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MATHEMATICS COMPETITIONS

JOURNAL OF THE

WORLD FEDERATION OF NATIONAL MATHEMATICS COMPETITIONS

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Please send articles to:

The Editor
Mathematics Competitions
World Federation of National Mathematics Competitions
University of Canberra Locked Bag 1
Canberra GPO ACT 2601
Australia

Fax:+61-2-6201-5052

or

Dr Jaroslav Švrček
Dept. of Algebra and Geometry
Faculty of Science
Palacký University
Tr. 17. Listopadu 12
Olomouc
772 02
Czech Republic

Email: svrcek@inf.upol.cz

MATHEMATICS COMPETITIONS VOLUME 25 NUMBER 1 2012

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World Federation of National Mathematics Competitions

Executive

President: Professor María Falk de Losada
Universidad Antonio Narino
Carrera 55 # 45-45
Bogotá
COLOMBIA

Senior Vice President: Professor Alexander Soifer
University of Colorado
College of Visual Arts and Sciences
P.O. Box 7150 Colorado Springs
CO 80933-7150
USA

Vice Presidents: Dr. Robert Geretschläger
BRG Kepler
Keplerstrasse 1
8020 Graz
AUSTRIA

Professor Ali Rejali
Isfahan University of Technology
8415683111 Isfahan
IRAN

Publications Officer: Dr Jaroslav Švrček
Dept. of Algebra and Geometry
Palacký University, Olomouc
CZECH REPUBLIC

North America: Professor Harold Reiter
Department of Mathematics
University of North Carolina at Charlotte
9201 University City Blvd.
Charlotte, NC 28223-0001
USA

Oceania: Professor Derek Holton
Department of Mathematics and Statistics
University of Otago
PO Box 56
Dunedin
NEW ZEALAND

South America: Professor Patricia Fauring
Department of Mathematics
Buenos Aires University
Buenos Aires
ARGENTINA

The aims of the Federation are:–

1. ***to promote excellence in, and research associated with, mathematics education through the use of school mathematics competitions;***
2. ***to promote meetings and conferences where persons interested in mathematics contests can exchange and develop ideas for use in their countries;***
3. ***to provide opportunities for the exchanging of information for mathematics education through published material, notably through the Journal of the Federation;***
4. ***to recognize through the WFNMC Awards system persons who have made notable contributions to mathematics education through mathematical challenge around the world;***
5. ***to organize assistance provided by countries with developed systems for competitions for countries attempting to develop competitions;***
6. ***to promote mathematics and to encourage young mathematicians.***

From the President

WFNMC and ICME-12

As ICME-12 draws near, WFNMC has finalized its plans for its second Miniconference which will be held just prior to ICME, at the same venue with a fascinating programme as follows.

Saturday, 7 July, COEX, Seoul

09:00–09:15	Installation of the mini-conference	María de Losada
09:15–10:30	Mathematics Seminar	Nicolai Konstantinov
11:00–11:45	Challenging Mathematics through the Improvement of Education	Ali Rejali
11:45–12:30	Technology and the Creation of Challenging Problems	Sergei Abramovich and Eun Kyeong Cho
14:00–14:45	A proposal for forming in-service primary school teachers for teaching mathematics through challenging mathematical problems	Flor Elva Jaime (presented by María de Losada)
14:45–15:30	The Development of Teachers' Ability to Think Mathematically	Toshimitsu Miyamoto
16:00–16:45	The Art of Presenting Ideas in Different Ways	Kiril Bankov

16:45–18:00	Problem Creation Workshop	Sergei Dorichenko, International Tournament of Towns, Alexander Soifer, Colorado Mathematics Olympiad, Jaroslav Švrček, Czech and Slovak Mathematical Olympiads
18:00–18:15	Closing of the miniconference	Alexander Soifer

Erdős Awards 2012

Another highlight are the 2012 Erdős Awards that will be presented at one of the regular meetings of WFNMC as part of the scientific programme of ICME.

The Paul Erdős Award was established to recognise the contributions of mathematicians which have played a significant role in the development of mathematical challenges at the national or international level and which have been a stimulus for the enrichment of mathematics learning.

Each recipient of the award is selected by the Executive and Advisory Committee of the World Federation of National Mathematics Competitions on the recommendation of the WFNMC Awards Subcommittee.

The recipients of the Paul Erdős awards for 2012 are professors Cecil Rousseau of the United States and Paul Vaderlind of Sweden.

Cecil Rousseau

In 1968, Cecil C. Rousseau received a Ph.D. in physics from Texas A&M University, with research under the direction of John L Gammel. Cecil was a member of the physics faculty at Baylor University from 1968 until 1970, and then he changed both disciplines and affiliations by taking a position in mathematics at what was then Memphis State University. He remained at that position until retirement in 2008.

His research record is dominated by contributions to graphical Ramsey theory; however it includes work in other areas of combinatorics, physics, applied analysis, and mathematical statistics. Most of his work is collaborative, including 35 joint papers with the most famous collaborator of all, Paul Erdős. The University of Memphis has recognized his contributions by awarding him a Meritorious Faculty Award in 1992 and a Dunavant Professorship in 1998.

Known as an avid problems creator and solver, over the years Cecil has contributed, solved, or edited problems for AMC competitions, MAA publications, *Fibonacci Quarterly*, SIAM *Problems and Solutions*, *PME Journal*, and others.

Cecil's Olympiad connection began with an invitation from Murray Klamkin to contribute problems for the then new exam, the USAMO. Cecil has been a member of the USAMO Committee for most of the intervening years. In the intervals (1992 to 1997) and (2004 to 2008) he was Chair of the Committee. In 1974, the first year the USA participated in the IMO, Cecil served as Deputy Leader. He served as Deputy Chief Coordinator when the USA hosted the IMO in 1981. He served as Coach and Leader from 1985 to 1987 and 1991 to 1993. At the invitation of the Canadian organizers, he served as a Problems Captain in 1995. When in 2001 the USA again hosted the IMO, he was there as Chair of the Problems Committee and Chief Coordinator.

Paul Vaderlind

Paul Vaderlind was born in Poland in 1948. His inclination to mathematics and problem solving became apparent very early. Since the age of 10 the book *The Moscow Puzzles* by Boris Kordemsky and other similar publications became frequent companions of his activities. This brought him to study mathematics at the University of Warsaw. In 1969 he continued his education in Sweden. He got his PhD degree in the field of Discrete Mathematics from Stockholm University and, since 1974, he has been teaching at this institution. Besides his research and lecturing activities, he has been strongly engaged in mathematics competitions and in the education of gifted young people. He has been visiting once a week a school with a special program in mathematics, offering many hours for training the pupils. Since 1999 he is the leader of the Swedish IMO team.

Paul Vaderlind is also well-known for his contributions to the Baltic Way competition with which he was strongly involved from the very beginning. He was attracted by the idea of team competition, and by the fact that this competition targeted the Baltic Sea countries. Sweden was one of the first "western" countries to join the Baltic Way. Paul worked tirelessly to get other countries join this competition too. In recognition of this involvement the Gold Medal of the Latvian Mathematical Society was given to him in connection with fiftieth anniversary of the organization.

Paul Vaderlind's international involvement in spreading mathematics education (on all levels) and mathematics competitions is truly admirable. For more than a decade, he has been involved with developing countries on a voluntary basis. This concerns mostly Master and PhD education but also mathematics competitions. Many projects, primarily in Africa, ranging from Madagascar through Rwanda to Senegal, are under the umbrella of SIDA (Swedish International Development Agency). However, he is also involved with other organizations such as the French CIMPA (in South-East Asia), the European Union (in Congo), London Mathematical Society (in Ivory Coast) and so on.

A particular feature of Paul Vaderlind's activities is that wherever he is and whatever he does, he always makes a point of promoting mathematics competitions. He has helped to organize some national competitions and has trained teachers. As a result some new countries were intoduced to PAMO (Pan African Math Olympiad), a competition in which Paul has been involved since 2003. There he served as a problem proposer, a coordinator and in whatever capacity needed for the success of this competition.

One of several projects in which Paul is currently strongly involved is PACM (Pan-African Centre for Mathematics), an ambitious project, creating a kind of Stockholm University branch (for Master and PhD in pure mathematics only) in Dar es Salaam, and serving gifted students from all Africa. After trying all different models of supporting Master and PhD students from Africa, it was decided to move the whole process to the continent itself. This would also serve to reduce the brain drain problem. Beginning in January 2013, 40 Master students will be accepted per year from all over Africa with the aim to get a Swedish Master and later Swedish PhD degree. Paul is the coordinator for this important project.

Paul Vaderlind has an important role also in SNAP Mathematics Foundation of Edmonton. The Foundation was established to promote Math Fairs. This is an activity for middle school students, who prepare booths featuring interactive mathematical puzzles for the audience. The movement has taken root in North America and Paul Vaderlind is spreading it to other parts of the world. He has held the world's largest Math Fair, with 147 booths, in Stockholm, and he has held two successful Math Fairs in Africa.

WFNMC proudly recognizes the outstanding contributions of these colleagues to the work of mathematical problem-solving competitions and its impact as a stimulus for the enrichment of mathematics learning.

Mathematics competitions: appreciation of their ethics and aesthetics

Traditionally philosophers have examined human values from two basic and fertile perspectives, that of ethics and that of aesthetics. In what follows we attempt to look at characteristics and issues of mathematics competitions from each of these viewpoints.

We take as given that mathematics problem-solving competitions represent a vital, creative scientific activity eliciting sound mathematical thinking, and, congruent to each level of challenge, inspired and original strategies to confront and conquer the problems posed.

Competitions are thus viewed as means to certain well-accepted ends, leading each student to strive to reach his personal best level of mathematical achievement, identifying and nurturing students with special interest and talent in mathematics, drawing young students to the study of mathematics thus assuring the future of the discipline and the sciences that depend on it, cementing scientific communities and contributing to development of new ideas that will transform the quality of human life.

However, when analyzed from the perspectives of ethics and aesthetics, we can attain new insights into the value and values of mathematics competitions, insights that we will explore here albeit all too briefly.

Ethics refers to well-founded standards of right and wrong that prescribe what humans ought to do, usually in terms of rights, obligations, benefits to society, fairness, or specific virtues, standards supported by consistent and well-founded reasons.

Secondly, ethics refers to the study and development of one's own ethical standards, and striving to ensure that we, and the institutions we help to shape, live up to standards that are reasonable and solidly-based.[1]

There are a host of ethical issues broached within the setting of mathematical competitions. One of these is the unparalleled experience of each participant when confronting a challenging problem with only his/her own resources, a moment of truth and great intellectual honesty. Furthermore, any situation of competition implies ethical standards of behavior, not receiving unauthorized aid, not misrepresenting any part of the solution one has devised, avoiding deliberate ambiguity, and so communicating precisely one's own strategies and advances, justifying all steps clearly, convincing those who will read the solution of the solidity of one's arguments only after one has convinced oneself.

