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MATHEMATICS COMPETITIONS

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Please send articles to:

The Editor
Mathematics Competitions
World Federation of National Mathematics Competitions
University of Canberra ACT 2601
Australia
Fax:+61-2-6201-5052

or

Dr Jaroslav Švrček
Dept. of Algebra and Geometry
Faculty of Science
Palacký University
Tr. 17. Listopadu 12
Olomouc
772 02
Czech Republic
Email: svrcek@inf.upol.cz

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Universidad Antonio Narino
Carrera 55 # 45-45
Bogota
COLOMBIA

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Palacký University, Olomouc
CZECH REPUBLIC

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Department of Mathematics
University of North Carolina at Charlotte
9201 University City Blvd.
Charlotte, NC 28223-0001
USA

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Dunedin
NEW ZEALAND

South America: Professor Patricia Fauring
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Buenos Aires
ARGENTINA

The aims of the Federation are:–

1. ***to promote excellence in, and research associated with, mathematics education through the use of school mathematics competitions;***
2. ***to promote meetings and conferences where persons interested in mathematics contests can exchange and develop ideas for use in their countries;***
3. ***to provide opportunities for the exchanging of information for mathematics education through published material, notably through the Journal of the Federation;***
4. ***to recognize through the WFNMC Awards system persons who have made notable contributions to mathematics education through mathematical challenge around the world;***
5. ***to organize assistance provided by countries with developed systems for competitions for countries attempting to develop competitions;***
6. ***to promote mathematics and to encourage young mathematicians.***

From the President

50 years of the IMO

It is a privilege to be able to address our readers in this the year that the IMO celebrates its 50th anniversary and WFNMC its 25th, and with this focal point, to reflect on the activities and impact of mathematics competitions in the XIX, XX and XXI centuries.

Anecdotes from the history of mathematics point to the existence of competitions in problem solving, for example, in the time of Fibonnaci and again a few centuries later when the great Italian algebraists of the Renaissance battled with the solution of polynomial equations. We are all aware of the fact that cash prizes have been offered and public recognition given for the solutions to longstanding unsolved problems. And from a different standpoint, among the most well-known facts of modern mathematics, Hilbert sought to revitalize the corps of mathematics and the interest of young people in studying mathematics with his famous list of 23 problems put forth at the International Congress of Mathematicians in Paris in 1900.

Where in this tradition did the IMO and the WFNMC spring from and what makes them essentially different from other efforts and activities that preceded them?

New explorations in elementary mathematics

Mathematical competitions—even those keyed for large populations—allow all students the chance to broaden their own mathematical horizons and enhance their mathematical experiences with problems that are fresh and challenging for them.

József Kürschák in his preface to the first edition of problems from the Hungarian Eötvös Competitions puts it simply: [Readers] “will find things of note and value here and will be gratified to see how much can be achieved with the elementary material to which high schools must restrict themselves.”

Nearly eighty years later former WFNMC president Petar Kenderov stated the case clearly: “Mathematics competitions, together with the people and organizations engaged with them, form an immense and vibrant global network today. This network has many roles. Competitions help identify students with higher abilities in mathematics. They motivate these students to develop their talents and to seek professional realization in science. Competitions have positive impact on education and on educational institutions. Last but not least, a significant part of the classical mathematical heritage known as 'Elementary Mathematics' is preserved, kept alive and developed through the network of competitions and competition-related activities." The IMO reaches young students at exactly the point in their development when choices are made that determine the direction their lives will take, informing and enriching their future and the future of mathematics.

Publication of problems, solutions and results

A look at the IMO website and the thousands of internet pages dedicated to problems and problem solving clearly illustrates that humankind has come a long way from the days where, in his introduction to the mathematics problems of the Rhind Papyrus, the Egyptian scribe Ahmed made a reference to its containing the secrets of "the occult"; or where battles were waged by Tartaglia and Cardano over the intellectual authorship and precedence in the discovery of formulas for solving cubic equations. Since the Enlightenment, the achievements of the modern mind have been shared openly. We have followed the master Gauss and his ethic of citing sources and attributing authorship of new ideas and knowledge, on the level of elementary mathematics or on the frontiers of research in mathematical science.

The IMO is the pinnacle of worldwide efforts, by problem posers, teachers, mathematicians, science academies and students themselves, to bring significant mathematical experiences to young people and to perpetuate the charm of mathematics as an elegant creation of the human mind.

Twenty-five years of WFNMC

WFNMC was founded at ICME V in Adelaide. It was my privilege to take part in the initial meetings; its co-founders include distinguished mathematicians and math educators from all continents around the globe.

Mathematics competitions in general run an entire gamut of formats, kinds of participants and level of difficulty of the problems, but it is my belief that the role of WFNMC complements the competitions themselves in that "it orients those who would orient young people, aspiring mathematicians and scientists."

As stated on its webpage, "the scope of activities of interest to the WFNMC, although centred on competitions *for students* of all levels (primary, secondary and tertiary), is much broader than the competitions themselves. The WFNMC aims to provide a vehicle for educators to exchange information on a number of activities related to mathematics and mathematics learning." These activities include competitions, enrichment activities, mathematical clubs or circles, mathematics days or camps, on the one hand, and research in math education pertaining to competitions and other challenging contexts, publications of journals for students and teachers containing problem sections, support for teachers who desire or require extra resources, *support for teachers*, *schools*, *regions and countries* who develop local, regional and national competitions.

For example, in those early days of Math Olympiads in Colombia, WFNMC was an important instrument in orienting our efforts, and with this support we were able to turn to Latin America as a whole and found the Iberoamerican Math Olympiads, helping to bring an entire continent (and its European counterparts Spain and Portugal) into the math Olympiad movement.

Furthermore, as Petar Kenderov has argued, "In many countries, year after year, some schools consistently 'produce' more competition winners than other schools... Very often...the prominent success of a particular school can be attributed to the dedicated efforts of a single teacher or a small group of teachers. For these excellent teachers, teaching is a vocation, a mission... Such special teachers are real assets for the school and for the whole country. They possess both the necessary scientific ability and the extraordinary personality needed to identify and motivate for hard work the future winners in competitions. Such teachers need special care, though. Their higher scientific ability is acquired. . . at the expense of great personal efforts.”

This WFNMC journal and the conferences held every four years are an essential part of the important support the federation gives worldwide.

In this vein, we extend a warm invitation to all to enjoy and benefit from the articles in this new issue of the journal and to take an active part in the WFNMC conference to be held in Riga, Latvia in July of 2010 (after 51st IMO).

María Falk de Losada
President of WFNMC
Bogotá, May 2009

From the Editor

Welcome to *Mathematics Competitions* Vol. 22, No. 1.

Again I would like to thank the Australian Mathematics Trust for continued support, without which each issue of the journal could not be published, and in particular Heather Sommariva, Bernadette Webster and Pavel Calábek for their assistance in the preparation of this issue.

Submission of articles:
The journal *Mathematics Competitions* is interested in receiving articles dealing with mathematics competitions, not only at national and international level, but also at regional and primary school level. There are many readers in different countries interested in these different levels of competitions.

- The journal traditionally contains many different kinds of articles, including reports, analyses of competition problems and the presentation of interesting mathematics arising from competition problems. Potential authors are encouraged to submit articles of all kinds.
- To maintain and improve the quality of the journal and its usefulness to those involved in mathematics competitions, all articles are subject to review and comment by one or more competent referees. The precise criteria used will depend on the type of article, but can be summarised by saying that an article accepted must be correct and appropriate, the content accurate and interesting, and, where the focus is mathematical, the mathematics fresh and well presented. This editorial and refereeing process is designed to help improve those articles which deserve to be published.

At the outset, the most important thing is that if you have anything to contribute on any aspect of mathematics competitions at any level, local, regional or national, we would welcome your contribution.

Articles should be submitted in English, with a black and white photograph and a short profile of the author. Alternatively, the article can be submitted on an IBM PC compatible disk or a Macintosh disk. We prefere LaTeX or TeX format of contributions, but any text file will be helpful.

Articles, and correspondence, can also be forwarded to the editor by mail to

The Editor, *Mathematics Competitions*
Australian Mathematics Trust
University of Canberra ACT 2601
AUSTRALIA

or to

Dr Jaroslav Švrček
Dept. of Algebra and Geometry
Palacky University of Olomouc
17. listopadu 12
772 02 OLOMOUC
CZECH REPUBLIC

svrcek@inf.upol.cz

Jaroslav Švrček,
June 2009

WFNMC 25 years on: some experiences

Peter Taylor

Peter Taylor graduated with a Ph.D. in Applied Mathematics in 1972 at the University of Adelaide. He is a Professor of Mathematics at the University of Canberra, Executive Director of the Australian Mathematics Trust and Chairman of the Education Advisory Committee of the Australian Mathematical Sciences Institute. He is a Past President of WFNMC and was Co-chair of ICMI Study 16 "Challenging Mathematics in and beyond the Classroom".

I first met Peter O'Halloran in February 1972. I had just graduated with a PhD and was fortunate to get a job at the Canberra College of Advanced Education, later to become the University of Canberra. I was the most junior of a mathematics department of five. Peter was the second most senior in this new institution, which was degree granting, but we were not allowed to do research. I proved this by giving a research paper I wrote to a typist and five minutes later was hauled in to the Head of School, who made it quite clear I should not do research. I took my manuscript to the Australian National University and paid a mathematical typist to prepare the paper, which was published in the *Journal of Ship Research*.

I looked for other outlets and in the 1970s found a nice involvement in the Open University, in England, where I went for sabbatical and wrote a couple of their books. But Peter had taken an interest in me and we spent many morning and afternoon teas together discussing the possibility of us founding an Australian Mathematics Competition similar to those he had seen on his sabbatical at the University of Waterloo, competitions run from there and the MAA competitions. I had seen the value of competitions in my home town of Adelaide, where my honours class seemed to be dominated by students from two schools who seemed to have benefited from a local Olympiad-type competition and the training associated with it.

So Peter had me on board. He also seemed to have a colleague Jo Edwards, and Warren Atkins from the School of Education, interested in the idea. (I should note that another important colleague, Peter Brown joined us the following year.) On 11 April 1976 he decided to hold an impromptu meeting, involving about 12 teachers and academics in Canberra, and we decided to run a competition which attracted entries from all high schools in Canberra. This event grew exponentially each year, becoming national in 1978 and soon generating hundreds of thousands of entries from Australia and other countries in our region.

In 1983 Peter took another sabbatical during which I was acting director of the competition, and I learned a lot of unexpected things. I had to deal with security problems, manage committees and communicate with the University hierarchy and significant personalities further afield. Peter returned at the end of 1983 and took me to dinner relating his newest idea. Why not have a World Federation of National Mathematics Competitions? We could ensure that significant organisers from other countries could be enticed to attend ICME-5 in Adelaide in 1984, and we could found the organisation then. Why not, indeed?

It is now history that WFNMC was founded by an international band of enthusiasts. The first activity was to be a newsletter. This job was allocated to Warren Atkins. The story of this can be told elsewhere, but I note this grew in popularity so that it soon became a journal. Warren remained its editor, and a member of the WFNMC through to 2004, when Jaroslav Švrček took over. The other main activities were to be the awards and the conferences. The awards were the brainchild of Peter O'Halloran. He first founded a David Hilbert Award based on popular voting on the best journal articles. A small number of these awards were made in 1991, but a new version of the award and a new Erdös Award based on general contribution to mathematics enrichment on a national or international scale became the high profile acknowledgement of WFNMC accepted internationally outside the organisation. By 1996 the distinction between the two awards became blurred and so were merged into just the Erdös Award and with a strict quota of three every two years. A strong refereeing system was also developed.

The other major activity became the conferences. Until 1990, WFNMC was meeting only every four years at ICME. It was the brainchild of WFNMC's second active President, Ron Dunkley of Canada, to initiate conferences to be held every four years from 1990.

I will now give some chronological recollections of significant and some light-hearted moments from each conference (and the related awards).

1988 ICME-6 Budapest

This was a wonderful event for those lucky enough to attend. Unfortunately Peter O'Halloran could not attend because he had the exciting job of hosting the IMO in Canberra, which almost overlapped. As a result I missed most of the IMO although delegates had started arriving when I left Canberra. I attended ICME instead. The WFNMC meetings were highly congenial. Strong bonds developed between many people from different countries.

The photo shows the WFNMC group, which if you were able to enjoy in high resolution would find great interest in identifying the characters. This photo can be found in better detail on the WFNMC website http://www.amt.edu.au/wfnmc.html.

1990 WFNMC 1 Waterloo

This was the initiative of Ron Dunkley and many of those who attended still say it was the best conference they ever attended. There was a busy workload of excellent lectures and social activities, with all participants enjoying the benefit of staying together at the one location, which has generally become a tradition, more or less a rule.

Again I have a group photo which I produce here, thanks to my friend Cheung Pak-Hong, but it can be found better annotated at the WFNMC website.

1992 ICME-7 Quebec

This conference saw the first awarding of Hilbert and Erdös Awards under the new criteria. Nikolay Konstantinov was a winner but he arrived after the normal ceremony so we convened a special ceremony where, as the photo below shows, he was seen by most for the only time in their lives wearing a collar and tie.

Nikolay Konstantinov—the second one from the left side of the photo

1994 WFNMC-2 Pravets

This excellent conference is memorable for two main reasons in addition to the Bulgarian hospitality. First it is memorable because it was attended by Paul Erdös, who participated in all activities and presented the awards. For some it was their only opportunity to spend time at all with this great man, who was there for the whole week.

On a sadder note, all could tell that Peter O'Halloran was very ill. He returned to Canberra and then disappeared for a few days. It seems he was undergoing serious medical testing. About a week after our return he telephoned me to say he had a serious form of cancer and had just six weeks to live. Then followed a remarkable period in which he held court at his home, receiving visitors and phone calls from his friends all over the world. During this period the Executive decided unanimously to present him with a Hilbert Award which he received at his home.

He had also, independently, been awarded the World Cultural Council's Jose Vasconcelos Award. This was to be presented to him in Switzerland. Being too ill to travel himself, two of his children went to Switzerland to receive it. He died the day after they had returned home to give it to him.

Peter O'Haloran receives Hilbert Award

1996 ICME-8 Seville

This conference was held in very hot, dry weather, similar to the summer heat of Adelaide, but seemingly even hotter. By this time Ron Dunkley had become President and at this time Ron, after wide consultation, devised the first WFNMC constitution. The constitution has since received two or three amendments, but the original structure survives, and it is of great necessity for the future development of WFNMC through personnel and other changes.

1998 WFNMC-3 Zhong Shan

This was a pleasant experience in which the participants met first in Hong Kong and then travelled together over the Pearl River estuary to Zhong Shan, an important city of China, the birthplace of Sun-Yat Sen.

This conference was held at a nice retreat and under the leadership of Professor Qiu Zhonghu, with his charming manner, all the ingredients of WFNMC conferences were present. This conference was attended by a large number of IMO Team Leaders, who were able to piggyback on the IMO which had just been held in Taipei.

2000 ICME-9 Tokyo

This was held at a major conference centre at Makuhari, between the Narita airport and Tokyo, and on the bay of Tokyo. Ron Dunkley stepped down as President and I was elected. The facilities were excellent and we ran a good program of lectures which attracted a loyal following.

2002 WFNMC-4 Melbourne

This was our opportunity to host the event in Australia. We invited John Conway to be our principal plenary speaker. He carried out a similar role to Paul Erdös in Pravets. John Conway's lecture was also a public lecture at the University of Melbourne and it attracted a large crowd. John diligently contributed to all the sessions and many enjoyed the opportunity to meet him and discuss mathematics with him.

2004 ICME-10 Copenhagen

At this conference we were fairly well spread out over various parts of the city. The conference centre was at a university a little distance out and, even for many westerners, was very expensive, but we overcame these obstacles. The facilities and the program were very good. We were not however able to run our lectures in the usual way in a Topic Group, and instead adapted to a Discussion Group in which individual lectures were not given. We moved a change to the Constitution to ensure that Presidents changed every four years. Petar Kenderov was elected.

2006 WFNMC-5 Cambridge

Tony Gardiner and some wellknown colleagues (Howard Groves, Bill Richardson and Adam McBride) hosted this conference. We all lived in

Robertson College, one of the newer ones, and this proved an excellent and appropriate venue. The opportunity to stay in one of the world's most pristine academic locations was enough to ensure success. There were some notable English speakers but María de Losada and Joszef Pelikan, among the plenary lecturers, gave compelling presentations.

2008 ICME-11 Monterrey

Once again, at these types of conferences we were well spread out. To make up for not having lectures in the normal format at the previous ICME, we had developed a new idea, to have our own mini conference the day before ICME started. Perhaps this was copied from other ICMI Affiliated Study Groups, but it was logical since it only cost an extra night's accommodation. The day we spent together was highly successful and is a tribute, again to María, and the excellent local support from within Mexico.

As in Copenhagen and other ICMEs the main venue was well away from the centre, but it was easily accessible by train and the facilities were excellent. There was also an excellent program within ICME to discuss competitions and related matters.

María de Losada was elected the new President and will be so until 2012. The presentation of awards was made by ICMI President Michelle Artigue. One of the Erdös Award winners, who could not attend two years previously was Ali Rejali, who went to great lengths to bring his wife and son from Iran to see him receive his well-earned award, a very big moment.

2010 WFNMC-6 Riga

We look forward to this meeting. Knowing Agnis Andjans' well-established organising ability, we all have high expectations of this conference to be held in the historic city of Riga.

I conclude this brief article, I exhort all readers to try to attend this conference to establish new friends and contacts in other countries and to exchange your ideas on not only how to organise competitions, but to discuss the mathematics and how we can improve the world by having more students enjoy our beloved subject, which surely benefits them in their everyday lives.

There are many challenges for the organisation as a whole, but it is evident that it is surviving personnel changes in recent years with a number of younger members on the new Executive, and I am sure they will make certain the next 25 years are just as productive.

Peter Taylor
Australian Mathematics Trust
University of Canberra ACT 2601
AUSTRALIA
email: pjt013@gmail.com

Ten Years of the Mathematical Olympiad of Central America and the Caribbean

José H. Nieto Said & Rafael Sánchez Lamoneda

Jose H. Nieto Said, MSc, is a Professor in the Faculty of Sciences, Department of Mathematics, Universidad del Zulia. He was Venezuelan OMCC Team Leader 2001–2002, Deputy Leader 2000 and Academic Support 2003. He is a member of the Venezuelan Association for Mathematical Competitions.

