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MATHEMATICS COMPETITIONS

JOURNAL OF THE
WORLD FEDERATION OF NATIONAL MATHEMATICS COMPETITIONS

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The aims of the Federation are:–

1. ***to promote excellence in, and research associated with, mathematics education through the use of school mathematics competitions;***
2. ***to promote meetings and conferences where persons interested in mathematics contests can exchange and develop ideas for use in their countries;***
3. ***to provide opportunities for the exchanging of information for mathematics education through published material, notably through the Journal of the Federation;***
4. ***to recognize through the WFNMC Awards system persons who have made notable contributions to mathematics education through mathematical challenge around the world;***
5. ***to organize assistance provided by countries with developed systems for competitions for countries attempting to develop competitions;***
6. ***to promote mathematics and to encourage young mathematicians.***

From the President

The recipients of the "Paul Erdös" Federation Award for the year 2008 are now known. The decision came shortly after the last volume of the journal was published in December 2007. According to the regulations of the Federation, the Paul Erdös Award is given every two years to up to three recipients. The procedure includes several phases: nomination, refereeing, assessment by the Awards Committee and final approval of the award recipients by the Executive Committee of the Federation. The Awards Committee chaired by Peter Taylor proposed and the Executive Committee approved the following persons as recipients of Paul Erdös Award for 2008:

- **Hans-Dietrich Gronau**, Rostock, Germany,
- **Bruce Henry**, Melbourne, Australia,
- **Leou Shian**, Kaohsiung, Taiwan.

Congratulations to the recipients for the well-deserved Award! More information about the outstanding achievements of these colleagues can be found at http://www.amt.canberra.edu.au/wfnmcann08.html . The Award Ceremony will take place during the International Congress on Mathematical Education in Monterrey, Mexico (July 6–13 2008).

This is the last time I address you as President of WFNMC. At the next business meeting of the Federation during ICME 11 in Monterrey, Mexico, new leadership of the Federation (including President, Vice-Presidents, Publications Officer, Secretary and members of Standing Committees) will be elected. This is why I cannot resist the temptation of presenting in brief my personal view on what the Federation is nowadays and what has been done in the period 2004–2008.

The World Federation of National Mathematics Competitions (WFNMC) appeared as a natural response to the need for international collaboration in the field of Mathematics Competitions. It was founded in 1984 during the Fifth International Congress on Mathematical Education (ICME 5) held in Adelaide, Australia, and became an Affiliated Study Group of ICMI in 1994.

The name of the Federation leaves the impression that its major goals are related to competitions only. To some extent, this might have been the case in the earlier stages of development of the organization. In 1988, on page 2 of Vol. 1, No. 1 of this journal, the following statement was published:

> The foundation members of the Federation hope that it will provide a focal point for people interested in, and concerned with, running national mathematics competitions; that it will become a resource centre for exchanging information and ideas on national competitions; and that it will create and cement professional links between mathematicians around the world.

Later the vision for the Federation's role gradually shifted to a broader understanding of the goals. The official viewpoint is now expressed in the preamble of the Federation's Constitution (approved in 1996 at ICME-8 in Seville, Spain, and amended at ICME-10 in Copenhagen, 2004):

> The World Federation of National Mathematics Competitions is a voluntary organization, created through the inspiration of Professor Peter O'Halloran of Australia, that aims to promote excellence in mathematics education and to provide those persons interested in promoting mathematics education through mathematics contests an opportunity of meeting and exchanging information.

An even wider viewpoint on the goals of the Federation is outlined in the Policy Statement adopted at the Federation's Conference in Melbourne, Australia in 2002:

> The scope of activities of interest to the WFNMC, although centered on competitions for students of all levels (primary, secondary and tertiary), is much broader than the competitions themselves. The WFNMC aims to provide a vehicle for educators to exchange information on a number of activities related to mathematics and mathematics learning...