Without strict ethical standards, neither competitors nor the competition itself can occupy places of importance before the groups and societies that harbor them.

[1] http://www.scu.edu/ethics/practicing/decision/whatisethics.html

Aesthetics on the other hand manifests itself in the appreciation of the beauty and power of each problem posed and of the arguments used in its solution.

The great German philosopher Immanuel Kant laid clear the idea that aesthetics treats each object and experience as an end in itself, drawing out its own intrinsic value.

Kant attributes the distinctive characteristics of aesthetics to our faculty of forming judgments, leading us to take certain positions before the objects of our experiences, separating them from our strictly scientific interests and our practical needs. The recipient of the aesthetic experience is, then, the rational being and rationality is not complete without the exercise of aesthetic judgment.

Each competition experience must draw heavily on the aesthetics of the problems, their genius, originality and power as ends in themselves, enriching the lives of organizers and participants, removed and distinguished from the compilation of scores and results, of comparisons and rivalries.

Thus, WFNMC and its members, persons dedicated to mathematical problem-solving events, can benefit enormously from these traditional facets of moral philosophy, maintaining strict surveillance of their ethics and aesthetics and thus elevating the quality of the experience for all involved.

María Falk de Losada
President of WFNMC
Bogotá, June 2012

From the Editor

Welcome to *Mathematics Competitions* Vol. 25, No. 1.

First of all I would like to thank again the Australian Mathematics Trust for continued support, without which each issue (note the new cover) of the journal could not be published, and in particular Heather Sommariva, Bernadette Webster and Pavel Calábek for their assistance in the preparation of this issue.

Submission of articles:
The journal *Mathematics Competitions* is interested in receiving articles dealing with mathematics competitions, not only at national and international level, but also at regional and primary school level. There are many readers in different countries interested in these different levels of competitions.

- The journal traditionally contains many different kinds of articles, including reports, analyses of competition problems and the presentation of interesting mathematics arising from competition problems. Potential authors are encouraged to submit articles of all kinds.
- To maintain and improve the quality of the journal and its usefulness to those involved in mathematics competitions, all articles are subject to review and comment by one or more competent referees. The precise criteria used will depend on the type of article, but can be summarised by saying that an article accepted must be correct and appropriate, the content accurate and interesting, and, where the focus is mathematical, the mathematics fresh and well presented. This editorial and refereeing process is designed to help improve those articles which deserve to be published.

At the outset, the most important thing is that if you have anything to contribute on any aspect of mathematics competitions at any level, local, regional or national, we would welcome your contribution.

Articles should be submitted in English, with a black and white photograph and a short profile of the author. Alternatively, the article can be submitted on an IBM PC compatible disk or a Macintosh disk. We prefere LaTeX or TeX format of contributions, but any text file will be helpful.

Articles, and correspondence, can also be forwarded to the editor by mail to

The Editor, *Mathematics Competitions*
Australian Mathematics Trust
University of Canberra Locked Bag 1
Canberra GPO ACT 2601
AUSTRALIA

or to

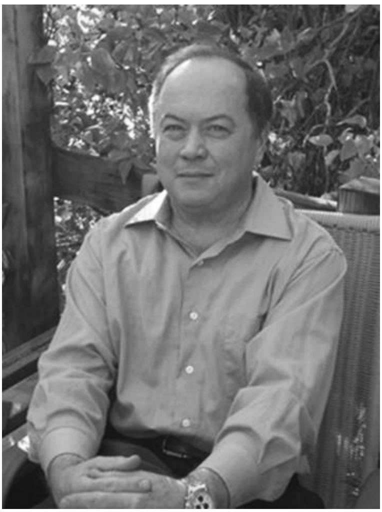
Dr Jaroslav Švrček
Dept. of Algebra and Geometry
Palacky University of Olomouc
17. listopadu 1192/12
771 46 OLOMOUC
CZECH REPUBLIC

jaroslav.svrcek@upol.cz

Jaroslav Švrček
June 2012

Mathematical Olympiads for Secondary Students: What is their effect on successful contestants? A Conversation in Five Parts

Alexander Soifer



Born and educated in Moscow, Alexander Soifer has for 33 years been a Professor at the University of Colorado, teaching math, and art and film history. He has published over 200 articles, and a good number of books. In the past 3 years, 6 of his books have appeared in Springer: The Mathematical Coloring Book: Mathematics of Coloring and the Colorful Life of Its Creators*;* Mathematics as Problem Solving: How Does One Cut a Triangle?*;* Geometric Etudes in Combinatorial Mathematics; Ramsey Theory Yesterday, Today, and Tomorrow*; and* Colorado Mathematical Olympiad and Further Explorations. *He has founded and for 29 years ran the Colorado Mathematical Olympiad. Soifer has also served on the Soviet Union Math Olympiad (1970–1973) and USA Math Olympiad (1996–2005).*

> *CMO is the best kind of math competition, because it tests creativity and intelligence rather than mere knowledge and training.*
>
> — James Carroll
>
> Winner of Literary Award 1999 & 2001, Third Prize 2000

1 My early experience

When I turned 6, my parents decided that I would be a composer. My life in music lasted eight years. It taught me discipline, concentration. It developed my intuition. Moscow University Olympiad that I started to attend at 12, showed me something school mathematics did not: it made epithets such as beautiful, elegant, surprising, humorous applicable to problems of mathematics. The Olympiads—and in equal measure the mathematical circle run by Nikolaj Nikolaevich Konstantinov on Saturdays at the Old Building of Moscow State University—were responsible for the loss of a composer. I chose mathematics at 14.

2 When participation ends

When we reach an old age of 17 or so, our Olympic career ends. But the flames started in us lead us to organizing Olympiads for the next generations of mathematicians. This is not a surprise that so many former Olympians become organizers, problem creators, and mentors.

I was no exception. After organizing Olympiads for just my gifted 25 students in the great magnet school No. 2 in Moscow, I was invited to represent *Kvant* (*Quantum*) magazine on the Jury of the Soviet Union National Mathematical Olympiad (1970–1973). Later there came service on the USA Mathematics Olympiad (1996–2005). But unquestionably the most rewarding experience has been to create in 1983-84 the Colorado Mathematical Olympiad (CMO), which is to be offered for the 29^{th} consecutive year in April 2012. (See a detailed 450-page report [1]).

3 Paul Erdős on Mathematical Contests

In March 1989, I was visited for a week by the great Paul Erdős. We worked on some problems, posed other problems. Paul gave two lectures at my university. He also appeared in our Colorado Mathematical Olympiad film. Let me repeat for you Paul Erdős's address to the winners of the Olympiad, captured in the film:

> It is important to learn more mathematics. The contest itself wouldn't be that important, but it creates new enthusiasm. From this point of view it is important. Also, it has good effect: it stimulates interest in mathematics...

> To all the winners of the Colorado Mathematical Olympiad I wish future successes and I hope they will become mathematicians, preferably pure mathematicians, since I am a pure mathematician, but my second choice would be computer science or various branches of applied mathematics, or possibly physics or engineering.

These are important words. Competitions are not a goal in themselves, but a means to the goal, the goal of attracting young talents to mathematics, and passing the baton to them. Moreover, not all winners become distinguished mathematicians. And not all leading mathematicians come from the ranks of Olympiad winners. There are many ways to success, and many hurdles that life throws in our way.

If a student excels even in a single Olympiad (I exclude here multiple choice entertainments as predictors), the student possesses talent, and we must encourage the student to take his talent with care and develop it with enjoyable but hard work. And if a student does not succeed even in a series of Olympiads, this does not at all imply the absence of talent. Some are slow in their development; it suffices to recall Albert Einstein.

4 The Winners Speak

While writing the book about CMO [1], a thought visited me. Let me offer a microphone (or is it a writing feather or a computer keyboard?) to the past CMO winners, and for them to share what they thought about CMO, how it affected their lives, and what their lives AD (*After Departure* from high school and the Olympiad) have been like. Read the full account there as well as the history of the event, problems, solutions, and "bridges" from CMO problems to mathematical research. Here I will present reminiscences of four winners.

David Hunter In 4 years of competing, 1985–1988, David won one Second Prize and three First Prizes in 1986, 1987 and 1988, a record that still stands. David holds Princeton University B.A. in mathematics and University of Michigan Ph.D. in statistics. He is now Professor of Statistics at Pennsylvania State University. David Hunter writes:

> My first experience with the Colorado Math Olympiad and with meeting Dr. Soifer was in the Olympiad's second year in 1985—

can it really have been almost a quarter century ago?—and, like most of the participating students, I had been involved with math contests of various sorts throughout junior high school. But I still had no idea what to expect of the Olympiad. Four hours for five questions? It seemed like an awfully strange format for a contest to me at the time, but I grew to love the Olympiad and I still recall it with great fondness. Of course, four hours for five questions is a lot closer to what I do today as a statistics professor—thinking about a single problem for days or weeks at a time, then carefully writing up what I've done—than the quick, tidy quizzes that most people think of when they think "math"; but that's not what I like about the Olympiad.

The point is that the Olympiad is a celebration of real mathematics, of elegance and creativity in reasoning. This is not a common occurrence in high school! I believe that there is something wonderful about an event at which hundreds of students voluntarily spend a whole day on five math problems (and then return a week later to hear the solutions and see who did well!).

I now have a job in which I do math a lot of the time, so the fact that I've always loved math means that I have a bit of built-in job satisfaction. As one of my jobs in the statistics department at Penn State, I am the chair of our undergraduate program, which means that I occasionally get to advice students prior to and during their undergraduate academic careers. My main piece of advise is this: Take as much math as you can. I was a math major in college and have always believed that a degree in math is an excellent basis for almost any career: It proves that a student can think. It lacks the vocational flavor of a degree like engineering or computer science, in which students learn particular skills; but such things can often be acquired later, either in graduate school or on the job, where it is often the case that job-specific knowledge is ultra-specialized anyway.

But I digress. When he contacted me about writing this essay, Dr. Soifer reminded me of a quotation of mine that appeared in the Gazette Telegraph one of the years I participated in the Olympiad, something along the lines of "I am not a nerd." This made me smile, because recently I have been known to announce to a lecture hall of hundreds of students in an introductory statistics class, when the situation calls for it, that I am a math nerd. Partly I do this to get their attention and elicit a bit of a laugh, and the students can tell exactly what I mean: I love mathematics, and even though few of them will ever be able to say that—my intro stats class is not generally a Math Olympiad kind of crowd—they can appreciate at least that I am trying to share my enthusiasm for statistics with them. Just like Dr. Soifer has been sharing his enthusiasm for mathematics through the Olympiad for all these years.

