathematics, geometry, mathematics education and mathematics recreations. He edits the Problem Corner of the MAA's magazine Math Horizons. *He was the Chair of the Problem Committee in the 1995 IMO in Canada. His contribution to the 1994 IMO in Hong Kong was a major reason for him being awarded a David Hilbert International Award by the World Federation of National Mathematics Competitions.*

Heaven and Earth is a team contest adapted from the Russian game Mathematical Carousel, as described by Professor Olga Zaitseva-Ivrii of the University of Toronto.

There is a set of ordered Earthly Problems and a set of ordered Heavenly Problems, all requiring numerical answers. The same sets are used for all the teams. Each set of problems is given out one at a time. When the solution to one problem is submitted, correct or otherwise, the next problem in the same set will be given out. A team may elect to pass on a problem. This is considered an incorrect answer.

Earthly Problems are not worth any points. Their sole purpose is to get team members to Heaven. The first correctly answered Heavenly Problem is worth 3 points. Each successive problem correctly answered is worth 1 point more than the preceding one. An incorrect answer is worth 0 points. Moreover, the worth of the next correctly answered problem is back to 3 points.

The contest ends for a team when either there are no more Heavenly Problems, or there are no more Earthly Problems and all team members are back on Earth. The whole contest ends when it has ended for each team, or when a prescribed time period has expired. The team with the highest score wins the contest.

Three supervisors are required. King Arthur governs Earth while Queen Mary governs Heaven. St. Peter guards the Pearly Gates between Heaven and Earth.

All members of each team start on Earth. The team will receive from King Arthur Earthly Problems one at a time in the given order, until there are no more. The members work on it together. When they agree upon an answer, one team member is elected to present it to King Arthur. If the answer is correct, this member receives a Ticket to Heaven, and may pass through the Pearly Gates. In any case, the team will receive the next available Earthly Problem from King Arthur as long as there are still members on Earth.

When at least one member of a team is in Heaven, the team will receive from Queen Mary the Heavenly Problem one at a time in the given order, until there are no more. The team members work on it together. When they agree upon an answer, one team member is elected to present it to Queen Mary. If the answer is incorrect, this member receives a Rejction Slip, and must pass through the Pearly Gates and return to Earth. In any case, the team will receive the next available Heavenly Problem from Queen Mary as long as there are still members in Heaven.

1 Sample Contest

There are 25 Earthly Problems and 25 Heavenly Problems. The team size is 5 and time allowed is 3 hours. Each of the 6 teams is assigned a color. The problems are printed one on each card, with a set of color-coded cards for each team. Color-coded answer slips are also prepared. Heaven and Earth are represented by two separate classrooms. They may also be represented by two halves of a partitioned auditorium.

Earthly Problems

1. Nicolas and his son and Peter and his son were fishing. Nicolas and his son caught the same number of fish while Peter caught three times as many fish as his son. All of them together caught 25 fish. Determine the number of fish caught by Peter's son.

2. Five circles are drawn on a plane, with exactly five points lying on at least two circles. Determine the least possible number of parts into which the plane is divided by these circles.

3. To colour the six faces of a $3 \times 3 \times 3$ cube, 6 grams of paint is used. When the paint dried, the cube was cut into 27 small cubes. Determine the amount of paint, in grams, needed to colour all the uncoloured faces of these cubes.

4. A Knight is placed on a square of an infinite chessboard. This Knight can move either 2 squares North and 1 square East, or 2 squares East and 1 square North. After several moves it lands on a square located 2009 squares North and 2008 squares East from its initial position. Determine the total number of moves.

5. Each of A, B and C writes down 100 words in a list. If a word appears on more than one list, it is erased from all the lists. In the end, A's list has 61 words left and B's list has 80 words left. Determine the least number of words that can be left on C's list.

6. Determine the number of different rectangles which may be added to a 7×11 rectangle and a 4×8 rectangle so that the three can be put together, without overlapping, to form one rectangle.

7. Joshua and his younger brother George attend the same school. It takes 12 minutes for Joshua and 16 minutes for George to reach the school from their home. If George leaves home 1 minute earlier than Joshua, determine the number of minutes Joshua will take to catch up to him.

8. In a box are 6 red balls, 3 white balls, 2 green balls and 1 black ball. Determine the least number of balls one needs to draw at random from the box in order to be sure that balls of at least three different colours are drawn.

9. Every inhabitant on an island is either a Knight who always tells the truth, or a Knave who always lies. Each of 7 inhabitants seated at a round table claims that exactly one of his two neighbours is a Knave. Determine the number of Knights among these 7 inhabitants.

10. To a number 345 add two digits to its right so that 5-digit number is divisible by 36. Determine the average value of all such 5-digit numbers.

11. A rectangle which is not a square has integer side lengths. Its perimeter is n centimetres and its area is n square centimetres. Determine n.

12. In a soccer tournament where each pair of teams played exactly once, a win was worth 3 points, a draw 1 point and a loss 0 points. If one-fifth of the teams finished the tournament with 0 points, determine the number of teams that participated in the tournament.

13. In a target, the centre is worth 10 points, the inner rim 8 points and the outer rim 5 points. A trainee hits the centre and the inner rim the same number of times, and misses the target altogether one-quarter of the time. His total score is 99 points. Determine the total number of shots he has fired.

14. Each of 100 fishermen caught at least 1 fish, but nobody caught more than 7 fish. There were 98 fishermen who caught no more than 6 fish each, 95 fishermen who caught no more than 5 fish each, 87 fishermen who caught no more than 4 fish each, 80 fishermen who caught no more than 3 fish each, 65 fishermen who caught no more than 2 fish each, and 30 fishermen who caught no more than 1 fish each. Determine the total number of fish caught by all fishermen.

15. There is a pile of candies on a table. The first boy takes 1/10 of the candies. The second boy takes 1/10 of leftovers plus 1/10 of what the first has. The third boy takes 1/10 of leftovers plus 1/10 of what the first two boys have together. This continues until nothing is left. Determine the number of boys who have taken candies from this pile.

16. Each of A, B and C had candies. A gave some of her candies to B and C so that each had three times as much candy as before.

Then B gave some of his candies to C and A so that each of them had three times as much candy as before. Finally, C repeated the same procedure. In the end each child had 27 candies. Determine the number of candies A had in the beginning.

17. Determine the greatest common divisor of all 9-digit numbers in which each of the digits 1, 2, . . . , 9 appears exactly once.

18. A farmer was selling milk, sour cream, and cottage cheese. He had one bucket of cottage cheese, and the total number of buckets with milk and sour cream is five. The volumes of the buckets are 15, 16, 18, 19, 20 and 31 litres respectively. The farmer has twice as much milk as he has sour cream. Determine the volume, in litres, of the bucket containing cottage cheese.

19. Determine the smallest positive integer which leaves a remainder of 22 when divided by the sum of its digits.

20. The sum of the first n positive integers is a 3-digit number with all its digits equal. Determine n.

21. In a sequence $\{a_n\}$, $a_1 = 19$, $a_2 = 99$ and for $n \geq 2$, $a_{n+1} = a_n - a_{n-1}$. Determine a_{2009}.

22. The distances between a point inside a rectangle and three of rectangle's vertices are 3, 4 and 5. Determine the distance between this point and the fourth vertex, given that it is the greatest of the four distances.

23. Determine the number of integer solutions of equation $x^2y^3 = 6^{12}$.

24. There are 30 pikes which can eat one another. A pike is sated if it has eaten 3 other pikes (sated or not). Determine the maximal number of pikes that could be sated, including those which have been eaten.

25. The positive integers x, y and z have been increased by 1, 2, and 3 respectively. Determine the maximal possible value by which the sum of their reciprocals can change.

Heavenly Problems

1. Determine $x + y$ if $x^3 + y^3 = 9$ and $yx^2 + xy^2 = 6$.

2. Let x and y be positive integers such than $\gcd\{x, y\} = 999$. Suppose $\text{lcm}\{x, y\} = n!$. Determine the least value of n.

3. Determine the average of all 3-digit numbers which read the same forward and backward.

4. Peter has 8 white $1\times1\times1$ cubes. He wants to construct a $2\times2\times2$ cube with all its faces being completely white. Determine the minimal number of faces of small cubes that Basil must paint black in order to prevent Peter from fulfilling his task.

5. $ABCD$ is a quadrilateral with AB parallel to DC. If $AD = 11$, $DC = 7$ and $\angle D = 2\angle B$, determine AB.

6. We write the integers from 1 to 1000 in order along a circle. Starting from 1, we mark every 15th number: 1, 16, 31, and so on. The next marked number after 991 is 6 and we go around the circle until no more numbers are marked. Determine the number of unmarked numbers.

7. On an $n \times n$ board, there are 21 dominoes. Each domino covers exactly two squares and no two dominoes touch one another, even at a corner. Determine the minimal value of n.

8. The four sides and one diagonal of a quadrilateral have lengths 1, 2, 2.8, 5 and 7.5, not necessarily in that order. Determine which number was the length of the diagonal.

9. Simplify $8(3^2 + 1)(3^4 + 1)\cdots(3^{2^{10}} + 1) + 1$.

10. Each monkey collected the same number of nuts and threw one of them at every other monkey. Nuts thrown were lost. Each monkey still had at least two nuts left, and among them, they have 33 nuts left. Determine the number of nuts collected by each monkey.

11. From an 8×8 chessboard, the central 2×2 block rises up to form a barrier. Queens cannot be placed on the barrier, and may not attack one another across this barrier. Determine the maximal number of Queens which can be placed on the chessboard so that no two attack each other.

12. Let P be an interior point of triangle ABC. Perpendiculars are dropped from P to F on the side AB and and E on the side AC. The triangles PAF and PCE are congruent, but the vertices are not necessarily in corresponding orders. Determine the ratio $\frac{AE}{AC}$.

13. In a kindergarten, 17 children made an even number of postcards. Any group of 5 children made no more than 25 postcards while any group of 3 children made no less than 14 postcards. Determine the total number of postcards made.

14. Determine the number of integers of the form $\overline{abc}+\overline{cba}$, where $\overline{abc}$ and $\overline{cba}$ are three-digit numbers with $ac \neq 0$.

15. M is the midpoint of side AC of triangle ABC. If $\angle ABM = 2\angle BAM$ and $BC = 2BM$, determine the measure, in degrees, of the largest angle of ABC.

16. A judge knows that among 9 coins that are identical in appearance, there are exactly 3 coins each weighing 3 grams, 3 coins each weighing 2 grams and 3 coins each weighing 1 gram. An expert wants to prove to the judge how much each of the coins weighs. He has a balance that shows which of two groups of coins is heavier, or that they have the same total weight. Determine the least number of weighings needed to persuade the judge.

17. On an island, each inhabitant is either a Knight who always tells the truth, or a Knave who always lies. Each inhabitant earns a different amount of money and works a different number of hours. Each inhabitant says, "Less than 10 inhabitants work more hours than I do." Each also said, "At least 100 inhabitants make more money than I do." Determine the number of inhabitants on this island.

18. Find the second smallest positive multiple of 11 with digit-sum 600.

19. In a chess tournament with an odd number of players, each player plays every other exactly once. A win is worth 1 point and a draw is worth $\frac{1}{2}$ point. The player in the fourth position from the bottom gets more points than the player in the third position from the bottom. Each player except the bottom three gets half of his or her points playing against the bottom three. Determine the number of players in the tournament.

20. $ABCD$ is a parallelogram. M and N are points on the sides AB and AD respectively, such that $AB = 4AM$ and $AD = 3AN$. Let K be the point of intersection of MN and AC. Determine the ratio $\frac{AC}{AK}$.

21. Determine the largest real number t such that the two polynomials $x^4 + tx^2 + 1$ and $x^3 + tx + 1$ have a common root.

22. Determine the smallest positive multiple of 777 such that the sum of its digits is the least.

23. Determine the minimum number of points marked on the surface of a cube so that no two faces of the cube contain the same number of marked points. A point at a corner or on a side of a face is considered to be in that face.

24. In a right triangle, the smallest height is one-quarter the length of the hypotenuse. Determine the measure, in degrees, of the smallest angle of this triangle.

25. The real numbers $x < y$ satisfy $(x^2 + xy + y^2)\sqrt{x^2 + y^2} = 185$ and $(x^2 - xy + y^2)\sqrt{x^2 + y^2} = 65$. Determine the smallest value of y.

Earthly Answer Key

1	5	2	7	3	12	4	1339	5	41
6	4	7	3	8	10	9	0	10	34560
11	18	12	5	13	20	14	245	15	10
16	55	17	9	18	20	19	689	20	36
21	-99	22	$\sqrt{32}$	23	18	24	9	25	$\frac{23}{12}$

Heavenly Answer Key

1	3	2	37	3	550	4	2	5	18
6	800	7	11	8	28	9	3^{2048}	10	13
11	10	12	$\frac{1}{2}$	13	84	14	170	15	2
16	120	17	110	18	34989 9...9 9...9	19	9	20	7
21	-2	22	10101	23	6	24	15	25	-3

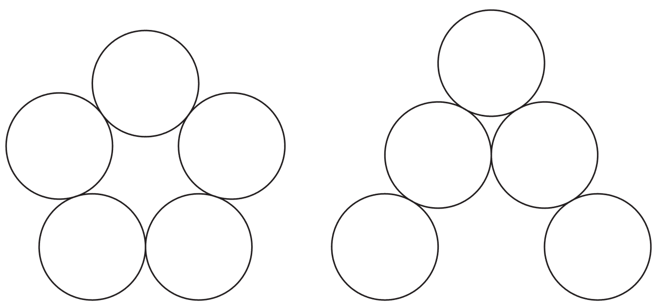
2 Solutions to Earthly Problems

1. Nicolas and his son and Peter and his son were fishing. Nicolas and his son caught the same number of fish while Peter caught three times as many fish as his son. All of them together caught 25 fish. Determine the number of fish caught by Peter's son.

Solution. Nicolas and his son caught an even number of fish, as did Peter and his son. If there were four people, the total number of fish caught could not have been 25. It follows that Nicolas was Peter's son, and it easy to see he caught 5 fish.

2. Five circles are drawn on a plane, with exactly five points lying on at least two circles. Determine the least possible number of parts into which the plane is divided by these circles.

Solution. We have at least 6 regions since the interior of each circle is one, and there is also the infinite region. Since there are five circles and five points lying on at least two circles, some of the circles must form a closed "ring". This forces an additional region. The diagram below shows that 7 regions can be attained in several ways.



3. To colour the six faces of a $3 \times 3 \times 3$ cube, 6 grams of paint is used. When the paint dried, the cube was cut into 27 small cubes. Determine the amount of paint, in grams, needed to colour all the uncoloured faces of these cubes.