Dr Rafael Sanchez Lamoneda is a Professor in the Faculty of Sciences, Department of Mathematics, Universidad Central de Venezuela. He is President of the Venezuelan Association for Mathematical Competitions. He has been Venezuelan IMO Team Leader in 1989 and since 1998 to the present, and Deputy Leader in 1981 and 1982. He has been a member of the Venezuelan Mathematical Olympiads Commission for 30 years and has been President of the Venezuelan Young Mathematical Olympiads Commission since 2004. He has been a member of the Kangourou Sans Frontiers Organisation since 2002.

Introduction

The *Mathematical Olympiad of Central America and the Caribbean* (OMCC) was created in 1999 in order to encourage the participation of the region in international mathematical competitions. A previous article in this journal [1] explains the characteristics of this olympiad and contains the proposed problems and statistics for years 1999 to 2004. The main goal of the present work is to update that information up to 2008.

For completeness we should mention that the OMCC is an IMO type competition. Each country participates with a delegation consisting of a Leader, a Deputy Leader and no more than three students. This contest is addressed to young high school students, up to sixteen years old. Other teachers may assist as observers, or to attend a seminar aimed to elevate the mathematical level of the participants, preparing them to be olympiad promoters and trainers in their own countries. The exam is sat over two consecutive days. Each day the participants are allowed four and a half hours for solving three problems, each one with a value of seven points. During the subsequent couple of days the contestants engage in several recreational activities, while their papers are being evaluated.

One of the recreational activities is a mathematical rally or a team contest. The students work together in groups of four on several mathematical problems with a recreational flavor. The teams are selected under the hypothesis that all the members of each group belong to different countries. The results of this contest are independent of the OMCC results and the winning team receives a special prize. The idea is to foster the relationship between the olympiad contestants and to give them the opportunity to work together on mathematical problems.

Medals are awarded to approximately one-half of the participants, in the proportion 1:2:3 between gold, silver and bronze. The students who do not obtain a medal but solve a problem perfectly, receive honorable mentions. There is also a special prize for the country which shows the best improvement on its performance during two consecutive years. It is a cup given by El Salvador and because of this, its name is *Copa El Salvador*. This prize was awarded for the first time in the year 2000 to

Cuba, during the second OMCC held in El Salvador. Table 6 at the end of this article shows all the winners from 2000 to 2008.

Proposed Problems

The problems proposed in the first six OMCC's may be found in [1]. Those interested in the original statements (in Spanish) may find them in [2] for the first three olympiads, in [3, 4, 5, 6] for olympiads IV to VII, at `http://www.viiiomcc.opm.org.pa/` for OMCC VIII, in [7] for OMCC IX and at `http://www.honmatsps.com/Resultados.htm` for OMCC X. Problems and solutions for the first eight olympiads may also be found at `http://www.oei.es/oim/omcc_problemas.htm`.

The problems proposed from 2005 to 2008 follow.

7th OMCC (San Salvador, El Salvador, 2005)

1. Among the positive integers which may be expressed as the sum of 2005 consecutive integers (not necessarily positive), which one occupies the 2005th position (in increasing order)?

2. Prove that the equation $a^2b^2 + b^2c^2 + 3b^2 - a^2 - c^2 = 2005$ has no integer solution.

3. Let P, Q and R be the points of tangency of the incircle of a triangle ABC with sides AB, BC and AC, respectively. Let L, M and N denote the feet of the altitudes of triangle PQR on the sides PQ, QR and PR, respectively.

Prove that the straight lines AN, BL and CM are concurrent at a point which belongs to the line passing through the orthocenter and the circumcenter of triangle PQR.

4. Two players named Blue and Red play alternately on a 10×10 board. Blue has a blue paint can, and Red a red paint can. Beginning with Blue, each player in his turn choses a row or column not previously chosen by any player and paints its 10 squares with his own color. If some square is already painted, the new color overlays the former one. After 20 turns the game ends and the squares of each color are counted. If the number of red squares surpasses by ten or more the number of blue squares, Red wins. Otherwise Blue wins. Determine which player has a winning strategy and explain it.

5. Let H be the orthocenter and M the midpoint of the side AC of an acute-angled triangle ABC. Let L be the parallel to the bisector of $\angle AHC$ passing through M. Prove that L divides the triangle ABC into two parts with the same perimeter.

6. n cards numbered from 1 to n must be placed in p boxes, with p prime. Determine the possible values of n for which the n cards may be distributed in the boxes in such a way that the sums of the numbers on the cards in each box are all the same.

8th OMCC (Ciudad de Panama, Panama, 2006)

1. Consider the positive integers

$$S_d = 1 + d + d^2 + \cdots + d^{2006},$$

with $d = 1, 2, 3, \ldots, 9$. Find the rightmost digit of

$$S_0 + S_1 + S_2 + \cdots + S_9.$$

2. Let Γ and Γ' be two circumferences with equal radius and centers O and O', respectively. Γ and Γ' intersect at two points, and A is one of them. An arbitrary point B is chosen on Γ. Let C be the other intersection point of the straight line AB with Γ', and let D be a point on Γ' such that $OBDO'$ is a parallelogram. Prove that the length of segment CD is constant, i.e. it does not depend on the choice of the point B.

3. For each natural number n let us define $f(n) = \lfloor n + \sqrt{n} + 1/2 \rfloor$. Prove that, for each $k \geq 1$, the equation

$$f(f(n)) - f(n) = k$$

has exactly $2k - 1$ solutions.

4. The product of several different positive integers is a multiple of $(2006)^2$. Determine the least possible value of the sum of these numbers.

5. The country *Olimpia* is formed by n islands. The most populated island is called *Panacenter*. Each island has a different number of inhabitants. It is desired to build bidirectional bridges between islands fulfilling the following conditions:

- At most one bridge joins each pair of islands.
- Any of the islands is reachable from Panacenter using the bridges.
- If a path goes from Panacenter to any other island, traversing each bridge at most once, then the number of inhabitants on visited islands is decreasing.

Determine the number of ways in which the bridges may be constructed.

6. Let $ABCD$ be a convex quadrilateral. Let I be the intersection point of diagonals AC and BD. Let E, H, F and G be points on the segments AB, BC, CD and DA, respectively, such that EF and GH meet at I. Let M be the intersection point of EG and AC, and let N be the intersection point of HF and AC. Prove that

$$\frac{AM}{IM} \cdot \frac{IN}{CN} = \frac{IA}{IC}.$$

9th OMCC (Mérida, Venezuela, 2007)

1. The OMCC is an annual mathematics olympiad. The ninth OMCC is held in the year 2007. Which positive integers n divide the year when the nth OMCC is held?

2. Let ABC be a triangle, and let D and E be points on the sides AC and AB, respectively, such that the lines BD, CE and the bisector from A meet at a point P interior to the triangle. Prove that there is a circumference tangent to the four sides of the quadrilateral $ADPE$ if and only if $AB = AC$.

3. Let S be a finite set of integers. Suppose that for any pair of elements p, q in S, with $p \neq q$, there are elements a, b and c in S, not necessarily different, with $a \neq 0$ and such that the polynomial

$F(x) = ax^2 + bx + c$ satisfies $F(p) = F(q) = 0$. Determine the maximum possible number of elements in S.

4. The inhabitants of a certain island speak a language whose words may all be written with the letters a, b, c, d, e, f and g. A word *produces* another one if it is possible to obtain the second from the first applying one or more times the following rules:

(a) A letter may be substituted by two letters, as follows:

$$a \to bc,\ b \to cd,\ c \to de,\ d \to ef,\ e \to fg,\ f \to ga,\ g \to ab.$$

(b) Two identical letters located to the right and left of another letter may be removed. For example: $dfd \to f$.

For example, $cafcd$ produces $bfcd$, since

$$cafcd \to cbcfcd \to bfcd.$$

Prove that in this language, any word produces any other word.

5. Given two non-negative integers m and n, with $m > n$, we say that m *ends* in n if it is possible to obtain n by erasing some digits of m, from left to right. For exemple, 329 ends in 9 and 29. Determine how many three-digit integers end in the product of their digits.

6. From a point P exterior to a circumference S, two tangents are drawn touching it at A and B. Let M be the midpoint of AB. The perpendicular bisector of AM intersects S at C (interior to ΔABP), line AC intersects line PM at G, and line PM intersects S at point D (exterior to ΔABP). If BD is parallel to AC, prove that G is the centroid of ΔABP.

10th OMCC (San Pedro Sula, Honduras, 2008)

1. Find the least positive integer N such that the sum of its digits is 100 and the sum of the digits of $2N$ is 110.

2. Let $ABCD$ be a convex quadrilateral inscribed in a circumference with center O and diameter $\overline{AC}$. Let E and F be points such that $DAOE$ and $BCOF$ are parallelograms. Prove that, if E and F belong to the circumference, then $ABCD$ is a rectangle.

3. 2008 bags are numbered from 1 to 2008. Each bag contains 2008 frogs. Two people play alternately. A move consists of choosing a bag k and removing the desired (positive) number of frogs from it. If in the chosen bag there remain x frogs, then from each bag with a label greater than k which contain more than x frogs, some frogs jump out until exactly x remain in the bag. The player who removes the last frog from bag 1 loses the game. Find and justify a winning strategy for some player.

4. Five friends own a little store that opens from Monday to Friday. Since two people suffice to attend the store each day, they decide to make a working plan for the entire week, specifying the pair of them that must work each day, and satisfying the following conditions:

a) Each person must work exactly two days a week.

b) The five pairs assigned for the week must be all different.

How many different working plans are possible?

Example: If the friends are A, B, C, D and E, a possible working plan is:
Monday A and B, Tuesday A and D, Wednesday B and E, Thursday C and E and Friday C and D.

5. Find a polynomial $p(x)$ with real coefficients such that

$$(x + 10)p(2x) = (8x - 32)p(x + 6)$$

for all real x, and $p(1) = 210$.

6. Let ABC be an acute-angled triangle. P and Q are points interior to the sides $\overline{AB}$ and $\overline{AC}$, respectively, such that B, P, Q and C are concyclic. The circumference circumscribed to $\triangle ABQ$ intersects BC again at S, the circumference circumscribed to $\triangle APC$ intersects BC again at R, PR and QS intersect at L. Prove that the intersection of AL and BC does not depend on the choice of P and Q.

Results and statistics

Tables 1 to 4 contain, for each olympiad from 2005 to 2008, the marks obtained by each student, the average mark for each problem and the medals and honorable mentions awarded.

Figures 1 to 4 are histograms which show the distribution of marks for each olympiad from 2005 to 2008.

Figure 5 shows the participation of countries and students in each olympiad.

Figure 6 shows the participation of students separated by sex. The female participation increased from an average of 14.7% during the first 5 years of the olympiad to 23.2% during the last 5 years, for an overall average of 19.5%.

Table 5 contains, for each olympiad, the number of participant countries and the total marks of each country.

Figure 7 is a graphic representation of the same information.

Figures 8 and 9 show the distribution of awards by year and for the ten Olympiads, respectively.

Figure 10 reflects the average marks for each proposed problem.

Figure 11 shows the distribution of the proposed problems of each olympiad in four mathematical areas: Algebra, Arithmetic, Combinatorics and Geometry.

Figure 12 shows the overall distribution for the ten years. As you can see this distribution shows an emphasis in Combinatorics, followed by Geometry and Arithmetic, with a small percentage of Algebra problems, just 6% of the total. Probably problems selection committees should work on this in the future, in order to have better balanced exams.

Table 6 lists the countries that have won the *Copa El Salvador*.

OMCC VII (2005)								
Code	Pr 1	Pr 2	Pr 3	Pr 4	Pr 5	Pr 6	Tot.	Award
COL1	7	7	1	7	7	6	35	Gold
COL2	7	6	0	7	4	1	25	Silver
COL3	7	2	3	7	1	5	25	Silver
CRC1	7	7	0	7	1	2	24	Bronze
CRC2	7	7	1	7	1	1	24	Bronze
CRC3	7	6	1	6	2	1	23	Bronze
CUB1	7	6	0	7	1	1	22	Bronze
CUB2	7	7	5	7	2	3	31	Silver
CUB3	7	7	0	6	1	3	24	Bronze
DOM1	0	1	1	0	0	2	4	
DOM2	0	0	0	0	0	0	0	
DOM3	0	0	0	1	0	0	1	
GUA1	4	0	0	1	1	0	6	
GUA2	5	0	0	6	1	0	12	
GUA3	7	0	0	7	0	1	15	Bronze
HON1	3	1	0	5	0	0	9	
HON2	0	2	0	1	0	1	4	
HON3	4	0	0	4	0	0	8	
MEX1	7	7	5	6	7	5	37	Gold
MEX2	7	5	4	7	7	1	31	Silver
MEX3	7	6	7	7	7	5	39	Gold
NIC1	4	2	0	6	1	0	13	
NIC2	0	0	0	0	0	0	0	
NIC3	7	5	0	1	1	0	14	Bronze
PAN1	7	7	1	6	1	1	23	Bronze
PAN2	7	5	0	7	3	3	25	Silver
PAN3	7	4	0	6	1	1	19	Bronze
PRC1	7	1	0	7	1	0	16	Bronze
PRC2	6	0	0	7	0	0	13	Mention
PRC3	0	1	0	3	0	1	5	
SAL1	7	7	0	1	7	2	24	Bronze
SAL2	7	7	0	7	6	1	28	Silver
SAL3	5	7	0	0	0	1	13	Mention
VEN1	7	1	0	1	0	1	10	Mention
VEN2	1	0	1	5	1	0	8	
VEN3	7	1	1	4	0	1	14	Bronze
Average	4.97	3.27	0.83	4.44	1.61	1.22	2.73	

Table 1: OMCC VII (El Salvador, 2005)

OMCC VIII (2006)								
Code	Pr 1	Pr 2	Pr 3	Pr 4	Pr 5	Pr 6	Tot.	Award
COL1	7	1	1	4	6	0	19	Bronze
COL2	7	7	7	2	0	0	23	Silver
COL3	6	0	1	2	0	0	9	
CRC1	6	0	1	7	1	0	15	Bronze
CRC2	7	7	2	6	0	0	22	Silver
CRC3	0	1	1	5	0	0	7	
CUB1	7	7	7	7	7	7	42	Gold
CUB2	7	7	7	7	7	0	35	Gold
CUB3	7	7	4	5	2	1	26	Silver
DOM1	6	0	0	0	0	0	6	
DOM2	4	1	0	0	0	0	5	
DOM3	0	0	0	0	0	0	0	
GUA1	7	0	0	7	6	0	20	Bronze
GUA2	7	7	1	6	0	0	21	Bronze
GUA3	6	0	0	3	0	0	9	
HON1	7	0	0	0	0	0	7	Mention
HON2	7	0	0	0	0	0	7	Mention
HON3	7	3	0	1	1	0	12	Mention
MEX1	7	7	7	7	7	0	35	Gold
MEX2	6	5	6	6	5	0	28	Silver
MEX3	7	7	7	7	7	7	42	Gold
NIC1	3	4	0	6	0	0	13	
NIC2	7	0	0	0	0	0	7	Mention
NIC3	5	7	0	0	0	1	13	Mention
PAN1	7	7	0	7	0	0	21	Bronze
PAN2	7	0	1	5	0	0	13	Mention
PAN3	7	1	0	5	0	0	13	Mention
PRC1	7	7	3	6	0	0	23	Silver
PRC2	7	0	1	0	0	0	8	Mention
PRC3	7	7	7	6	0	0	27	Silver
SAL1	7	7	0	1	0	0	15	Bronze
SAL2	7	7	1	2	4	0	21	Bronze
SAL3	7	7	0	1	0	0	15	Bronze
VEN1	7	7	0	4	2	0	20	Bronze
VEN2	4	1	0	1	0	0	6	
VEN3	6	1	0	3	0	0	10	
Average	6.11	3.61	1,81	3.58	1.53	0.44	2,85	

Table 2: OMCC VIII (Panama, 2006)

OMCC IX (2007)								
Code	Pr 1	Pr 2	Pr 3	Pr 4	Pr 5	Pr 6	Tot.	Award
COL1	4	7	0	7	6	2	26	Silver
COL2	7	7	0	7	3	1	25	Bronze
COL3	7	3	0	7	7	2	26	Silver
CRC1	7	7	1	1	7	2	25	Bronze
CRC2	7	7	0	0	6	0	20	Bronze
CRC3	7	7	0	0	4	2	20	Bronze
CUB1	7	7	6	7	7	2	36	Gold
CUB2	7	7	0	7	4	2	27	Silver
CUB3	7	7	2	6	6	2	30	Silver
DOM1	7	4	1	0	1	0	13	Mention
DOM2	3	0	0	0	7	0	10	Mention
DOM3	0	0	0	1	3	2	6	
GUA1	7	0	1	3	6	2	19	Mention
GUA2	7	0	2	5	7	1	22	Bronze
GUA3	0	0	0	1	2	1	4	
HON1	7	5	0	2	2	2	18	Mention
HON2	6	5	0	5	0	2	18	
HON3	7	0	0	0	7	0	14	Mention
MEX1	6	7	0	6	6	6	31	Gold
MEX2	7	7	7	7	6	7	41	Gold
MEX3	7	3	0	7	7	3	27	Silver
NIC1	7	7	0	0	0	2	16	Mention
NIC2	6	4	0	0	0	2	12	
NIC3	0	3	0	0	6	0	9	
PAN1	7	7	0	7	4	0	25	Bronze
PAN2	3	1	0	6	3	0	13	
PAN3	7	0	0	6	7	1	21	Bronze
PRC1	7	6	0	7	6	2	28	Silver
PRC2	7	7	0	2	4	1	21	Bronze
SAL1	7	7	0	7	7	2	30	Silver
SAL2	7	7	0	6	4	2	26	Silver
SAL3	5	7	0	6	4	0	22	Bronze
VEN1	7	0	0	0	3	1	11	Mention
VEN2	6	1	0	7	7	0	21	Bronze
VEN3	3	3	0	0	4	1	11	
Average	5.8	4.29	0.57	3.8	4.66	1.57	3.45	

Table 3: OMCC IX (Venezuela, 2007)