Gideon Yaffe won Third Prizes as a sophomore in 1986 and a junior in 1987, and First Prize as a senior in 1988. Gideon also won the 1988 Creativity Award. He is Professor of Philosophy and Professor of Law at the University of Southern California. Gideon writes:

Dear Alexander,

Nice to hear from you. I've had news of you now and again from Paul Zeitz. I'm glad to hear that you are thriving.

I wandered from major to major at Harvard. I never actually majored in drama, although I had intended to when I went to college. I acted in some plays, but never majored. I majored in math for a time, in biology for a time, in applied math for a time, and in various blank-and-blank combinations. In the end, I majored in philosophy largely because I wanted to write a senior thesis in the area. My initial interest in philosophy came from my interest in math. I was very taken by logic and by the way in which formal argument could be used to make progress on philosophical questions. After some time living in the Bay Area and working some stupid jobs, I went to graduate school in philosophy at Stanford and finished my Ph.D. in 1998.

I wrote my dissertation on the free will problem and on the seventeenth century philosopher John Locke's answer to it. In 1999, I started on the tenure track at the University of Southern California [USC].

I published two books between 2000 and 2004, one on Locke and one on a fairly obscure 18th century philosopher named Thomas Reid. Also in that period I developed interests in legal issues, particularly the application of philosophical thought about free will and responsibility to criminal law. In the 2004–05 school year, I was a law student at USC, and two years later the law school hired me part time. So now I split my time between the law school and the philosophy department. For the last few years, I've been publishing in philosophy of law and I'm currently working on a book about criminal attempts—the crime of attempting a crime. I'm also involved with the Macarthur Foundation's law and neuroscience project. The Macarthur has dedicated $10 million to study the use of neuroscience to the law and I'm one of the people developing projects for that.

My family doesn't live in Colorado Springs anymore, so it's not on my usual trek any longer, unfortunately. However, if I find myself there, I'll be sure to look you up.

Thanks for the Olympiad. It made a big difference to me when I was in high school. You might remember that the year that David Hunter and I won it, I also won a creativity prize. That probably meant more to me than the victory in the Olympiad itself. It made me realize that the brain that I used to do math could be used for lots of things, anything that required creativity, and so it prompted me to try out lots of things in college.

Best,

— Gideon

Aaron Parsons comes from the tiny town of Rangely in the northwestern corner of the state of Colorado. He won Second Prize in 1997, and First Prize in 1998. Aaron earned B.A. in mathematics from Harvard University and Ph.D. in astronomy from the University of California Berkeley. He is now a Professor of Astronomy at Berkeley. Aaron writes:

Dear Prof. Soifer,

How wonderful to hear from you! ... I would be more than happy to provide a short note about myself and CMO:

I participated in the Colorado Math Olympiad [CMO] from 1994 to 1998, representing Rangely High School [RHS] and coached by the generous and committed Melvin Oliver, who single-handedly developed and supported the math program at RHS. At a time when most math contests focused on speed, numbers, and arithmetic tricks, CMO stood out as something completely different.

The first time I took a CMO test, I was flabbergasted—I was so tuned to the "other" type of contest, I felt I could hardly solve a single problem! No one on our team qualified for the awards ceremony that year, and one of the problems just drove me nuts—one about polygons of unit area on a grid. Resolving to qualify for the "answer session" next year, I set to work. I did manage an "Honorable Mention" the next year, and after visiting UCCS and meeting Prof. Soifer for the annual recapitulation of the contest, both Mel and I came away with a new understanding of a broader, more abstract, and altogether much more fun side of mathematics than we had previously seen.

I distinctly remember the following year, when, having qualified for the awards ceremony, I found myself unfortunately needing to skip the ceremony in order to participate in a state track meet (held just an hour away). I met with Prof. Soifer to excuse myself and to apologize for necessity of my departure. "Oh, that's quite alright," he said. "You know, I was a sprinter myself in high school." And a good one at that, as I found out. I began to understand his bounding energy in front of his students, to guess at a joie de vivre that could be expressed both academically and athletically.

After graduating from high school, I studied physics and mathematics (and ran track) at Harvard. There, I discovered that math—the real math that mathematicians do—was really much more like the bounding, gleeful CMO math than any other math I had been exposed to. I grew much better versed in mathematical reasoning, but it still wasn't until my second year away at college that the solution to that demonic polygon problem finally came to me. Liberated at long last, I moved on to astrophysics. I am currently finishing my doctorate at [the University of California] Berkeley, working to discover the first stars that formed in the universe 10 billion years ago.

— Aaron Parsons

Mathew Kahle won one Third Prize in 1988 and two First Prizes in 1990 and 1991. He was not motivated at his high school, earning a C- in geometry, with his teacher too busy for individual work with him. And so I offered mentorship to Matt. Many years later, he remembered this on the pages of *Geombinatorics* quarterly (see it in [2]):

> Perhaps the most important thing that I learned from him is that we are free to ask our own mathematical questions and pursue our own interests, that we can trust our sense of aesthetics and our intuition. This was a very empowering idea to me, particularly when I was young and struggling with boredom and frustration with school.

The University of Colorado refused to admit Matt due to his low school grade point average (C-). And so he got his B.A. and M.A. in mathematics from Colorado State University. Matt then earned his Ph.D. degree in mathematics from the University of Washington, Seattle. In 2007 he was awarded a Post-Doctoral Fellowship at Stanford University, and in 2010 a Post-Doctoral Fellowship at the historic Institute for Advanced Study in Princeton, at one point the seat of Albert Einstein. Matt is now Professor of Mathematics at Ohio State University.

Perhaps, more than any contestant, Matt absorbed the Olympiad spirit. He competed in 5 Olympiads, and came back 7 times to serve as the Olympiad's judge, to participate in passing the flaming Olympic torch to the next generations of young mathematicians. In order to help with judging, Matt traveled several times the 240-mile round trip from Fort Collins. In 2008, Matt traveled 1200 miles each way to come from Stanford University to Colorado Springs. Matthew writes:

> Professor Alexander "Sasha" Soifer told me that for the new edition of the Colorado Mathematical Olympiad book, he wanted to include a chapter written by past winners, about the role of the Olympiad in their lives, their view of math, and their future careers. I am very happy to contribute an essay for this.
>
> I competed in many math competitions growing up—in several statewide competitions sponsored by the local universities, as well as in well-known national contests—MathCounts, USAMO [USA Mathematics Olympiad], Putnam, etc. The first thing that comes to mind was that these competitions helped me realize that I was good at math, and perhaps even more importantly, they helped me realize how much I enjoy it. And they helped instill me with a sense of con-

fidence when I was an angst filled adolescent with low self-esteem. So for those things I will always be very grateful.

Given the topic of this essay, however, I want to emphasize that there are many features of the CMO [Colorado Mathematical Olympiad] that distinguish it in my mind from the other contests which I competed in. First, the format itself was different from most competitions—five questions and four hours! It is true that there are other Olympiad-style contests that I would come across years later (USAMO, Putnam, . . .), but I first sat for the CMO as an 8th grader, when our teacher and coach Betty Daniels took a few of us from the middle school to UCCS [University of Colorado at Colorado Springs] for the Olympiad.

I had never seen such an exam before! I remember Sasha coming around to the rooms at the beginning of the day, asking if anyone had questions. I think I asked him if we could use calculators. He smiled impishly, shrugged his shoulders and said, “Sure, why not?” (Perhaps, needless to say, calculators weren’t much help.) I am pretty sure that I spent the whole four hours every year I took it, even in eighth grade, and I never solved all of the problems. But it was not an ordeal for me to take a four hour math test; I could leave any time I wanted. I stayed until the end because I was having fun. This might be one of the most important qualities that separate the CMO in my mind—the spirit of fun, and even the sense of humor.

Not to say that the Olympiad was not serious business, not at all. In fact I would say it was some of the hardest tests that I ever took, and I am more proud of my performances in the CMO than in any other math contest. But it was always a pleasure to work on the problems. And I would continue thinking about the ones that I hadn’t gotten all week, and then come back for the award ceremony to see the answers revealed. In hindsight, Soifer’s presentations at the award ceremony feel like some of the first real math lectures I ever saw. They were so far beyond what they were teaching us in school, so different in style, yet somehow accessible at the same time. It felt like he was giving us a peek of an entire other world that we had never been exposed to.

I grew up to be a mathematician, but my road was a little bit of a long and winding one. In short, I was always much more excited about mathematics itself than I ever was about school, so much so that I barely made it out of high school and dropped out of college twice. But after a few false starts I set my sights on getting a math Ph.D., which I finished at the University of Washington in 2007. Since then, I've been a postdoctoral fellow at Stanford University. I greatly enjoy both research and teaching, and I hope to find a tenure track job as a professor that will allow me to continue both.

I think the Olympiad influenced me in many ways. It introduced me to the idea that there are many more math problems unsolved than there are solved. (I remember Professor Soifer saying, "To nearest percent, 0 % of all math problems are solved."). It helped me find out that I am capable of obsessing about math problems for hours, or days, or longer. (Needless to say, this is an important trait for a math researcher.) And that which problems we work on, and how we work on them, is not only a matter of ability, but of aesthetics and taste. (The CMO awarded special prizes for creative solutions, as well as literary prizes for clever poems and stories.) I think it also helped me realize how much I enjoy just talking with people about math—some of the first "serious" math conversations I remember having were with Soifer.

I appreciate that Sasha treated me as a friend and a peer even when he first met me as a 14 year old, and I have stayed in touch with him and connected to the Olympiad since then. I have contributed three research articles so far to his journal *Geombinatorics*, and helped judge the CMO several times. I was honored to be able to help judge for the 25th annual CMO last year. I think Sasha and I are just kindred spirits. We may naturally have some similar tastes mathematically, but I also think we recognized and appreciated the mischief and humor in each other's eyes from when we first met. For some reason, right now I am remembering winning the Olympiad, and shaking hands with Professor Soifer and one of the deans, and Soifer turning to the dean and saying, "See! This is why we should admit C-students!"

This is, by necessity, a bit of a personal note, and the particulars of my story might be unique. But something that I really appreciate, especially after having seen things from the other side as a judge, is just how many students have been affected. Several hundred students compete annually, and prizes are awarded to over a hundred students almost every year. Now multiply that by 25 years. So on behalf of all the lives and minds you've touched, many thanks Sasha. And here's to 25 more years of the Colorado Mathematical

Olympiad!

5 Farewell Thoughts

Mathematics can be easily made dull, reduced to mind-numbing drill. No wonder so many people around the world proudly declare “I hate mathematics.”

As Paul Erdős observed, competitions bring a new excitement to mathematics. But competitions should not be goal in themselves. The goal ought to be the enjoyment of doing mathematics.

We ought to stop discrimination based on age. We ought to allow and encourage the engagement in research to high school Olympians, and surely to all undergraduate majors.