Solution. The faces of the original cube consist of $6 \times 3 \times 3 = 54$ faces of small cubes. The total number of faces of the small cubes is $27 \times 6 = 162$. The number of coloured faces is $162 - 54 = 158 = 2 \times 54$. Hence $2 \times 6 = 12$ grams of paint are needed.

4. A Knight is placed on a square of an infinite chessboard. This Knight can move either 2 squares North and 1 square East, or 2 squares East and 1 square North. After several moves it lands on a square located 2009 squares North and 2008 squares East from its initial position. Determine the total number of moves.

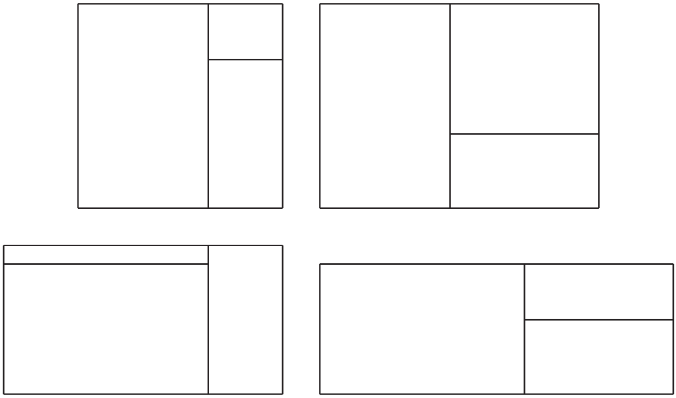
Solution. In each move, the Knight covers three squares towards the North and the East. At the end, $2008 + 2009 = 4017$ squares are covered. Hence the number of moves is $4017 : 3 = 1339$.

5. Each of A, B and C writes down 100 words in a list. If a word appears on more than one list, it is erased from all the lists. In the end, A's list has 61 words left and B's list has 80 words left. Determine the least number of words that can be left on C's list.

Solution. A has 39 words crossed off while B has 20. Even if these are all against C, C will still have $100 - 39 - 20 = 41$ words left on the list.

6. Determine the number of different rectangles which may be added to a 7×11 rectangle and a 4×8 rectangle so that the three can be put together, without overlapping, to form one rectangle.

Solution. Since there are four different ways of abutting two rectangles, the third rectangle can be chosen in at most 4 ways. The following diagram shows that they are indeed different, being 3×4, 7×8, 1×11 and 3×8 respectively.



7. Joshua and his younger brother George attend the same school. It takes 12 minutes for Joshua and 16 minutes for George to reach the school from their home. If George leaves home 1 minute earlier than Joshua, determine the number of minutes Joshua will take to catch up to him.

Solution. The ratio of the brothers' speeds is 4:3. Joshua will catch up with George after 3 minutes because George will have been moving for 4 minutes.

8. In a box are 6 red balls, 3 white balls, 2 green balls and 1 black ball. Determine the least number of balls one needs to draw at random from the box in order to be sure that balls of at least three different colours are drawn.

Solution. The largest number of balls drawn without getting balls of at least three different colours is 9, by drawing all 6 red balls and 3 white balls. Hence if we draw 10 balls, we are guaranteed to have balls of at least three different colours.

9. Every inhabitant on an island is either a Knight who always tells the truth, or a Knave who always lies. Each of 7 inhabitants seated at a round table claims that exactly one of his two neighbours is a Knave. Determine the number of Knights among these 7 inhabitants.

Solution. Each Knight must be seated between a Knight and a Knave and each Knave must be seated either between two Knights or two Knaves. If there is at least one Knight, then the Knights and Knaves must occupy alternate seats, but 7 is an odd number. Hence all 7 are Knaves.

10. To a number 345 add two digits to its right so that 5-digit number is divisible by 36. Determine the average value of all such 5-digit numbers.

Solution. To be divisible by 2, the last digit must be even number.In order to be divisible by 9, the sum of the five digits must be a multiple of 9. Hence the sum of the last two digits must be 6 or 15. In the first case, these two digits form the number 06, 24, 42 or 60. This number must be divisible by 4, which eliminates 06 and 42. In the second case, these two digits must be 78 or 96, with 78 eliminated. The average of 34524, 34560 and 34596 is 34560.

11. A rectangle which is not a square has integer side lengths. Its perimeter is n centimetres and its area is n square centimetres. Determine n.

Solution. Let the dimensions of the rectangle be $a \times b$ with $a < b$. The perimeter is $2(a+b)$ and the area is ab. Hence $ab = 2a + 2b$, so that

$$(a-2)(b-2) = ab - 2a - 2b + 4 = 4.$$

Hence $a - 2 = 1$ and $b - 2 = 4$, yielding $a = 3,\ b = 6$ and $n = 18$.

12. In a soccer tournament where each pair of teams played exactly once, a win was worth 3 points, a draw 1 point and a loss 0 points. If one-fifth of the teams finished the tournament with 0 points, determine the number of teams that participated in the tournament.

Solution. In any tournament where each pair of teams played exactly once, at most one team can have no wins and no draws. Since one-fifth of the teams finished with 0 points, the number of teams is 5.

13. In a target, the centre is worth 10 points, the inner rim 8 points and the outer rim 5 points. A trainee hits the centre and the inner rim the same number of times, and misses the target altogether one-quarter of the time. His total score is 99 points. Determine the total number of shots he has fired.

Solution. The total score from hitting the centre and the inner rim is 18, 36, 54, 72 or 90. The total score from hitting the outer rim will then be 81, 63, 45, 27 or 9 respectively. Of these, only 45 is a multiple of 5. Hence the outer rim is hit 9 times and each of the centre and the inner rim is hit 3 times. Since 15 shots hit the target, 5 shots miss. Hence the total number of shots fired is 20.

14. Each of 100 fishermen caught at least 1 fish, but nobody caught more than 7 fish. There were 98 fishermen who caught no more than 6 fish each, 95 fishermen who caught no more than 5 fish each, 87 fishermen who caught no more than 4 fish each, 80 fishermen who caught no more than 3 fish each, 65 fishermen who caught no more than 2 fish each, and 30 fishermen who caught no more than 1 fish each. Determine the total number of fish caught by all fishermen.

Solution. The numbers of fishermen who caught 7, 6, 5, 4, 3, 2 and 1 fish are respectively $100 - 98 = 2$, $98 - 95 = 3$, $95 - 87 = 8$, $87 - 80 = 7$, $80 - 65 = 15$, $65 - 30 = 35$ and $30 - 0 = 30$. Hence the total number of fish caught was $2 \times 7 + 3 \times 6 + 8 \times 5 + 7 \times 4 + 15 \times 3 + 35 \times 2 + 30 \times 1 = 245$.

15. There is a pile of candies on a table. The first boy takes 1/10 of the candies. The second boy takes 1/10 of leftovers plus 1/10 of what the first has. The third boy takes 1/10 of leftovers plus 1/10 of what the first two boys have together. This continues until nothing is left. Determine the number of boys who have taken candies from this pile.

Solution. Each boy, including the first, takes 1/10 of what has already been taken plus 1/10 of what is left behind. In other words, each boy takes 1/10 of everything. The number of boys needed to take everything is 10.

16. Each of A, B and C had candies. A gave some of her candies to B and C so that each had three times as much candy as before. Then B gave some of his candies to C and A so that each of them had three times as much candy as before. Finally, C repeated the same procedure. In the end each child had 27 candies. Determine the number of candies A had in the beginning.

Solution. Note that the total number of candies, namely 81, remained constant throughout. Suppose A started with n candies. Then B and C started with $81-n$ together. After A's gift, she had $81-3(81-n)=3n-162$ candies left. After B's and C's gifts, she had $9(3n-162)=27$. Hence $n=55$.

17. Determine the greatest common divisor of all 9-digit numbers in which each of the digits 1, 2, . . . , 9 appears exactly once.

Solution. Each of these numbers is divisible by 9 since so is its digit-sum. Their greatest common divisor must divide their pairwise differences, one of which is $123456798-123456789=9$. Hence their greatest common divisor is 9.

18. A farmer was selling milk, sour cream, and cottage cheese. He had one bucket of cottage cheese, and the total number of buckets with milk and sour cream is five. The volumes of the buckets are 15, 16, 18, 19, 20 and 31 litres respectively. The farmer has twice as much milk as he has sour cream. Determine the volume, in litres, of the bucket containing cottage cheese.

Solution. The total volume, in litres, of milk and sour cream is a multiple of 3. The volumes of the buckets, in litres, are congruent modulo 3 to 0, 1, 0, 1, 2 and 1 respectively. The total volume is congruent modulo 3 to 2. Hence the volume of cottage cheese must also be congruent modulo 3 to 2, so that it must be 20 litres. Indeed, the farmer had $15+18=33$ litres of sour cream and $16+19+31=66$ litres of milk.

19. Determine the smallest positive integer which leaves a remainder of 22 when divided by the sum of its digits.

Solution. The minimal value of the digit-sum is 23, with digits (5,9,9), (6,8,9), (7,7,9) or (7,8,8). As it happens, the remainder is indeed 22 when 689 is divided by 23, but not when 599 is divided by 23. Hence 689 is the smallest such number.

20. The sum of the first n positive integers is a 3-digit number with all its digits equal. Determine n.

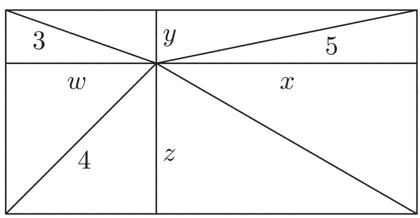
Solution. A 3-digit number with all its digits equal is divisible by 37. The sum of the first n positive integer is $\frac{n(n+1)}{2}$. Since 37 is prime, it divides n or $n+1$. To minimize the sum, we take $n = 36$ so that $\frac{n(n+1)}{2} = 666$.

21. In a sequence $\{a_n\}$, $a_1 = 19$, $a_2 = 99$ and for $n \geq 2$, $a_{n+1} = a_n - a_{n-1}$. Determine a_{2009}.

Solution. Iteration yields $a_3 = 80$, $a_4 = -19$, $a_5 = -99$, $a_6 = -80$, $a_7 = 19$ and $a_8 = 99$. Hence the sequence is of period 6. Since the remainder when 2009 is divided by 6 is 5, we have $a_{2009} = a_5 = -99$.

22. The distances between a point inside a rectangle and three of rectangle's vertices are 3, 4 and 5. Determine the distance between this point and the fourth vertex, given that it is the greatest of the four distances.

Solution. The fourth vertex is opposite to the one at the smallest distance from the given point, namely 3. Let the distances from the given point to the four sides of the rectangle be w, x, y and z as shown in the diagram below. Then $w^2 + y^2 = 9$, $w^2 + z^2 = 16$ and $x^2 + y^2 = 25$. It follows that $x^2 + z^2 = 16 + 25 - 9 = 32$, and the distance from the given point to the fourth vertex is $\sqrt{32}$.



23. Determine the number of integer solutions of equation $x^2y^3 = 6^{12}$.

Solution. Since 2 and 3 are relatively prime, each of 2^{12} and 3^{12} is the product of a square and a cube. In the former case, the two factors may be $(2^0)^2(2^4)^3$, $(2^3)^2(2^2)^3$ or $(2^6)^2, (2^0)^3$, with three possibilities for the latter case. Since x can be positive or negative, the total number of integer solutions is $2 \times 3^2 = 18$.

24. There are 30 pikes which can eat one another. A pike is sated if it has eaten 3 other pikes (sated or not). Determine the maximal number of pikes that could be sated, including those which have been eaten.

Solution. Each pike is eaten by at most one other pike, so that the 29 pikes being eaten can make 9 pikes sated. We may have as many as 9 sated pikes by having the 30 pikes line up in a row. Pikes 1, 2, and 3 will be eaten by pike 4, pikes 4, 5 and 6 will be eaten by pike 7, and so on. The sated pikes are therefore pikes 4, 7, 10, 13, 16, 19, 22, 25 and 28, for a total number of 9.

25. The positive integers x, y and z have been increased by 1, 2, and 3 respectively. Determine the maximal possible value by which the sum of their reciprocals can change.

Solution. We have

$$\frac{1}{x} - \frac{1}{x+1} = \frac{1}{x(x+1)}, \qquad \frac{1}{y} - \frac{1}{y+2} = \frac{2}{y(y+2)}$$

and

$$\frac{1}{z} - \frac{1}{z+3} = \frac{3}{z(z+3)}.$$

These are decreasing functions of x, y and z respectively, so that their minima occur at $x = y = z = 1$. Thus the maximum possible value of change is $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} = \frac{23}{12}$.

3 Solutions to Heavenly Problems

1. Determine $x + y$ if $x^3 + y^3 = 9$ and $yx^2 + xy^2 = 6$.

Solution. We have $(x + y)^3 = x^3 + y^3 + 3(yx^2 + xy^2) = 27$. Hence $x + y = 3$.

2. Let x and y be positive integers such than $\gcd\{x, y\} = 999$. Suppose $\operatorname{lcm}\{x, y\} = n!$. Determine the least value of n.

Solution. Since $999 = 3^3 \times 37$, the least value of n is 37. This can be attained if $x = 999$ and $y = 37!$.

3. Determine the average of all 3-digit numbers which read the same forward and backward.

Solution. Any digit can be the tens-digit so that they appear with equal likelihood. The average of the tens-digit is $(0 + 10 + 20 + \cdots + 90) : 10 = 45$. Any non-zero digit can be the units-digit. It follows that the average is $(1 + 2 + \cdots + 9) : 9 = 5$. Similarly, the average of the hundreds-digit is $(100 + 200 + \cdots + 900) : 9 = 500$. The average of the numbers is $500 + 45 + 5 = 550$.

4. Peter has 8 white $1 \times 1 \times 1$ cubes. He wants to construct a $2 \times 2 \times 2$ cube with all its faces being completely white. Determine the minimal number of faces of small cubes that Basil must paint black in order to prevent Peter from fulfilling his task.

Solution. If Basil paints only one face, Peter can hide it easily. However, if Basil paints two opposite faces of a small cube, it is not possible for Peter to hide both. Hence the minimal number of faces Basil must paint is 2.

5. $ABCD$ is a quadrilateral with AB parallel to DC. If $AD = 11$, $DC = 7$ and $\angle D = 2\angle B$, determine AB.