OMCC X (2008)								
Code	Pr 1	Pr 2	Pr 3	Pr 4	Pr 5	Pr 6	Tot.	Award
COL1	7	7	1	7	3	3	28	Bronze
COL2	6	7	7	5	3	7	35	Silver
COL3	7	7	7	5	7	7	40	Gold
CRC1	4	7	7	0	0	1	19	Bronze
CRC2	1	7	7	0	0	3	18	Mention
CRC3	0	2	0	0	0	1	3	
CUB1	7	7	7	7	7	7	42	Gold
CUB2	7	7	7	0	2	7	30	Silver
CUB3	7	7	0	1	3	0	18	Mention
DOM1	1	1	0	1	0	0	3	
DOM2	4	1	0	3	0	0	8	
DOM3	0	1	0	7	0	0	8	Mention
GUA1	7	7	1	4	7	0	26	Bronze
GUA2	0	4	0	2	0	1	7	
GUA3	1	7	0	6	0	0	14	Mention
HON1	7	7	0	3	7	7	31	Silver
HON2	3	7	7	2	0	2	21	Bronze
HON3	1	5	7	1	0	1	15	Mention
MEX1	6	7	1	7	2	5	28	Bronze
MEX2	7	7	1	7	2	6	30	Silver
MEX3	7	7	7	7	7	7	42	Gold
NIC1	6	7	0	0	0	3	16	Mention
NIC2	0	5	0	1	0	1	7	
NIC3	0	7	0	4	2	3	16	Mention
PAN1	5	7	7	7	0	2	28	Bronze
PAN2	1	6	0	2	7	0	16	Mention
PRC1	5	5	3	1	0	0	14	
PRC2	3	7	0	7	0	3	20	Bronze
PRC3	7	7	7	7	3	0	31	Silver
SAL1	3	7	0	5	1	7	23	Bronze
SAL2	7	7	0	5	0	7	26	Bronze
SAL3	1	7	1	0	0	0	9	Mention
VEN1	0	7	0	1	0	0	8	Mention
VEN2	7	2	1	6	0	0	16	Mention
VEN3	7	1	7	7	0	0	22	Bronze
Average	4,06	5,74	2,66	3,66	1,8	2,6	3,42	

Table 4: OMCC X (Honduras, 2008)

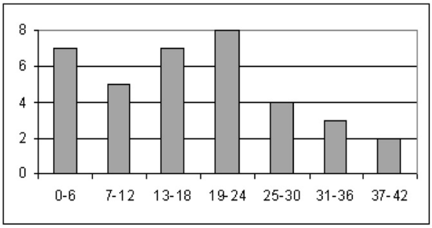

Figure 1: Distribution of Marks in OMCC VII

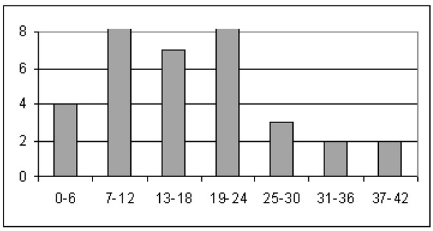

Figure 2: Distribution of Marks in OMCC VIII

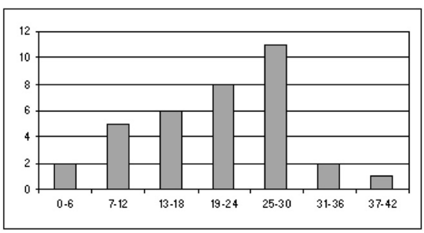

Figure 3: Distribution of Marks in OMCC IX

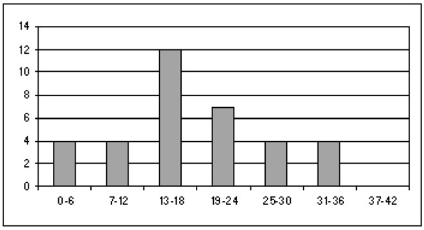

Figure 4: Distribution of Marks in OMCC X

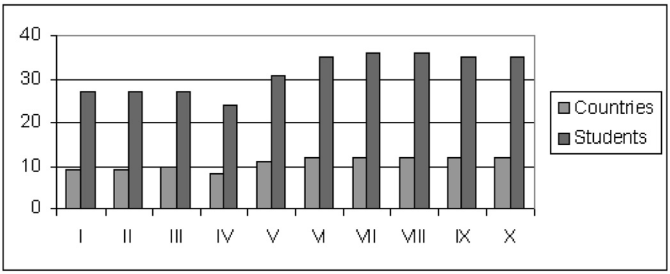

Figure 5: Participation by Year

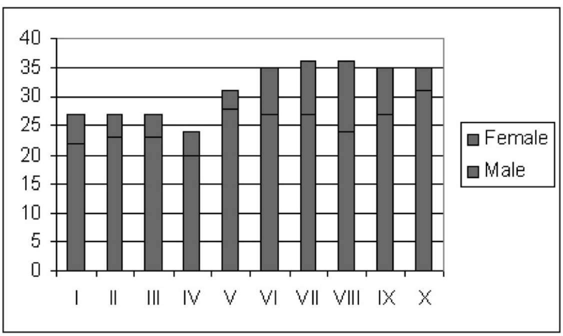

Figure 6: Students by Sex

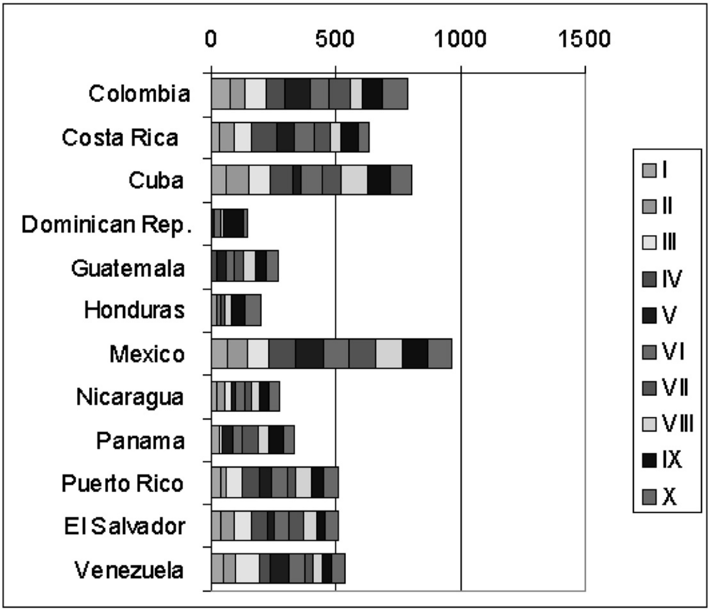

Figure 7: Total Marks by Country and Year

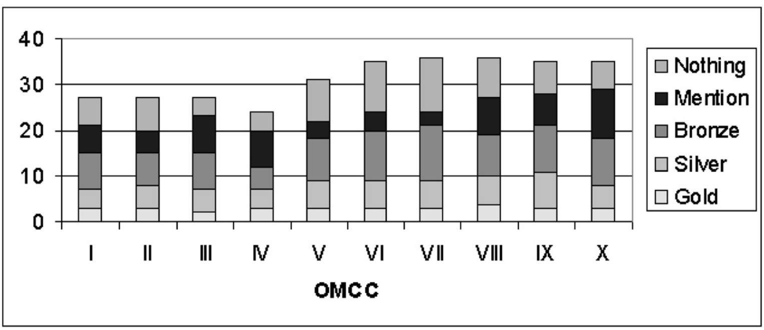

Figure 8: Distribution of Awards by Year

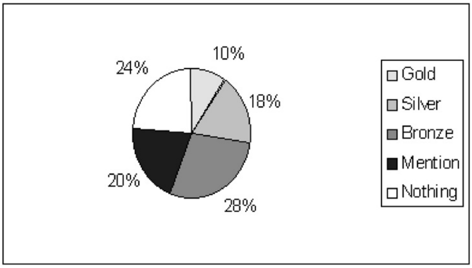

Figure 9: Distribution of Awards (average 10 years)

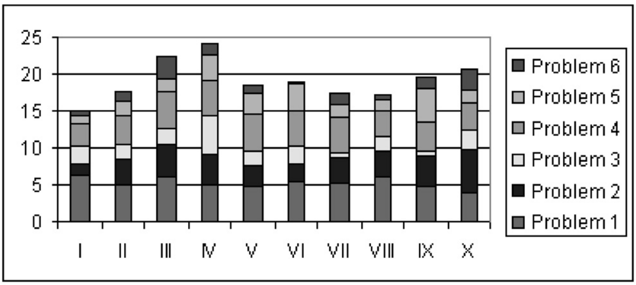

Figure 10: Average Marks by Problem and Year

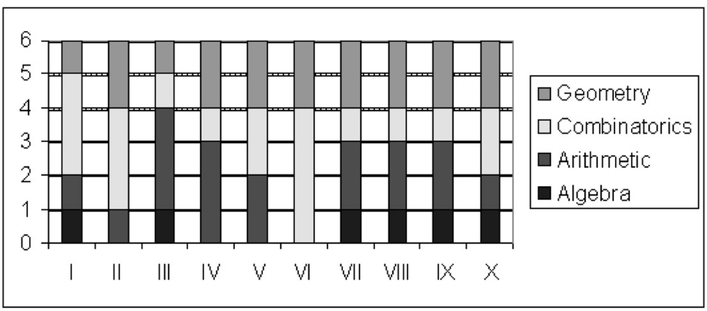

Figure 11: Distribution of Problems by Area and Year

	Olympiad									
Country	I	II	III	IV	V	VI	VII	VIII	IX	X
Colombia	70	62	85	80	95	80	85	51	77	103
Costa Rica	32	59	74	96	76	73	71	44	65	40
Cuba	58	95	83	91	34*	84	77	103	93	90
Dominicana					15	19	5	11	78	19
Guatemala			2*	25	34	32	33	50	45	47
Honduras		26				8†	21	26	50	67
Mexico	64	79	90	110	108	104	107	105	99	100
Nicaragua	25	28	26†		18	39	27	33	37	39
Panama	30		15		38	36	67	47	59	44
Puerto Rico	39	22	61	72	51	61	34	58	49	65†
El Salvador	38	54	70	63	29	54	65	51	29†	58
Venezuela	47	52	94	41	74	69	32	36	43	46
No. of countries	9	9	10	8	11	12	12	12	12	12
* with only one student; † with only two students										

Table 5: Total Marks by Country and Year

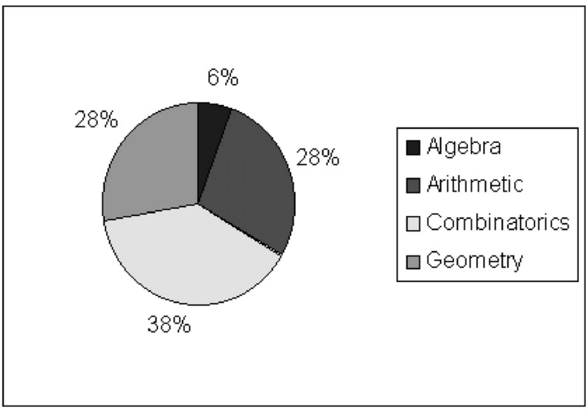

Figure 12: Problems by Area (average 10 years)

Conclusions

During its first ten years the OMCC has proved to be an excellent way to initiate the high school students of the region in international

Copa El Salvador		
OMCC	Year	Winning Country
I	1999	Not awarded
II	2000	Cuba
III	2001	Puerto Rico
IV	2002	Puerto Rico
V	2003	Colombia
VI	2004	Venezuela
VII	2005	Panama
VIII	2006	Cuba
IX	2007	República Dominicana
X	2008	Colombia

Table 6: Copa El Salvador

mathematical competitions, preparing them for more demanding events such as the Iberoamerican, Asian-Pacific and International Mathematical Olympiads. It has fostered friendly relationships among students and teachers of the participant countries, creating many opportunities for the exchange of information and experience on the teaching of mathematics in countries with similar cultures and problems. Also the OMCC has provided to the participant countries an international experience in this kind of competition and it has helped most of them to raise the level of their own national mathematical olympiads.

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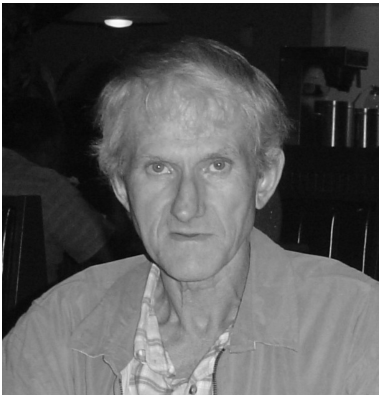
José H. Nieto Said
Escuela de Matemáticas
Facultad de Ciencias
Universidad del Zulia
Maracaibo. Edo. Zulia.
VENEZUELA
Email: jhnieto@cantv.net

Rafael Sánchez Lamoneda
Escuela de Matemáticas
Facultad de Ciencias
Universidad Central de Venezuela
Caracas 1020-A
VENEZUELA
Email: rsanchez@euler.ciens.ucv.ve

On a Problem of Arthur Engel

Arthur Holshouser & Harold Reiter



Arthur Holshouser does math all the time. He enjoys doing mathematics, walking, and eating blueberries. He's especially fond of working problems and building generalizations of his and others' theorems.

Harold B. Reiter enjoys creating challenges for high school students and middle and high school teachers though the Charlotte Math Club and the Charlotte Teachers' Circle. He also enjoys reading, travel, kenken, and solving problems with his students and with Arthur Holshouser.

1 Introduction

Problem 21, page 10 of [1] states

> Three integers a, b, c are written on a blackboard. Then one of the integers is erased and replaced by the sum of the other

two diminished by 1. This operation is repeated many times with the final result $17, 1967, 1983$. Could the initial numbers be (a) $2, 2, 2$, (b) $3, 3, 3$?

This paper develops a mathematical context for a class of problems that includes this one and solves them. We deal with Arthur Engel's problem in section 8.

A set F of triplets of integers is said to be a *Fibonacci set* if

1. Each $t \in F$ is a triplet of the form $t = \{x, y, x+y\}$ where x and y are positive integers and $x = y$ is allowed.

2. If $t = \{x, y, x+y\} \in F$ then $\{x, x+y, 2x+y\} \in F$ and $\{y, x+y, x+2y\} \in F$.

The main purpose of this paper is to compute in a closed form the smallest Fibonacci set $F_t = g(t)$ that contains the single element $t = \{a, b, a+b\}$ where $a \le b$ and $a, b \in \{1, 2, 3, \ldots\}$. It is also possible to think of a, b as purely algebraic symbols. In the end, we devise two algorithms for determining if $\{\overline{a}, \overline{b}, \overline{a}+\overline{b}\} \in g(\{a, b, a+b\})$ when $\overline{a}, \overline{b}, a, b$ have specific numerical values.

Throughout this paper, we use the notation $\mathbb{N} = \{0, 1, 2, 3, \ldots\}, \mathbb{N}^+ = \{1, 2, 3, \ldots\}$. The main result is that if $a, b \in \mathbb{N}^+$ and $a \le b$, then

$$g(a, b, a+b) = \left\{ \{x, y, x+y\} : \begin{pmatrix} x \\ y \end{pmatrix} = M \cdot \begin{pmatrix} a \\ b \end{pmatrix}, M \in \overline{M} \right\},$$

where $M \cdot \begin{pmatrix} a \\ b \end{pmatrix}$ is matrix multiplication and

$$\overline{M} = \left\{ \begin{bmatrix} \theta & \phi \\ \psi & \pi \end{bmatrix} : \theta, \phi, \psi, \pi \in \mathbb{N}, \begin{vmatrix} \theta & \phi \\ \psi & \pi \end{vmatrix} = \pm 1, \theta + \phi \le \psi + \pi \right\}.$$

2 Preliminary Concepts

For our purposes we will modify the definition of a Fibonacci set as follows:

Definition 1 *A set F is said to be a Fibonacci set if 1. and 2. are true.*

1. *Each $t \in F$ is an ordered triple of the form $t = (x, y, x + y)$ where $x, y \in \mathbb{N}^+$ and $x \le y$.*
2. *If $(x, y, x + y) \in F$ then $(x, x + y, 2x + y) \in F$ and $(y, x + y, x + 2y) \in F$.*

Notation 1 *If $(x, y, x + y)$, where $x \le y, x, y \in \mathbb{N}^+$, is a member of a Fibonacci set F we will often use an abbreviated notation and write $(x, y, x + y) = (x, y)$.*

Definition 2 *Suppose $(x, y) = (x, y, x + y) \in F$ where F is a Fibonacci set and $x \le y, x, y \in \mathbb{N}^+$.*

Let $(x, x + y) = (x, x + y, 2x + y)$ and $(y, x + y) = (y, x + y, x + 2y)$ be the *immediate successors* of $(x, y) = (x, y, x + y)$ in F. Of course, if $x = y$ then the two immediate successors of (x, y) are equal, and if $x < y$ then the two immediate successors of (x, y) are unequal. We denote them by $(x, y) \to (x, x + y)$ and $(x, y) \to (y, x + y)$. Of course, we could also denote them by $(x, y, x + y) \to (x, x + y, 2x + y)$ and $(x, y, x + y) \to (y, x + y, x + 2y)$. Call $(x, y) = (x, y, x + y)$ the *immediate predecessor* of $(x, x + y) = (x, x + y, 2x + y)$ and call $(x, y) = (x, y, x + y)$ the immediate predecessor of $(y, x + y) = (y, x + y, x + 2y)$.

Note that if $(\overline{x}, \overline{y})$ is an immediate successor of (x, y) in F then $\overline{x} < \overline{y}$.

Lemma 1 *Suppose F is a Fibonacci set and $(x, y, x + y) \in F$ where $x \le y, x, y \in \mathbb{N}^+$. If $(x, y, x + y)$ has an immediate predecessor $(\theta, \phi, \theta + \phi)$ in F, where $\theta \le \phi$ and $\theta, \phi \in \mathbb{N}^+$, then $(\theta, \phi, \theta + \phi)$ is unique.*

Proof Suppose $(\theta, \phi, \theta + \phi)$ is an immediate predecessor of $(x, y, x + y)$ in F where $\theta \le \phi$ and $\theta, \phi \in \mathbb{N}^+$. Since $(x, y, x + y)$ must be an immediate successor of $(\theta, \phi, \theta + \phi)$ in F, we must have (1): $(\theta, \theta + \phi, 2\theta + \phi) = (x, y, x + y)$ or (2): $(\phi, \theta + \phi, \theta + 2\phi) = (x, y, x + y)$.

Suppose (1). Then $(\theta, \theta+\phi, 2\theta+\phi) = (x, y, x+y)$. Then $\theta = x, \theta+\phi = y, 2\theta+\phi = x+y$. Therefore, $\theta = x, \phi = y-x$ and we require $x \le y-x$.