The major activities of the Federation in the period 2004–2008 were in full compliance with the mentioned goals. More specifically, the activities included:

- regular conferences (every even-numbered year after ICME) with the last one taking place in Cambridge, England, 22–28 July 2006
- active participation in Discussion Groups and Topic Study Groups at ICMEs
- recognition (through the Paul Erdös Award) of persons with outstanding achievements in the detection, motivation and development of talented young people
- regular publication (twice yearly) of this Journal and continuous development and maintenance of the highly informative website of the Federation (see http://www.amt.canberra.edu.au/wfnmc.html): in particular, the site contains the above mentioned versions of Federation's Constitution and the Policy Statement
- participation in projects initiated and supported by other organizations: recent examples are the just completed ICMI Study 16, and the project MATHEU supported by the European Community within the framework of the Socrates Programme (see correspondingly http://www.amt.edu.au/icmis16.html and http://www.matheu.eu)
- regular business meetings of the Federation where organizational matters are considered
- opportunities for competition-related activities in the field of Informatics.

Further information about the goals, the essence and the history of WFNMC could be found in the official website of the Symposium on the occasion of the 100th Anniversary of ICMI (Rome, 5–8 March 2008) http://www.icmihistory.unito.it/wfnmc.php. Further information (arguments on the role of competitions for mathematics education, for attracting talent to science, for educational institutions and for the society as a whole) is contained in the invited lecture presented at Section 19 "Mathematics Education and Popularization of Mathematics" at the International Congress of Mathematicians in Madrid, 2006 .

Finally, I would like to express my deepest gratitude to all the members of the current leadership of the Federation for their support, dedication and engagement with the realization of the goals of the Federation in the period 2004–2008. Special thanks are due also to the Australian

Mathematics Trust. Without its firm and permanent support the Federation would have never reached its current status.

Petar S. Kenderov
President of WFNMC
June 2008

From the Editor

Welcome to *Mathematics Competitions* Vol. 21, No. 1.

Again I would like to thank the Australian Mathematics Trust for continued support, without which each issue of the journal could not be published, and in particular Heather Sommariva, Bernadette Webster and Pavel Calábek for their assistance in the preparation of this issue.

Submission of articles:
The journal *Mathematics Competitions* is interested in receiving articles dealing with mathematics competitions, not only at national and international level, but also at regional and primary school level. There are many readers in different countries interested in these different levels of competitions.

- The journal traditionally contains many different kinds of articles, including reports, analyses of competition problems and the presentation of interesting mathematics arising from competition problems. Potential authors are encouraged to submit articles of all kinds.
- To maintain and improve the quality of the journal and its usefulness to those involved in mathematics competitions, all articles are subject to review and comment by one or more competent referees. The precise criteria used will depend on the type of article, but can be summarised by saying that an article accepted must be correct and appropriate, the content accurate and interesting, and, where the focus is mathematical, the mathematics fresh and well presented. This editorial and refereeing process is designed to help improve those articles which deserve to be published.

At the outset, the most important thing is that if you have anything to contribute on any aspect of mathematics competitions at any level, local, regional or national, we would welcome your contribution.

Articles should be submitted in English, with a black and white photograph and a short profile of the author. Alternatively, the article can be submitted on an IBM PC compatible disk or a Macintosh disk. We prefere LaTeX or TeX format of contributions, but any text file will be helpful.

Articles, and correspondence, can also be forwarded to the editor by mail to

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Jaroslav Švrček,
June 2008

World Youth Mathematics Intercity Competition

Simon Chua, Andy Liu & Bin Xiong

Simon Chua is the founder and president of the Mathematics Trainers' Guild, Philippines. He was the first Filipino to win the Paul Erdös Award *from the World Federation of National Mathematics Competitions for his distinguished and sustained contribution to the enrichment of mathematics education in the Philippines. His ambition is to transform all mathematics teachers in the Philippines to good mathematics educators, so that more Filipino students can achieve international recognition in the field of mathematics.*

Andy Liu is a professor of mathematics at the University of Alberta in Canada. His research interests span discrete mathematics, geometry, mathematics education and mathematics recreations. He edits the Problem Corner of the MAA's magazine Math Horizons*. He was the Chair of the Problem Committee in the 1995 IMO in Canada. His contribution to the 1994 IMO in Hong Kong was a major reason for him being awarded a David Hilbert International Award by the World Federation of National Mathematics Competitions. He has trained students in all six continents.*

Bin Xiong is a professor of mathematics at the East China Normal University. His research interest is in problem solving and gifted education. He is a member of the Chinese Mathematical Olympiad committee and the Problem Subcommittee. He had served as the leader of the national team at the 2005 International Mathematical Olympiad in Mexico. He had been a trainer at the national camp many times. He is involved in the National Junior High School Competition, the National High School Mathematics Competition, the Western China Mathematical Olympiad and the Girls' Mathematical Olympiad.