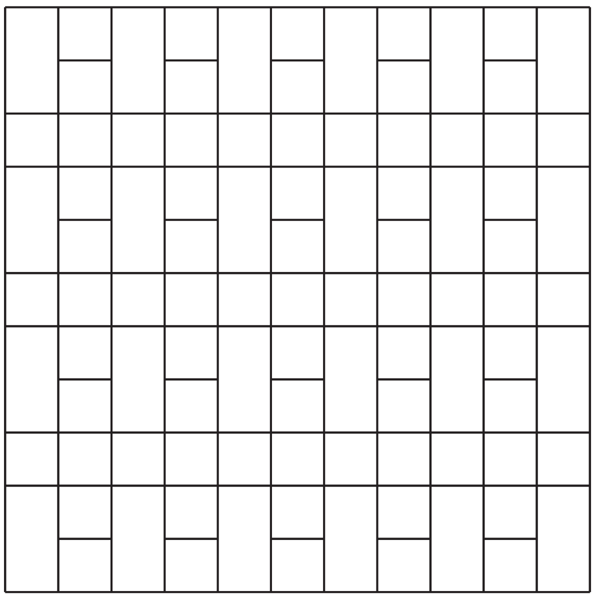
Solution. Let the bisector of $\angle D$ cut AB at E. Then $BCDE$ is a parallelogram since it has a pair of parallel opposite edges and a pair of equal opposite angles. Hence $BE = CD = 7$. Moreover, BC is parallel to ED, so that $\angle AED = \angle B = \angle ADE$. Hence $AE = AD = 11$ so that $AB = AE + BE = 18$.

6. We write the integers from 1 to 1000 in order along a circle. Starting from 1, we mark every 15th number: 1, 16, 31, and so on. The next marked number after 991 is 6 and we go around the circle until no more numbers are marked. Determine the number of unmarked numbers.

Solution. We extend 1000 to 3000 so that the last marked number is 2986. Then $3000 : 15 = 200$ numbers have been marked. We now replace 1001 and 2001 by 1, 1001 and 2002 by 2, and so on, so that every number from 1 to 1000 appears 3 times. A number cannot be marked more than once since all marked numbers are congruent to 1 modulo 5. Hence the number of unmarked numbers is $1000 - 200 = 800$.

7. On an $n \times n$ board, there are 21 dominoes. Each domino covers exactly two squares and no two dominoes touch one another, even at a corner. Determine the minimal value of n.

Solution. Expand each domino by half a unit on all four sides so that each becomes a 2×3 tile. Expand the board similarly so that it becomes $(n+1) \times (n+1)$. Any placement of the dominoes on the original board with no touching corresponds to a placement of the tiles on the expanded board with no overlapping, and vice versa. The area of the expanded board is $(n+1)^2$ while the total area of the 21 tiles is 126. It follows that $n \geq 11$. The following diagram shows that an 11×11 board can hold 24 dominoes without touching.



8. The four sides and one diagonal of a quadrilateral have lengths 1, 2, 2.8, 5 and 7.5, not necessarily in that order. Determine which number was the length of the diagonal.

Solution. Each of the five segments with given lengths is in a triangle with two others. The segment of length 7.5 must be in a triangle with the segments of lengths 2.8 and 5 because only they are the only ones whose total length exceeds 7.5, as required by the Triangle Inequality. Now the segments of length 1 and 2 must form a triangle with the segment of length 2.8 since it is the only one of length less than 1+2, again as required by the Triangle Inequality. It follows that the diagonal is of length 2.8.

9. Simplify $8(3^2+1)(3^4+1)\cdots(3^{2^{10}}+1)+1$.

Solution. We have

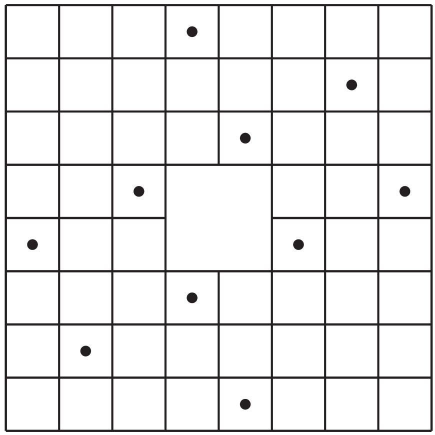
$$\begin{aligned} &8(3^2+1)(3^4+1)\cdots(3^{2^{10}}+1)+1 \\ &\qquad = (3^2-1)(3^2+1)(3^4+1)\cdots(3^{2^{10}}+1)+1 \\ &\qquad = (3^4-1)(3^4+1)\cdots(3^{2^{10}}+1)+1 \\ &\qquad = (3^8-1)(3^8+1)\cdots(3^{2^{10}}+1)+1 \\ &\qquad = \cdots = 3^{2^{11}}-1+1 = 3^{2048}. \end{aligned}$$

10. Each monkey collected the same number of nuts and threw one of them at every other monkey. Nuts thrown were lost. Each monkey still had at least two nuts left, and among them, they have 33 nuts left. Determine the number of nuts collected by each monkey.

Solution. We either have 11 monkeys each with 3 nuts left or 3 monkeys each with 11 nuts left. Since each monkey throws a nut at every monkey, it must have collected $3+11-1=13$ nuts.

11. From an 8×8 chessboard, the central 2×2 block rises up to form a barrier. Queens cannot be placed on the barrier, and may not attack one another across this barrier. Determine the maximal number of Queens which can be placed on the chessboard so that no two attack each other.

Solution. Each of the clear rows can hold at most 1 Queen while each of the divided rows can hold at most 2 Queens. It follows that the number of Queens is at most $6\times 1+2=10$. The diagram below shows that 10 non-attacking Queens can be placed.



12. Let P be an interior point of triangle ABC. Perpendiculars are dropped from P to F on the side AB and and E on the side AC. The triangles PAF and PCE are congruent, but the vertices are not necessarily in corresponding orders. Determine the ratio $\frac{AE}{AC}$.

Solution. Since congruent right triangles must have equal hypotenuses, we have $PA = PC$. Since PE is perpendicular to AC, it must bisect it so that $\frac{AE}{AC} = \frac{1}{2}$.

13. In a kindergarten, 17 children made an even number of postcards. Any group of 5 children made no more than 25 postcards while any group of 3 children made no less than 14 postcards. Determine the total number of postcards made.

Solution. Consider the two children with the smallest total of cards made. Suppose this total is at most 8. Since any group of children made no less than 14, everyone else makes at least 6. This yields a total of at least $8 + 15 \times 6 = 98$, for an average exceeding 5. This contradicts the condition that any group of 5 children made no more than 25 postcards. It follows that the smallest total is at least 9. If it is at least 10, then everyone made 5 postcards for a total of 85, an odd number. Hence the smallest total is 9, with the two children making 4 and 5 postcards respectively. It follows that every other child made 5 postcards, for a total of 84.

14. Determine the number of integers of the form $\overline{abc}+\overline{cba}$, where $\overline{abc}$ and $\overline{cba}$ are three-digit numbers with $ac \neq 0$.

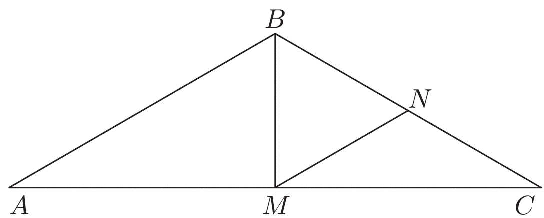
Solution. The digit b can be any of $0, 1, 2, \ldots, 9$ while the sum of the digits a and c can be any number of $2, 3, 4, \ldots, 18$. Hence the total number of integers of the form $101(a + c) + 10b$ is $(9 - 0 + 1)(18 - 2 + 1) = 170$.

15. M is the midpoint of side AC of triangle ABC. If $\angle ABM = 2\angle BAM$ and $BC = 2BM$, determine the measure, in degrees, of the largest angle of ABC.

Solution. Let N be the midpoint of BC so that $BN = NC = BM$. Moreover, MN is parallel to AB. Now

$$\begin{aligned} 2\angle BAM = \angle ABM = \angle BMN = \angle BNM& \\ = \angle NMC + \angle NCM = 2\angle NCM&. \end{aligned}$$

It follows that $AB = BC$ so that BM is perpendicular to AC. Hence $90° = \angle BAM + \angle ABM = 3\angle BAM$ so that $\angle ABC$ is the largest angle of triangle ABC, and its measure in degrees is 120.



16. A judge knows that among 9 coins that are identical in appearance, there are exactly 3 coins each weighing 3 grams, 3 coins each weighing 2 grams and 3 coins each weighing 1 gram. An expert wants to prove to the judge how much each of the coins weighs. He has a balance that shows which of two groups of coins is heavier, or that they have the same total weight. Determine the least number of weighings needed to persuade the judge.

Solution. Clearly, 1 weighing is not sufficient. However, 2 weighings are. Let A, B and C be the coins each weighing 3 grams, D, E and F be the coins each weighing 2 grams, and G, H and I be the coins each weighing 1 gram. In the first weighing, show that A is equal in weight to G, H and I combined. This is only possible if A indeed weighs 3 grams while each of G, H and I weighs 1 gram. In the second weighing, show that B and C combined are equal in weight to D, E and F combined. This shows that each of B and C weighs 3 grams while each of D, E and F weighs 2 grams.

17. On an island, each inhabitant is either a Knight who always tells the truth, or a Knave who always lies. Each inhabitant earns a different amount of money and works a different number of hours. Each inhabitant says, "Less than 10 inhabitants work more hours than I do." Each also said, "At least 100 inhabitants make more money than I do." Determine the number of inhabitants on this island.

Solution. There may be exactly 110 inhabitants on this island. Let inhabitant n work n hours and make n dollars. For $1 \le n \le 10$, both statements of inhabitant n are true, making them Knights. For $11 \le n \le 110$, both statements of inhabitant n are false, making them Knaves. Suppose there are at least 111 inhabitants. Then the second statement is true for at least 11 of them, making them Knights. However, for the one working the highest number of hours among them, the first statement will be false, which is a contradiction. Suppose there are at most 100 inhabitants. Then the second statement is false for all of them, making them Knaves. However, for the one working the highest number of hours, the first statement will be true, which is a contradiction. Finally, suppose the number of inhabitants is at least 101 but at most 109. Then the first statement is true for 10 of them, making them Knights. However, for the one making the least amount of money among them, the second statement will be false, which is a contradiction. It follows that the number of inhabitants must be 110.

18. Find the second smallest positive multiple of 11 with digit-sum 600.

Solution. To minimize the multiple, we should use as many copies of the digit 9 as we can. Since $600 = 9 \times 66 + 6$, we can use 66 copies of 9 plus some digits adding up to 6. Thus the smallest such multiple of 11 is $33999\ldots9$ and the second smallest is $34989\ldots9$, where $9\ldots9$ denotes 64 copies of the digit 9.

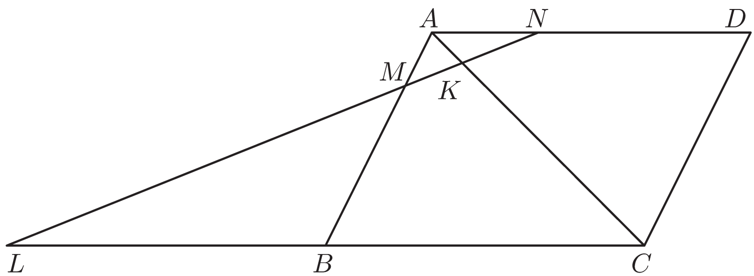
19. In a chess tournament with an odd number of players, each player plays every other exactly once. A win is worth 1 point and a draw is worth $\frac{1}{2}$ point. The player in the fourth position from the bottom gets more points than the player in the third position from the bottom. Each player except the bottom three gets half of his or her points playing against the bottom three. Determine the number of players in the tournament.

Solution. Let there be $n+3$ players. Each of the n players not in the bottom three gets at most 3 points from the bottom three, so that among them, they can get at most $3n$ points this way. Among themselves, $\frac{1}{2}n(n-1)$ games are played, so that this many points are obtained. From the given condition, $\frac{1}{2}n(n-1) \leq 3n$ so that $n \leq 7$. Note that n is even. Suppose $n = 6$. Let all the games among themselves be draws, so that each gets $2\frac{1}{2}$ points this way. Each can also get the same number of points from the bottom three with 2 wins and 1 draw. If each of the bottom three has 2 draws against them, and 2 draws among themselves, each will have 2 points, which is less than $2\frac{1}{2}$ points. Suppose $n = 4$. These players get 6 points from one another, so that they must also get 6 points from the bottom three. Since there are 12 games played between the two groups, the bottom three also get 6 points from the top players. Moreover, they get 3 points among themselves. The total score of the top four players is 6+6 for an average of 3. The total score of the bottom three players is 6+3 for an average of 3 also. Thus the player in the fourth position from the bottom cannot get more points than the player in the third position from the bottom. Finally, $n = 2$ also leads to a contradiction. Hence the total number of players is 6+3=9.

20. $ABCD$ is a parallelogram. M and N are points on the sides AB and AD respectively, such that $AB = 4AM$ and $AD = 3AN$. Let K be the point of intersection of MN and AC. Determine the ratio $\frac{AC}{AK}$.

Solution. Extend NM and CB, intersecting at L. We have $BM = 3AM$. Since triangles BLM and ANM are similar, $BL = 3AN$.

Note that we have $BC = AD = 3AN$, so that $LC = 6AN$. Since triangles CLK and ANK are similar, $CK = 6AK$ so that $\frac{AC}{AK} = 7$.



21. Determine the largest real number t such that the two polynomials $x^4 + tx^2 + 1$ and $x^3 + tx + 1$ have a common root.

Solution. Any common root of the two polynomials is also a root of

$$x^4 - x^3 + tx^2 - tx = x(x-1)(x^2+t).$$

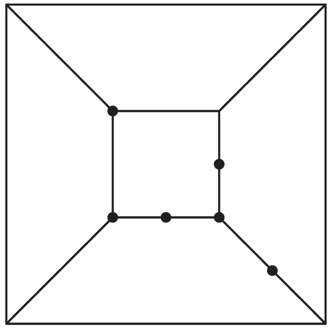
We cannot have $x^2 + t = 0$ since $x^2(x^2+t) + 1 = 0$. We cannot have $x = 0$ since 0 is not a root of $x^3 + tx + 1 = 0$. Hence $x = 1$. From $1^3 + t + 1 = 0$, we have $t = -2$.

22. Determine the smallest positive multiple of 777 such that the sum of its digits is the least.

Solution. Since 777 is divisible by 3, the digit-sum of any of its positive multiples is at least 3. It is routine to verify that $777 \times 13 = 10101$ is the smallest such multiple of 777.

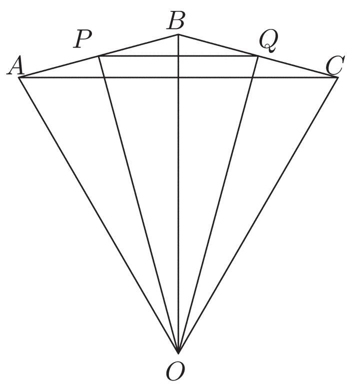
23. Determine the minimum number of points marked on the surface of a cube so that no two faces of the cube contain the same number of marked points. A point at a corner or on a side of a face is considered to be in that face.