Next, suppose (2). Then $(\phi, \theta+\phi, \theta+2\phi) = (x, y, x+y)$. Then $\phi = x, \theta+\phi = y, \theta+2\phi = x+y$. Therefore, $\phi = x, \theta = y-x$ and we require $1 \le y-x \le x$.

Of course, if $x = y-x$, then $(\theta, \phi, \theta+\phi) = (x, y-x, y)$ and $(\theta, \phi, \theta+\phi) = (y-x, x, y)$ are the same for both (1) and (2) and this implies that $(\theta, \phi, \theta+\phi)$ is unique.

Also, if $x < y-x$ we have $(\theta, \phi, \theta+\phi) = (x, y-x, y)$ where $\theta < \phi$ and $\theta, \phi \in \mathbb{N}^+$ and if $1 \le y-x < x$, we have $(\theta, \phi, \theta+\phi) = (y-x, x, y)$ where $\theta < \phi$ and $\theta, \phi \in \mathbb{N}^+$. Therefore, $(\theta, \phi, \theta+\phi)$ is uniquely determined from $(x, y, x+y)$ if $(x, y, x+y)$ has an immediate predecessor $(\theta, \phi, \theta+\phi)$ in F. ■

Definition 3 *Suppose A is any set such that for every $t \in A, t$ satisfies the condition $t = (x, y, x+y) = (x, y)$ where $x \le y$ and $x, y \in \mathbb{N}^+$. Then $F_A = g(A)$ is the smallest Fibonacci set such that $A \subseteq F$. We say that $F_A = g(A)$ is generated by A and we generate $F_A = g(A)$ in a standard way by first insuring that $A \subseteq F$ and then insuring that for all t in F_A the two immediate successors of t are also in F_A. Also, if $t = (x, y, x+y) = (x, y)$, where $x \le y$ and $x, y \in \mathbb{N}^+$, we define $F_t = F_{\{t\}} = g(\{t\}) = g(t)$, and we say that F_t is the Fibonacci set generated by the single element t.*

It is fairly easy to convince yourself that

$$F_{\{t_1, t_2, \cdots, t_k\}} = \bigcup_{i=1}^{k} F_{t_i}.$$

We do not give a formal proof of this since it is not used in this paper. Also, see Section 8 for another property.

In Fig. 1 we illustrate $F_{(1,1)} = F_{(1,1,2)}$. Again note that if $(\overline{x}, \overline{y})$ is an immediate successor of (x, y), then we must have $\overline{x} < \overline{y}$. Therefore, from Definition 2 and Lemma 1, we easily see that $F_{(1,1)}$ is a binary tree since each vertex in $F_{(1,1)}$, (except the initial vertex $(1,1,2)$), has two immediate successors and one immediate predecessor in $F_{(1,1)}$. We later explain why $F_{(1,1)}$ is the basic or universal Fibonacci set.

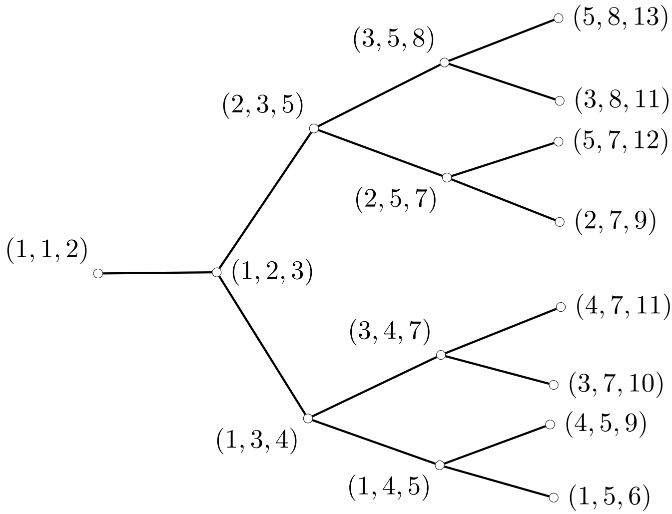


Figure 1: The binary tree $F_{(1,1)} = g\,(1,1)$.

Lemma 2 *Suppose* $a \in \mathbb{N}^+$. *Then*

$$F\,(a,a) = g\,(a,a) = \{(ax, ay) : (x,y) \in g\,(1,1)\}.$$

Also, suppose t *is the greatest common divisor of* a, b *where* $a < b$, *and* $a, b \in \mathbb{N}^+$. *Then* $F_{(a,b)} = g\,(a,b) = \left\{(tx, ty) : (x,y) \in g\left(\frac{a}{t}, \frac{b}{t}\right)\right\}$.

Proof This is obvious. ■

3 Statement of the Two Problems

Main Problem Suppose that a, b are algebraic literal numbers where we agree that $a < b$ and $a, b \in \mathbb{N}^+$. The secondary problem will take care of the case where $a = b$. We wish to compute in a closed form the Fibonacci set $F_{(a,b)} = F_{(a,b,a+b)} = g(a,b)$. We will use the following easy secondary problem to help us solve the main problem.

Secondary Problem Suppose that a is an algebraic literal number in $\mathbb{N}^+$. We wish to compute in a closed form the Fibonacci set $F_{(a,a)} = F_{(a,a,2a)}$.

4 The Solution to the Secondary Problem

If $a, b \in \mathbb{N}^+$, the notation $(a,b) = 1$ means that a and b are relatively prime.

Solution of the Secondary Problem If $a \in \mathbb{N}^+$ is arbitrary but fixed, then $F_{(a,a)} = g(a,a) = \{(\theta a, \phi a) : \theta, \phi \in \mathbb{N}^+, \theta \le \phi, (\theta, \phi) = 1\}$.

Note 1 Of course, this solution implies that

$$F_{(1,1)} = F_{(1,1,2)} = \left\{(\theta, \phi) : \theta, \phi \in \mathbb{N}^+, \theta \le \phi, (\theta, \phi) = 1\right\}.$$

Proof of the Solution Of course, $(a,a) \in F_{(a,a)}, (a,a) = (1 \cdot a, 1 \cdot a)$ and $(1,1) = 1$. Now $F_{(a,a)}$ is the Fibonacci set generated by (a,a) and we observe that if $(\theta a, \phi a) \in F_{(a,a)}$, where $\theta \le \phi$ and $\theta, \phi \in \mathbb{N}^+, (\theta, \phi) = 1$, then the two immediate successors of $(\theta a, \phi a)$ are $(\theta a, (\theta + \phi) a)$ and $(\phi a, (\theta + \phi) a)$. We see that $\theta < \theta + \phi$ and $\theta, \theta + \phi \in \mathbb{N}^+$ and $(\theta, \theta + \phi) = 1$ since $(\theta, \phi) = 1$. Also, $\phi < \theta + \phi$ and $\phi, \theta + \phi \in \mathbb{N}^+$ and $(\phi, \theta + \phi) = 1$ since $(\theta, \phi) = 1$.

From this it follows that each $(x,y) \in F_{(a,a)}$ must be of the form $(x,y) = (\theta a, \phi a)$ where $\theta \le \phi$ and $\theta, \phi \in \mathbb{N}^+$ and $(\theta, \phi) = 1$.

We now reverse directions and show that any arbitrary (x,y) that satisfies $(x,y) = (\theta a, \phi a)$, $\theta \le \phi$, $\theta, \phi \in \mathbb{N}^+$, $(\theta, \phi) = 1$ must be in $F_{(a,a)} = g(a,a)$. We do this by mathematical induction on $\theta + \phi = n$.

Now if $n = 2$, then $\theta = \phi = 1$ and $(\theta a, \phi a) = (1 \cdot a, 1 \cdot a) = (a,a) \in F_{(a,a)}$. So we have started the induction on n, and we now suppose that the conclusion is true for each $\theta + \phi = \overline{n}$ where $\overline{n} \in \{1, 2, 3, \ldots, n-1\}$ and $n \ge 3$. We now show that the conclusion is true for any (θ, ϕ) when $\theta + \phi = n$, $\theta \le \phi$ and $\theta, \phi \in \mathbb{N}^+$, $(\theta, \phi) = 1$.

We consider three cases.

Case 1. $\theta = \phi - \theta$.

Case 2. $\theta < \phi - \theta$.

Case 3. $\phi - \theta < \theta$.

We first observe that the conditions $\theta, \phi \in N^+, \theta \le \phi, (\theta, \phi) = 1$ and $\theta + \phi \ge 3$ together imply that $\theta < \phi$. Thus $1 \le \phi - \theta$.

Case 1. Now $\theta = \phi - \theta$ implies $2\theta = \phi$ which implies $\theta = 1, \phi = 2$ since $(\theta, \phi) = 1$. Therefore, $(\theta a, \phi a) = (a, 2a)$. Also $(a, a) \in F_{(a,a)}$ implies $(a, 2a) \in F_{(a,a)}$.

Case 2. By induction $(\theta a, (\phi - \theta)\, a) \in F_{(a,a)}$ since $\theta, \phi - \theta \in \mathbb{N}^+, \theta < \phi - \theta, (\theta, \phi - \theta) = 1$ and $\theta + (\phi - \theta) < \theta + \phi = n$.

Also, $(\theta a, (\phi - \theta)\, a,) = (\theta a, (\phi - \theta)\, a, \phi a) \in F_{(a,a)}$ implies $(\theta a, \phi a) \in F_{(a,a)}$.

Case 3. Now by induction $((\phi - \theta)a, \theta a) \in F_{(a,a)}$ since (1) $\theta < \phi$ implies $\phi - \theta, \theta \in N^+$, (2) $\phi - \theta < \theta$, (3) $(\phi - \theta, \theta) = 1$, and (4) $(\phi - \theta) + \theta < \theta + \phi = n$. Also, $((\phi - \theta)a, \theta a) = ((\phi - \theta)a, \theta a, \phi a) \in F_{(a,a)}$ implies $(\theta a, \phi a) \in F_{(a,a)}$. ■

Observation 1 From Lemma 2, we know that if t is the greatest common divisor of a, b where $a \le b$ and $a, b \in \mathbb{N}^+$, then $F_{(a,b)} = g\,(a, b) = \left\{(tx, ty) : (x, y) \in g\left(\frac{a}{t}, \frac{b}{t}\right)\right\}$.

Also, $\left(\frac{a}{t}, \frac{b}{t}\right) = 1$ and from Note 1 this implies that $\left(\frac{a}{t}, \frac{b}{t}\right) \in F_{(1,1)}$. Thus, $\left(\frac{a}{t}, \frac{b}{t}\right)$ is a member of the binary tree $F_{(1,1)}$ which is shown in Fig. 1. Therefore, $F_{\left(\frac{a}{t}, \frac{b}{t}\right)}$ consists of all of those vertices on the binary tree $F_{(1,1)}$ that are generated by the single vertex $\left(\frac{a}{t}, \frac{b}{t}\right)$. Thus, the binary tree $F_{(1,1)}$ contains embedded in itself sufficient information to compute all $F_{(a,b)}$ where $a, b \in \mathbb{N}^+, a \le b$. This is why we call $F_{(1,1)}$ the basic or universal Fibonacci set.

Before we solve the Main Problem, we will first develop the very basic matrix machinery that we will need.

5 Basic Matrix Machinery

Lemma 3 *Suppose* $\theta, \phi, \psi, \pi \in \mathbb{N}$ *and* $\det \begin{bmatrix} \theta & \phi \\ \psi & \pi \end{bmatrix} = \begin{vmatrix} \theta & \phi \\ \psi & \pi \end{vmatrix} = \pm 1.$

Then $(\theta, \psi) = 1, (\phi, \pi) = 1, (\theta, \phi) = 1, (\psi, \pi) = 1, (\theta + \phi, \psi + \pi) = 1$ *and* $(\theta + \psi, \phi + \pi) = 1.$

Proof $(\theta, \psi) = (\phi, \pi) = (\theta, \phi) = (\psi, \pi) = 1$ is obvious. We show that $(\theta + \phi, \psi + \pi) = 1$. The proof of $(\theta + \psi, \phi + \pi) = 1$ is the same. Suppose p is a prime such that $p|m, p|n$ where $\theta + \phi = m, \psi + \pi = n$.

Now $\begin{vmatrix} \theta & \phi \\ \psi & \pi \end{vmatrix} = \begin{vmatrix} \theta & m - \theta \\ \psi & n - \psi \end{vmatrix} = \begin{vmatrix} \theta & m \\ \psi & n \end{vmatrix}.$

Now $\begin{vmatrix} \theta & \phi \\ \psi & \pi \end{vmatrix} = \pm 1$ is impossible since $p|m, p|n$. ■

Lemma 4 *Suppose* $1 \leq m \leq n$ *are relatively prime positive integers. Then there exists a unique* 2×2 *matrix* $\begin{bmatrix} \theta & \phi \\ \psi & \pi \end{bmatrix}$ *that satisfies the following conditions.*

1. θ, ϕ, ψ, π *are non-negative integers.*
2. $\begin{vmatrix} \theta & \phi \\ \psi & \pi \end{vmatrix} = 1.$
3. $\theta + \phi = m, \psi + \pi = n.$

Proof First suppose that $1 = m \leq n$.

Now $\begin{vmatrix} \theta & \phi \\ \psi & \pi \end{vmatrix} = \theta\pi - \psi\phi = 1$ implies $\theta \neq 0, \pi \neq 0$. Therefore, $\theta + \phi = m = 1$ implies $\theta = 1, \phi = 0$. Therefore, $\theta = 1, \phi = 0, \theta\pi - \psi\phi = 1$ implies $\pi = 1$. Therefore, $\psi + \pi = n, \pi = 1$ implies $\psi = n - 1$. Therefore, $\begin{bmatrix} \theta & \phi \\ \psi & \pi \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ n - 1 & 1 \end{bmatrix}.$

Secondly, suppose that $2 \le m \le n$. From $\theta+\phi = m, \psi+\pi = n, \begin{vmatrix} \theta & \phi \\ \psi & \pi \end{vmatrix} = 1$ we have the following:

$$\phi = m - \theta, \pi = n - \psi$$

and $\begin{vmatrix} \theta & \phi \\ \psi & \pi \end{vmatrix} = \begin{vmatrix} \theta & m-\theta \\ \psi & n-\psi \end{vmatrix} = \begin{vmatrix} \theta & m \\ \psi & n \end{vmatrix} = n\theta - m\psi = 1.$

Now obviously, $\theta \neq 0$. Also, $2 \le n$ implies $\psi \neq 0$ since $\psi = 0$ would imply $n|1$. Suppose $\phi = 0$. Then $\theta + \phi = m$ implies $\theta = m$ and $n\theta - m\psi = nm - m\psi = 1$ is impossible since $m \ge 2$.

Therefore, $\phi \neq 0$.

Suppose $\pi = 0$. Then $\psi+\pi = n$ implies $\psi = n$ and $n\theta - m\psi = n\theta - mn = 1$ is impossible since $2 \le n$.

Therefore, $\pi \neq 0$. Therefore, $\theta + \phi = m, \psi + \pi = n, \theta \neq 0, \phi \neq 0, \psi \neq 0, \pi \neq 0$ imply $1 \le \theta \le m-1, 1 \le \phi \le m-1, 1 \le \psi \le n-1$ and $1 \le \pi \le n-1$.

Since $2 \le n, 2 \le m$ and n, m are relatively prime we know from number theory (the Euclidean algorithm) that there exists a unique (θ, ψ) with $1 \le \theta \le m-1, 1 \le \psi \le n-1$ that satisfies $n\theta - m\psi = 1$. From this unique (θ, ψ) and from $\theta + \phi = m, \psi + \pi = n$ we see that (ϕ, π) is also unique. Therefore $\begin{bmatrix} \theta & \phi \\ \psi & \pi \end{bmatrix}$ is unique. ■

Corollary 1 *Suppose* $1 \le m \le n$ *are relatively prime positive integers. Then, there exists a unique* 2×2 *matrix* $\begin{bmatrix} \overline{\theta} & \overline{\phi} \\ \overline{\psi} & \overline{\pi} \end{bmatrix}$ *that satisfies the following conditions.*

1. $\overline{\theta}, \overline{\phi}, \overline{\psi}, \overline{\pi} \in \mathbb{N}$.

2. $\begin{vmatrix} \overline{\theta} & \overline{\phi} \\ \overline{\psi} & \overline{\pi} \end{vmatrix} = -1$.

3. $\overline{\theta} + \overline{\phi} = m, \overline{\psi} + \overline{\pi} = n$.

Also, the unique matrix $\begin{bmatrix} \overline{\theta} & \overline{\phi} \\ \overline{\psi} & \overline{\pi} \end{bmatrix}$ *of Corollary 1 and the unique matrix* $\begin{bmatrix} \theta & \phi \\ \psi & \pi \end{bmatrix}$ *of Lemma 4 are related by* $\begin{bmatrix} \overline{\theta} & \overline{\phi} \\ \overline{\psi} & \overline{\pi} \end{bmatrix} = \begin{bmatrix} \phi & \theta \\ \pi & \psi \end{bmatrix}$.

Proof We use Lemma 4 with the matrix $\begin{bmatrix} \phi & \theta \\ \pi & \psi \end{bmatrix}$. ■

Corollary 2 *The conclusion of Lemma 4 remains true if we drop* $1 \leq m \leq n$ *and simply assume that* m, n *are any relatively prime positive integers.*

Proof If $1 \leq n < m$ we use Corollary 1 with the matrix $\begin{bmatrix} \psi & \pi \\ \theta & \phi \end{bmatrix}$. ■

Corollary 3 *The conclusion of Corollary 1 remains true if we drop* $1 \leq m \leq n$ *and simply assume that* m, n *are any relatively prime positive integers.*

6 Solving the Main Problem

In the main problem we consider a, b to be algebraic literal numbers and we assume that $a < b$ and $a, b \in \mathbb{N}^+$. We can assume that $a < b$ since the case where $a = b, a \in \mathbb{N}^+$ was solved in the secondary problem.

Starting with $(a, b) = (a, b, a + b)$ in Fig. 2 we show a few of the branches in the binary tree that represents the Fibonacci set $F(a, b) = g(a, b)$.

Of course, $F_{(a,b)}$ must be a binary tree since it is a binary tree for specific values of a, b.