The World Youth Mathematics Intercity Competition (WYMIC) was founded in 1999 by Prof. Hsin Leou of the Kaohsiung National Normal University in Taiwan. It was designed as an International Mathematical Olympiad for junior high school students. Each team consists of a leader, a deputy leader and four contestants. They represent their city rather than their country, downplaying politics.

For the first two years, the contest was actually held in Kaohsiung, and the participating teams were all from South East Asia. The hosts in 2001 and 2002 were the Philippines and India, respectively. It was not held in 2003 during the height of the SARS scare. In 2004, the host was Macau.

In 2005, the contest returned to Kaohsiung. Much progress has been made since the beginning. China had joined the ranks, and the Chinese city of Wen Zhou hosted the 2006 contest. In 2007, when the contest was held in Chang Chun, China, there were sixty-five teams, including those from Canada, Iran, South Africa and the United States of America.

There is now a permanent WYMIC board. The president is Mr. Wen-Hsien Sun of Chiu Chang Mathematics Foundation in Taipei, and the secretary is Mr. Simon Chua of the Mathematics Trainers' Guild in Zamboanga. Added to the board each year are the local organizing committee and the local problem committee. Prof. Zonghu Qiu of Academia Sinica in Beijing is the advisor for the former, and Prof. Andy Liu and Prof. Bin Xiong are advisors for the latter.

The event typically lasts five days in late July. Day 1 is for arrival. Day 2 is taken up with registration, the opening ceremony and team leader meetings. Day 3 is the actual competition, with an individual contest in the morning and a team contest in the afternoon. Day 4 is set aside for excursion, with a banquet in the evening. Following the closing ceremony on Day 5, the students depart. Slight variation occur from year to year.

Gold, silver and bronze medals are awarded for the individual contest, as well as honorable mentions. The teams are drawn into a number of groups. Within each group, gold, silver and bronze medals are awarded for the team contest. A second set of medals are awarded on the basis of the sum of the best three scores in the individual contest. Finally, a team is declared the grand champion, based on the sum of these two team scores. The next two teams also receive trophies. In addition, there are two non-academic team awards, one for the best behaviour and one for popularity.

A most special feature of the WYMIC is the Cultural Evening. Each team must perform on stage for about five minutes, to showcase their ethnic identities. The performances vary from instrumental and choral music to power-point presentations of scenery. This cements international friendship and fosters mutual understanding. Two more team awards are handed out, one for the best performance and one for innovation.

In 2009, the contest will be hosted by South Africa in Durban. It is a golden opportunity for European and other African teams to join in. For further information, contact Prof. Gwen Williams at gwilliams@telkomsa.net. To conclude this paper, we append the problems used in the 2007 contest.

1 Individual Contest

Time limit: 120 minutes 2007/7/23 Changchun, China

Section I

In this section, there are 12 questions, fill in the correct answers in the spaces provided at the end of each question. Each correct answer is worth 5 points.

1. Let A_n be the average of the multiples of n between 1 and 101. Which is the largest among A_2, A_3, A_4, A_5 and A_6?

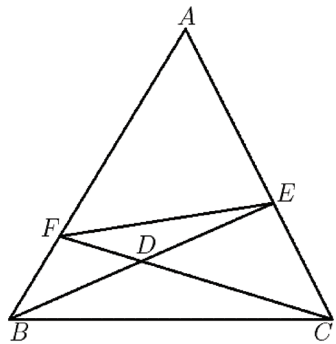
Solution. The smallest multiple of n is of course n. Denote by a_n the largest multiple of n not exceeding 101. Then $A_n = (n+a_n)/2$. Hence $A_2 = A_3 = 51$, $A_4 = 52$, $A_5 = 52.5$ and $A_6 = 51$, and the largest one is A_5.

2. It is a dark and stormy night. Four people must evacuate from an island to the mainland. The only link is a narrow bridge which allows passage of two people at a time. Moreover, the bridge must be illuminated, and the four people have only one lantern among them. After each passage to the mainland, if there are still people on the island, someone must bring the lantern back. Crossing the bridge individually, the four people take 2, 4, 8 and 16 minutes respectively. Crossing the bridge in pairs, the slower speed is used. What is the minimum time for the whole evacuation?