Solution. To minimize the total number of points, the numbers of points on the six faces should be 0, 1, 2, 3, 4 and 5. Hence 5 points in total is necessary. This is however not sufficient, because all 5 points must be on the same face. The opposite face will have 0 points, but the other four faces must have 1+2+3+4=10 points in all. Hence each point must belong to two of these four faces, so that they are all corners of the face containing them. This is impossible since a face has only four corners. The following diagram shows a placement of 6 points satisfying the condition of the problem.



24. In a right triangle, the smallest height is one-quarter the length of the hypotenuse. Determine the measure, in degrees, of the smallest angle of this triangle.

Solution. Make four copies of the right triangles OPA, OPB, OQB and OQC as shown in the diagram below. Then PQ is twice the length of the smallest height, and $AC = 2PQ$. It follows that $AC = OB$ and OAC is equilateral. Hence the smallest angle of the right triangle, in degrees, is $60 : 4 = 15$.



25. The real numbers $x < y$ satisfy $(x^2+xy+y^2)\sqrt{x^2+y^2} = 185$ and $(x^2-xy+y^2)\sqrt{x^2+y^2} = 65$. Determine the smallest value of y.

Solution. Subtraction yields $2xy\sqrt{x^2+y^2} = 120$. Adding $xy\sqrt{x^2+y^2} = 60$ to the first equation, we have $(x+y)^2\sqrt{x^2+y^2} = 245$. Subtracting from the second yields $(x-y)^2\sqrt{x^2+y^2} = 5$. Division yields

$$\frac{x+y}{x-y} = -7$$

so that $4x = 3y$. Substituting into $xy\sqrt{x^2+y^2} = 60$ yields $(x, y) = (-4, -3)$ or (3,4). Hence the smallest value of y is -3.

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Isosceles Triangles on Vertices of a Regular Polygon

Jaromír Šimša

Jaromír Šimša graduated from Kharkov State University in 1979. He received his Ph.D. degree at Masaryk University (Czech Rep.) in 1984, where he has worked as Associate Professor from 1993 till now. Besides his scientific interests (differential and functional equations), he has been intesively involved with the Czech and Slovak Mathematical Olympiad. Over the last two decades, he proposed tens of problems for this competition, lectured at the annual training camps for the Czech IMO team, and acted for many times as leader. One of his proposals was accepted as Problem 3 of the 1995 IMO in Toronto.

The first Czech-Polish-Slovak Mathematics Junior Competition was held in Mszana Dolna (Poland) in May 2012. In the individual part of the competition the contestants were asked to solve five problems in 3 hours. One of them was posed by the author of this contribution and reads as follows.

Statement of the problem. *Prove that if we choose arbitrarily 51 vertices of a regular 101-gon, then three chosen points are vertices of an isosceles triangle.*

Most of the successful solvers developed a procedure which might originate as an attempt to solve the problem by standard using the well-known *pigeonhole principle.* Thus we first show how this principle "works" in proving a related conjecture with an integer parameter k, $3 \le k \le 101$.

Conjecture. *If we choose arbitrarily k vertices of a regular 101-gon, then three chosen points are vertices of an isosceles triangle.*

Of course, the conjecture is false if k is sufficiently small. The reader can easily find some counterexamples, e.g. for $k = 5$. On the other hand, it is clear that if the conjecture is true for some k ($k < 101$), then the same holds for any k' such that $k < k' \le 101$.

To focus on the conjecture with a given k, suppose that V is any k-element set of vertices of a regular 101-gon $A_1A_2 \dots A_{101}$. We can clearly assume that $A_1 \in V$. Since all the triangles

$$A_1A_2A_{101},\ A_1A_3A_{100},\ \dots,\ A_1A_{51}A_{52}$$

are isosceles, the conjecture holds if the set V contains both vertices of some pair from the following list

$$\{A_2, A_{101}\},\ \{A_3, A_{100}\},\ \dots,\ \{A_{51}, A_{52}\}. \qquad (1)$$

The condition from the last sentence is fulfilled, as we now explain, under the assumption that $k - 1 > 50$. In fact, the pairs in (1) contain all the vertices A_i with $i \ne 1$, so there are exactly $k - 1$ points from V that occur in the all 50 pairs (1). The inequality $k - 1 > 50$ thus ensures the existence of a pair in (1) with both elements from V. The last fact is so obvious that there is even no need to refer to any principle. Although we have used the pigeonhole principle in such a simple way, we have succeeded in proving the conjecture for any $k \ge 52$.

One can obviously ask: Is it possible (in some way) to benefit from pairs (1) also in the case $k = 51$, the case which was assigned to young contestants in Dolna Mszana? It is clear that they could restrict their attention only to the case when a 51-element V contains (besides A_1) exactly one vertex from each of the 50 pairs in (1). Fortunately, this condition is so simple and rich in content that it enables to prove the existence of an isosceles triangle with vertices in V. The contestants did it in several ways, let us now describe one of them. Preserving the previous notations and assumptions, we use the first pair $\{A_2, A_{101}\}$ in (1) to conclude that either $A_2 \in V$, or $A_{101} \in V$. This can be specified as $A_2 \in V$ and $A_{101} \notin V$ (otherwise the set V can be reflected with respect to the perpendicular bisector of the segment A_2A_{101}). Suppose that each of the isosceles triangles $A_1A_2A_3$ and $A_1A_{98}A_{100}$ has a vertex not lying in V (otherwise the proof is done). Thus from $A_1, A_2 \in V$ we have $A_3 \notin V$ and hence $A_{100} \in V$ (because of the pair $\{A_3, A_{100}\}$ in (1)). Furthermore, from $A_1, A_{100} \in V$ we have $A_{98} \notin V$ and hence $A_5 \in V$ (because of the pair $\{A_5, A_{98}\}$ in (1)). Consequently, $A_2A_5A_{100}$ is an isosceles triangle with vertices in V, which completes the proof of the conjecture in the required case $k = 51$.

Only one contestant in Mszana Dolna, namely *Pavel Turek* from the Czech Republic, showed that the stated problem (i.e. the conjecture with $k = 51$) can be solved by direct method using the pigeonhole principle. Let us present his solution now. Suppose that V is any 51-element set of vertices of a regular 101-gon $A_1A_2 \ldots A_{101}$. Denote by S the set of all segments A_iA_j with endpoints $A_i \neq A_j$ lying in V. It is clear that the total number of segments in S equals $\frac{1}{2} \cdot 51 \cdot 50$ and that each segment in S has the same length as one of the 50 segments

$$A_1A_2,\ A_1A_3,\ \ldots, A_1A_{52}.$$

Consequently, if we separate segments from S into groups of congruent segments (i.e. segments of the same length), we get at most 50 groups. By the pigeonhole principle, the number of segments in some group will be $(\frac{1}{2} \cdot 51 \cdot 50) : 50 = \frac{1}{2} \cdot 51$ at least. Thus there are 26 congruent segments in S. Since all their endpoints lie in the 51-element set V, there are two congruent segments in S which have a common endpoint, which proves the existence of an isosceles triangle with vertices in V. Note that the proved result can be stated as an inequality

$$k(101) < 51,$$

where $k = k(n)$ denotes the greatest number of vertices of a regular n-gon which can be chosen so that no three chosen points are vertices of an isosceles triangle. While the exact value of $k(101)$ is unknown to us[1], the values $k(n)$ for $n \leq 72$ have been determined on a computer in spring 2012 by *Bohuslav Zmek*, a Bc. student of Masaryk University in Brno (Czech Republic). We list them in the following tables.

[1] The recent result is that $k(101) = 17$.

n	3	4	5	6	7	8	9	10	11	12
$k(n)$	2	2	2	4	3	4	4	4	4	4

n	13	14	15	16	17	18	19	20	21	22
$k(n)$	4	6	4	6	5	8	6	8	6	8

n	23	24	25	26	27	28	29	30	31	32
$k(n)$	6	8	7	8	8	8	8	8	8	9

n	33	34	35	36	37	38	39	40	41	42
$k(n)$	8	10	9	10	10	12	10	11	9	12

n	43	44	45	46	47	48	49	50	51	52
$k(n)$	9	12	10	12	10	13	10	14	11	14

n	53	54	55	56	57	58	59	60	61	62
$k(n)$	11	16	11	16	12	16	12	16	13	16

n	63	64	65	66	67	68	69	70	71	72
$k(n)$	13	16	14	16	13	16	14	18	14	16

As you can see, the function $k = k(n)$ is rather complicated. However, it obeys a simple rule

$$k(2n) = 2k(n) \quad \text{for any odd } n \geq 3. \tag{2}$$

We establish (2) it together with estimates

$$k(n) \leq k(2n) \leq 2k(n) \quad \text{for any even } n \geq 4. \tag{3}$$

(Confront (3) with the known values $k(6) = k(12) = 4$ and $k(24) = 8$.)

First we prove that (3) holds for any (odd or even) $n \geq 3$. As to the right-hand inequality, we need to show that if we choose any $2k(n) + 1$ vertices of a regular $2n$-gon $A_1A_2 \ldots A_{2n}$, then three chosen points are vertices of an isosceles triangle. However, this simply follows from the fact that some $k(n)+1$ chosen points are vertices of one of the two regular n-gons $A_1A_3\dots A_{2n-1}$ or $A_2A_4\dots A_{2n}$ and hence, by definition of $k(n)$, three of these $k(n)+1$ points are vertices of an isosceles triangle. This completes the proof of $k(2n)\le 2k(n)$. On the other hand, if we choose $k(n)$ vertices of the n-gon $A_1A_3\dots A_{2n-1}$ so that no three chosen points are vertices of an isosceles triangle, then, in the original $2n$-gon $A_1A_2\dots A_{2n}$, the chosen points are $k(n)$ vertices having the property which implies that $k(2n)\ge k(n)$.

Now we prove that $k(2n)\ge 2k(n)$ (and hence (2) holds) for any odd $n\ge 3$. This parity of n ensures that the above mentioned regular n-gons $A_1A_3\dots A_{2n-1}$ and $A_2A_4\dots A_{2n}$ are mutually symmetric with respect to the center S of the original $2n$-gon $A_1A_2\dots A_{2n}$. Therefore, by definition of $k(n)$, we can choose $k(n)$ points A_i with odd indices i and $k(n)$ points A_i with even indices i such that these two $k(n)$-tuples of points are mutually symmetric with respect to S and three chosen points A_i, A_j, A_k are not vertices of an isosceles triangle whenever the indices i, j, k are of the same parity. If we verify that the last holds without any condition on indices i, j, k, we get an example of $2k(n)$ vertices of the $2n$-gon $A_1A_2\dots A_{2n}$ which shows that $k(2n)\ge 2k(n)$. Suppose on the contrary that $|A_iA_j|=|A_iA_k|$ for some three chosen points $A_i\ne A_j\ne A_k$. We can assume that i is odd and hence both j and k are even (because A_j and A_k are supposed to be symmetric with respect to the line SA_i, the indices j and k must be of the same parity). Denote by A_l the point symmetric to A_i with respect to S. Then A_l is chosen, l is even and $|A_lA_j|=|A_lA_k|$; therefore the chosen points A_l, A_j, A_k with even l, j, k are vertices of an isosceles triangle, which is a contradiction. The proof of (2) and (3) is complete.

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Algorithms and Problem-Solving Lab

Yahya Tabesh & Abbas Mousavi

Yahya Tabesh is a faculty member of Sharif University of Technology (Tehran, Iran). He used to be director and team leader of the Iranian Mathematical Olympiad. Tabesh is Erdős award winner in 2010.

Abbas Mousavi has graduated from Department of Mathematical Sciences, Sharif University of Technology in 2007, since then he has designed, implemented and worked on many online education systems including several online competitions.

Polya's *How to Solve It* after decades is still a classic and proper reference in problem solving, but in the information era which through cloud computing and social networks we are connected anytime, anywhere, what improvement could be done in problem solving methodology and skills? We intend to introduce a new system such as "How to iSolve It!" iSolve is a reference system for developing problem-solving skills over a competitive social network.

In this paper we introduce the iSolve system and some pilot results will be discussed.

1 Introduction

iSolve is a repository for problem solving which saves and retrieves problems, solutions, and also a process of solving problems on an online system. Contributors around the globe would develop it as a social problem-solving system. On a complicated network of ideas, problems, solutions, and process of problem solving, new items will be developed and also old items may be retrieved as role models.

2 The Model

In this section we discuss outlines of the iSolve model. Section 2.1 explains the structure that iSolve uses to organize and categorize the problem-solving ideas.

Section 2.2 introduces the "Social Problem Solving" concept and explains the features designed to inspire and engage people to make an active and productive community and help them to communicate to each other and manage their team works.

Section 2.3 describes iSolve as a source of knowledge, a database of problem-solving ideas that answers semantic questions. It finds similar ideas, suggests ideas that may help you to solve your problem, finds problems that you can solve and finds generalizations or special cases of your ideas. It searches with high level criteria and presents search results as structured data.

2.1 Content Structure

The atomic unit of content in iSolve is a "Solute". A Solute in iSolve is a minor component in the knowledge solution. It may be a simple idea, a problem, a definition, a solution to a problem, a generalization of an idea, a theorem or lemma, a proof or a proof sketch, a problem-solving technique or trick or any other bit of knowledge. A Solute may have different forms, a piece of text, a picture, video or voice, a computer simulation or an interactive experience.

To organize the Solutes and relate them to each other, iSolve saves them in a directed, labeled graph. In this graph Solutes are represented as labeled vertices and relations between Solutes are represented as labeled, directed edges between vertices.

To categorize Solutes and their relations, iSolve tags each vertex and edge with some labels or tags. Tags on vertices show their type (problem, solution, idea, ...), their form (e.g. text, video, simulation, ...) and their topics (e.g. Geometry, Number Theory, Algebra, ...) and tags on edges show the type of relation between two Solutes that they connect, relations such as "second Solute is a solution to the first Solute", "second Solute is a generalization of the idea in first Solute", "second Solute is a theorem that is used to solve the problem in first Solute" and so on.

These tags will be added to Solutes and edges between them by users when creating or editing content and also by the system using some smart algorithms. Extra structures and patterns on the tags (e.g. the AMS classification tree structure on topic tags), relations between users and contents and collected statistics from existing content support these algorithms.