As always, each vertex on the binary tree $F_{(a,b)}$ has exactly two immediate successors on the tree and each vertex except (a, b) has exactly one immediate predecessor on the tree. As always, from this it follows that all of the vertices shown on the binary tree $F_{(a,b)}$ must be distinct. Also, the successive levels of the tree have $1, 2, 4, 8, 16, \ldots$ vertices respectively.

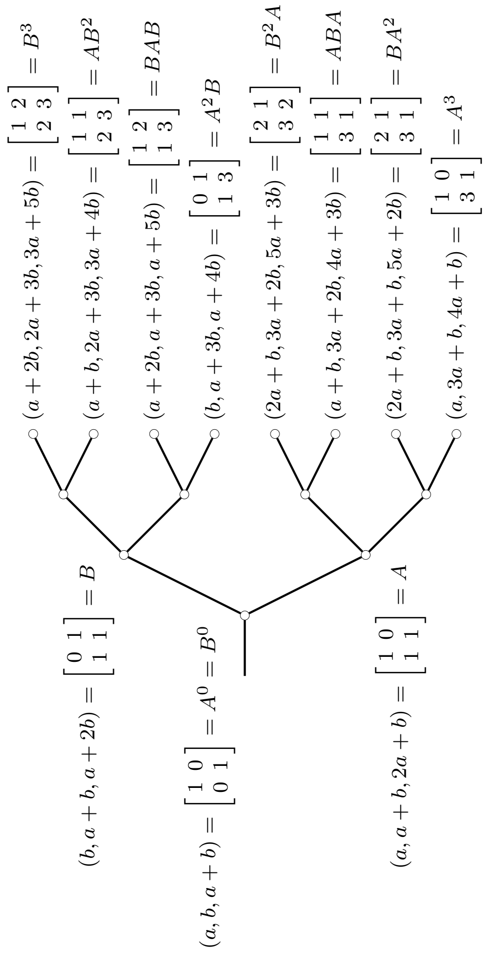
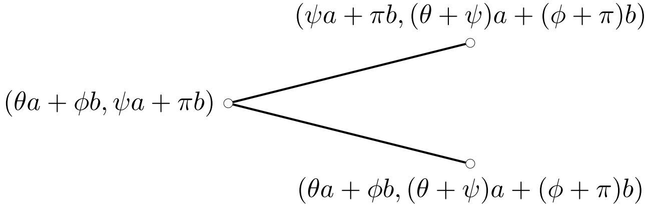


Figure 2: The binary tree $F_{(a,b)} = g(a,b)$.

The following statement $(*)$ is easy to prove by mathematical induction.

$(*)$ If $(\theta a + \phi b, \psi a + \pi b) = (\theta a + \phi b, \psi a + \pi b, (\theta + \psi)a + (\phi + \pi)b)$ is any vertex on the binary tree $F_{(a,b)}$ except (a,b), then $\theta, \psi, \phi, \pi \in$

$\mathbb{N}, \theta+\phi \in \mathbb{N}^+, \psi+\pi \in \mathbb{N}^+, \theta \le \psi, \phi \le \pi$ and at least one of $\theta < \psi, \phi < \pi$. Therefore, $\theta + \phi < \psi + \pi$. $(*)$ follows by induction because each vertex $(\theta a + \phi b, \psi a + \pi b)$ of the binary tree $F_{(a,b)}$ has two immediate successors namely



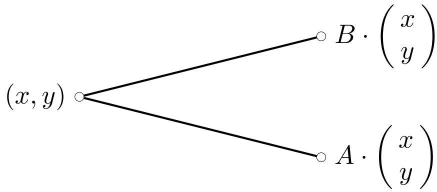
Suppose $(\theta a + \phi b, \psi a + \pi b)$ satisfies the above conditions $(*)$. Also, suppose we wish to decide whether

$$(\theta a + \phi b, \psi a + \pi b) = (\theta a + \phi b, \psi a + \pi b, (\theta + \psi)\, a + (\phi + \pi)\, b)$$

lies on the binary tree $F_{(a,b)}$. To do this, let us first define $ka+hb < \overline{k}a+\overline{h}b$ if $k, h, \overline{k}, \overline{h} \in \mathbb{N}, k+h \in \mathbb{N}^+, \overline{k}+\overline{h} \in \mathbb{N}^+, k \le \overline{k}, h \le \overline{h}$ and at least one of $k < \overline{k}, h < \overline{h}$. We next assume that $(\theta a + \phi b, \psi a + \pi b)$ lies on $F_{(a,b)}$. Then since each vertex of $F_{(a,b)}$ except (a, b) has exactly one immediate predecessor in $F_{(a,b)}$, we work backwards from $(\theta a + \phi b, \psi a + \pi b)$ one step at a time, using the above definition of the relation $<$, until we either reach (a, b) or else reach a point where an immediate predecessor does not exist. By using the above definition of the relation $<$, these immediate predecessors can be computed exactly as we did in the proof of Lemma 1. We now derive a lemma that will tell us directly whether $\left(\overline{a}, \overline{b}\right) = (\theta a + \phi b, \psi a + \pi b)$ lies on $F_{(a,b)}$ or not.

Notation 2 *Let* $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$. *Then* $A^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$ *and* $B^{-1} = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$.

Lemma 5 *Suppose* $(x, y) = (x, y, x + y)$, *where* $x, y \in \mathbb{N}^+$, $x \le y$, *is an element in a Fibonacci set* F. *Then the two immediate successors of* $(x, y) = \begin{pmatrix} x \\ y \end{pmatrix}$ *in* F *are the following.*



Proof This is obvious. ■

Observations 2 It follows from Lemma 5 that each element $(x, y) = \begin{pmatrix} x \\ y \end{pmatrix}$ of the binary tree $F_{(a,b)} = g(a, b)$ can be written $(x, y) = \begin{pmatrix} x \\ y \end{pmatrix} = T \cdot \begin{pmatrix} a \\ b \end{pmatrix}$ where $T = C_1 \cdot C_2 \cdots C_t$ with each $C_i \in \{A, B\}$ and where we also include $T = A^\circ = B^\circ = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Since $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ and since we are also including $A^\circ = B^\circ = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ we immediately see that $T = \begin{bmatrix} \theta & \phi \\ \psi & \pi \end{bmatrix}$ satisfies the following conditions which are slightly weaker than the conditions $(*)$ mentioned earlier.

1. $\det T = \pm 1$.
2. $\theta, \phi, \psi, \pi \in \mathbb{N}$.
3. $1 \leq \theta + \phi \leq \psi + \pi$.

From Lemma 3, conditions (1), (2) also imply that $(\theta + \phi, \psi + \pi) = 1$.

If we include $T = A^\circ = B^\circ = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, then we will soon show that these three properties (1), (2), (3) exactly determine all of the 2×2 matrices T such that $T = C_1 \cdot C_2 \cdots C_t$ with each $C_i \in \{A, B\}$. This will be the complete solution to the Main Problem.

Also, suppose $T = C_1 \cdot C_2 \cdots C_r, \overline{T} = \overline{C}_1 \cdot \overline{C}_2 \cdots \overline{C}_s$ where each $C_i \in \{A, B\}$ and each $\overline{C}_i \in \{A, B\}$.

Starting at $(a,b) = \begin{pmatrix} a \\ b \end{pmatrix}$ on the binary tree. $F_{(a,b)} = g\,(a,b)$, where $a < b$ and $a, b \in \mathbb{N}^{+}$, we see that $T \cdot \begin{pmatrix} a \\ b \end{pmatrix}$ and $\overline{T} \cdot \begin{pmatrix} a \\ b \end{pmatrix}$ will be the same vertex on the tree $F_{(a,b)}$ if and only if $r = s$ and for every $i \in \{1, 2, \cdots, r = s\}, C_i = \overline{C}_i$.

Therefore, it follows that $T = \overline{T}$ if and only if $r = s$ and for every $i \in \{1, 2, \cdots, r = s\}, C_i = \overline{C}_i$.

Lemma 6 *Suppose* $(x,y) = (x, y, x+y)\,, x, y \in \mathbb{N}^{+}, x < y$, *is an element in a Fibonacci set* F.

If $(x,y) = \begin{pmatrix} x \\ y \end{pmatrix}$ *has an immediate predecessor* $(\overline{x}, \overline{y})$ *in* F, *then either*

$$(A) \qquad (\overline{x}, \overline{y}) = \begin{pmatrix} \overline{x} \\ \overline{y} \end{pmatrix} = A^{-1} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ -x+y \end{pmatrix}$$

or

$$(B) \qquad (\overline{x}, \overline{y}) = \begin{pmatrix} \overline{x} \\ \overline{y} \end{pmatrix} = B^{-1} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x+y \\ x \end{pmatrix}.$$

Proof The proof is obvious. ■

Definition 4 *Suppose* $\begin{bmatrix} \theta & \phi \\ \psi & \pi \end{bmatrix}, \begin{bmatrix} \overline{\theta} & \overline{\phi} \\ \overline{\psi} & \overline{\pi} \end{bmatrix}$ *are* 2×2 *matrices. We say that* $\begin{bmatrix} \theta & \phi \\ \psi & \pi \end{bmatrix} \sim \begin{bmatrix} \overline{\theta} & \overline{\phi} \\ \overline{\psi} & \overline{\pi} \end{bmatrix}$ *if* $\begin{bmatrix} \theta & \phi \\ \psi & \pi \end{bmatrix} = \begin{bmatrix} \overline{\phi} & \overline{\theta} \\ \overline{\pi} & \overline{\psi} \end{bmatrix}$.

Lemma 7 *Suppose* R, S *are* 2×2 *matrices, and* $R \sim S$. *Then* $AR \sim AS$ *and* $BR \sim BS$.

Proof Let $R = \begin{bmatrix} \theta & \phi \\ \psi & \pi \end{bmatrix}$, $S = \begin{bmatrix} \phi & \theta \\ \pi & \psi \end{bmatrix}$.

Then $AR = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \theta & \phi \\ \psi & \pi \end{bmatrix} = \begin{bmatrix} \theta & \phi \\ \theta + \psi & \phi + \pi \end{bmatrix}$.

Also, $AS = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \phi & \theta \\ \pi & \psi \end{bmatrix} = \begin{bmatrix} \phi & \theta \\ \phi + \pi & \theta + \psi \end{bmatrix}$

Therefore, $AR \sim AS$. Likewise $BR \sim BS$. ■

Lemma 8 *Let T be the matrix product $T = C_1 \cdot C_2 \cdots C_t$ where each $C_i \in \{A, B\}$. Then* $\det T = \pm 1$.

Also, $TA \sim TB$.

Proof Since $\det A = 1, \det B = -1$ it follows that $\det T = \pm 1$. Also, since $A \sim B$ it follows from repeated use of Lemma 7 that $TA \sim TB$. ■

Observations 3 In the Fig. 2 Fibonacci tree $F_{(a,b)}$, we observe that $A^3 \sim A^2B, BA^2 \sim BAB, ABA \sim AB^2, B^2A \sim B^3$. From Lemma 8, we note in general that the 2^n elements in level $n+1$ of the Fibonacci tree $F_{(a,b)}$ must occur in symmetric pairs $\begin{bmatrix} \theta & \phi \\ \psi & \pi \end{bmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \theta a + \phi b \\ \psi a + \pi b \end{pmatrix}$ and $\begin{bmatrix} \phi & \theta \\ \pi & \psi \end{bmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \phi a + \theta b \\ \pi a + \psi b \end{pmatrix}$. For example, in the 4th level of the Fig. 2 Fibonacci tree, we note that $(a+b, 2a+3b) = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = AB^2 \begin{pmatrix} a \\ b \end{pmatrix} \in F_{(a,b)}$ and $(a+b, 3a+2b) = \begin{bmatrix} 1 & 1 \\ 3 & 2 \end{bmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = ABA \begin{pmatrix} a \\ b \end{pmatrix} \in F_{(a,b)}$.

This is because $AB \cdot B \sim AB \cdot A$.

Lemma 9 *Suppose $m, n \in \mathbb{N}^+$ are any arbitrary members of $\mathbb{N}^+$ that satisfy $m \leq n$ and $(m,n) = 1$. Then there exists at least one matrix T of the form $T = C_1 \cdot C_2 \cdots C_t = \begin{bmatrix} \theta & \phi \\ \psi & \pi \end{bmatrix}$, where each $C_i \in \{A, B\}$, such that $\theta + \phi = m, \psi + \pi = n$. This includes $T = A^\circ = B^\circ = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Also, $\theta, \phi, \psi, \pi \in \mathbb{N}$ and* $\det T = \pm 1$.

Proof From the solution of the Secondary Problem, we know that

$$g(a,a) = \left\{ \left(\overline{\theta}a, \overline{\phi}a\right) : \overline{\theta}, \overline{\phi} \in \mathbb{N}^+, \overline{\theta} \le \overline{\phi}, \left(\overline{\theta}, \overline{\phi}\right) = 1 \right\}.$$

Let $\overline{\theta} = m, \overline{\phi} = n$. By letting $a = b$ and using the properties of the binary tree $F_{(a,b)}$ it follows that there exists $T = C_1 \cdot C_2 \cdots C_t = \begin{bmatrix} \theta & \phi \\ \psi & \pi \end{bmatrix}$ with each $C_i \in \{A, B\}$ such that

$$\begin{bmatrix} \theta & \phi \\ \psi & \pi \end{bmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{bmatrix} \theta & \phi \\ \psi & \pi \end{bmatrix} \begin{pmatrix} a \\ a \end{pmatrix} = \begin{pmatrix} (\theta + \phi)\, a \\ (\psi + \pi)\, a \end{pmatrix} = \begin{pmatrix} ma \\ na \end{pmatrix}.$$

■

Corollary 4 *Suppose $m, n \in \mathbb{N}^+, m \le n$ and $(m, n) = 1$. Then from Lemma 8 and Observation 3 we know that there exists at least two distinct matrices $T, \overline{T}$ that satisfy the conclusion of Lemma 9.*

Also, by Lemma 8 and Observation 3 we can call $T = C_1 \cdot C_2 \cdots C_t \cdot A$ and call $\overline{T} = C_1 \cdot C_2 \cdots C_t \cdot B$. Since $\det T = -\det \overline{T}$ we also conclude that $\left\{\det T, \det \overline{T}\right\} = \{-1, 1\}$.

Proof The proof is obvious. ■

Lemma 10 *Define $\overline{T} = \{C_1 C_2 \cdots C_t : t \in \mathbb{N},$ each $C_i \in \{A, B\}\}$ where we agree that $C_1 \cdot C_2 \cdots C_t = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ when $t = 0$.*

Also, $\overline{M} = \left\{ \begin{bmatrix} \theta & \phi \\ \psi & \pi \end{bmatrix} : \theta, \phi, \psi, \pi \in \mathbb{N}, \begin{vmatrix} \theta & \phi \\ \psi & \pi \end{vmatrix} = \pm 1, \theta + \phi \le \psi + \pi \right\}.$

Then $\overline{T} = \overline{M}$.

Erratum. Technically, $\overline{T} = \overline{M} \setminus \{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\}$. We patch this up by agreeing that $\overline{M} = \overline{M} \setminus \{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\}$.

Proof First we show that $\overline{T} \subseteq \overline{M}$.

Let $T = \begin{bmatrix} \theta & \phi \\ \psi & \pi \end{bmatrix} = C_1 \cdot C_2 \cdots C_t$ where each $C_i \in \{A, B\}$. We show $T \in \overline{M}$. Now obviously $\theta, \phi, \psi, \pi \in \mathbb{N}$ and $\begin{vmatrix} \theta & \phi \\ \psi & \pi \end{vmatrix} = \pm 1$. Also, since $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ it is obvious by induction that $\theta + \phi \leq \psi + \pi$. Also, when $t = 0$, we define $T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Therefore, $T \in \overline{M}$.

Next we show that $\overline{M} \subseteq \overline{T}$, where we consider $\overline{M} = \overline{M} \setminus \{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\}$.

Therefore, suppose $\begin{bmatrix} \theta & \phi \\ \psi & \pi \end{bmatrix} \in \overline{M}$ is any fixed member of $\overline{M}$.

Now since $\begin{vmatrix} \theta & \phi \\ \psi & \pi \end{vmatrix} = \pm 1$, we know from Lemma 3 that $(\theta + \phi, \psi + \pi) = 1$.

Let us call $\theta + \phi = m, \psi + \pi = n$ where m, n are fixed, $m, n \in \mathbb{N}^+, m \leq n$ and $(m, n) = 1$. Since the case $m = n = 1$ is trivial, we suppose that $1 \leq m < n$.

From Lemma 4 and Corollary 1, we now know the following.

(a). If $\begin{vmatrix} \theta & \phi \\ \psi & \pi \end{vmatrix} = 1$, then there is only one possible matrix $\begin{bmatrix} \overline{\theta} & \overline{\phi} \\ \overline{\psi} & \overline{\pi} \end{bmatrix}$ that satisfies $\overline{\theta}, \overline{\phi}, \overline{\psi}, \overline{\pi} \in \mathbb{N}, \begin{vmatrix} \overline{\theta} & \overline{\phi} \\ \overline{\psi} & \overline{\pi} \end{vmatrix} = 1$ and $\overline{\theta} + \overline{\phi} = m, \overline{\psi} + \overline{\pi} = n$, namely $\begin{bmatrix} \overline{\theta} & \overline{\phi} \\ \overline{\psi} & \overline{\pi} \end{bmatrix} = \begin{bmatrix} \theta & \phi \\ \psi & \pi \end{bmatrix}$.

(b). If $\begin{vmatrix} \theta & \phi \\ \psi & \pi \end{vmatrix} = -1$, then there is only one possible matrix $\begin{bmatrix} \overline{\theta} & \overline{\phi} \\ \overline{\psi} & \overline{\pi} \end{bmatrix}$ that satisfies $\overline{\theta}, \overline{\phi}, \overline{\psi}, \overline{\pi} \in \mathbb{N}, \begin{vmatrix} \overline{\theta} & \overline{\phi} \\ \overline{\psi} & \overline{\pi} \end{vmatrix} = -1$ and $\overline{\theta} + \overline{\phi} = m, \overline{\psi} + \overline{\pi} = n$ namely, $\begin{bmatrix} \overline{\theta} & \overline{\phi} \\ \overline{\psi} & \overline{\pi} \end{bmatrix} = \begin{bmatrix} \theta & \phi \\ \psi & \pi \end{bmatrix}$.