In my opinion, Olympiads for university students and competitions like Putnam are not really needed. University is the time to compete with the field of mathematics, not with each other. And if any competition is needed to wake up to sleepy souls, let it be a competition of research works of young mathematicians, such as the Westinghouse Talent Search, now called The Intel Science Talent Search, and other similar research work contests, organized by a university for its own students. (My university had such a contest, where all fields, not just mathematics, competed together. My third undergraduate year research paper won second prize.)

I enthusiastically applaud Nikolaj Konstantinov for creating, with his colleagues the International Tournament of Towns, and most especially its “conferences,” where young high school mathematicians work on open-ended and open problems for two weeks, individually or in groups, and present results of their research to their peers.

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Alexander Soifer
University of Colorado at Colorado Springs
P. O. Box 7150, Colorado Springs, CO 80933
USA
E-mail: asoifer@uccs.edu
http://www.uccs.edu/˜asoifer/

Competition Aftermath (a Case Study)

Borislav Lazarov

Borislav Lazarov is an associate professor in the Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, where he received his PhD degree in 1998. As a student he participated in the 19th IMO. He is the chairman of the Chernorizec Hrabar Tournament—the first multiple-choice math competition in Bulgaria.

1 Prologue

Mathematics competitions are dedicated to advanced students in mathematics (whatever it means). The Bulgarian schedule of national mathematics competitions includes 29 contests per scholastic year. This makes 1 competition weekly. Another 10 competitions in informatics and the same number in linguistics and information technologies as well as reasonable number of local competitions make the big picture of such mathematics activities for Bulgarian secondary school students. However, the performance of the advanced students becomes more and more modest at any level. So the question about didactical dimensions of mathematical competitions stands on agenda.

In this paper we will share our experience in turning a single participation in a particular mathematics competition into medium term students' activity in the inquiry-based style. The details we are going to present bear a personal imprint. Nevertheless the individual approach we will present could be easily extended in the context of any small group of students.

2 Beyond Polya's 4^{th} step

The fourth step in the famous George Polya's procedure for solving a problem is review/extend [1]. It enables a student to elaborate the strategy used in the solution and apply it in future problems that are related to the given one. It is crucial this step is passed during the preparation for any traditional competition as far as one expects that there will be similar problems in the next issue of the competition. But sometimes after taking this 4^{th} step students could make a step in another direction—heuristics: to state and to solve a more general problem. Guided by the teacher, a student could go quite far in developing more and more general problems. So the teaching-learning evolves into a real research process.

Below we give detailed description of a didactical process for extending the inertia of a particular mathematical competition in the next few months. We make no claim to originality of the technology—just share our experience which is unique with respect to the particular educational case. Moreover, the style we adopted is rather old.

3 Socratic style

The Socratic style (a synonym is inquiry-based approach) of getting the truth via inquiry is a process of inductive questioning that leads the student to knowledge through small steps [2]. Here we do not advocate the original Socratic Method that uses questioning to dismantle and discard the pre-existing ideas of the student, but we emphasize on the ultimate goal of this style: to increase understanding through inquiry. The modern Socratic style usually refers to the paradigm: the teacher knows in advance where (s)he leads the student; the teacher only chooses appropriate types of activity according to the context of the class with a clear view to the final goals.

Further in the article we will describe how we apply the Socratic style to awaken student's curiosity on a particular math topic and then to develop this curiosity to a kind of research process. The start position was: we do not know in advance anything about the topic under consideration, except a definition given in a contest task; trial-and-error is the common way of findings. So the didactic paradigm in our case was: the teacher organizes educational process in real time having nothing prepared in advance; the teacher's erudition remains the only didactical tool.

4 How it all begins

Chernorizec Hrabar Tournament is a Bulgarian traditional competition. The following problem appears as test item 29 in the theme for 11–12 graders of the 19^{th} issue of this Tournament [3].

Problem 1 We will call a number *summarywise* if it is a sum of two consecutive positive integers as well as a sum of three consecutive positive integers. Which arithmetic operation preserves the set of the summarywise numbers, i.e. the result of the operation upon any two summarywise numbers is a summarywise number?

Prolet was an 11^{th} grade student. She took part in the Tournament but did not solve Problem 1 during the competition. Soon after the competition has passed, she was urged by the author to try again with solving this problem—she failed again. Then she was given to solve the two related problems which were included in the tournament themes for lower grades.

Problem 2 How many numbers not exceeding 10 are a sum of two consecutive positive integers and also a sum of three consecutive positive integers?

Problem 3 How many numbers not exceeding 2010 are a sum of two consecutive positive integers and also a sum of three consecutive positive integers simultaneously?

Prolet solved Problem 2 quickly but succeeded with Problem 3 only after a while when she managed to find the general form of the summarywise numbers. Thereafter solving Problem 1 took her just a few more minutes.

5 Light the fire

Passing the Polya's four steps in solving the set of problems 1–3 Prolet was encouraged to think about what will happen if these problems are stated about the sums of three and four consecutive natural numbers instead of two and three consecutive natural numbers respectively. Initially it was just a simple speculation: is it reasonable to state such kind of questions. Than came the interest. It appeared that preserving properties of the arithmetic operations differ in summarywise-likely sets. But let us follow the chronology. Finding proper and motivated notations is crucial for the success of any mathematical search. The notation S was chosen for the set of the summarywise numbers because it refers to the name of the summarywise numbers. Prolet separated the general description of the summarywise numbers in the following way.

Theorem 1 $S = \{6n + 3 : n \in \mathbb{N}\}$.

An important observation made in the beginning is that the question about operation preserving properties of S is sensible only for addition and multiplication. Then this result was extended to the other kind of summarywise-likely sets of numbers. And the idea for such sets comes quickly.

Definition 1 Let l_1 and l_2 be positive integers greater than 1. We call a number l_1, l_2-summarywise, if it is a sum of l_1 consecutive positive integers as well as a sum of l_2 consecutive integers. We denote by S_{l_1,l_2} the set of the l_1, l_2-summarywise numbers.

It is clear that $S = S_{l_1,l_2}$. Nevertheless the notation S was kept to highlight the special role of the summarywise numbers as the origin of all further results. The first steps in the twilight zone of extensions and generalizations were timid, the formulations were heavy, the examples were a little overdetailed. In fact Theorem 1 was formulated separately from the solutions of the problems 1 and 2 at a later stage.

Example 1 The set $S_{3,4}$ of the $3, 4$-summarywise numbers consists of the numbers which are a sum of 3 consecutive positive integers as well as a sum of 4 consecutive positive integers. We claim that $S_{3,4} = \{6(2n+1) : n \in \mathbb{N}\}$ and $S_{3,4}$ is neither sum-preserving nor product-preserving. The art of formulation was in progress and the following statements sound quite better.

Example 2 $S_{3,5}$ is sum-preserving and product-preserving.

Example 3 $S_{2,4} = \emptyset$.

This variety of operation preserving properties indicates that the matter is not trivial and appears as stimulus for further study.

6 Further generalization

Prolet comes to the most general cases quite easily after passing through Examples 1–3. She needs just a small hint—to consider the set $S_{3,4,5}$ of the $3, 4, 5$-summarywise numbers: the numbers that are at the same time a sum of 3 consecutive positive integers, a sum of 4 consecutive positive integers, and a sum of 5 consecutive positive integers.

After that the following definition was given almost immediately.

Definition 2 Let $l_1, l_2, \ldots, l_k$ be integers greater than 1. A number is called $l_1, l_2, \ldots, l_k$-summarywise if it is a sum of l_i ($l_i \in \mathbb{N}$) consecutive positive integers for every $i = 1, 2, \ldots, k$. We denote the set of the $l_1, l_2, \ldots, l_k$-summarywise numbers by $S_{l_1, l_2, \ldots, l_k}$.

Questions like 'what is going on if ... ' are typical for the Socratic style, e.g. what happens if $k = 1$? The above definition also includes the most primitive $l_1, l_2, \ldots, l_k$-summarywise numbers: the ones which are a sum of l consecutive positive integers only, e.g. S_2 the set of the 2-summarywise numbers consists of the numbers which are a sum of 2 consecutive positive integers.

Theorem 2 S_l consists of the members of an arithmetic progression with the first member equal to $\frac{l(l+1)}{2}$ and the difference in magnitude of l.

The role of examples is crucial for a deeper understanding of a math topic. This is why Prolet was urged to interpret some of the sets S_l from the perspective of arithmetic progressions. Moreover, the arithmetic progression was the topic that she has just studied at school and the connection of the extracurricular activity with classroom practice was an additional motivator.

Example 4 S_2 is the series of the odd numbers greater than or equal to 3; S_3 is the series of the multiples of 3 which are greater than or equal to 6; S_5 is the series of the multiples of 5 which are greater than or equal to 15.

The following two theorems appears soon after Example 4.

Theorem 3 If l is odd then $S_l = \{ln : l \in \mathbb{N}, n \geq \frac{l+1}{2}\}$.

Theorem 4 If l is even then $S_l = \{\frac{l}{2}(2n+1) : l \in \mathbb{N}, n \geq \frac{l}{2}\}$.

Keeping in mind the starting point there are two main questions about $l_1, l_2, \ldots, l_k$-summarywise numbers: to describe the structure of the set $S_{l_1,l_2,\ldots,l_k}$ and to study its operation preserving properties. Prolet was recommended to look at Definition 2 from another perspective: to atomize $S_{l_1,l_2,\ldots,l_k}$ and she made it.

Theorem 5 (Main Theorem) $S_{l_1,l_2,\ldots,l_k} = S_{l_1} \cap S_{l_2} \cap \cdots \cap S_{l_k}$.

Having the Main Theorem and Theorem 2 she presented $S_{l_1,l_2,\ldots,l_k}$ as the set of common members of arithmetic progressions. To give a more detailed picture she needed the next technical result.

Lemma 1 Given are the positive integers a_1', a_1'', d', d''. Consider the arithmetic progressions

$$A' = \{a_1' + (n-1)d' : n \in \mathbb{N}\}, \qquad A'' = \{a_1'' + (n-1)d'' : n \in \mathbb{N}\}.$$

If there exists a common member of A' and A'' then there are infinitely many common members of A' and A'' that form an arithmetic progression with difference the least common multiple $[d', d'']$ of d' and d''.

An immediate corollary from the Main Theorem and Lemma 1 is

Theorem 6 Let $k \geq 2$ be a positive integer and $S_{l_1,l_2,\ldots,l_k} \neq \emptyset$. Then $S_{l_1,l_2,\ldots,l_k}$ is an arithmetic progression with difference the least common multiple of $l_1, l_2, \ldots, l_k$.

The results from Theorem 1 to Theorem 6 have been finished for about a month: from 1 Nov till 3 Dec when was the registration deadline for the Bulgarian annual student math conference. Prolet prepared an article which was included in the conference program and she had to prepare a 15-minute presentation of the article.