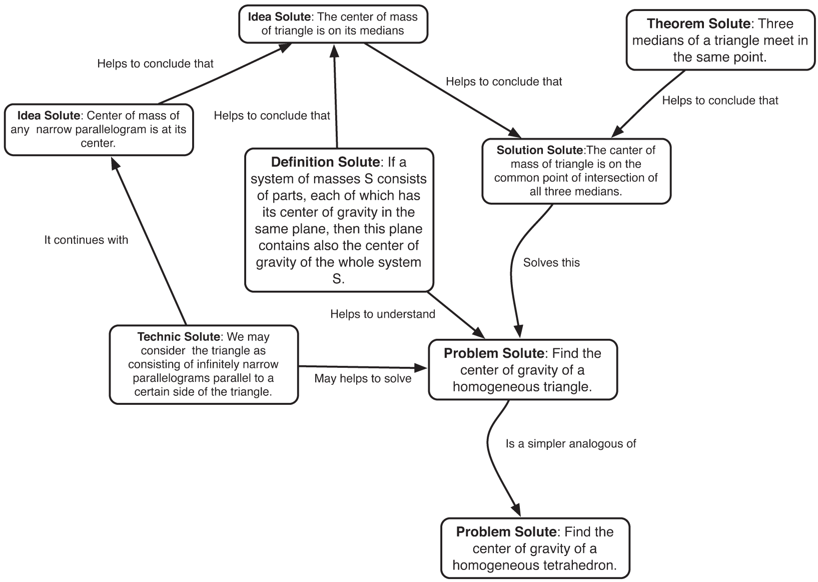
We use a problem from page 38 of Polya's book [1] as an example to illustrate the idea of a Solute and to show how iSolve can help problem solvers:

> Find the center of gravity of a homogeneous tetrahedron.

In figure 1 you can see a neighborhood of a problem Solute (Find the center of gravity of a homogeneous tetrahedron) in the Solute graph (We show only connections between shown Solutes. Each Solute may have many connections to Solutes out of this set that is not shown). Seeing this problem one user adds another problem Solute to propose a simpler analogous problem that may help to solve the original problem, another user adds a definition Solute and a technique Solute that he thinks may help to solve the simpler problem, then another user using the definition and technique Solutes, some connecting idea and a theorem from another part of the Solute graph finds a solution to the simpler problem. Since there is complete documentation of the process of solving simpler problem, one can solve the harder problem by following the documented steps. This is how iSolve uses small pieces of knowledge from contributors to solve a problem. The collected knowledge on iSolve solves new problems and solving of new problems collects more knowledge in iSolve.

2.2 Social Problem Solving

To inspire and engage people to collaborate in iSolve, there are some features to support the so-called Social Problem Solving. iSolve leverages



the different abilities and talents of the community members to solve problems.

Members of iSolve community have profiles to introduce themselves. They can tag themselves with the same topic tags used for Solutes to show their own fields of interest, and probably with other tags to show different interests and backgrounds.

They can connect to other members as friends or colleagues. To encourage more contributions, iSolve tags members automatically to identify members that generate more advanced content, are more active and are more supportive to the community. iSolve also assigns credits and reputations to members to reward their efforts.

The iSolve community consists of individuals or teams who are creating Solutes or projects and developing them. A project consists of some Solutes which could be grouped manually or automatically via some criteria, (e.g. all Solutes with specified tags that are connected to these

Solutes through certain length). Projects allow their members to focus on a limited set of Solutes and manage the teamwork.

Each project has an introduction page and an issue tracker to help group members to present their project to others, discuss Solutes, assign jobs to each other and track project issues.

iSolve has the so-called Project version control system which manages group members' collaborations on the projects. It keeps the complete change history of the Project Solutes and relations between them, containing the time of change, the change itself and the member committed the change. Using this system, group members can access different versions of project content, make different branch of contents, work on branches and then merge the branches.

iSolve members can fork any project, a fork is a clone of the forked project. The fork owner can make changes in it independently. He can also send a request to the owner of main project to merge changes (pull request) to the main project.

iSolve has a notification system that notifies members about changes in projects such as pull request and other important events.

In Figure 2 you can see a schematic diagram explaining the idea of projects and project version control system. At the left a partial view of the Solute graph can be seen. We see three projects, and two free solutes that are not in any projects and we can see that projects can have Solutes in common, changes of these Solutes in different projects will be managed by the project version control system. Also we can see a part of the history of project C. It shows change of project C over time and it shows that this project has two branches. Branches of a project can co-exist. Branches can be merged with each other. And we can return to any point in the history of project at any time and compare each two point and see the differences.

Finally, mobile clients will help us simplify and spread the use of iSolve between problem solvers around the world.

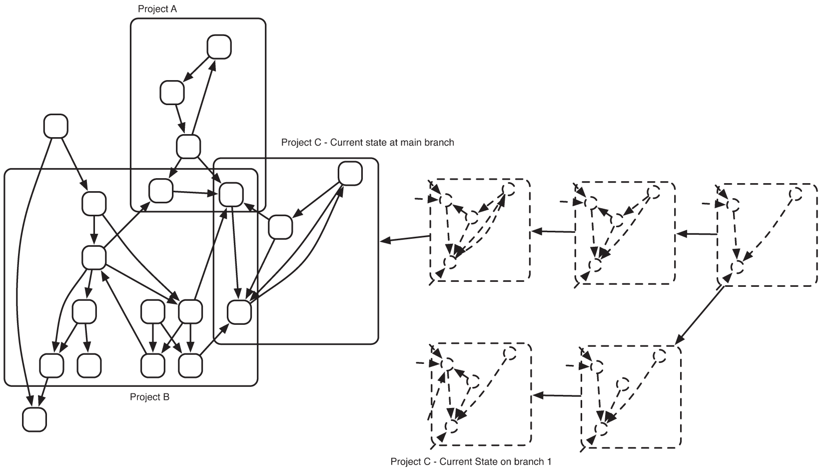


Figure 2. A schematic diagram explaining the idea of projects and project version control system

2.3 Content searching, retrieving and visualizing

iSolve's main functionality is presenting problem-solving ideas in a structured manner and helping people search not only by words but also by ideas and relations, topics and other semantic criteria. Solutes could be searched with criteria such as tags, relations, projects, authors, etc.

iSolve can display a visual representation of Solutes graph in different scales. Zooming in to a Solute, one can see the Solute, its tags and neighborhood. In this representation, different types of Solutes and edges are coded with different colors. One can walk on the graph by clicking on a neighbor Solute. Zooming out, one can see a large scale picture of the graph, this picture visualizes the density of graph in different regions, important Solutes that have many relations with other Solutes, distance between specific Solutes and other information. This visualization can be filtered by type and topic of Solutes and by type of relations between them.

iSolve can generate much information about Solutes by performing different graph algorithms on the Solute graph. iSolve can find all the paths from one Solute to another Solute, shortest path between two solutes, Solutes with highest degree in a neighborhood, spanning trees of Solutes and other information that can be extracted from Solutes graph. iSolve also can help math education researchers to extract statistical data about mathematical ideas and the process of problem solving.

3 Reference Implementation

We are working on a reference implementation of iSolve. This implementation will be done as a web application using Python programming language. We also will implement mobile clients on iOS and Android platforms as well.

One of challenges in implementing iSolve is storing and retrieving the Solute graph and related metadata and running different graph algorithms on it in real time. For this we will adapt and use Python graph and network libraries like NetworkX.

Another challenging task is visualizing different aspects of iSolve like Solute graph, projects and their history. For this we will use HTML5 techniques and JavaScript libraries like InfoVis Toolkit.

References

[1] Polya, G. (1957), *How to Solve It: A New Aspect of Mathematical Method,* Princeton University Press.

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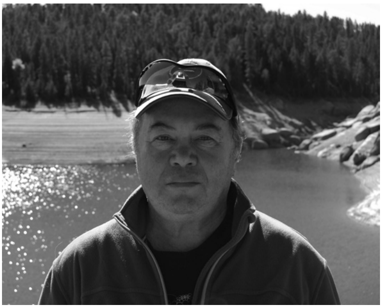
The Goal of Mathematics Education, Including Competitions, Is to Let Student Touch "Real" Mathematics: We Ought to Build that Bridge

Alexander Soifer



Born and educated in Moscow, Alexander Soifer has for 33 years been a Professor at the University of Colorado, teaching math, and art and film history. He has published over 200 articles, and a good number of books. In the past 3 years, 6 of his books have appeared in Springer: The Mathematical Coloring Book: Mathematics of Coloring and the Colorful Life of Its Creators*;* Mathematics as Problem Solving: How Does One Cut a Triangle?*;* Geometric Etudes in Combinatorial Mathematics; Ramsey Theory Yesterday, Today, and Tomorrow*; and* Colorado Mathematical Olympiad and Further Explorations. *He has founded and for 29 years ran the Colorado Mathematical Olympiad. Soifer has also served on the Soviet Union Math Olympiad (1970–1973) and USA Math Olympiad (1996–2005).*

> *There should be no wall between Mathematical Olympiads problems and real problems of mathematics – indeed, there shall be a two-way BRIDGE between the two! As in "real" mathematics, Olympiad problems ought to include not just deductive reasoning, but also experimentation, construction of examples, synthesis in a single problem of ideas from various branches of mathematics, open ended problems, and even open problems. Olympiad problems should merit such epithets as beautiful and counterintuitive.*

Mathematical Olympiads convey what mathematics is, and thus extract from mathematics not ashes but flames.

1 The Goal of Mathematics Education

It seems natural for any discipline to show in the classroom what that discipline is and what its professionals do. However, it is seldom done! Surprisingly, instruction often boils down to teaching skills, and thus obscuring this main goal.

What amazes me the most about much of mathematics education is its superficiality, its limited relevance to the field of mathematics, its frequent emphasis on mindless drill, and its relentless theorem-proof-theorem-proof style of narrative. As if one robot pre-programs other robots to crunch standard exercises. No wonder that so few people like mathematics!

I do not call here for moving to applied mathematics, for more often than not it carries its applicability on a shallow level. Furthermore, one ought to learn mathematics first and only then see how to apply it.

In fact, I do not favor the partition of mathematics into pure and applied. This is like partitioning literature into children's and grown-ups'. Where, for example, would you put *Through the Looking Glass, and What Alice Found There?* At my grown-up age, I am still in love with Kipling's *Just So Stories.* I prefer partitioning literature into good and not so good. Likewise mathematics ought to be divided into good and not so good.

What is the goal of mathematics education? Many would suggest that "problem solving" is that goal, even though most would disagree with each other on what problem solving means. *I submit, problem solving is a means, and the question is, a means to what end?*

Shouldn't "truth in advertising" require us to show in our classrooms *what mathematics is* and *what mathematicians do*? And if so, we ought to build a bridge between mathematics and mathematics education. And the best location for building this two-way bridge between "real" mathematical research and secondary education is mathematics competitions, and first of all Mathematical Olympiads.

2 Creating Problems and Envisioning the Bridge

The Russian mathematician Boris Nikolaevich Delone once said, as the great Andrej Nikolaevich Kolmogorov recalls, that "a major scientific discovery differs from a good Olympiad problem only by the fact that a solution of the Olympiad problem requires 5 hours whereas obtaining a serious scientific result requires 5,000 hours."

It is natural to offer "real" open problems to young high school Olympians, but maybe not at Mathematical Olympiads, for they do not offer 5,000 hours. In fact, when in the mid-1960s, as a high school student, I attended the Award Presentation Ceremony of the Moscow Mathematical Olympiad, the Chairman of the Olympiad's Jury A. N. Kolmogorov paid us, young Olympians, an elegant compliment. "Perhaps, the only way to receive a proof of Fermat's' Last Theorem is to offer it at the Moscow Mathematical Olympiad," he said.

We ought to stop discrimination of high school mathematicians based on their tender age. However, we can offer true research problems only to a small percent of high school mathematicians. What can we do for others, who are not ready for research? We can offer them inspiring problems to solve.

Problem solving requires the supply of problems. Somebody has to create them. But how does one create original suitable Olympiad problems? Here is how it works for me. Every early March I attend the Southeastern International Conference on Combinatorics, Graph Theory, and Computing at the Florida Atlantic University (the 43^{rd} Conference was held in March 2012). Listening to talks there accelerates the gears of my mind. Ideas of research mathematics inspire my problem creation.

While reading or creating research mathematics, I caught myself many times thinking how beautiful, Olympiad-like certain ideas were. Of course, research mathematics often deals with a sophisticated topic matter, often uses break-through-the-wall-by-all-available-means solutions, and so the first task is to extract these "naked" beautiful ideas out of it. Some of these striking ideas have given birth to problems I have created for the Olympiads. Thus, I first notice a fragment of the research, which utilizes a beautiful, better yet surprising idea.

The second task is to translate thus found mathematical gem into the language of secondary mathematics. Finally, the third task is to dress up these ideas and present them to the Olympians in a form of enjoyable stories. Consequently, some of the problems of the Colorado Mathematical Olympiad have special titles, such as "Football for 23", "Chess 7×7", "Old Glory", "Stone Age Entertainment", "King Arthur and the Knights of the Round Table", "Crawford Cowboy Had a Farm, Ee i ee i oh!" (I depicted here the most famous Crawford resident, who at the time was the U.S. President, George W. Bush), etc. Sometimes I imitate a "real" mathematical train of thought by offering in the Colorado Mathematical Olympiad a series of problems, increasing in difficulty, and leading to generalizations and deeper results. It is also important to realize that "real" mathematics cannot be reduced to just analytical reasoning, for about 50 % mathematics is about construction of counterexamples. I try to reflect this dichotomy in our Olympiad problems, many of which require not only analytical proofs but also construction of examples.

The bridge we are building can be walked in the opposite direction as well: it is worthwhile for professionals to take a deeper look at problems of Mathematical Olympiads. Those problems just might inspire exciting generalizations and new directions for mathematical research.

My books [4], [5], [6], [7], [8], [9] are bricks in building a bridge between problems of Mathematical Olympiads and problems of "real" mathematics. In them, I try to show that problems of competitions and research problems of mathematics stem from the same root, made of the same cloth, have no natural boundaries to separate them. I am grateful to my publisher Springer for its service as a mason of this bridge: in the recent years, Springer has contracted nine books with me.

I would like to illustrate in this essay how a problem is created with the freshest example dated April, 2012. You will witness the construction of a two-way bridge between problems of mathematical Olympiads and research problems of mathematics. The leading American mathematician Ronald L. Graham likes to say that every talk must include one proof. I will present just that—and more—in the next, main part of my essay.

3 Birth of a Problem: The Story of Creation in Seven Stages

I can envision a thought-provoking, exciting course introducing cardinal numbers to secondary students. However, Olympiads cannot afford long introductions; their approach ought to be more immediate. In the 29$^{\text{th}}$ Colorado Mathematical Olympiad (April 20, 2012), I wanted once again to let my Olympians "touch" the infinity through two problems in our 5-problem 4-hour Colorado Mathematical Olympiad. In this chapter I will show you the second of these problems.

Moreover, while memory holds, I would like to show here how this most difficult problem of the XXIX Colorado Mathematical Olympiad (April 20, 2012) *Beyond the Finite II* came about. I will share here its genesis, the story of creation of mathematical kind, if you will. Our train of thought will take us through seven stages, like in a children's alphabet song: A, B, C, D, E, F, G. Fasten your belts; our train is leaving the station. I first created the following problem in early March 2012 while at the conference at Florida Atlantic University.