Also, from Lemma 9 and Corollary 4, we know that this unique matrix $\begin{bmatrix} \overline{\theta} & \overline{\phi} \\ \overline{\psi} & \overline{\pi} \end{bmatrix} = \begin{bmatrix} \theta & \phi \\ \psi & \pi \end{bmatrix}$ of cases (a), (b) must lie in $\overline{T}$. This completes the proof. ■

Solution to the Main Problem

We are required to compute $F_{(a,b)} = g(a,b)$ in a closed form.

Calling $(x,y) = \begin{pmatrix} x \\ y \end{pmatrix}$ we know that $F_{(a,b)} = \left\{ T \cdot \begin{pmatrix} a \\ b \end{pmatrix} : T \in \overline{T} \right\}$ where $\overline{T}$ is defined in Lemma 10.

Now $\overline{T} = \overline{M}$, where $\overline{M}$ is also defined in Lemma 10. Therefore

$$F_{(a,b)} = \left\{ M \cdot \begin{pmatrix} a \\ b \end{pmatrix} : M \in \overline{M} \right\}$$

$$= \left\{ \begin{bmatrix} \theta & \phi \\ \psi & \pi \end{bmatrix} \begin{pmatrix} a \\ b \end{pmatrix} : \theta, \phi, \psi, \pi \in \mathbb{N}, \begin{vmatrix} \theta & \phi \\ \psi & \pi \end{vmatrix} = \pm 1, \theta + \phi \leq \psi + \pi \right\}. \blacksquare$$

Note 2 It is easy to compute members $M \in \overline{M}$. Suppose, for example, that $\theta, \phi, \psi, \pi \in \mathbb{N}$, $\begin{vmatrix} \theta & \phi \\ \psi & \pi \end{vmatrix} = \pm 1$ and $\theta + \phi > \psi + \pi$. We just reverse the two rows and we have $\begin{vmatrix} \psi & \pi \\ \theta & \phi \end{vmatrix} = \pm 1$ with $\psi + \pi < \theta + \phi$.

7 Solving Specific Numerical Problems

Suppose $a, b \in \mathbb{N}^+, a < b, \overline{a}, \overline{b} \in \mathbb{N}^+, \overline{a} < \overline{b}$ are specific positive integers and we wish to decide whether $\left(\overline{a}, \overline{b}\right) \in g(a,b)$ when $\left(\overline{a}, \overline{b}\right) \neq (a,b)$.

Define $t = \gcd(a,b), \overline{t} = \gcd\left(\overline{a}, \overline{b}\right)$ where gcd denotes the greatest common divisor. From Lemma 2 (or by induction) it is easy to prove that if $\left(\overline{a}, \overline{b}\right) \in g(a,b)$ then $t = \overline{t}$ is a necessary condition.

Also, if $a < b$ and $\left(\overline{a}, \overline{b}\right) \in g(a,b)$, then the inequalities of Fig. 3 are also easily proved necessary conditions.

$$\begin{array}{ccc} a\,\bullet & \leq & \bullet\,\overline{a} \\ \bigwedge & & \bigwedge \\ b\,\bullet & < & \bullet\,\overline{b} \end{array}$$

Figure 3: Inequalities when $\left(\overline{a},\overline{b}\right) \in g\,(a,b)$ and $a<b$.

From Lemma 2, it is also easy to show that $(\overline{a},\overline{b}) \in g(a,b)$ if and only if $\left(\frac{\overline{a}}{t},\frac{\overline{b}}{t}\right) \in g\left(\frac{a}{t},\frac{b}{t}\right)$ where $t = \gcd\left(\overline{a},\overline{b}\right) = \gcd\,(a,b)$.

We will now develop two algorithms for deciding if $\left(\overline{a},\overline{b}\right) \in g\,(a,b)$ when $\left(\overline{a},\overline{b}\right) \neq (a,b)\,, \gcd\,(a,b) = \gcd\left(\overline{a},\overline{b}\right) = 1$ and the necessary inequalities of Fig. 3 are met. We will suppose $a<b$ since the secondary problem has already taken care of the easy case where $a=b$. Since $\gcd\,(a,b) = \gcd\left(\overline{a},\overline{b}\right) = 1$, we know that (a,b) and $\left(\overline{a},\overline{b}\right)$ are vertices on the basic Fibonacci tree $F_{(1,1)}$. Therefore, it makes sense to talk about immediate predecessors on the tree.

(A) One way to numerically decide if $\left(\overline{a},\overline{b}\right) \in g\,(a,b)$ is to work backwards from $\left(\overline{a},\overline{b}\right)$ by finding consecutive immediate predecessors until we either arrive at (a,b) or else arrive at a contradiction to the necessary inequalities of Fig. 3.

(B) We will now develop a matrix solution that uses the solution to the Main Problem. We know that $\left(\overline{a},\overline{b}\right) \in g\,(a,b)$ is true if and only if there exists a matrix $\begin{bmatrix} \theta & \phi \\ \psi & \pi \end{bmatrix}$ that satisfies the following conditions.

1. $\theta,\phi,\psi,\pi \in \mathbb{N}$.

2. $\begin{vmatrix} \theta & \phi \\ \psi & \pi \end{vmatrix} = \pm 1$.

3. $\theta+\phi \leq \psi+\pi$.

4. $\begin{bmatrix} \theta & \phi \\ \psi & \pi \end{bmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \overline{a} \\ \overline{b} \end{pmatrix}$.

Each matrix $\begin{bmatrix} \theta & \phi \\ \psi & \pi \end{bmatrix}$ that satisfies conditions (1), (2), (3) can be written as $\begin{bmatrix} \theta & \phi \\ \psi & \pi \end{bmatrix} = C_1 \cdot C_2 \cdots C_t$ with each $C_i \in \{A, B\}$.

Also, each distinct $C_1 \cdot C_2 \cdots C_t$ places $(C_1 \cdot C_2 \cdots C_t) \begin{pmatrix} a \\ b \end{pmatrix}$ at a different vertex on the Fibonacci tree $F_{(a,b)}$. Thus, if $\begin{bmatrix} \theta & \phi \\ \psi & \pi \end{bmatrix}$ exists that satisfies conditions (1), (2), (3) and (4), then it is unique.

From (1), (2), (3), (4) we have the following.

$(1')$ $\theta\pi - \psi\phi = \pm 1$.

$(2')$ $\theta a + \phi b = \overline{a}$.

$(3')$ $\psi a + \pi b = \overline{b}$.

Therefore, $(2'')$ $\theta = \frac{\overline{a} - \phi b}{a}$.

$(3'')$ $\psi = \frac{\overline{b} - \pi b}{a}$.

Therefore, $(1'')$ $\left[\frac{\overline{a} - \phi b}{a}\right] \pi - \left[\frac{\overline{b} - \pi b}{a}\right] \phi = \pm 1$.

Therefore, $\overline{a}\pi - b\pi\phi - \overline{b}\phi + b\pi\phi = \pm a$.

Thus $(**)$ $\overline{a}\pi - \overline{b}\phi = \pm a$. From $(3')$, we see that $0 \leq \pi < \overline{b}$ since $b \geq 2$. Also, from $(2')$ we see that $0 \leq \phi < \overline{a}$ since $b \geq 2$. From $(**)$ we have $(***)\, \pi = \frac{\pm a + \overline{b}\phi}{\overline{a}}$ subject to $0 \leq \phi < \overline{a}, 0 \leq \pi < \overline{b}$.

Since $\left(\overline{a}, \overline{b}\right) = 1$, it is easy to see that if solutions (ϕ, π) exist for $(***)$ then they must be unique for each $\pm a$.

Therefore, the matrix solution requires us to first find these unique solutions (ϕ, π) to $(***)$ subject to the side conditions $0 \leq \pi < \overline{b}, 0 \leq \phi < \overline{a}$ if such solutions exist.

If a solution (ϕ, π) to $(***)$ exists, for either $\pm a$, then (θ, ψ) can be uniquely computed from (ϕ, π). We then check to see if the matrix

$\begin{bmatrix} \theta & \phi \\ \psi & \pi \end{bmatrix}$ satisfies the conditions $\theta, \psi \in \mathbb{N}$ and $\theta + \phi \leq \psi + \pi$. The other conditions in (1), (2), (3), (4) are automatically satisfied.

8 Some Concluding Remarks

It is possible to prove more properties of Fibonacci sets than we have proved in this paper.

As an example, suppose $a, b, \overline{a}, \overline{b} \in \mathbb{N}^+$ are $a < b, \overline{a} < \overline{b}$. We say that (a, b) and $\left(\overline{a}, \overline{b}\right)$ are independent if $(a, b) \notin g\left(\overline{a}, \overline{b}\right)$ and $\left(\overline{a}, \overline{b}\right) \notin g(a, b)$. If (a, b) and $\left(\overline{a}, \overline{b}\right)$ are independent, then we can show that $g(a, b) \cap g\left(\overline{a}, \overline{b}\right) = \phi$, the empty set.

As a further extension, the reader might like to use the isomorphism $f : (\mathbb{Z}, 0, +) \to (\mathbb{Z}, 1, *)$, where $\mathbb{Z}$ is the set of all integers, $f(x) = x + 1, a * b = a + b - 1$, and then substitute this operator $*$ for $+$ and study para-Fibonacci sets $\overline{F}$ that satisfy both (1) for all $t \in \overline{F}$, $t = \{x, y, x * y\}$ where $x, y, x * y \in \mathbb{N}^+$ and (2) if $t = \{a, b, a * b\} \in \overline{F}$, then $\{a, a * b, a * (a * b)\} \in \overline{F}$ and $\{b, a * b, b * (a * b)\} \in \overline{F}$.

References

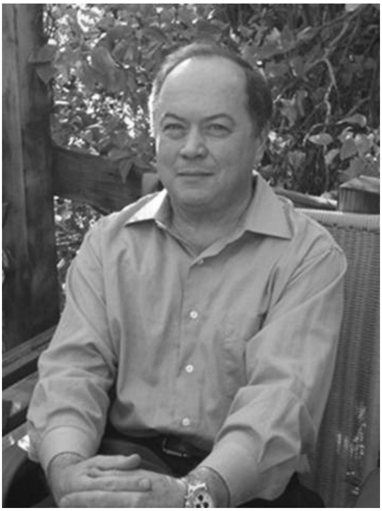
[1] Engel, Arthur, *Problem-Solving Strategies*, Springer, 1998.

Arthur Holshouser
3600 Bullard St.
Charlotte, NC 28208, USA

Harold Reiter
Department of Mathematics
University of North Carolina Charlotte
Charlotte, NC 28223, USA
hbreiter@uncc.edu

Building a Bridge II: from Problems of Mathematical Olympiads to Open Problems of Mathematics

Alexander Soifer



Born and educated in Moscow, Alexander Soifer has for 29 years been a Professor at the University of Colorado, teaching math, and art and film history. He has written six books and some 200 articles, including The Mathematical Coloring Book, *Springer, September 2008. In fact, Springer has contracted Soifer for seven books, five of which are coming out shortly and two are to be written in 2009–10. He founded and for 25 years ran the Colorado Mathematical Olympiad. Soifer has also served on the Soviet Union Math Olympiad (1970–1973) and USA Math Olympiad (1996–2005). He has been Secretary of WFNMC (1996–present), and a recipient of the Paul Erdös Award (2006). Soifer was the founding editor of the research quarterly* Geombinatorics *(1991–present), whose other editors include Ronald L. Graham, Branko Grünbaum, Heiko Harborth, Peter D. Johnson Jr., and Janos Pach. Paul Erdös was an editor and active author of this journal. Soifer's Erdös number is 1.*

1 Introduction

This is part II of the triptych of "Building a Bridge" papers. In part I, I showed an example of how a mathematical researcher can keep an eye open for ideas that could be used in Mathematical Olympiads.

In part I, I made the following observation:

> The Bridge we are building can be walked in the opposite direction as well: it is worthwhile for professionals to take a deeper look at problems of Mathematical Olympiads. Those problems just might inspire exciting generalizations and new directions for mathematical research.

I will illustrate this idea here in the context of a problem I created in August 1987.

2 An Olympiad Problem

In August 1987, I taught a problem-solving course for gifted high school students from several countries: Japan, Israel, USA, Canada, Switzerland, France and others. For their exam I wanted to create an easy, but original pigeonhole principle problem of geometric flavor. I looked at the familiar to everyone picture (Figure 1), and came up with the following problem.

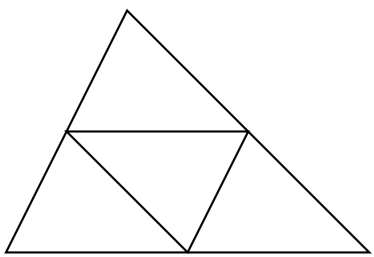


Figure 1

Problem 1 Given 9 points in a triangle (i.e. in the interior or on the boundary) of area 1, prove that three of them form a triangle of area at most $1/4$.

Proof Midlines partition the given triangle into four congruent triangles of area $1/4$ (Figure 1). These congruent triangles are our pigeonholes. The given points are our pigeons. Now nine pigeons are sitting in four pigeonholes. Since $9 = 2 \cdot 4 + 1$, there is at least one pigeonhole containing at least three pigeons. ■

If you feel that nine points are excessive to guarantee the result in problem 1, you are quite right. But, in order to prove the following stronger statement, we need to allow the pigeonholes to differ in size and shape.

Problem 2 Given 7 points in a triangle of area 1, prove that three of them form a triangle of area at most 1/4.

Proof Since $7 = 2 \cdot 3 + 1$, it would be nice to have three pigeonholes: then at least one of them would contain at least three pigeons, and we are done! Okay, let us draw only two midlines in the given triangle (Figure 2).

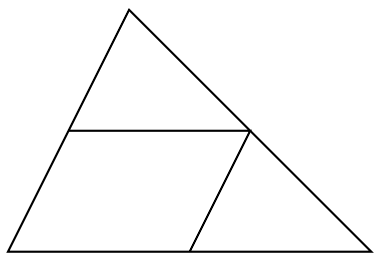


Figure 2

If the parallelogram contains three given points, then all we have left to prove is a simple tool (which I am leaving to the reader): The maximum area of a triangle inscribed in a parallelogram of area 1/2 is equal to 1/4. ■

All right, and how far can we push down the number of points? Not very far, as for 4 points the result is not true—prove it on your own or read in [3]. Thus the best result, if true, would require 5 points.

The Best Result 3 Given 5 points in a triangle of area 1, prove that three of them form a triangle of area at most 1/4.

Proof You will find my proof in [3]. I will quote from [3] the proof that is essentially due to Royce Peng, who at the time was in high school, and took my summer course in a California international program for gifted students—I just simplified and shortened it. Midlines (again!) partition the given triangle into four pigeonholes (Figure 1). At least one of the pigeonholes must contain at least two of the five given points. If the midlines triangle MNK contains two given points v_1 and v_2 (Figure 4), then we are done. Indeed, one of the corner triangles, say MBN, must contain at least one of the three remaining given points v_3, and we can surround the three points v_1, v_2, and v_3 by a parallelogram $MBNK$ of area 1/2. This guarantees that the area of the triangle $v_1v_2v_3$ is at most 1/4.

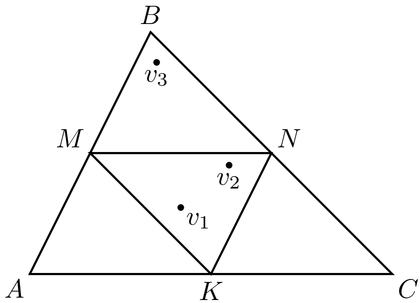


Figure 4

Assume now that one of the corner triangles, say AMK, contains at least two given points v_1 and v_2 (Figure 5).

The locus of all vertices K of triangles v_1Kv_2 of area 1/4 is a pair of lines L_1 and L_2 parallel to $\overline{v_1v_2}$. But where can L_1 and L_2 lie? Can they intersect MN, for example?

Since each of the triangles v_1Mv_2, v_1Nv_2, v_1Kv_2, v_1Av_2, is contained in the parallelogram $AMNK$ of area 1/2. Each of these triangles has area at most 1/4. Therefore, the points A, M, N, and K must all lie between or on the lines L_1 and L_2. What part of the triangle ABC can possibly lie outside of the strip bounded by the lines L_1 and L_2? *Only* a piece P of the triangle MBN *or* the triangle NCK! (Can you prove that pieces of *both* triangles MBN and NCK may not be outside of the strip?)

Now we are done:

Either one more of the given points v_3 lies between the lines L_1 and L_2 (or on L_1 or L_2), and then the area of the triangle $v_1v_2v_3$ is 1/4 or less (can you tell why?), or else all three remaining given points v_3, v_4, and

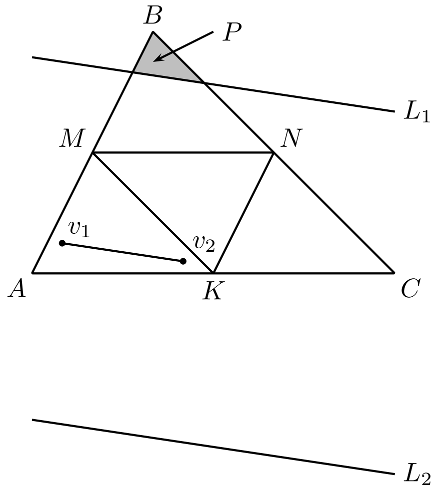


Figure 5

v_5 lie in the piece P of the triangle MBN (or triangle NCK), and then the area of triangle $v_3v_4v_5$ is surely less than or equal to the area of triangle MBN, which is exactly 1/4! ■

(Problems 1, 2, 3 and the assertion that four points do not suffice for the required conclusion were offered at the V Colorado Mathematical Olympiad on April 22, 1988.)

3 Crossing the Bridge into Research

A perfect result is achieved; a series of problems is solved. What can we do now? Well, a good solution to a good problem generates many new questions. We found an interesting property of a triangle: in any set of five points in a triangle of area S there are three points which form a triangle of area at most $S/4$. Moreover, five is the minimal number of points in a triangle guaranteeing this result. But why did we study a triangle alone? I came up with a generalization.

Notation For a geometric figure F, symbol $|F|$ will denote the *area* of F. For example, $|ABC|$ would denote the area of the triangle ABC; $|ABCD|$ would stand for the area of the quadrilateral $ABCD$.

Definition 4 Given a figure F, let $\mathrm{S}(F)$ denote the minimal positive integer n, such that among any n points located inside or on the boundary of F there are always three points which form a triangle of area at most $|F|/4$.