She made several iterations to come to the final version of the presentation. Her activeness was pretty high during the period between submitting the paper and the conference. She gathered her classmates to present her results and improved her TEX skills. But most important was the new results that appeared during this period.

Attending the conference gave another impulse to Prolet for continuing the study of the $l_1, l_2, \ldots, l_k$-summarywise numbers. Meeting students with the same attitude to mathematics also motivated her for going deeper in the matter.

7 Entering deep waters

The preparation for a mathematical activity close to real research process has been done. Now Prolet was able to discuss the details and to raise ideological ideas. She was encouraged to study the operation preserving properties of $l_1, l_2, \ldots, l_k$-summarywise numbers in more general form with respect to the Main Theorem.

Theorem 7 In case $S_{l_1,l_2,\ldots,l_k} \neq \emptyset$ it is sum-preserving if and only if S_{l_i} is sum-preserving $\forall i \in \{1, 2, \ldots, k\}$.

Theorem 8 In case $S_{l_1,l_2,\ldots,l_k} \neq \emptyset$ it is product-preserving if and only if S_{l_i} is product-preserving $\forall i \in \{1, 2, \ldots, k\}$.

To prove the above two theorems Prolet needed to get to know the congruencies. Her abstract thinking was lifted to the level on which she recognized results that are similar for two operations. This reflects into the next technical result, where $\circ$ stands for either addition or multiplication.

Lemma 2 Let A be an arithmetic progression of positive integers and difference d. If there exist $a, b, c \in A$ such that $a \circ b \equiv c \pmod{d}$ then $a \circ b \equiv c \pmod{d}$ for all $a, b, c \in A$.

Further speculation brings two more theorems.

Theorem 9 If S_l is sum-preserving, then it is product-preserving.

Theorem 10 S_l is sum-preserving for any odd l.

An opportunity for presenting the new results appeared—the international conference EUROMATH-2011. Attending this conference was a stimulus for Prolet to deal with questions about the existence of some particular types of $l_1, l_2, \ldots, l_k$-summarywise numbers.

Theorem 11 $S_{2^k, q2^{k+m}} = \emptyset$ for all $k, m, q \in \mathbb{N}$.

Theorem 12 If l_i, $i = 1, 2 \ldots, k$, are pairwise coprime, then the set $S_{l_1,l_2,\ldots,l_k} \neq \emptyset$.

Theorem 13 If l_i, $i = 1, 2 \ldots, k$, are odd then $S_{l_1,l_2,\ldots,l_k} \neq \emptyset$.

The proof of the last theorem required the Chinese remainder theorem. As far as Theorems 11–13 are not necessary and sufficient conditions, some examples and counterexamples that state delimitation are welcome. But finding examples and counterexamples for existence is hard to be done by hand.

Example 5 $S_{4,9,12} = \{126 + 36(n - 1) : n \in \mathbb{N}\}$.

Validation of Example 5 was done by a computer algebra system (CAS). This example comes to show that $S_{4,9,12}$ is not empty and 4, 9 and 12 are neither pairwise coprime nor odd in total.

8 Open questions

Example 5 shows that none of the Theorems 12–13 could be turned into criteria for the existence $l_1, l_2, \ldots, l_k$-summarywise numbers. An open question is to state such criteria. A plausible conjecture is that $S_{l_1,l_2,\ldots,l_k} \neq \emptyset$ if and only if all even indices are multiples of the same power of 2. This question was left standing by position. But meeting the CAS was an experience that allows finding more examples in a new area. In order to state the question a couple of new concepts were introduced. Let $\circ$ stand for either addition (a) or multiplication (m).

Definition 3 Given $S \subset \mathbb{N}$ which is $\circ$-preserving. We call a number $s \in S$ $\circ$-prime in S if it cannot be decomposed in the manner $s = s_1 \circ s_2$ for any $s_{1,2} \in S$ and we call s $\circ$-composed in S if it is not $\circ$-prime.

We will say that S is an $\circ$-u-p-d-p set if it satisfies the following condition: every time when $s = s'_1 \circ s'_2 = s''_1 \circ s''_2$ and $s'_{1,2}$ as well as $s''_{1,2}$ are $\circ$-primes in S then $\{s'_1, s'_2\} = \{s''_1, s''_2\}$ (the abbreviation $\circ$-u-p-d-p stands for *unique-prime-decomposition property* with respect to $\circ$).

Convention. In case $\circ$ is addition we will talk about a-primes and a-composed numbers in a sum-preserving set S; in case $\circ$ is multiplication we will talk about m-primes and m-composed numbers in a product-preserving set S.

Open Question. Does there exist a $\circ$-u-p-d-p set $S_{l_1,l_2,\ldots,l_k}$ for which the unique-prime-decomposition property with respect to $\circ$ holds in a non-trivial manner?

Example 6 There is ambiguous m-decomposition in S.

The validation of Example 6 was done again by a CAS: the input

FindInstance[(6x+3)(6y+3)==285,{x,y},Integers]

gives the output {}, i.e. $\emptyset$, which means 285 is m-prime in S. Now $4275 = 15 \cdot 285 = 57 \cdot 75$ shows that S is not a m-u-p-d-p set.

Example 7 $S_{3,5}$ is not an m-u-p-d-p set and it is an a-u-p-d-p set.

But $S_{3,5}$ satisfies the unique-prime-decomposition property about addition in a trivial manner. Indeed, it occurs 15 is the only a-prime in $S_{3,5}$. Hence the representation of any 3,5-summarywise number as a sum of a-primes is unique, i.e. $S_{3,5}$ is an a-u-p-d-p set.

9 Concluding remarks

Our standing point is that the Socratic style of teaching is most effective in small groups of students where the individual approach could be applied. There is a long tradition in national mathematics and sciences high schools in Bulgaria to organize classroom inquiry-based teaching-learning process. But as reported by two of the teachers that apply such type of teaching every time only a small fraction of the students are involved in a creative activity. In fact teachers organize individual education for them and the rest of the class stays passive.

Another reason to organize inquiry-based education beyond the classroom is that any kind of inquiry takes far more time then a lesson. E.g. it took about five months for Prolet to accomplish the small theory we presented here. Her activities were complex:

- competing in a tournament
- solving problems beyond the curriculum
- learning new mathematical concepts and putting them in use
- dealing with a text-formatting computer system
- examining cases by a CAS
- creating new knowledge
- preparing presentations and giving talks at conferences.

Our impression is that in the described case the inquiry-based approach adopted has been pushed to its limits as educational resource. Questions were stated without knowing not only the answers, but also the direction where the answers could be found. At this stage the education turns into a scientific research.

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Appendix

Proof of Theorem 1 Suppose a is positive integer. The sum $a+(a+1)=2a+1$ represents the general form of a number which is a sum of two consecutive positive integers. Analogously, if $b \geq 2$ is a positive integer than $(b-1)+b+(b+1)$ is the general form of a number which is a sum of three consecutive positive integers.

Hence any summarywise number is an odd multiple of 3 which implies that the set S of the summarywise numbers is a subset of the set of the odd multiples of 3. Now we will establish that $6n+3, n \in \mathbb{N}$ is the general form of the summarywise numbers. Let $s=6n+3, n \in \mathbb{N}$. Then

- $s = 6n + 3 = (3n + 1) + (3n + 2)$, which means s is a sum of two consecutive positive integers.
- $s = 6n + 3 = (2n) + (2n + 1) + (2n + 2)$, which means s is a sum of three consecutive positive integers.

Solution of Problem 1 Let $s_{1,2} \in S$. Having the general form of the summarywise numbers we can write that $s_{1,2} = 6n_{1,2} + 3$. Since $S \subset \mathbb{N}$ the condition $s_1 < s_2$ implies $s_1 - s_2 \notin \mathbb{N}$ and $\frac{s_1}{s_2} \notin \mathbb{N}$. Thus $s_1 - s_2 \notin S$ and $\frac{s_1}{s_2} \notin S$.

Let us check whether or not S preserves the result of addition: $s_1 + s_2 = (6n_1 + 3) + (6n_2 + 3) = 6(n_1 + n_2 + 1)$. As far as $6(n_1 + n_2 + 1)$ is an even number it is not of the form $6n + 3$ and we conclude S is not sum-preserving.

Now let us check whether S preserves the result of multiplication: $s_1 \cdot s_2 = (6n_1 + 3) \cdot (6n_2 + 3) = 6[6n_1n_2 + 3(n_1 + n_2) + 1] + 3$. The product $s_1 \cdot s_2$ is of the form $6n + 3$ and we conclude S is product-preserving.

Proof of Theorems 7 and 8 Suppose $S_{l_1,l_2,\ldots,l_k}$ is $\circ$-preserving, i.e. $s_{1,2} \in S_{l_1,l_2,\ldots,l_k} \Rightarrow s_1 \circ s_2 \in S_{l_1,l_2,\ldots,l_k}$. According to the Main theorem we have $s_1 \circ s_2 \in S_{l_i}$ for all $i = 1, 2, \ldots, k$. By Lemma 2 we conclude that for all $i = 1, 2, \ldots, k$ we have $s'_{1,2} \in S_{l_i}$ which means S_{l_i} is $\circ$-preserving for all $i = 1, 2, \ldots, k$.

Suppose any of the sets S_{l_i}, $i = 1, 2, \ldots, k$, is $\circ$-preserving. According to the Main theorem, for any $s_{1,2} \in S_{l_1,l_2,\ldots,l_k}$ and for all $i = 1, 2, \ldots, k$ it follows $s_{1,2} \in S_{l_i}$. Since S_{l_i} is $\circ$-preserving, we have $s_{1,2} \in S_{l_i}$ for all $i = 1, 2, \ldots, k$. Now by the Main Theorem we have $s_{1,2} \in S_{l_1,l_2,\ldots,l_k}$.

Proof of Theorem 11 The first member of the progression S_{2^k} is $a_{2^k,1} = \frac{2^k(2^k+1)}{2} = 2^{k-1}(2^k + 1)$ and its difference is 2^k. Since $a_{2^k,1} \equiv 2^{k-1} \pmod{2^k}$ then $a_{2^k,n} \equiv 2^{k-1} \pmod{2^k}$ $\forall n \in \mathbb{N}$. The first member of the progression $S_{q2^{k+m}}$ is $a_{q2^{k+m},1} = \frac{q2^{k+m}(q2^{k+m}+1)}{2} = q2^{k+m-1}(q2^{k+m}+1)$ and the difference is $q2^{k+m}$. Therefore $a_{q2^{k+m},n} \equiv 0$

(mod 2^k) $\forall n \in \mathbb{N}$. Thus for any two $n_{1,2} \in \mathbb{N}$ a_{2^k,n_1} and a_{q2^{k+m},n_2} are not congruent modulo 2^k.