Stage A**.** Infinitely many circular disks of diameter 1/700 are given inside a unit square. Prove that there is a closed circular disk of diameter 1/2012 that is contained in infinitely many of the given disks.
A *circular disk* consists of all points on and inside a circle.

Solution. My solution was based on creating a 2012×2012 square grid on the given unit square grid. The numbers given in the problem conveniently deliver the inequality: $700 \times 2\sqrt{2} < 2012$. (We can even replace 700 by 711, but not by 712). This inequality shows that the radius of the given disks is greater than the diagonal of the unit square of the grid. Using this observation, you can easily prove (do) that every given disk contains completely at least one of the unit cells of the grid (see Figure 0).

Assume that there is no unit cell in our grid that is contained by infinitely many given disks. Then for each of the (finitely many) unit cells c of the grid G we get finitely many disks $f(c)$ containing this cell c, so the sum Σ of $f(c)$ over all cells c is finite. This sum includes all the given disks

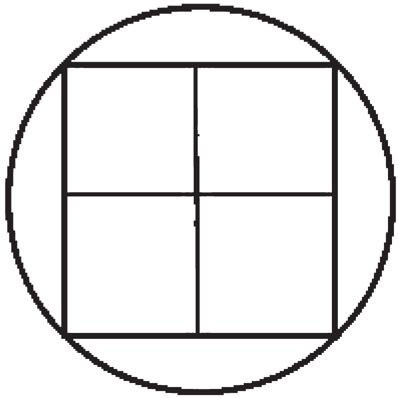


Figure 0

(some possibly counted more than once), for we have shown that each disk contains at least one unit cell. We have arrived at the contradiction, for we were given infinitely many disks. Therefore, there is a cell c_1 that is contained in infinitely many of the given disks.

All that is left is to inscribe the required circular disk in the unit cell c_1.

Having created and solved this problem, I thought: *infinity is a rather large number. Why should the size of the given square matter?* It really should not, and we can get rid of the size of the square and get to the next stage in our genesis of the problem.

Stage B**.** Infinitely many circular disks of diameter $1/711$ are given inside a square. Prove that there is a closed circular disk of diameter $1/2012$ that is contained in infinitely many of the given disks.

Of course, with the size of the square gone, we can increase and thus simplify all the remaining numbers.

Stage B'**.** Infinitely many circular disks of radius 1 are given inside a square. Prove that there is a closed circular disk of diameter $1/\sqrt{2}$ that is contained in infinitely many of the given disks.

My next thought was about the shape. *Must we embed our flock of circular disks in a square? Does the shape matter?* I came to the conclusion that it really does not. We can get rid of the shape of the square and thus arrive at the next station of our ride on the train of thought.

Stage C. Infinitely many circular disks of radius 1 are given inside a bounded figure in the plane. Prove that there is a circular disk of diameter $1/\sqrt{2}$ that is contained in infinitely many of the given disks.

The next question I posed to myself was inspired by a typical mathematics search for the best possible result: *is $1/\sqrt{2}$ the maximum diameter of a disk we can guarantee to exist inside infinitely many of the given disks?* It was clear to me that $1/\sqrt{2}$ was not the maximum. And the new problem was born: find that maximum.

Stage D. Infinitely many circular disks of radius 1 are given inside a bounded figure in the plane. Find the maximum radius of a circular disk which is contained in infinitely many of the given disks.

We decided to use a version of Stage D problem in the April 20, 2012 Colorado Mathematical Olympiad. Here is that version:

Beyond the Finite II (Alexander Soifer, 2012). Infinitely many circular disks of radius 1 are given inside a bounded figure in the plane. Prove that there is a circular disk of radius 0.9 that is contained in infinitely many of the given disks.

Robert Ewell's Solution in Alexander Soifer's Exposition. We start by visualizing the problem (Figure 1). Since the given figure F is bounded, it can be surrounded by a square S. Create on S a square grid (Figure 2) of a "small" unit side a, whose value we will determine later.

There are infinitely many centers of the given disks located in finitely many cells of the grid. By the Infinite Pigeonhole Principle, there is a

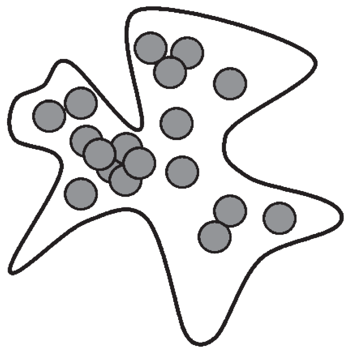


Figure 1

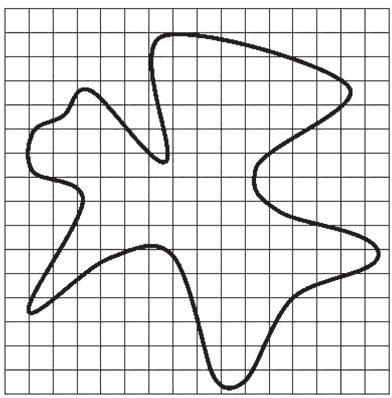


Figure 2

cell c_1 that contains infinitely many centers of the given disks (Figure 3). Denote by S this infinite set of the disks whose centers lie in c_1.

Now we need "a piece" of plane geometry, which I contributed to Robert Ewell's solution. Let us calculate how small the side a of the cell c_1 should be so that we can find a disk D_1 of radius 0.9 that is contained in each of disks of the family S. In Figure 4, I depict the cell c_1 with center O; A is the center of the shown disk from the family S, and B an

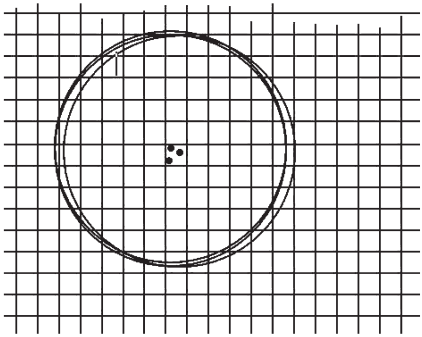


Figure 3

arbitrary point on the boundary of that disk.

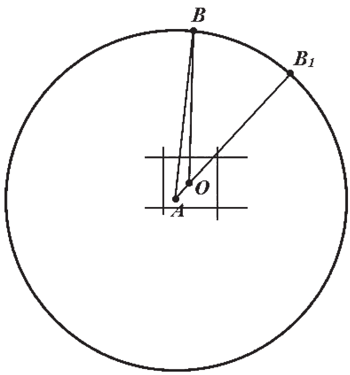


Figure 4

According to the triangle inequality,

$$|OB| \geq |AB| - |OA| = 1 - |OA| \geq 1 - a/\sqrt{2}$$

where vertical bars denote the length of a segment, and the equality is attained when the point B is in the position B_1 and the triangle AOB is degenerate. Therefore, for any position of B on the circle, we have $|OB| \geq 1 - a/\sqrt{2}$.

Now we can choose a to satisfy the condition $1 - a/\sqrt{2}$. We can thus let $a = 0.1\sqrt{2}$. We are done, for the circular disk of radius 0.9 and center O is contained in all of the disks of the infinite family S.

Note: In the statement of this problem, we can replace 0.9 by any positive number less than 1. We can come however close to 1, but the maximum radius does not exist.

The problem solved gives birth to new problems. We were embedding circular disks, *is the circularity essential to our "games"? Probably not!* To proceed, we need a definition.

Translates $F_1, F_2, \ldots$ of a convex figure F, are just that: figures obtained from F by translations (Figure 5). We will also use the term *translate* when we shrink the size (but do not rotate); in the latter case, we will state the coefficient of homothety.

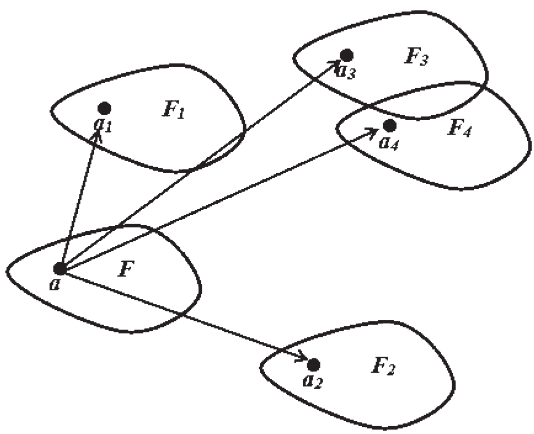


Figure 5

We are ready for Stage E, where *circularity of disks vanishes, and we may get the same result as in the previous stage, but now for all convex figures. Diameter* of a convex figure is the largest distance between its two points.

Stage E: Translates Conjecture. Given a convex figure F of diameter 1. Infinitely many translates of F are given inside a bounded figure in the plane. Then for any however small $\varepsilon > 0$ there is a translate of F of diameter $1 - \varepsilon$ that is contained in infinitely many of the given translates (Figure 6).

Figure 6

Can we relax the requirement of packing in only translates? As Presidential candidate Barak Obama used to say, *Yes we can!*

Given a convex figure F; a *clone* of F is a figure congruent to F. We will also use the term *clone* for figures similar to F, in which case we will state the coefficient of similarity.

Stage F: New Open Problem. Let F be a convex figure of diameter 1. Infinitely many clones of F are given inside a bounded figure in the plane. Find the maximum $\phi(F)$ such that for any ε there is a clone $F(\varepsilon)$ of F of diameter $\phi(F) - \varepsilon$ that is contained in infinitely many of the given clones (Figure 7). Is the upper bound attainable, i.e., can $\phi(F) - \varepsilon$ be replaced by $\phi(F)$ in this statement?

Having created this train of thought, the Translates Conjecture E and the Open Problem F, on March 23, 2012, I asked the great geometer

Figure 7

Professor Branko Grünbaum for comment. He replied (long live e-mail!) the same day:

> Dear Sasha,
>
> Your question for arbitrary convex figure is interesting, and I think it is new. It seems to me that the proof can be completed by using for the "center" of F the (area) centroid of F. Then it is well known (Minkowski's measure of asymmetry) that every chord of F through this center is divided by the center in a ratio that that is between 1/2 and 2. This should be enough to show that there is a homothetic copy of F, for any given $\varepsilon > 0$. Notice that the whole thing generalizes to all dimensions d, by the same argument, because then Minkowski's ratio is between $1/d$ and d.
>
> I hope this is satisfactory.
>
> Best wishes,
>
> Branko

Let me share with you the meanings of the terms used by Professor Grünbaum in his e-mail (you may wish to consult him on that [1]).

The *centroid* of a plane figure F is the intersection of all straight lines that divide F into two parts of equal moment about the line. Physically speaking, the centroid of a plane figure is the point on which it would balance when placed on a needle.

Minkowski's Measure of Asymmetry Theorem [3] states that every chord of a convex figure F through its centroid is divided by the centroid in a ratio that is between $1/2$ and 2.

On April 21, 2012, the day after the 29th Colorado Mathematical Olympiad, I proved the Translates Conjecture E (without using Minkowski's Theorem). The Colorado Mathematical Olympiad veteran judge Shane Holloway soon after proved it too.

Proof of the Translates Conjecture E. Let F be a convex figure with the centroid C; F' an image of F under the homothety with center in C and the coefficient $1 - \varepsilon$. Let the minimum distance between a point of F and a point of F' be δ. Let K be a circular disk of radius δ with center C (Figure 8).

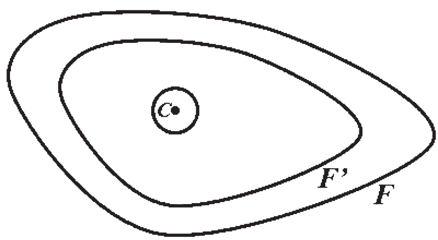


Figure 8

Since the given figure, call it G is bounded, it can be covered (possibly with overlapping) by finitely many disks congruent to K. Since infinitely many centroids of the given translates lie in these covering disks, by the Infinite Pigeonhole Principle there is a disk (call it K, so that we can use Figure 8 as an illustration) contains infinitely many centroids of translates forming the infinite set S.

Let F be a translate from S. Due to the choice of δ, the translate of F in any direction through the distance not exceeding δ will contain the figure

F'. Therefore, F' is contained in *each* of the infinitely many translates from S.

Having proved the Translates Conjecture E, I moved on to Open Problem F. I thought: now that we can rotate the clones of F, surely, we can reduce significantly the size of the guaranteed in the intersection F'. And so I introduced a new invariant for convex figures.

Given a convex figure F and its incircle (Figure 9). Inscribe in the incircle the largest clone F' of F. The *circularity* $\phi(F)$ of F I define as the ratio of the diameters of F' and F.

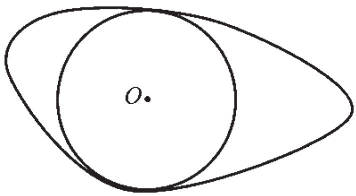


Figure 9

For example, circularity of a circle ◯ is 1: $\phi(\bigcirc) = 1$. Circularity of a square □ is $\phi(\square) = 1/\sqrt{2}$ (prove it on your own).

I thought that the circularity delivers the answer to Open Problem F. We can rotate F about O (Figure 10), so that the intersection of the rotated images is the incircle, and then inscribe a figure F' similar to F in the incircle (Figure 11). We can then "spoil" the intersection and "slightly" reduce the incircle (as we did in the *Beyond the Finite II*), and thus get the result $\phi(F) - \varepsilon$, just as in the latter Olympiad problem. However, there was a flaw in my argument. Finding the flow is your home work! :-).

I will now show you a solution of Open Problem F that I found on April 26, 2012. Having noticed a flaw in the previous argument (have you found it yet?), I managed to use my intuition and see the answer. Even though at first I did not know how to prove it, the feeling that I finally had a plausible conjecture put a smile on my face. Then there came a proof.

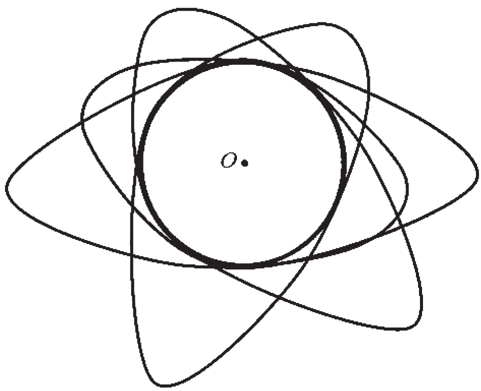


Figure 10

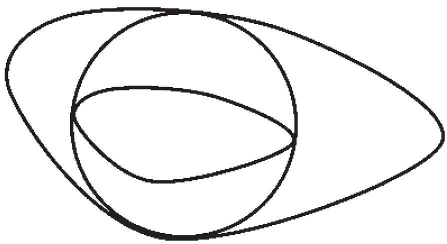


Figure 11

General Theorem F**.** Let F be a convex figure of diameter 1. Infinitely many clones of F are given inside a bounded figure G in the plane. Then for any $\varepsilon > 0$ there is a clone $F(\varepsilon)$ of diameter $1 - \varepsilon$ that is contained in infinitely many of the given clones.