In this terminology, the result we obtained early can be stated as follows:

$$\text{For any triangle } \triangle,\ \mathrm{S}(\triangle) = 5.$$

Problem 5 For any convex figure F, $\mathrm{S}(F) \leq 6$.

Proof: Assume that points $v_1, v_2, \ldots, v_6$ lie in a figure F (including its boundary, of course) of area S. There is a straight line through v_1 dividing F into two pieces F_1 and F_2 of equal area $|F|/2$. By the pigeonhole principle, one of these pieces contains at least three of the five points $v_2, v_3, \ldots, v_6$. Without loss of generality we can assume that the piece F_1 contains the points v_2, v_3, and v_4. (Please note, both F_1 and F_2 contain the point v_1 on their boundary.)

Now we can throw away the piece F_2 and apply the above reasoning to the piece F_1: there is a straight line through v_1 dividing F_1 into two pieces F_{11} and F_{12} of equal area $|F|/4$. By the pigeonhole principle, one of these pieces contains at least two of the three points v_2, v_3, and v_4. Without loss of generality we can assume that the piece F_{11} contains the points v_2 and v_3 (see Figure 6). We are done: the area of the triangle $v_1v_2v_3$ does not exceed the area of the piece F_{11} which is equal to $|F|/4$. ■

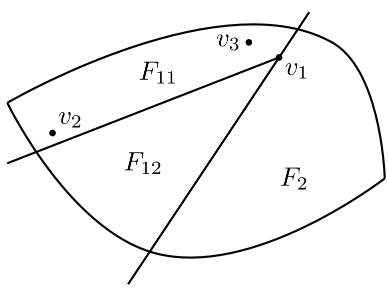


Figure 6

Can we improve the inequality of problem 6? Not really; there exists a figure F, such that $\mathrm{S}(F) = 6$. Can you find one such figure?

Problem 6 Prove that for a regular pentagon F, $\mathrm{S}(F) = 6$.

Please note that in problem 5 we proved more than we stated:

Problem 7 If six points $v_1, v_2, \ldots, v_6$ are given in a convex figure F, then every given point v_i is a vertex of a triangle $v_i v_j v_k$ of area at most $|F|/4$ formed by v_i and two other given points v_j and v_k.

Problem 8 If five points $v_1, v_2, \ldots, v_5$ are given in a triangle of area 1, then is it true that *every* given point v_i is a vertex of a triangle $v_i v_j v_k$ of area at most $1/4$ formed by v_i and two more given points v_j and v_k?

The answer is no. Can you prove it, i.e., construct a counter-example?

You will enjoy reading the proof of the following result in [3]:

Problem 9 For any convex figure F, $\mathrm{S}(F) > 4$.

Problems 5 and 9 show that over all convex figures the function $\mathrm{S}(F)$ takes on exactly 2 values, 5 and 6! It prompted me to pose the $50 Problem.

The Fifty-Dollar Problem Find (classify) all convex figures F, such that $\mathrm{S}(F) = 6$.

You will find much more on this train of thought in the new Springer edition [3] of my book *How Does One Cut a Triangle?* that is coming out in the fall 2008. In fact, much exciting material will appear in my books [2]–[8]. Books [5] and [6] contain 10 and 20 essays, each of which builds a bridge from Olympiads to open problems. *Mathematical Coloring Book* [7], that has just been offered for pre-ordering at Springer's website, will introduce much of new mathematics in a unique historical setting. As to the $50 Problem, it is still open, although there has been relevant development presented in [3]!

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Alexander Soifer
University of Colorado at Colorado Springs
P. O. Box 7150, Colorado Springs, CO 80933,
USA.
email: asoifer@uccs.edu
http://www.uccs.edu/˜asoifer/

About a Chain of Geometric Inequalities and its Crucial Role for Activating Gifted Students to Work with Mathematics

Svetoslav Jordanov Bilchev

Svetoslav Bilchev is Professor and Head of Department of Algebra and Geometry at the University of Rousse, Bulgaria. He studied applications of differential geometry to the system of partial differential equations for his Ph.D., and for many years has been interested in oscillations of differential equations with delay, game theory, mathematical models in economics, transformations theory, inequalities, geometry and mathematics education. He has been awarded many national and international prizes for his strong and enthusiastic work as a mathematician and as a lecturer.

1 Introduction

My native city of Rousse, a small but a very nice European city indeed, situated on the river Danube, celebrated its 20 centuries jubilee in 2002. In Bulgaria the city of Rousse is known as "City of Mathematics" and as a "Special Place of Intellect in Bulgaria." But why is Rousse known as the "City of Mathematics?"

Gifted math students from Rousse have performed exceptionally at International Mathematical Olympiads (IMO)—the most representative competition in the World.

During the first thirty IMOs:

- Four times Rousse has had three members on the Bulgarian team at the IMO,

- Nine times Rousse has had two members on the Bulgarian Team at the IMO,
- Twelve times Rousse has had one member on the Bulgarian Team at the IMO,
- One student took part in three consecutive IMOs,
- 13 other students took part in two consecutive IMOs.

Of course, we had many more such successes during the following 18 IMOs up to now. Nevertheless, the feeling at the start was bittersweet.

The performance of students from Rousse at the IMO is very notable as well. Almost all students win awards. It is only fair to say that the author of this paper took part in two consecutive IMOs and his love for mathematics and math competitions—professionally and as a hobby—has endured for over 40 years now. Working with talented math students has proven to be his destined profession, in other words his "participation" in mathematics competitions began in the early 1970s and continues to this day.

It is necessary to state the continuous excellent performance in the field of mathematics by a relatively small city in a very little country. Great love of the subject matter and a great deal of enthusiasm are required from both teachers and students. But excellent results take more than love and enthusiasm. Detailed knowledge and profound experience come into the equation as well. We gained knowledge and experience through hard work and focus on the most important things.

Moreover, students and teachers get to know each other well at the same time.

2 Rousse Mathematical Circles

In the early 1970s "Rousse Mathematical Circles" were established to combine the efforts on learning mathematics of many leading teachers, students and university students, together, as well university professors and enthusiastic parents. The extent of their joint efforts was centered around the following big chain of mathematical activities:

$\Rightarrow$ previous background $\Leftrightarrow$ school everyday experience $\Leftrightarrow$ work at home with journals and literature $\Leftrightarrow$ additional training at school, extra classes $\Leftrightarrow$ mathematical evenings, excursions, etc. $\Leftrightarrow$ extracurricular classes at "Rousse Mathematical Circles" on special subjects of elementary and higher mathematics $\Leftrightarrow$ hard work in specialized groups of mathematics in "Rousse Mathematical Circles" $\Leftrightarrow$ problem solving $\Leftrightarrow$ problem posing $\Leftrightarrow$ lectures by visiting renowned mathematicians $\Leftrightarrow$ control paper on every specifically taken subject $\Leftrightarrow$ general control papers $\Leftrightarrow$ summer and winter schools $\Leftrightarrow$ taking part in a wide range of mathematics competitions $\Leftrightarrow$ taking part in Junior and Senior Balkan Mathematical Olympiads $\Leftrightarrow$ taking part in International Mathematics Competitions $\Leftrightarrow$ taking part in Kangaroo Competitions and Tournament of Towns $\Leftrightarrow$ taking part in Junior and Senior International Mathematical Olympiads.

The main aspects of the above-mentioned mathematical activities realized in Rousse are:

- continuously education
- quickly increasing the difficulty of the solved problems and learned ideas and methods
- working in specialized groups at "Rousse Mathematical Circles" where students in different forms and different schools in Rousse are gathered—for example: from 9, 10 and 11 forms. The realized system of joint work in Rousse on Olympiad problems of mathematics sped up the preparation of each student with regard to using many different methods for solving hard Olympiad problems, despite their various backgrounds and the various current levels of school. The students in every specialized group work together and increase their level of education together helping each other with ideas, methods, literature etc.
- joint efforts of leading teachers, students, university students, university professors and enthusiastic parents.

The above-mentioned number of activities brought to fame the name and authority of the "Rousse Mathematical Circles" not only within the framework of Bulgaria but all over the world (see [3]).

3 Example

Below we will give an example of a chain of geometric inequalities to demonstrate the students' strengths in some specialized subject matter in the field of mathematics. Its crucial role in encouraging gifted students to work with mathematics will be clarified. The area of *Geometric Inequalities* (GI) is a very special part of elementary mathematics in teaching gifted students and for creating many, often times difficult, problems at many mathematical competitions and olympiads.

We shall use a special part of the GI area in supporting the idea behind the paper at present. This special part of GI is the part with triangle inequalities with (R, r, s) (see [5], pp. 49–63, 278–339).

An arbitrary triangle ABC is given with semiperimeter s and radii R and r of the circumcircle and incircle respectively. Then the following chain of inequalities is "almost" well known (see [5], p. 166 and [2] pp. 35–36):

$$
\begin{aligned}
&\frac{3r\left(4R+r\right))}{\left(7R-5r\right)^2}\\
&\leq \frac{r}{7\left(4R+r\right)^2}\left(16R+49r+2\sqrt{316\,R^2+518\,Rr+616\,r^2}\right)\\
&\leq \frac{r}{2R-r} \leq \frac{3r}{4R+r} \leq \frac{9r}{10R+7r} \leq \frac{3r\left(5R-r\right)}{\left(4R+r\right)^2} \leq \frac{r\left(4R+r\right)}{\left(2R-r\right)\left(2R+5r\right)}\\
&\leq \frac{r\left(16R+3r\right)}{\left(4R-r\right)\left(4R+7r\right)} \leq \frac{r}{R+r} \leq \frac{r\left(8R-r\right)}{\left(2R+r\right)\left(4R+r\right)} \leq \frac{r\left(16R-5r\right)}{\left(4R+r\right)^2}\\
&\leq \frac{4r\left(12R^2-11Rr+r^2\right)}{\left(3R-2r\right)\left(4R+r\right)^2} \leq \frac{s^2}{\left(4R+r\right)^2} \leq \frac{27R^2}{27R^2-8r^2}\cdot\frac{\left(2R+r\right)^2}{\left(4R+r\right)^2}\\
&\leq \frac{R}{2\left(2R-r\right)} \leq \frac{1}{\left(4R+r\right)^2}\left[4R^2+4Rr+\frac{25}{9}r^2+\frac{64r^4}{9\left(9R^2-4r^2\right)}\right]\\
&\leq \frac{4R^2+4Rr+3r^2}{\left(4R+r\right)^2} \leq \frac{1}{3} \leq \frac{4R+r}{27r} \leq \frac{R^2}{4r\left(R+r\right)} \qquad (1)
\end{aligned}
$$

Let us denote the inequalities from the left side or from the right side of the middle expression $\frac{s^2}{(4R+r)^2}$ respectively with (1)–L–i, where $i = 1, 2, \ldots, 12$ and with (1)–R–j, where $j = 1, 2, \ldots, 7$.

It is necessary to emphasize the fact that many of the inequalities in (1) are obtained from "working out" *the formulas for the distances between some remarkable points in the triangle.*

For example, the inequality (1)–L–4 can be obtained from the formula for the distance

$$N\Gamma^2 = 16R\,(R+r)\left[\frac{s^2}{(4R+r)^2} - \frac{r}{R+r}\right],$$

where N is the *Nagel point* and Γ is the *Gergonne point* of the triangle ABC (see Mitrinović et al, p. 282). The same inequality is possible to be obtained from the distance IK^2, where I is the *incentre* and K is the *Lhuilier–Lemoine point.* Analogically, the inequalities (1)–L–i, ($i = 6, 5, 2$), and (1)–R–j, ($j = 2, 4, 5, 7$) can be obtained respectively from the distances: $G\Gamma^2$, $S\Gamma^2$, GI^2, SI^2, SG^2 and $4 \cdot SN^2 = NI^2 = s^2 - 16Rr + 5r^2$; and $H\Gamma^2$; HI^2 and $4 \cdot SO^2 = 4R^2 + 4Rr + 3r^2 - s^2$; $I\Gamma^2$, SH^2 and the inequality GI 5.5; $O\Gamma^2$, where the used remarkable points are G the *centroid*, O the *circumcentre*, H the *orthocenter*, S the centre of the *Spieker's circle* (see [5], pp. 282, 285).

The mathematicians Bager and Reuter obtained the inequality (1)–L–10 in 1973–1974. The inequality (1)–L–9 is a T_{l^2}-image from the inequality GI 4.7 (Finsler–Hadwiger): $\sum a^2 - \sum (b-c)^2 \geq 4F\sqrt{3}$ (see [7], p. 48, (11)).

The lowest limit (1)–L–1 is a $GT_l(3)$-image of the inequality GI 5.13: $\sum a^2 \leq 9R^2$ because the formula for the distance OH^2 is as follows $OH^2 = 9R^2 - \sum a^2$ (see [7], p. 72, (10) and [1], p. 168, 3.14).

The highest limit (1)–R–1 was found independently by Bilchev and Velikova by applying the transformation T_l^{-1} on the inequality GI 5.12: $27Rr \leq 2s^2$ (see [9], p. 25, (30)), and Mitrinović and Pečarić, (see [11], p. 283), (see also [6], p. 61, (3), (3')). So (1)–R–1 is a T_l^{-1}-image of GI 5.12. Bilchev and Velikova established that the inequality (1)–R–1 is equivalent to the following one: $27R^4 \cos\alpha \cos\beta \cos\gamma \leq 2F^2$, where α, β, γ are the angles of a triangle ABC (see also [2], p. 35).

The inequality (1)–R–2 is equivalent to the inequality:

$$\sum a^2 \leq \frac{27R^4}{9R^2 - 4r^2}$$

and it is due to Mushkarov–Simeonov–Bilchev–Kontogiannis (see [2], p. 33).

The right inequality (1)–R–6 is KK-image of GI 4.20:

$$27\Pi\left(b^2 + c^2 - a^2\right)^2 \leq (4F)^6 ,$$

where Π is a cyclic product over the sides a, b, c (see [9], p. 17).

The other inequalities in (1) are obtained by using a new method named "An unexpected transition from algebra to geometry" as developed by the author (see Bilchev Svetoslav, [14], [15] and [16]).

Hence, the inequality (1)–L–3 does "not follow in a straightforward way" from the following obvious inequality for the sides of any triangle:

$$\begin{aligned}&\sum c(a-b)^2(b+c-a)(c+a-b) \geq 0 \Leftrightarrow \\ &\qquad \sum a^2\left(\sum a^3 + 6abc\right) \geq (a+b+c)\sum a^4 + 6abc\sum ab \Leftrightarrow \\ &\qquad\qquad s^2 \geq \frac{r}{2R+r}(r+4R)(8R-r). \qquad (2)\end{aligned}$$

The inequality (1)–L–7 follows from the obvious inequality for any triangle:

$$\begin{aligned}&\sum (a+b-2c)^2(a+b-c) \geq 0 \Leftrightarrow \\ &\qquad 3(a+b+c)\left(\sum a^2 + \sum ab\right) \geq 5\sum a^3 + 39abc \Leftrightarrow \\ &\qquad\qquad s^2 \geq 15Rr - 3r^2. \qquad (3)\end{aligned}$$

Analogically, (1)–L–8 and (1)–L–11 are also "not straightforward" consequences of the following obvious inequalities respectively for the sides of any triangle:

$$\begin{aligned}&\sum c\,(a+b-2c)^2\,(a+b-c)\geq 0 \Leftrightarrow\\ &9\,(a+b+c)\sum a^3 \geq 13\sum a^4+10\sum a^2b^2+4abc\,(a+b+c)\Leftrightarrow\\ &\qquad\qquad s^2\geq \frac{9r}{7r+10R}\,(r+4R)^2 \qquad (4)\end{aligned}$$

and

$$\begin{aligned}&\sum c\,(a+b-2c)^2\,(4a+4b+3c)\geq 0 \Leftrightarrow\\ &4\sum a^4+8\,(a+b+c)\sum a^3\geq 26\sum a^2b^2+2abc\,(a+b+c)\Leftrightarrow\\ &14s^4-4r\,(49r+16R)\,s^2-18r^2\,(r+4R)^2\geq 0\Leftrightarrow\\ &s^2\geq \frac{r}{7}\left(16R+49r+2\sqrt{316R^2+518Rr+616r^2}\right). \qquad (5)\end{aligned}$$

Of course, it is possible to show similar examples and make the chain of geometric inequalities bigger on both sides—left and right.

Nevertheless, all of the above-mentioned inequalities give a good basis for gifted students in analysing possible methods for establishing, improving and creating a very interesting chain of geometric inequalities connecting the elements (R, r, s) of any given triangle.

The presented inequalities in (1) attracted the attention of the gifted students to the given special subject of mathematics and activate their interest to work in this field because here it is really possible to obtain new results.

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Assoc. Prof. Svetoslav Jordanov Bilchev, Ph.D.
Head of the Department of Algebra and Geometry
Faculty of Natural Sciences and Education
University of Rousse
8 Studentska Str., 7017 Rousse
BULGARIA
Cell phones: +359-886 735 536, +359-8 999 555 28
E-mail: slavy@ami.ru.acad.bg, slavy_bilchev@yahoo.com

Tournament of Towns

Andy Liu

Andy Liu is a professor of mathematics at the University of Alberta in Canada. His research interests span discrete mathematics, geometry, mathematics education and mathematics recreations. He edits the Problem Corner of the MAA's magazine Math Horizons. *He was the Chair of the Problem Committee in the 1995 IMO in Canada. His contribution to the 1994 IMO in Hong Kong was a major reason for him being awarded a David Hilbert International Award by the World Federation of National Mathematics Competitions. He has trained students in all six continents.*

As of this issue, I am replacing as editor of this column my friend Andrei Storozhev, who is pursuing other interest.

Since there are two rounds of the Tournament per year, and two issues of this magazine per year, I will use each column to feature my personal favourites from a particular round. Since there are four papers altogether, Junior and Senior as well as O-Level and A-Level, there are plenty of choices.

Since this column primarily serves readers whose countries do not take part in the Tournament, timeliness is not an issue, and to avoid rushing to meet deadlines, I will start with the Fall 2008 round in this column. Below are seven lovely problems:

1. Can the product of all terms of a non-constant arithmetic progression be equal to a^{2008} for some positive integer a, if the length of the progression is
(a) three;
(b) five?

2. On a straight track are several runners, each running at a different constant speed. They start at one end of the track at the same time. When they reach either end of the track, they turn around and continue to run indefinitely. Some time after the start, all runners meet at the same point. Prove that this will happen again.