Proof of Theorem 12 If l_i are parewise coprime then the simultaneous congruencies $x \equiv \frac{l_i(l_i+1)}{2} \pmod{l_i}$, $i = 1, 2, \ldots, k$, have a solution x^* with respect to the Chinese remainder theorem. In the case $x^* \notin S_{l_i}$ for some i (this could happen iff $x^* < \frac{l_i(l_i+1)}{2}$) we can add to x^* a common multiple of $l_1, l_2, \ldots, l_k$ in order to obtain a number x^{**} greater than $\frac{l_i(l_i+1)}{2}$. According to Chinese remainder theorem x^{**} is also solution of the simultaneous congruencies $x \equiv \frac{l_i(l_i+1)}{2} \pmod{l_i}$, $i = 1, 2, \ldots, k$, hence it is a common member of S_{l_i}, for all $i = 1, 2, \ldots, k$.

Borislav Lazarov
Bulgarian Academy of Sciences
Acad. G. Bonchev-Street, Block 8
1113 Sofia
BULGARIA
E-mail: lazarov@math.bas.bg

To be continued

Andy Liu

Andy Liu is a professor of mathematics at the University of Alberta in Canada. His research interests span discrete mathematics, geometry, mathematics education and mathematics recreations. He edits the Problem Corner of the MAA's magazine Math Horizons. *He was the Chair of the Problem Committee in the 1995 IMO in Canada. His contribution to the 1994 IMO in Hong Kong was a major reason for him being awarded a David Hilbert International Award by the World Federation of National Mathematics Competitions.*

Most discussions about competitions focus on what leads up to the event, primarily about training methods. In this paper, we discuss what can be done after the event is over. Many competition problems can be generalized or extended in various ways, providing fertile ground for explorations.

The main event in the **Saturday Mathematical Activities, Recreations & Tutorials (SMART)** program in Edmonton, Canada, which has been running continuously since 1981, is the **International Mathematics Tournament of the Towns**. We list below eight sample problems from this wonderful competition, along with four problems from other competitions.

From the Alberta High School Mathematics Competition:

1. An isosceles triangle is called an **amoeba** if it can be divided into two isosceles triangles by a straight cut. How many different (that is, not similar) amoebas are there? (see [5])

From the International Mathematical Olympiad (short-listed problems):

2. In a multiple-choice test there were 4 questions and 3 possible answers for each question. A group of students was tested and it turned out that for any 3 of them, there was a question that the three students answered differently. What is the maximal possible number of students tested? (see [4])

From the Kürschák Mathematics Competition:

3. Which positive integers cannot be expressed as sums of two or more consecutive positive integers? (see [2])

From the Chinese Mathematical Olympiad:

4. Bottle A contains 1997 pills. Bottle B and C are empty. Each pill has a potency rating of 100 points. Whenever a bottle is opened, each pill inside loses 1 point. The patient takes 1 pill per day, and may redistribute the pills among the three bottles each time he takes a pill. When he finishes all the pills, what is the minimum value of the total number of points lost? (National Team Selection Test 1997)

From the International Mathematics Tournament of the Towns:

5. Show how to cut an isosceles right triangle into a number of triangles similar to it in such a way that every two of these triangles are of different sizes. (see [15])

6. Consider a polyhedron having 100 edges.

(a) Find the maximum possible number of its edges which can be intersected by a plane (not containing any vertices of the polyhedron) if the polyhedron is convex.

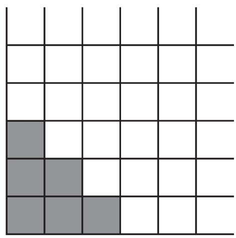
(b) Prove that for a non-convex polyhedron, this number can be as great as 96.

(c) Prove that for a non-convex polyhedron, this number cannot be as great as 100. (See [17])

7. On an infinite "squared" sheet, six squares are shaded as in the diagram. On some squares there are pieces. It is possible to transform the positions of the pieces according to the following rule. If the neighbour squares to the right and above a given piece are free, it is possible to remove this piece and put pieces on these two squares. The goal is to have all the shaded squares free of pieces. Is it possible to reach the goal if in the initial position,

(a) there are 6 pieces and they are placed on the 6 shaded squares;

(b) there is only one piece, located in the bottom-left shaded square? (see [14])



8. There are 32 boxers in a tournament. Each boxer can fight no more often than once per day. It is known that the boxers are of different strengths, and the stronger man always wins. Prove that a 15 day tournament can be organized so as to determine their classification (put them in the order of strength). The schedule of fights for each day is fixed in advance and cannot be changed during the tournament. (see [16])

9. The angle $\angle A$ of an isosceles triangle ABC ($AB = AC$) equals α. Let D be a point on the side AB such that $AD = \frac{1}{n}AB$. Find the sum of the $n-1$ angles $\angle AP_1D$, $\angle AP_2D$, $\ldots$, $\angle AP_{n-1}D$, where $P_1, P_2, \ldots, P_{n-1}$ are the points dividing the side BC into n equal parts, if

(a) $n = 3$;

(b) n is an arbitrary integer, $n > 2$. (see [18])

10. In a $f(x) = \frac{x^2+ax+b}{x^2+cx+d}$, the quadratics $x^2 + ax + b$ and $x^2 + cx + d$ have no common roots. Prove that the next two statements are equivalent.

(A) There is a numerical interval without any values of $f(x)$.

(B) The function $f(x)$ can be represented in the form $f(x) = f_1(f_2(\cdots f_{n-1}(f_n(x))\cdots))$, where each of the functions f_j is one of the three forms $k_jx + b_j$, $\frac{1}{x}$, x^2. (see [13])

11. Seated in a circle are 11 wizards. A different positive integer not exceeding 1000 is pasted onto the forehead of each. A wizard can see the numbers of the other 10, but not his own. Simultaneously, each wizard puts up either his left hand or his right hand. Then each declares the number on his forehead at the same time. Is there a strategy on which the wizards can agree beforehand, which allows each of them to make the correct declaration? (Spring 2008 Senior A-Level paper)

12. At the entrance to a cave is a rotating round table. On top of the table are n identical barrels, evenly spaced along its circumference. Inside each barrel is a herring either with its head up or its head down. In a move, Ali Baba chooses from 1 to n of the barrels and turns them upside down. Then the table spins around. When it stops, it is impossible to tell which barrels have been turned over. The cave will open if the heads of the herrings in all n barrels are all up or are all down. Determine all values of n for which Ali Baba can open the cave in a finite number of moves. (Fall 2009 Senior A-Level paper)

After the competitions, various members of the **SMART** program, as well as some associated members from China and Taiwan, conducted further investigations which led to the following publications. The last five were based on solutions found by the members during the actual competition.

1. Daniel Robbins (Grade 12 student), Sudhakar Sivapalan (Grade 12 student) and Matthew Wong (University student), Dissecting Tri-

angles into Isosceles Triangles, *Crux Mathematicorum* 22, (1996) 97–100.

2. Graham Denham (University student) and Andy Liu, Competitions, Matrices, Geometry and Circuits, *Austral. Math. Gaz.* 24 (1997), 109–113.

3. Robert Barrington-Leigh (Grade 7 student) and Richard Travis Ng (Grade 8 student), Zigzag, *Mathematics Competitions* 10(2) (1997), 38–43. A Hungarian version of this paper appears in *Abacus* 4 (1998), 318–320, under the title Cikcakk.

4. Robert Barrington-Leigh (Grade 9 student) and Richard Travis Ng (Grade 10 student), Minimizing Aroma Loss, *College Mathematics Journal* 30 (1999), 356–358.

5. Byung-Kyu Chun (Grade 12 student), Andy Liu and Daniel van Vliet (University student), Dissecting Squares into Similar Rectangles, *Crux Mathematicorum* 22 (1996), 241–248.

6. Fusang Leng (University student) and Andy Liu, How many edges of a polyhedron can a plane cut? *Mathematics Competitions*, 13(1) (2000), 20–30.

7. Jerry Lo (Grade 8 student), Two Great Escapes, *Delta-K*, 4(2) (2006), 23–27.

8. Calvin Li (Grade 11 student) and Andy Liu, The Coach's Dilemma, *Mathematics and Informatics Quarterly* 2 (1992), 155–157.

9. Hubert Chan (Grade 12 student), Andy Liu and Andrei Storozhev, Induction in Geometry, *Mathematics Competitions* 10(1) (1997), 61–68.

10. Xin Li (Grade 12 student) and Andy Liu, Some properties of functions of the form $f(x) = \frac{x^2+ax+b}{x^2+cx+d}$, *Mathematics Competitions*, 14(2) (2001), 35–41.

11. Jonathan Zung (Grade 10 student), A Magic Trick with Eight Wizards, *G4G8 Exchange Book* 1 (2010), 143–145.

12. Hsin Po Wang (Grade 11 student), Congo Bongo, *Math Horizons*, September (2010), 18–21.

In the remaining part of this paper, we take the readers along on one of these explorations, based on problem 8. It has led to fruitful discussions in meetings of the SMART program. We repeat its statement here.

There are 32 boxers in a tournament. Each boxer can fight no more often than once per day. It is known that the boxers are of different strengths, and the stronger man always wins. Prove that a 15 day tournament can be organized so as to determine their classification (put them in the order of strength). The schedule of fights for each day is fixed in advance and cannot be changed during the tournament.

First, we must get a complete understanding of the problem. The reference to a tournament is perhaps unfortunate, as it conjures up images of upsets, whereas the statement of the problem clearly precludes such events. Also, we want to determine more than just the champion. We need to sort all the players. Finally, the significance of the last sentence is not immediately clear. We just ignore this sentence for now and address the issue later.

Let us formulate a plan. A standard approach for solving this type of problem is down-sizing, working with a smaller number of boxers. We have to guess the group of integers for which 32 is a representative. Here, the reference to a tournament is beneficial, as it leads us to the correct conclusion that we are dealing with powers of 2. So we will work our way progressively up from 1 through 2, 4, 8 and 16 to get to 32.

Clearly, with only 1 boxer, 0 days will do. With 2 boxers, we can get by with 1 day. A bold conjecture is that with 2^n boxers, n days are sufficient.

The next case involves 4 boxers. There is no better way than have them fight one another in two pairs on the first day. We may assume that a_1 beats a_2 and b_1 beats b_2. On the second day, we may have a_1 fight b_1 and a_2 fight b_2. We would have found the strongest and the weakest boxers. However, if the other two boxers have not yet fought each other, the sorting is incomplete. An extra day is then required to decide second and third places.

This seems to put an irrepairable dent in our conjecture. However, it can be salvaged if we add a condition. With 2^n boxers, n days are sufficient if the boxers have been divided into two groups and each group has been sorted. In the case with 4 boxers, now we do not count the first day when the two halves are being sorted. It takes only 2 days to merge the two sorted pairs into a sorted quartet.