Proof. The idea of the proof is to find two converging to their limits sequences, one "locational" and one "angular." Thus, you can say that the soul of the proof comes from mathematical analysis. I find applications of analysis to geometry very exciting. See other examples in [7].

1. Assign a direction to the figure F by, say, drawing a line from its centroid to a point on its boundary. This induces direction on all clones of F. Partition the $360°$ angle (comprising all conceivable directions) into 360 pigeonholes. Since infinitely many clones of F are given, there are infinitely many clones with directions inside a $1°$ pigeonhole. We can now partition this pigeonhole into 10 pigeonholes of size $0.1°$ and find infinitely many clones of F with directions within a pigeonhole of size $0.1°$; etc. Continuing this process, we obtain the infinite series of enclosed angular intervals. These series of enclosed diminishing in size intervals converge to exactly one point (direction), its intersection; call this limit direction d. We proved that for any angle α there are infinitely many clones of F with directions within α from d.

Essentially, we proved that we can replace our unruly clones with infinitely many "near parallel" clones!

2. Let F be a convex figure with centroid C, F' a clone of F obtained through homothety with center in C and the coefficient $1-\varepsilon$ which is then rotated through an angle α small enough so that F' still lies inside the boundary of F (see Figure 12). Let the minimum distance between a point of F and a point of F' be δ. Let K be a circular disk of radius δ with its center in C.

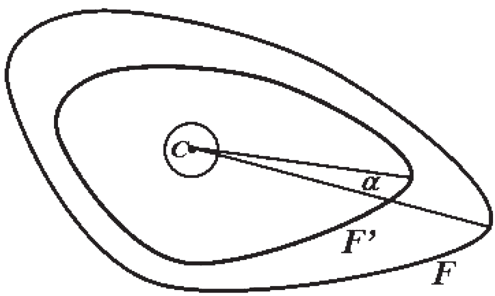


Figure 12

3. Since the given figure G is bounded, it can be covered (possibly with overlapping) by finitely many disks congruent to K. Since infinitely many centroids of the given clones lie in these covering disks, by the Infinite Pigeonhole Principle there is a disk (call it K, so that we can use Figure 12 as an illustration) that contains infinitely many centroids of the given clones.

Among the clones with their centroids in K, there are infinitely many clones, whose directions lie within an angle α (which we chose in part 1) from the direction of a fixed clone F' of direction d; denote by S the set of these clones.

Let F be a clone from S. Due to the choice of δ, any rotation of F through an angle smaller than α followed by a translation in any direction through the distance not exceeding δ will contain the figure F'. Therefore, F' is contained in *each* of the infinitely many clones comprising the set S. F' is the required in the theorem $F(\varepsilon)$.

We gained nothing by allowing rotations! *This proof provides a counter-intuitive result, which demonstrates how different infinity is from finite objects that we are used to!*

Our train of thought is ready to take us beyond the two-dimensional entertainment: recall, in his e-mail, *Branko Grünbaum suggested n-dimensional generalizations of the Conjecture E and Open Problem F.*

Stage G: Theorem in Space. Let F be a convex solid of diameter 1 in an n-dimensional Euclidean space. Infinitely many clones of F are given inside a bounded solid. Then for any $\varepsilon > 0$ there is a clone $F(\varepsilon)$ of F of diameter $1 - \varepsilon$ that is contained in infinitely many of the given clones.

The proof faithfully follows the arguments of the two-dimensional case. In part 1 of the proof, instead of choosing a small angle, we choose a small cone in n-dimensional space.

Paul Erdős once told me, that he can usually solve the infinite case of a problem; it is the finite case that holds the greatest difficulty. Indeed, finite case here ranges from intriguing problems of packing to problems of intersections, which are so beautifully represented by the Helly Theorem.

I hope I have convinced you that there is no wall separating problems of mathematical Olympiads from research problems of mathematics.

Just the opposite is true: there is a *bridge*, a two-way bridge connecting the two.

Not all Olympiads are created equal. What separates the Colorado Mathematical Olympiad is our commitment to this Bridge.

4 The Unmistakable Truth about Value of Mistakes

Yes, I know, I know. Mistakes have a bad reputation, especially in mathematics. In 1879 Alfred Bray Kempe was first to publish an attempted proof of the Four-Color Theorem. He discovered new territories, found a new approach that became known as "The Kempe Chain Method." Yet, even over a century later, in 1993, Richard Steinberg [10] ridicules Kempe:

> The most notorious paper in the history of graph theory: the 1879 work by A. B. Kempe that contains the fallacious proof of the Four Color Theorem.

Tomas L. Saati, right in the title of his 1967 paper, calls the attempt "The Kempe Catastrophe." I cannot disagree more. Yes, Alfred B. Kempe did not succeed in his goal, but what a fine try it had been, far exceeding anything his celebrated predecessors De Morgan and Cayley achieved in years of toying with the Four-Color Conjecture (4CC)! Moreover, both known today proofs of the Four-Color Theorem had Kempe's ideas in their foundation. Kempe came up with beautiful ideas; his chain argument was used many times by fine 20$^{\text{th}}$ century professionals, such as Dénes Kőnig in his 1916 work on the chromatic index of bipartite graphs, and Vadim Vizing in his famous 1964 Chromatic Index Theorem. Read more about these exciting episodes of mathematical history in [4].

In early August 2000, near the end of ICME-9, I attended the great Russian animator Yuri Norstein's master class offered to some 200–300

Japanese professionals in a completely filled Tokyo theater. A Japanese animator asked Norstein, "Everyone uses computers for creating animation; why don't *you*?" Yuri's reply surprized everyone:

Because computers make no mistakes.

Yuri explained: "When I create my animations by hand, I make mistakes, which often open new and surprizing approaches in my work." I will soon share more of Yuri Norstein's wisdom [9].

Ever since our childhood, we learn lessons of wisdom, morality, and heroism from our fairytales, in spite of their mistakes and exaggerations. The great Russian poet Aleksandr Pushkin ends his "Fairytale about the Gold Cockerel" ("Сказка о золотом петушке") with the words:

A fairytale is a lie, but with a hint, a lesson for a good lad.[1]

I learn from my mistakes—at least I try. And since learning should be shared, I offered you here an opportunity to see my mistake. I hope you noticed how from committing and then finding a mistake, I was then able to discover an unmistakable solution of Open Problem F.

Aspiring to discover new knowledge and making mistakes in the process is surely superior to the laziness of doing nothing. But even laziness can be a virtue. The inventor of a bicycle was too lazy to walk! But that is a topic for another day.

5 Passing the Flaming Torch

In June 1932, in the preface to his classic book *Geometry and Imagination*, the great David Hilbert wrote [2]:

> It is true, generally speaking, that mathematics is not a popular subject, even though its importance may be generally conceded.

Indeed, we compete with such fabulous sources of entertainment as sports, oceans, mountains, movies, television, computer games, etc. In order to win such a tough competition, Olympiad problems should merit such epithets as beautiful and counterintuitive.

[1] Translated from the Russian by Maya Soifer. The original rhymed Russian text is: "Сказка ложь, да в ней намек! Добрым молодцам урок."

The epigraph of this essay has been inspired by the profound line by one of my favorite writers Lion Feuchtwanger:

[Historical novelist] wants to understand the present,
and thus searches in history not for ashes but for flames.[2]

In our setting I have translated it to read:

Mathematical Olympiads convey what mathematics is, and thus
extract from mathematics not ashes but flames!

Acknowledgements. I am grateful to Professor Branko Grünbaum for a valuable feedback and Col. Dr. Robert Ewell for creating computer-aided illustrations based on my sketches.

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The 53rd International Mathematical Olympiad, Mar del Plata, Argentina, 2012

The 53rd International Mathematical Olympiad (IMO) was held July 4–16 in Mar Del Plata, Argentina. This was the second time that Argentina has hosted the IMO. The first time was in 1997 when 460 high school students from 82 countries competed in the 38th IMO. Since that time the IMO has grown. In 2012 a total of 548 high school students from 100 countries competed.

Each country sends a team of up to six students, a Team Leader and a Deputy Team Leader. At the IMO the Team Leaders, as a collective, form what is called the *Jury*.

The first major task facing the Jury is to set the two competition papers. During this period the Leaders and their observers are trusted to keep all information about the contest problems completely confidential. The local Problems Selection Committee had already shortlisted 30 problem proposals submitted by participating countries from around the world. During the Jury meetings six of the shorlisted problems had to be discarded from consideration due to being too similar to material already in the public domain. Eventually, the Jury finalised the exam questions and then made translations into all the languages required by the contestants.

The six questions are described as follows.

1. A very easy classical geometry problem proposed by Greece.
2. A medium classical inequality proposed by Australia.
3. A difficult combinatorics problem proposed by Canada. Called the *Liars Guessing Game*, this very interesting information problem has the twist that the respondent is permitted to lie almost of the time if he wishes.
4. An easy functional equation with unexpected sporadic solutions proposed by South Africa.
5. A medium classical geometry problem proposed by the Czech Republic.

6. A difficult number theory problem proposed by Serbia. This problem requires absolutely no advanced technical background, only clear and creative thinking.

These six questions were posed in two exam papers held on consecutive days. Each paper has three of the problems in which the contestants were allowed $4\frac{1}{2}$ hours to attempt them. Each problem is scored out of a maximum of seven points. This year the competition days were Tuesday July 10 and Wednesday July 11.

The Opening Ceremony occurred the day prior to the first day of competition. Nazar Argakanov, Chairman of the IMO Advisory Board, emphasized that the IMO is the most important mathematical competition of the year in the world. He also highlighted the principle of fair play by students and leaders alike. Following this was something new introduced by the IMO Ethics Committee. Both students and then Leaders were required to stand and take the IMO oath. Hans-Dietrich Gronau of the Ethics Committee read the following statement:

> *The IMO is an honest and fair competition between the young mathematicians of the world. We promise to uphold both the rules and the spirit of the IMO.*

To which those taking the IMO responded:

> *Yes.*

Following the formal speeches was the parade of the Teams. At the conclusion the IMO song was sung tango style in Spanish and then confetti filled the auditorium. Thus the 2012 IMO was declared open!

After the exams the Leaders and their Deputies spent about two days assessing the work of the students from their own countries, guided by marking schemes discussed earlier. A local team of markers called *Coordinators* also assessed the papers. They too were guided by the marking schemes but are allowed some flexibility if, for example, a Leader brings something to their attention in a contestant's exam script which is not covered by the marking scheme. The Team Leader and Coordinators have to agree on scores for each student of the Leader's country in order to finalise scores.

Question 1 turned out to be very easy as expected. It averaged 5.6 points, the highest in the last five IMOs. Questions 3 and 6 were very difficult, averaging 0.4 and 0.3, respectively. There were 277 (= 50.5%) medals awarded. The distributions being 138 (= 25.2%) Bronze, 88 (= 16.1%) Silver and 51 (= 9.3%) Gold. One student, Jeck Lim from Singapore, achieved the most excellent feat of a perfect score of 42. The medal cuts were set at 28 for Gold, 21 for Silver and 14 for Bronze. These awards were presented at the Closing Ceremony. Of those who did not get a medal, a further 148 contestants received an Honourable Mention for solving at least one question perfectly.

The 2012 IMO was organized by the Argentine Mathematical Olympiad Foundation.

Venues for future IMOs have been secured up to 2017 as follows:

2014	Capetown, South Africa
2015	Thailand
2016	Hong Kong
2017	Brazil

The 2013 IMO is scheduled to be held in Santa Marta, Colombia.

Much of the statistical information found in this report can also be found at the official website of the IMO.

www.imo-official.org

Angelo Di Pasquale
Australian IMO Team Leader
AUSTRALIA

1 IMO Papers

Tuesday, July 10, 2012
Language: English

First Day

Problem 1. Given triangle ABC the point J is the centre of the excircle opposite the vertex A. This excircle is tangent to the side BC at M, and to the lines AB and AC at K and L, respectively. The lines LM and BJ meet at F, and the lines KM and CJ meet at G. Let S be the point of intersection of the lines AF and BC, and let T be the point of intersection of the lines AG and BC. Prove that M is the midpoint of ST.

(The *excircle* of ABC opposite the vertex A is the circle that is tangent to the line segment BC, to the ray AB beyond B, and to the ray AC beyond C.)

Problem 2. Let $n \geq 3$ be an integer, and let $a_2, a_3, \ldots, a_n$ be positive real numbers such that $a_2 a_3 \cdots a_n = 1$. Prove that

$$(1 + a_2)^2 (1 + a_3)^3 \cdots (1 + a_n)^n \ > \ n^n.$$

Problem 3. The *liar's guessing game* is a game played between two players A and B. The rules of the game depend on two positive integers k and n which are known to both players.

At the start of the game A chooses integers x and N with $1 \leq x \leq N$. Player A keeps x secret, and truthfully tells N to player B. Player B now tries to obtain information about x by asking player A questions as follows: each question consists of B specifying an arbitrary set S of positive integers (possibly one specified in some previous question), and asking A whether x belongs to S. Player B may ask as many such questions as he wishes. After each question, player A must immediately answer it with *yes* or *no*, but is allowed to lie as many times as she wants; the only restriction is that, among any $k+1$ consecutive answers, at least one answer must be truthful.

After B has asked as many questions as he wants, he must specify a set X of at most n positive integers. If x belongs to X, then B wins; otherwise, he loses. Prove that:

a) If $n \geq 2^k$, then B can guarantee a win.

b) For all sufficiently large k, there exists an integer $n \geq 1.99^k$ such that B cannot guarantee a win.

Time allowed: 4 hours 30 minutes
Each problem is worth 7 points

Wednesday, July 11, 2012
Language: English

Second Day

Problem 4. Find all functions $f\colon \mathbb{Z} \to \mathbb{Z}$ such that, for all integers a, b, c that satisfy $a + b + c = 0$, the following equality holds:

$$f(a)^2 + f(b)^2 + f(c)^2 = 2f(a)f(b) + 2f(b)f(c) + 2f(c)f(a).$$

(Here $\mathbb{Z}$ denotes the set of integers.)

Problem 5. Let ABC be a triangle with $\angle BCA = 90°$, and let D be the foot of the altitude from C. Let X be a point in the interior of the segment CD. Let K be the point on the segment AX such that $BK = BC$. Similarly, let L be the point on the segment BX such that $AL = AC$. Let M be the point of intersection of AL and BK. Show that $MK = ML$.