3. Each of four stones weighs an integral number of grams. Available for use is a balance which shows the difference of the weights between the objects in the left pan and those in the right pan. Is it possible to determine the weight of each stone by using this balance four times, if it may make a mistake of 1 gram either way in at most one weighing?

4. Baron Münchausen asserts that he has a map of Oz showing five towns and ten roads, each road connecting exactly two cities. A road may intersect at most one other road once. The four roads connected to each town are alternately red and yellow. Can this assertion be true?

5. Space is dissected into non-overlapping unit cubes. Is it necessarily true that for each of these cubes, there exists another one sharing a common face with it?

6. In the infinite sequence $\{a_n\}$, $a_0 = 0$. For $n \geq 1$, if the greatest odd divisor of n is congruent modulo 4 to 1, then $a_n = a_{n-1} + 1$, but if the greatest odd divisor of n is congruent modulo 4 to 3, then $a_n = a_{n-1} - 1$. The initial terms are 0, 1, 2, 1, 2, 3, 2, 1, 2, 3, 4, 3, 2, 3, 2 and 1.
(a) Prove that the number 1 appears infinitely many times in this sequence.
(b) Prove that every positive integer appears infinitely many times in this sequence.

7. A contest consists of 30 true or false questions. Victor knows nothing about the subject matter. He may write the contest several times, with exactly the same questions, and is told the number of questions he has answered correctly each time. How can he be sure that he will answer all 30 questions correctly on his
(a) 30th attempt;
(b) 25th attempt?

Upcoming Conferences

Updates WFNMC conference, Riga, Latvia, July 2010

The congress of WFNMC in Latvia will be held few days after the ending of the 51st IMO in Kazakhstan.

Call for Candidates WFNMC miniconference at ICME-12 Seoul, Korea, July 2012

Looking to continue the activity begun in Monterrey in 2008, the Executive Committee of WFNMC wishes to call for candidates to organize its miniconference in Seoul one day before the start of ICME-12.

Call for Proposals WFNMC conference, July 2014

The Executive Committee of WFNMC calls for proposals to organize its 2014 conference in the tradition established by Waterloo, Pravetz, Zhong Shan, Melbourne, Cambridge, as well as the conference to be held in Riga in 2010.

The proposal should include the organization responsible, the city and probable venue, the names of the three to four chief members of the organizing team, as well as preliminary dates and estimates of registration fees.

Send proposals to
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Math and Science Olympiads
Methodology of mathematics and different natural sciences (ecology, geology, astronomy, biotechnology, environmental sciences, biomedical sciences, etc.)
Popularization of mathematics and natural sciences

The congress will include keynote and plenary lectures, parallel symposia, workshops and poster sessions.

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Jenaro Guisasola (Universidad del País Vasco, Spain). Research in Physics Education: achievement, challenges and new approaches

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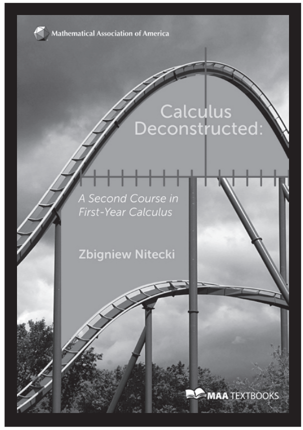

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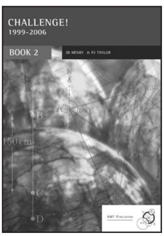
To attain an appropriate level of achievement in mathematics, students require talent in combination with commitment and self-discipline. The following books have been published by the AMT to provide a guide for mathematically dedicated students and teachers.

NEW

Australian Mathematics Competition Primary Problems & Solutions Book 1 2004–2008
W Atkins & PJ Taylor
This book consists of questions and full solutions from past AMC papers and is designed for use with students in Middle and Upper Primary. The questions are arranged in papers of 10 and are presented ready to be photocopied for classroom use.

NEW



Challenge! 1999—2006 Book 2
JB Henry & PJ Taylor
This is the second book of the series and contains the problems and full solutions to all Junior and Intermediate problems set in the Mathematics Challenge for Young Australians, exactly as they were proposed at the time. They are highly recommended as a resource book for classes from Years 7 to 10 and also for students who wish to develop their problem-solving skills. Most of the problems are graded within to allow students to access an easier idea before developing through a few levels.

Bundles of Past AMC Papers

Past Australian Mathematics Competition papers are packaged into bundles of ten identical papers in each of the Junior, Intermediate and Senior divisions of the Competition. Schools find these sets extremely valuable in setting their students miscellaneous exercises.

AMC Solutions and Statistics
Edited by PJ Taylor

This book provides, each year, a record of the AMC questions and solutions, and details of medallists and prize winners. It also provides a unique source of information for teachers and students alike, with items such as levels of Australian response rates and analyses including discriminatory powers and difficulty factors.

Australian Mathematics Competition Book 1 1978-1984
Edited by W Atkins, J Edwards, D King, PJ O'Halloran & PJ Taylor

This 258-page book consists of over 500 questions, solutions and statistics from the AMC papers of 1978-84. The questions are grouped by topic and ranked in order of difficulty. The book is a powerful tool for motivating and challenging students of all levels. A must for every mathematics teacher and every school library.

Australian Mathematics Competition Book 2 1985-1991
Edited by PJ O'Halloran, G Pollard & PJ Taylor

Over 250 pages of challenging questions and solutions from the Australian Mathematics Competition papers from 1985-1991.

Australian Mathematics Competition Book 3 1992-1998
W Atkins, JE Munro & PJ Taylor

More challenging questions and solutions from the Australian Mathematics Competition papers from 1992-1998.

Australian Mathematics Competition Book 3 CD
Programmed by E Storozhev

This CD contains the same problems and solutions as in the corresponding book. The problems can be accessed in topics as in the book and in this mode is ideal to help students practice particular skills. In another mode students can simulate writing one of the actual papers and determine the score that they would have gained. The CD runs on Windows platform only.

Australian Mathematics Competition Book 4 1999-2005
W Atkins & PJ Taylor

More challenging questions and solutions from the Australian Mathematics Competition papers from 1999-2005.

Problem Solving via the AMC
Edited by Warren Atkins

This 210-page book consists of a development of techniques for solving approximately 150 problems that have been set in the Australian Mathematics Competition. These problems have been selected from topics such as Geometry, Motion, Diophantine Equations and Counting Techniques.

Methods of Problem Solving, Book 1
Edited by JB Tabov & PJ Taylor

This book introduces the student aspiring to Olympiad competition to particular mathematical problem solving techniques. The book contains formal treatments of methods which may be familiar or introduce the student to new, sometimes powerful techniques.

Methods of Problem Solving, Book 2
JB Tabov & PJ Taylor

After the success of Book 1, the authors have written Book 2 with the same format but five new topics. These are the Pigeon-Hole Principle, Discrete Optimisation, Homothety, the AM-GM Inequality and the Extremal Element Principle.

Mathematical Toolchest
Edited by AW Plank & N Williams

This 120-page book is intended for talented or interested secondary school students, who are keen to develop their mathematical knowledge and to acquire new skills. Most of the topics are enrichment material outside the normal school syllabus, and are accessible to enthusiastic year 10 students.

International Mathematics — Tournament of Towns (1980–1984)
Edited by PJ Taylor

The International Mathematics Tournament of the Towns is a problem-solving competition in which teams from different cities are handicapped according to the population of the city. Ranking only behind the International Mathematical Olympiad, this competition had its origins in Eastern Europe (as did the Olympiad) but is now open to cities throughout the world. This 115-page book contains problems and solutions from past papers for 1980-1984.

International Mathematics — Tournament of Towns (1984–1989)
Edited by PJ Taylor

More challenging questions and solutions from the International Mathematics Tournament of the Towns competitions. This 180-page book contains problems and solutions from 1984-1989.

International Mathematics — Tournament of Towns (1989–1993)
Edited by PJ Taylor

This 200-page book contains problems and solutions from the 1989-1993 Tournaments.

International Mathematics — Tournament of Towns (1993–1997)
Edited by PJ Taylor

This 180-page book contains problems and solutions from the 1993-1997 Tournaments.

International Mathematics — Tournament of Towns (1997–2002)
Edited by AM Storozhev

This 214-page book contains problems and solutions from the 1997-2002 Tournaments.

Challenge! 1991 – 1998
Edited by JB Henry, J Dowsey, AR Edwards, L Mottershead, A Nakos, G Vardaro & PJ Taylor

This book is a major reprint of the original Challenge! (1991-1995) published in 1997. It contains the problems and full solutions to all Junior and Intermediate problems set in the Mathematics Challenge for Young Australians, exactly as they were proposed at the time. It is expanded to cover the years up to 1998, has more advanced typography and makes use of colour. It is highly recommended as a resource book for classes from Years 7 to 10 and also for students who wish to develop their problem-solving skills. Most of the problems are graded within to allow students to access an easier idea before developing through a few levels.

USSR Mathematical Olympiads 1989 – 1992
Edited by AM Slinko

Arkadii Slinko, now at the University of Auckland, was one of the leading figures of the USSR Mathematical Olympiad Committee during the last years before democratisation. This book brings together the problems and solutions of the last four years of the All-Union Mathematics Olympiads. Not only are the problems and solutions highly expository but the book is worth reading alone for the fascinating history of mathematics competitions to be found in the introduction.

Australian Mathematical Olympiads 1979 – 1995
H Lausch & PJ Taylor

This book is a complete collection of all Australian Mathematical Olympiad papers from the first competition in 1979. Solutions to all problems are included and in a number of cases alternative solutions are offered.

Chinese Mathematics Competitions and Olympiads Book 1 1981–1993
A Liu

This book contains the papers and solutions of two contests, the Chinese National High School Competition and the Chinese Mathematical Olympiad. China has an outstanding record in the IMO and this book contains the problems that were used in identifying the team candidates and selecting the Chinese team. The problems are meticulously constructed, many with distinctive flavour. They come in all levels of difficulty, from the relatively basic to the most challenging.

Asian Pacific Mathematics Olympiads 1989-2000
H Lausch & C Bosch-Giral

With innovative regulations and procedures, the APMO has become a model for regional competitions around the world where costs and logistics are serious considerations. This 159 page book reports the first twelve years of this competition, including sections on its early history, problems, solutions and statistics.

Polish and Austrian Mathematical Olympiads 1981–1995
ME Kuczma & E Windischbacher

Poland and Austria hold some of the strongest traditions of Mathematical Olympiads in Europe even holding a joint Olympiad of high quality. This book contains some of the best problems from the national Olympiads. All problems have two or more independent solutions, indicating their richness as mathematical problems.

Seeking Solutions
JC Burns

Professor John Burns, formerly Professor of Mathematics at the Royal Military College, Duntroon and Foundation Member of the Australian Mathematical Olympiad Committee, solves the problems of the 1988, 1989 and 1990 International Mathematical Olympiads. Unlike other books in which only complete solutions are given, John Burns describes the complete thought processes he went through when solving the problems from scratch. Written in an inimitable and sensitive style, this book is a must for a student planning on developing the ability to solve advanced mathematics problems.

101 Problems in Algebra from the Training of the USA IMO Team
Edited by T Andreescu & Z Feng

This book contains one hundred and one highly rated problems used in training and testing the USA International Mathematical Olympiad team. The problems are carefully graded, ranging from quite accessible towards quite challenging. The problems have been well developed and are highly recommended to any student aspiring to participate at National or International Mathematical Olympiads.

Hungary Israel Mathematics Competition
S Gueron

The Hungary Israel Mathematics Competition commenced in 1990 when diplomatic relations between the two countries were in their infancy. This 181-page book summarizes the first 12 years of the competition (1990 to 2001) and includes the problems and complete solutions. The book is directed at mathematics lovers, problem solving enthusiasts and students who wish to improve their competition skills. No special or advanced knowledge is required beyond that of the typical IMO contestant and the book includes a glossary explaining the terms and theorems which are not standard that have been used in the book.

Chinese Mathematics Competitions and Olympiads Book 2 1993–2001
A Liu
This book is a continuation of the earlier volume and covers the years 1993 to 2001.

Bulgarian Mathematics Competition 1992–2001
BJ Lazarov, JB Tabov, PJ Taylor & A Storozhev

The Bulgarian Mathematics Competition has become one of the most difficult and interesting competitions in the world. It is unique in structure combining mathematics and informatics problems in a multi-choice format. This book covers the first ten years of the competition complete with answers and solutions. Students of average ability and with an interest in the subject should be able to access this book and find a challenge.

Mathematical Contests – Australian Scene
Edited by PJ Brown, A Di Pasquale & K McAvaney

These books provide an annual record of the Australian Mathematical Olympiad Committee's identification, testing and selection procedures for the Australian team at each International Mathematical Olympiad. The books consist of the questions, solutions, results and statistics for: Australian Intermediate Mathematics Olympiad (formerly AMOC Intermediate Olympiad), AMOC Senior Mathematics Contest, Australian Mathematics Olympiad, Asian-Pacific Mathematics Olympiad, International Mathematical Olympiad, and Maths Challenge Stage of the Mathematical Challenge for Young Australians.

Mathematics Competitions
Edited by J Švrcek

This bi-annual journal is published by AMT Publishing on behalf of the World Federation of National Mathematics Competitions. It contains articles of interest to academics and teachers around the world who run mathematics competitions, including articles on actual competitions, results from competitions, and mathematical and historical articles which may be of interest to those associated with competitions.

Problems to Solve in Middle School Mathematics
B Henry, L Mottershead, A Edwards, J McIntosh, A Nakos, K Sims, A Thomas & G Vardaro

This collection of problems is designed for use with students in years 5 to 8. Each of the 65 problems is presented ready to be photocopied for classroom use. With each problem there are teacher's notes and fully worked solutions. Some problems have extension problems presented with the teacher's notes. The problems are arranged in topics (Number, Counting, Space and Number, Space, Measurement, Time, Logic) and are roughly in order of difficulty within each topic. There is a chart suggesting which problem-solving strategies could be used with each problem.

Teaching and Assessing Working Mathematically Book 1 & Book 2
Elena Stoyanova

These books present ready-to-use materials that challenge students understanding of mathematics. In exercises and short assessments, working mathematically processes are linked with curriculum content and problem solving strategies. The books contain complete solutions and are suitable for mathematically able students in Years 3 to 4 (Book 1) and Years 5 to 8 (Book 2).

A Mathematical Olympiad Primer
G Smith

This accessible text will enable enthusiastic students to enter the world of secondary school mathematics competitions with confidence. This is an ideal book for senior high school students who aspire to advance from school mathematics to solving olympiad-style problems. The author is the leader of the British IMO team.

ENRICHMENT STUDENT NOTES

The Enrichment Stage of the Mathematics Challenge for Young Australians (sponsored by the Dept of Innovation, Industry, Science and Research) contains formal course work as part of a structured, in-school program. The Student Notes are supplied to students enrolled in the program along with other materials provided to their teacher. We are making these Notes available as a text book to interested parties for whom the program is not available.

Newton Enrichment Student Notes
JB Henry

Recommended for mathematics students of about Year 5 and 6 as extension material. Topics include polyominoes, arithmetricks, polyhedra, patterns and divisibility.

Dirichlet Enrichment Student Notes
JB Henry

This series has chapters on some problem solving techniques, tessellations, base five arithmetic, pattern seeking, rates and number theory. It is designed for students in Years 6 or 7.

Euler Enrichment Student Notes
MW Evans & JB Henry

Recommended for mathematics students of about Year 7 as extension material. Topics include elementary number theory and geometry, counting, pigeonhole principle.

Gauss Enrichment Student Notes
MW Evans, JB Henry & AM Storozhev

Recommended for mathematics students of about Year 8 as extension material. Topics include Pythagoras theorem, Diophantine equations, counting, congruences.

Noether Enrichment Student Notes
AM Storozhev

Recommended for mathematics students of about Year 9 as extension material. Topics include number theory, sequences, inequalities, circle geometry.

Pólya Enrichment Student Notes
G Ball, K Hamann & AM Storozhev

Recommended for mathematics students of about Year 10 as extension material. Topics include polynomials, algebra, inequalities and geometry.

T-SHIRTS

T-shirts of the following six mathematicians are made of 100% cotton and are designed and printed in Australia. They come in white, Medium (Turing only) and XL.

Leonhard Euler T-shirt

The Leonhard Euler t-shirts depict a brightly coloured cartoon representation of Euler's famous Seven Bridges of Königsberg question.

Carl Friedrich Gauss T-shirt

The Carl Friedrich Gauss t-shirts celebrate Gauss' discovery of the construction of a 17-gon by straight edge and compass, depicted by a brightly coloured cartoon.

Emmy Noether T-shirt

The Emmy Noether t-shirts show a schematic representation of her work on algebraic structures in the form of a brightly coloured cartoon.

George Pólya T-shirt

George Pólya was one of the most significant mathematicians of the 20th century, both as a researcher, where he made many significant discoveries, and as a teacher and inspiration to others. This t-shirt features one of Pólya's most famous theorems, the Necklace Theorem, which he discovered while working on mathematical aspects of chemical structure.

Peter Gustav Lejeune Dirichlet T-shirt

Dirichlet formulated the Pigeonhole Principle, often known as Dirichlet's Principle, which states: "If there are p pigeons placed in h holes and $p>h$ then there must be at least one pigeonhole containing at least 2 pigeons." The t-shirt has a bright cartoon representation of this principle.

Alan Mathison Turing T-shirt

The Alan Mathison Turing t-shirt depicts a colourful design representing Turing's computing machines which were the first computers.

ORDERING

All the above publications are available from AMT Publishing and can be purchased on-line at:

www.amt.edu.au/amtpub.html or contact the following:

AMT Publishing

Australian Mathematics Trust

University of Canberra ACT 2601

Australia

Tel: +61 2 6201 5137

Fax: +61 2 6201 5052

Email: mail@amt.edu.au

The Australian Mathematics Trust

The Trust, of which the University of Canberra is Trustee, is a non-profit organisation whose mission is to enable students to achieve their full intellectual potential in mathematics. Its strengths are based upon:

- a network of dedicated mathematicians and teachers who work in a voluntary capacity supporting the activities of the Trust;
- the quality, freshness and variety of its questions in the Australian Mathematics Competition, the Mathematics Challenge for Young Australians, and other Trust contests;
- the production of valued, accessible mathematics materials;
- dedication to the concept of solidarity in education;
- credibility and acceptance by educationalists and the community in general whether locally, nationally or internationally; and
- a close association with the Australian Academy of Science and professional bodies.