During the actual competition, Calvin Li followed largely this train of thought, except that in merging the two sorted pairs, he used a different approach. On the first day of the merger, a_1 fights b_2 and a_2 fights b_1. The ranking is complete if either a_2 or b_2 wins. Obviously they cannot both win. If neither of them wins, a_1 fights b_1 on the second day of the merger to decide first and second places, while a_2 fights b_2 to decide third and fourth places.

We will carry out one more case study, that of 8 boxers in two sorted quartets $a_1 \to a_2 \to a_3 \to a_4$ and $b_1 \to b_2 \to b_3 \to b_4$. We now merge them into a single sorted octet in 3 days. Calvin's method is easier to generalize. On the first day, a_1 fights b_4, a_2 fights b_3, a_3 fights b_2 and a_4 fights b_1. For obvious reasons, we refer to this approach as the Upside Down Merge Sort.

If a_4 or b_4 wins, the sorting is completed. So we may as well assume that a_1 and b_1 win. If a_2 and b_2 also win, we can complete the sorting by merging $a_1 \to a_2$ with $b_1 \to b_2$ to yield the top four, as well as $a_3 \to a_4$ with $b_3 \to b_4$ to yield the bottom four. This takes 2 more days.

We now assume that on the first day, a_3 beats b_2, so that in the other fight, a_2 beats b_3. Then the top four are $a_1,\ a_2,\ a_3$ and b_1, while the bottom four are $b_2,\ b_3,\ b_4$ and a_4. On the second day, b_1 fights a_2 while a_4 fights b_3. On the third day, b_1 fights a_1 following a victory or a_3 following a defeat, while a_4 plays b_2 following a victory or b_4 following a defeat. This completes the sorting.

The generalization of the other method is more subtle. Beyond having a_i fight b_i for $1 \leq i \leq 4$ on the first day, it is not immediately clear how to proceed on the second day. Going back to the case with 4 boxers, a_1 and b_1 go into one group, while a_2 and b_2 go into another group.

With 8 boxers, $a_1,\ b_1,\ a_3$ and b_3 go into one group while $a_2,\ b_2,\ a_4$ and

b_4 go into another group. The first group consists of those boxers with odd subscripts while the second group consists of those boxers with even subscripts. Thus this approach is called the Odd Even Merge Sort, due to K. Batcher (see [3]).

On the first two days, we merge the sorted pairs $a_1 \to a_3$ and $b_1 \to b_3$ into one sorted quartet, and at the same time merge the sorted pairs $a_2 \to a_4$ and $b_2 \to b_4$ into another sorted quartet. This takes 2 days, and we call the result $c_1 \to c_2 \to c_3 \to c_4$ for the odd group and $d_1 \to d_2 \to d_3 \to d_4$ for the even group.

Clearly, c_1 is the strongest and d_4 is the weakest. On the third day, c_2 fight d_1, c_3 fights d_2 and c_4 fights d_3. We claim that the first fight determines second and third places. By symmetry, the third fight would determine sixth and seventh places, so that the second fight will determine fourth and fifth places.

By symmetry, we may assume that $c_1 = a_1$. Then $c_2 = b_1$ or a_3 while $d_1 = a_2$ or b_2. Now only a_1 and possibly a_2 can be ahead of c_2. Similarly, only a_1 and possibly b_1 can be ahead of d_1. This justifies our claim and completes the sorting.

At this point, we are convinced that we have two solutions to our problem. We shall prove formally by mathematical induction that with 2^n boxers, n days are sufficient if the boxers have been divided into two groups and each group has been sorted. The basis has been established for both approaches. Suppose the result holds for some $n \geq 1$. Consider the next case where the number of boxers is 2^{n+1}. We wish to show that the sorting can be accomplished in $n + 1$ days.

Note that the Upside Down Merge Sort does not stipulate that the two groups should have the same size, as long as their combined size is 2^n. So let $a_1 \to a_2 \to \cdots \to a_s$ be one group and $b_1 \to b_2 \to \cdots \to b_t$ be another group, where $s + t = 2^{n+1}$. We may assume that $1 \leq s \leq t$. For $1 \leq i \leq k$, let a_i fight b_j on the first day, where $i + j = 2^n + 1$. We consider two cases.

Case 1. All the a's lose.
In particular, b_{2^n} beats a_1. The top 2^n players are $b_1, b_2, \ldots, b_{2^n}$. To complete the sorting, we only need to merge $a_1 \to a_2 \to \cdots \to a_s$ with

$b_{2^n+1} \to b_{2^n+2} \to \cdots \to b_t$. Since $s + (t - 2^n) = 2^n$, at most n more days are needed by the induction hypothesis.

Case 2. At least one a wins.
Let i be the largest subscript such that a_i wins. Then the top 2^n players are $a_1, a_2, \ldots, a_i$ along with $b_1, b_2, \ldots, b_{2^n-i}$, while the bottom 2^n players are $a_{i+1}, a_{i+2}, \ldots, a_k$ along with $b_{2^n-i+1}, b_{2^n-i+2}, \ldots, b_t$. At most n more hours are needed by the induction hypothesis.

In the Odd Even Merge Sort, we merge $a_1 \to a_3 \to \cdots \to a_{2^n-1}$ with $b_1 \to b_3 \to \cdots \to b_{2^n-1}$ into $c_1 \to c_2 \to \cdots \to c_{2^n}$, and merge $a_2 \to a_4 \to \cdots \to a_{2^n}$ with $b_2 \to b_4 \to \cdots \to b_{2^n}$ into $d_1 \to d_2 \to \cdots \to d_{2^n}$. By the induction hypothesis, this can be done in n days.

We have only one more day to complete the sorting. At this point, we know that c_1 is the strongest boxer while d_{2^n} is the weakest. For $1 \le k \le 2^n - 1$, c_{k+1} fights d_k on the last day to determine the $2k$th and the $(2k+1)$st places. This is justified by the following two results.

Lemma 1 For $1 \le k \le 2^n$, d_k is never ahead of c_k.

Proof:
By symmetry, we may assume that $d_k = a_{2m}$ for some m. Among the d's, there are $m-1$ other a's ahead of d_k, namely, $a_2, a_4, \ldots, a_{2m-2}$. Hence there are $k-m$ b's ahead of d_k, namely, $b_2, b_4, \ldots, b_{2k-2m}$. Among the c's, $a_1, a_3, \ldots, a_{2m-1}, b_1, b_3, \ldots, b_{2k-2m-1}$ are ahead of d_k. These consist of m a's and $k-m$ b's, so that $m + (k-m) = k$ c's are ahead of d_k. Since c_k is in the kth place among the c's, c_k is ahead of d_k.

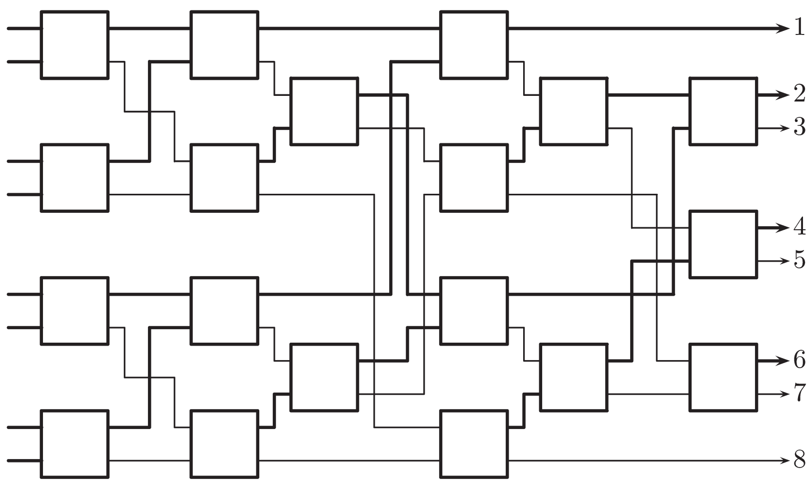
Lemma 2 For $2 \le k \le 2^n - 1$, c_{k+1} is never ahead of d_{k-1}.

Proof:
By symmetry, we may assume that $c_{k+1} = a_{2m-1}$ for some m. Among the c's, there are $m-1$ other a's ahead of c_{k+1}, namely, $a_1, a_3, \ldots, a_{2m-3}$. Hence there are $k-m+1$ b's ahead of c_{k+1}, namely, $b_1, b_3, \ldots, b_{2k-2m+1}$. Among the d's, $a_2, a_4, \ldots, a_{2m-2}, b_2, b_4, \ldots, b_{2k-2m}$ are ahead of c_{k+1}. These consist of $m-1$ a's and $k-m$ b's, so that $(m-1)+(k-m) = k-1$

d's are ahead of c_{k+1}. Since d_{k-1} is in the $k-1$st place among the d's, d_{k-1} is ahead of c_{k+1}.

Finally, we take into consideration the requirement that the schedule of fights for each day is fixed in advance and cannot be changed during the tournament. It rules out the Upside Down Merge Sort, as in some cases the schedule for a day depends on what transpires on the preceding day. Such a solution is called an, em, adaptive solution.

The Odd Even Merge Sort, on the other hand, is *non-adaptive*, which is what is required. We give a graphical illustration in the case of 8 boxers. The 6-day tournament can be represented by the following diagram. The 8 boxers enter from the left. When they fight, the winners proceed along the dark lines and the losers along the light lines. Once they go through the pre-programmed tournament and emerge from the right, they will be sorted by strength as indicated. We have now carried out



three of the four steps in George Pólya's famous four-step method in problem-solving (see [6]). The fourth step is looking back. Comparing the two approaches, the Upside Down Merge Sort seems more flexible as it can handle unequal groups, and its justification is more transparent.

However, the Odd Even Merge Sort has a big advantage in being non-adaptive.

Another issue we have not addressed is whether the number of days for the sorting is actually necessary. Perhaps yet another approach may yield a shorter tournament. However, this is another exploration for another day, and the story is to be continued.

A version of this sorting problem appears in a delightful puzzle book [7] on computing related mathematics. For more intriguing problems from the same author, see [8], [9]. [10], [11] and [12]. For more information on sorting problems, consult standard textbooks on parallel algorithms such as [1].

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Andy Liu
University of Alberta
CANADA
E-mail: aliumath@telus.net

Tournament of the Towns Selected Problems, Fall 2011

Andy Liu

1. From the 9×9 chessboard, all 16 unit squares whose row numbers and column numbers are both even have been removed. Dissect the punctured board into rectangular pieces, with as few of them being unit squares as possible.

2. Positive integers $a < b < c$ are such that $b+a$ is a multiple of $b-a$ and $c+b$ is a multiple of $c-b$. If a is a 2011-digit number and b is a 2012-digit number, exactly how many digits does c have?