Problem 6. Find all positive integers n for which there exist non-negative integers $a_1, a_2, \ldots, a_n$ such that

$$\frac{1}{2^{a_1}} + \frac{1}{2^{a_2}} + \cdots + \frac{1}{2^{a_n}} = \frac{1}{3^{a_1}} + \frac{2}{3^{a_2}} + \cdots + \frac{n}{3^{a_n}} = 1.$$

Time allowed: 4 hours 30 minutes
Each problem is worth 7 points

2 Results

Some Country Scores		
Rank	Country	Score
1	South Korea	209
2	China	195
3	U.S.A.	194
4	Russia	177
5	Canada	159
5	Thailand	159
7	Singapore	154
8	Iran	151
9	Vietnam	148
10	Romania	144
11	India	136
12	North Korea	128
12	Turkey	128
14	Taiwan	127
15	Serbia	126

Some Country Scores		
Rank	Country	Score
16	Peru	125
17	Japan	121
18	Poland	119
19	Brazil	116
19	Bulgaria	116
19	Ukraine	116
22	Netherlands	115
22	U.K.	115
24	Belarus	114
25	Croatia	110
26	Greece	107
27	Australia	106
27	Hong Kong	106
29	Saudi Arabia	105
30	Moldova	104

Mark Distribution by Question						
Mark	Q1	Q2	Q3	Q4	Q5	Q6
0	41	263	481	53	349	474
1	37	83	11	65	17	39
2	15	8	4	95	29	12
3	24	5	31	74	45	3
4	16	8	7	48	15	9
5	11	2	6	26	4	0
6	2	7	0	44	3	1
7	402	172	8	143	86	10
Total	548	548	548	548	548	548
Mean	5.63	2.56	0.41	3.77	1.66	0.34

The medal cuts were set at 28 for gold, 21 for silver and 14 for bronze.

Distribution of Awards at the 2012 IMO

Country	Total	Gold	Silver	Bronze	H.M.
Argentina	74	0	0	2	4
Armenia	80	0	1	2	2
Australia	106	0	2	4	0
Austria	79	0	0	4	1
Azerbaijan	60	0	0	2	2
Bangladesh	74	0	1	2	2
Belarus	114	0	4	1	1
Belgium	93	0	2	1	2
Bolivia	6	0	0	0	0
Bosnia and Herzegovina	84	0	1	2	2
Brazil	116	1	1	3	1
Bulgaria	116	1	2	2	1
Canada	159	3	1	2	0
Chile	59	0	0	1	4
China	195	5	0	1	0
Colombia	83	0	0	3	3
Costa Rica	80	0	0	3	2
Croatia	110	1	1	3	1
Cuba	8	0	0	0	0
Cyprus	39	0	0	0	3
Czech Republic	80	0	1	1	4
Denmark	60	0	0	1	4
Ecuador	47	0	0	1	2
El Salvador	28	0	0	0	2
Estonia	50	0	0	0	4
Finland	57	0	1	0	3
France	93	0	1	4	1
Georgia	68	0	0	1	3
Germany	102	0	2	3	1
Greece	107	1	1	3	1
Guatemala	11	0	0	0	1
Honduras	33	0	0	1	2
Hong Kong	106	0	3	1	2
Hungary	93	0	2	1	3
Iceland	21	0	0	0	0

Distribution of Awards at the 2012 IMO					
Country	Total	Gold	Silver	Bronze	H.M.
India	136	2	3	0	1
Indonesia	100	0	1	3	1
Iran	151	3	2	1	0
Ireland	34	0	0	0	2
Israel	102	0	3	1	1
Italy	93	0	2	1	2
Ivory Coast	29	0	0	0	3
Japan	121	0	4	1	1
Kazakhstan	101	0	1	4	1
Kosovo	9	0	0	0	0
Kuwait	0	0	0	0	0
Kyrgyzstan	50	0	0	0	6
Latvia	55	0	0	0	5
Liechtenstein	5	0	0	0	0
Lithuania	69	0	0	3	1
Luxembourg	36	0	0	1	2
Macau	40	0	0	0	3
Macedonia (FYR)	59	0	0	2	2
Malaysia	100	0	2	3	1
Mexico	102	1	1	2	2
Moldova	104	0	2	3	1
Mongolia	90	1	0	2	3
Montenegro	5	0	0	0	0
Morocco	49	0	0	2	0
Netherlands	115	2	0	3	1
New Zealand	75	0	0	2	4
Nigeria	52	0	0	1	3
North Korea	128	2	1	3	0
Norway	33	0	0	0	1
Pakistan	41	0	1	0	1
Panama	17	0	0	0	1
Paraguay	31	0	0	0	2
Peru	125	0	3	2	1
Philippines	41	0	0	2	1
Poland	119	0	2	4	0

Distribution of Awards at the 2012 IMO					
Country	Total	Gold	Silver	Bronze	H.M.
Portugal	96	1	1	2	1
Puerto Rico	32	0	0	1	1
Romania	144	2	3	1	0
Russia	177	4	2	0	0
Saudi Arabia	105	0	2	3	0
Serbia	126	1	2	1	2
Singapore	154	1	3	2	0
Slovakia	85	1	0	2	2
Slovenia	71	0	0	2	4
South Africa	71	0	0	2	3
South Korea	209	6	0	0	0
Spain	64	0	1	0	3
Sri Lanka	30	0	0	1	1
Sweden	47	0	0	1	1
Switzerland	76	0	0	3	1
Syria	19	0	0	0	1
Taiwan	127	1	3	0	2
Tajikistan	91	0	0	4	2
Thailand	159	3	3	0	0
Trinidad and Tobago	26	0	0	0	1
Tunisia	25	0	1	0	0
Turkey	128	1	3	2	0
Turkmenistan	78	0	1	2	2
Uganda	2	0	0	0	0
Ukraine	116	0	3	2	1
United Kingdom	115	1	1	4	0
United States of America	194	5	1	0	0
Uruguay	30	0	0	0	2
Venezuela	17	0	0	0	1
Vietnam	148	1	3	2	0
Total (100 teams, 548 contestants)		**51**	**88**	**138**	**148**

N.B. Not all countries sent a full team of six students.

Tournament of the Towns Selected Problems, Spring 2012

Andy Liu

1. RyNo, a little rhinoceros, has 17 scratch marks on its body. Some are horizontal and the rest are vertical. Some are on the left side and the rest are on the right side. If RyNo rubs one side of its body against a tree, two scratch marks, either both horizontal or both vertical, will disappear from that side. However, at the same time, two new scratch marks, one horizontal and one vertical, will appear on the other side. If there are less than two horizontal and less than two vertical scratch marks on the side being rubbed, then nothing happens. If RyNo continues to rub its body against trees, is it possible that at some point in time, the numbers of horizontal and vertical scratch marks have interchanged on each side of its body?

Solution

Let a, b, c and d be the numbers of scratch marks which are horizontal and on the left side, vertical and on the left side, horizontal and on the right side, and vertical and on the right side. Suppose the initial values of a and b have been interchanged, and so are those of c and d, then $a+b$ and $c+d$ are unchanged. Since each of these two sums changes by 2 after a rubbing, the total number of rubbings must be even. If we allow negative values temporarily, the order of the rubbings is immaterial, and we can assume that they occur alternately on the left side and on the right side. After each pair of rubbings, the parity of each of a, b, c and d has changed. Suppose initially $a + b$ is odd so that $c + d$ is even. After an odd number of pairs of rubbings, the final values of a and b may have interchanged from their initial values, the odd one becomes even and the even one becomes odd. However, this is not possible for c and d, as they either change from both even to both odd, or from both odd to both even. Similarly, after an even number of pairs of rubbings, the final values of c and d may have interchanged from their initial values, but this is not possible for a and b. Thus the desired scenario cannot occur.

2. In an 8×8 chessboard, the rows are numbers from 1 to 8 and the columns are labelled from a to h. In a two-player game on this chessboard, the first player has a white rook which starts on the square $b2$, and the second player has a black rook which starts on the square $c4$. The two players take turns moving their rooks. In each move, a rook lands on another square in the same row or the same column as its starting square. However, that square cannot be under attack by the other rook, and cannot have been landed on before by either rook. The player without a move loses the game. Which player has a winning strategy?

Solution

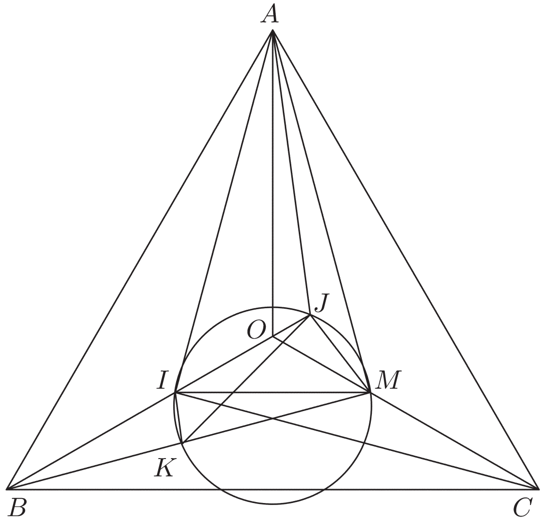
The second player has a winning strategy. Divide the eight rows into four pairs (1,3), (2,4), (5,7) and (6,8), and the eight columns also into four pairs (b, c), (d, e), (f, g) and (h, a). Then divide the sixty-four squares into thirty-two pairs. Two squares are in the same pair if and only if they are on two different rows which form a pair, and on two different columns which also form a pair. Thus the starting squares of the two rooks form a pair. The second player's strategy is to move the black rook to the square which forms a pair with the square where the white rook has just landed. First, this can always be done, because if the white rook stays on its current row, the black rook will do the same, and if the white rook stays on its current column, the black rook will do the same.

Second, since the square on which the white rook has just landed cannot have been landed on before, the square to which the black rook is moving has never been landed on before, since the squares are occupied by the two rooks in pairs. Third, the black rook will not be under attack by the white rook since the two squares in the same pair are on different rows and on different columns. Hence the second player always has a move, and can simply wait for the first player to run out of moves.

3. Let AH be an altitude of an equilateral triangle ABC. Let I be the incentre of triangle ABH, and let L, K and J be the incentres of triangles ABI, BCI and CAI respectively. Determine $\angle KJL$.

Solution by Central Jury

Since K is the incentre of triangle BCI, $\angle BKI = 90° + \frac{1}{2}\angle BCI = 97.5°$. Let O be the centre of triangle ABC and let M be the point symmetric to I about OA. Note that K lies on BM. We have $\angle MAJ = 7.5° = \angle OAJ$. Since $\angle AOJ = 60° = \angle MOJ$, J is the incentre of triangle MAO. Hence $\angle MJO = 90° + \frac{1}{2}\angle MAO = 97.5° = \angle BKI$. Hence $IJMK$ is a cyclic quadrilateral and $\angle IJK = \angle IMK = 15°$. Since L is symmetric to K about BO, $\angle KJL = 2\angle IJK = 30°$.



4. Konstantin has a pile of 100 pebbles. In each move, he chooses a pile and splits it into two smaller ones until he gets 100 piles each with a single pebble.

(a) Prove that at some point, there are 30 piles containing a total of exactly 60 pebbles.

(b) Prove that at some point, there are 20 piles containing a total of exactly 60 pebbles.

(c) Prove that Konstantin may proceed in such a way that at no point, there are 19 piles containing a total of exactly 60 pebbles.

Solution

(a) **Solution by Daniel Spivak**
At some point in time, we must have exactly 70 piles. At least

40 of them contain exactly 1 pebble each, as otherwise the total number of pebbles is at least $39+2\times 31 = 101$. Removing these 40 piles leave behind exactly 30 piles containing exactly 60 pebbles among them.

(b) **Solution by Peter Xie**
We call k piles containing a total of exactly $2k+20$ pebbles a good collection. We claim that if $k \geq 23$, a good collection contains either 1 pile with exactly 2 pebbles or 2 piles each with exactly 1 pebble. Otherwise, the total number of pebbles in the collection is at least $1 = 3(k-1) = 3k-2$, which is strictly greater than $2k+20$ when $k \geq 23$. Now any partition of the original pile into 40 piles results in a good collection with $k = 40$. From this, we can obtain a good collection with $k = 39$ by either removing 1 pile with exactly 2 pebbles or removing 2 piles each with exactly 1 pebble and then subdividing any other pile with at least 2 pebbles. In the same way, we can obtain good collections down to $k = 22$, with a total of exactly 64 pebbles. We claim that there exist 2 or more piles containing a total of exactly 4 pebbles. Suppose this is not the case. If there are no piles with exactly 1 pebble, then the total number of pebbles in the collection is at least $2+3\times 21 > 64$. If there are piles with exactly 1 pebble, then the total is at least $3+4\times 19 > 64$. Thus the claim is justified. We now remove these 4 pebbles, obtaining 60 pebbles in at most 20 piles. Eventual subdivision of these piles will bring the number of piles to 20 while keeping the total number of pebbles at 60.

(c) **Solution by Peter Xie**
Separate out piles of 3 until we have 32 piles of 3 and are left with 1 pile of 4. Throughout this process, exactly one pile contains a number of pebbles not divisible by 3. If we include this pile, the total cannot be 60. If we exclude this pile, the total of 19 piles of 3 is only 57. We now separate the pile of 4 into 2 piles of 2. Now every pile contains at most 3 pebbles, so that 19 piles can contain at most 57 pebbles. Further separation will not change this situation.

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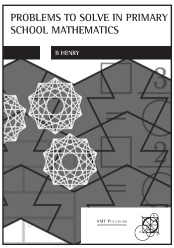

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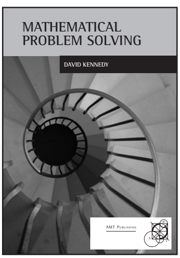

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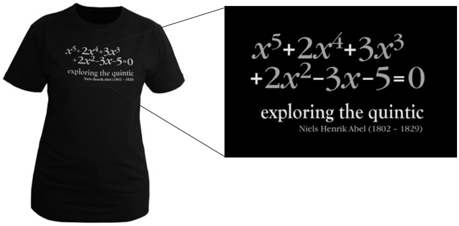

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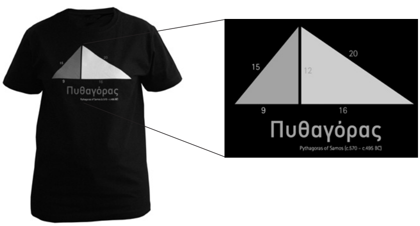

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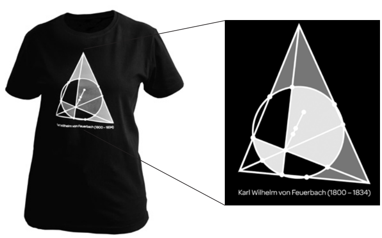

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