3. Pete has marked at least 3 points in the plane such that all distances between them are different. A pair of marked points A and B will be called *unusual* if A is the furthest marked point from B, and B is the nearest marked point to A (apart from A itself). What is the largest possible number of unusual pairs that Pete can obtain?

4. A set of at least two objects with pairwise different weights has the property that for any pair of objects from this set, we can choose a subset of the remaining objects so that their total weight is equal to the total weight of the given pair. What is the minimum number of objects in this set?

5. We will call a positive integer *good* if all its digits are nonzero. A good integer will be called *special* if it has at least k digits and their values are strictly increasing from left to right. Let a good integer be given. In each move, one may insert a special integer into the digital expression of the current number, on the left, on the right or in between any two of the digits. Alternatively, one may also delete a special number from the digital expression of the current number.What is the largest k such that any good integer can be turned into any other good integer by a finite number of such moves?

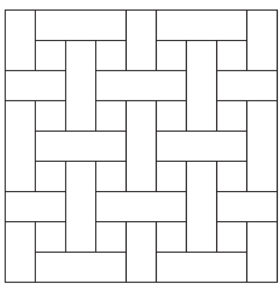
6. Let $0 < a, b, c, d < 1$ be real numbers such that $abcd = (1 - a)(1 - b)(1 - c)(1 - d)$. Prove that $(a + b + c + d) - (a + c)(b + d) \geq 1$.

7. A car goes along a straight highway at the speed of 60 kilometres per hour. A 100 metre long fence is standing parallel to the highway. Every second, the passenger of the car measures the angle of vision of the fence. Prove that the sum of all angles measured by him is less than $1100°$.

8. On a highway, a pedestrian and a cyclist were going in the same direction, while a cart and a car were coming from the opposite direction. All were travelling at constant speeds, not necessarily equal to one another. The cyclist caught up with the pedestrian at 10 o'clock. After a time interval, the cyclist met the cart, and after another time interval equal to the first, she met the car. After a third time interval, the car met the pedestrian, and after another time interval equal to the third, the car caught up with the cart. If the pedestrian met the car at 11 o'clock, when did he meet the cart?

Solution to Selected Problems, Fall 2011

1. The following diagram shows a dissection of the punctured chessboard into rectangular pieces, none of them being unit squares.



2. Since $c > b$, c has at least 2012 digits. We have $b + a = k(b - a)$ for some integer $k > 1$. Hence $a(k + 1) = b(k - 1)$, so that

$\frac{b}{a} = \frac{k+1}{k-1} = 1 + \frac{2}{k-1} \leq 3$, with equality if and only if $k = 2$. Similarly, $\frac{c}{b} \leq 3$, so that $\frac{c}{a} = \frac{c}{b} \cdot \frac{b}{a} \leq 9$. Hence $c < 10a$. Since a has 2011 digits, c has at most 2012 digits. It follows that c has exactly 2012 digits.

3. First, we show by example that we may have one unusual pair when there are at least 3 points. Let A and B be chosen arbitrarily. Add some points within the circle with centre B and radius AB but outside the circle with centre A and radius AB. Then B is the point nearest to A while A is the point furthest from B. We now prove that if there is another pair of unusual points, we will have a contradiction. We consider two cases.

Case 1 The additional unusual pair consists of C and D such that D is the point nearest to C while C is the point furthest from D. Now $DA > AB$ since B is the point nearest to A, $AB > BC$ because A is the point furthest from B, $BC > CD$ since D is the point nearest to C, and $CD > DA$ because C is the point furthest from D. Hence $DA > DA$, which is a contradiction.

Case 2 The additional unusual pair consists of B and C such that C is the point nearest to B and B is the point furthest from C. Now $CA > AB$ since B is the point nearest to A, $AB > BC$ since A is the point furthest from B, and $BC > CA$ because B is the point furthest from C. Hence $CA > CA$, which is a contradiction.

4. Clearly, the set cannot have 2, 3 or 4 objects, as it would not be possible to balance the heaviest two objects. Suppose it has only 5 objects, of respective weights $a > b > c > d > e$. Clearly, we must have $a + b = c + d + e$. Since $a + c > b + d$, we must also have $a + c = b + d + e$, which implies $b = c$. The set may have 6 objects, of respective weights 8, 7, 6, 5, 4 and 3. Then 8+7=6+5+4, 8+6=7+4+3, 8+5=7+6, 8+4=7+5, 8+3=7+4=6+5, 7+3=6+4, 6+3=5+4, 5+3=8 and 4+3=7.

5. Solution by Daniel Spivak
We cannot have $k = 9$ as the only special number would be 123456789. Adding or deleting it does not change anything. We may have $k = 8$. We can convert any good number into any other good number by adding or deleting one digit at a time. We give below the procedures for adding digits. Reversing the steps allows

us to delete digits.

Adding 1 or 9 anywhere.

Add 123456789 and delete 23456789 or 12345678.

Adding 2 anywhere.

Add 23456789 and then add 1 between 2 and 3. Now delete 13456789.

Adding 8 anywhere.

Add 12345678 and then add 9 between 7 and 8. Now delete 12345679.

Adding 3 anywhere.

Add 23456789 and delete 2. Now add 1 and 2 between 3 and 4 and delete 12456789.

Adding 7 answhere.

Add 12345678 and delete 8. Now add 8 and 9 between 6 and 7 and delete 12345689.

Adding 4 anywhere.

Add 23456789 and delete 2 and 3. Now add 1, 2 and 3 between 4 and 5 and delete 12356789.

Adding 6 anywhere.

Add 12345678 and delete 7 and 8. Now add 7, 8 and 9 between 5 and 6 and delete 12345789.

Adding 5 anywhere.

Add 23456789, delete 2, 3 and 4 and add 1, 2, 3 and 4 between 5 and 6. Alternately, add 12345678, delete 6, 7 and 8 and add 6, 7, 8 and 9 between 4 and 5. Now delete 12346789.

6. First Solution.

From $abcd = (1-a)(1-b)(1-c)(1-d)$, we have

$$\begin{aligned}
& a+b+c+d-(a+c)(b+d) \\
= \ & 1+ac(1-b-d)+bd(1-a-c) \\
= \ & 1+ac(1-b)(1-d)+bd(1-a)(1-c)-2abcd \\
\geq \ & 1+2\sqrt{ac(1-b)(1-d)bd(1-a)(1-c)}-2abcd \\
= \ & 1+2abcd-2abcd=1.
\end{aligned}$$

Second Solution by Adrian Tang

From $abcd = (1-a)(1-b)(1-c)(1-d)$, we have

$$\frac{ac}{(1-a)(1-c)} = \frac{(1-b)(1-d)}{bd}.$$

It follows that

$$\frac{a+c-1}{(1-a)(1-c)} = \frac{ac}{(1-a)(1-c)} - 1$$
$$= \frac{(1-b)(1-d)}{bd} - 1 = \frac{1-b-d}{bd}.$$

Now

$$\frac{(a+c-1)(1-b-d)}{(1-a)(1-c)bd} \geq 0$$

since it is the product of two equal terms. From

$$(1-a)(1-c)bd > 0,$$

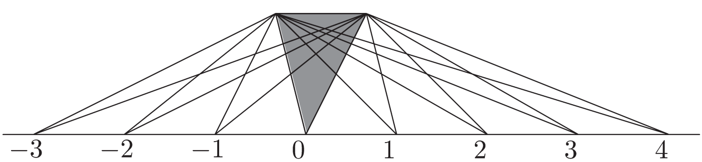
we have $(a+c-1)(1-b-d) \geq 0$. Expansion yields

$$a - ab - ad + c - cb - cd - 1 + b + d \geq 0,$$

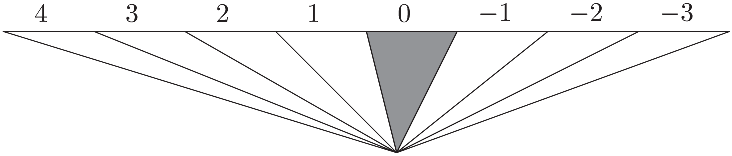
which is equivalent to $a+b+c+d-(a+c)(b+d) \geq 1$.

7. Solution by Central Jury

Divide the points of observation into six groups cyclically, so that the points in each group are 100 metres apart, the same as the length of the billboard.The diagram below shows the angles of visions from the points of a group.

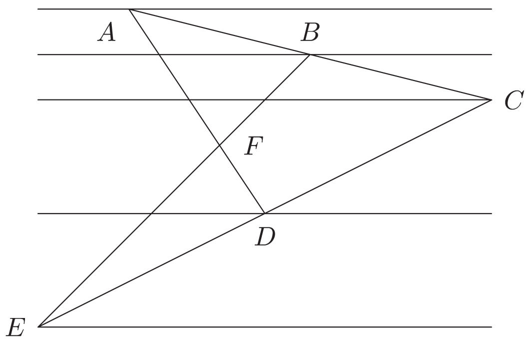


We now parallel translate all these points to a single point, along with their fences and angles of vision, as shown in the diagram below.



The sum of all these angles is clearly at most $180°$. Since there are six groups of points of observations, the sum of all angles of vision is at most $6 \times 180° < 1100°$.

8. The diagram below shows five snapshots of the highway. Since all speeds are constant, the motions can be represented by straight lines, AD for the pedestrian, AC for the cyclist, BE for the cart and CE for the car. The equality of time intervals yield $AB = BC$ and $CD = DE$. Hence F, which represents the moment the pedestrian met the cart, is the centroid of triangle ACE, so that $AF = \frac{2}{3}AD$. Since A is at 10 o'clock and D is at 11 o'clock, F is at 10:40.



Andy Liu
University of Alberta
CANADA
E-mail: aliumath@telus.net

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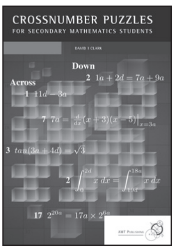

Crossnumber Puzzles
D I Clark

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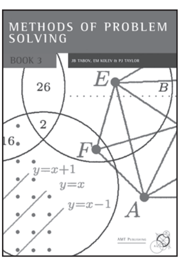
A unique feature of this book is that each puzzle comes with a solution guide which gives one possible order for solving the puzzle. The solution guide shows which cell or cells to tackle next, without showing how to do it. Many of the clues in the puzzles have operations that go beyond simple addition, subtraction, multiplication, division and averaging, but they are still accessible to solvers who are unfamiliar with them.

Full solutions to each puzzle are also included as well as a glossary of mathematical terms used in the book.



Methods of Problem Solving – Book 3
JB Tabov, EM Kolev and PJ Taylor

This book provides senior students aspiring to Olympiad competition with an introduction to particular mathematical problem solving techniques, following the style of Books 1 and 2 by the same authors and introducing a new author to the series. The book contains formal treatments of methods with which students may be familiar, or may introduce the student to new, sometimes powerful techniques.

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