Solution. Exactly five passages are required, three pairs to the mainland and two individuals back to the island. Let the fastest two people cross first. One of them brings back the lantern. Then the slowest two people cross, and the fastest person on the mainland brings back the lantern. The final passage is the same as the first. The total time is $4+2+16+4+4 = 30$ minutes. To show that this is minimum, note that the three passages in pairs take at least $16 + 4 + 4 = 24$ minutes, and the two passages individually take at least $4 + 2 = 6$ minutes.

3. In triangle ABC, E is a point on AC and F is a point on AB. BE and CF intersect at D. If the areas of triangles BDF, BCD

and CDE are 3, 7 and 7 respectively, what is the area of the quadrilateral $AEDF$?



Solution. Since triangles BCD and CDE have equal areas, $BD{=}DE$. Hence the area of triangle DEF is also 3. Let the area of triangle EFA be x. Then $\frac{x}{6} = \frac{AF}{BF} = \frac{x+3+7}{3+7}$. It follows that $10x = 6x + 60$ so that $x = 15$. The area of the quadrilateral $AEDF$ is $15 + 3 = 18$.

4. A regiment had 48 soldiers but only half of them had uniforms. During inspection, they form a 6×8 rectangle, and it was just enough to conceal in its interior everyone without a uniform. Later, some new soldiers joined the regiment, but again only half of them had uniforms. During the next inspection, they used a different rectangular formation, again just enough to conceal in its interior everyone without a uniform. How many new soldiers joined the regiment?

Solution. Let the dimensions of the rectangle be x by y, with $x \leq y$. Then the number of soldiers on the outside is $2x + 2y - 4$ while the number of those in the interior is $(x - 2)(y - 2)$. From $xy - 2x - 2y + 4 = 2x + 2y - 4$, we have $(x - 4)(y - 4) = xy - 4x - 4y + 16 = 8$. If $x - 4 = 2$ and $y - 4 = 4$, we obtain the original 6×8 rectangle. If $x - 4 = 1$ and $y - 4 = 8$, we obtain the new 5×12 rectangle. Thus the number of new soldiers is $5 \cdot 12 - 6 \cdot 8 = 12$.

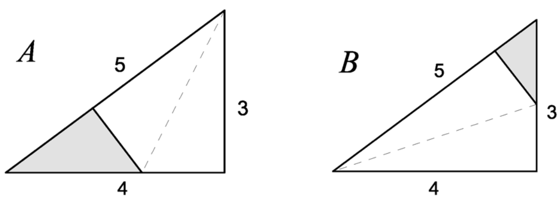
5. The sum of 2008 consecutive positive integers is a perfect square. What is the minimum value of the largest of these integers?

Solution. Let a be the smallest of these integers. Then

$$a + (a+1) + (a+2) + \ldots + (a+2007) = 251 \cdot (2a+2007) \cdot 2^2.$$

In order for this to be a perfect square, we must have $2a+2007 = 251n^2$ for some positive integer n. For $n = 1$ or 2, a is negative. For $n = 3$, we have $a = 126$ so that $a + 2007 = 2133$ is the desired minimum value.

6. The diagram shows two identical triangular pieces of paper A and B. The side lengths of each triangle are 3, 4 and 5. Each triangle is folded along a line through a vertex, so that the two sides meeting at this vertex coincide. The regions not covered by the folded parts have respective areas S_A and S_B. If $S_A + S_B = 39$ square centimetres, find the area of the original triangular piece of paper.



Solution. In the first diagram, the ratio of the areas of the shaded triangle and one of the unshaded triangles is $(5-3) : 3$ so that S_A is one-quarter of the area of the whole triangle. In the second diagram, the ratio of the areas of the shaded triangle and one of the unshaded triangles is $(5-4) : 4$ so that S_B is one-ninth of the area of the whole triangle. Now $\frac{1}{4} + \frac{1}{9} = \frac{13}{36}$. Hence the area of the whole triangle is $\frac{36}{13} \cdot 39 = 108$ square centimetres.

7. Find the largest positive integer n such that $3^{1024} - 1$ is divisible by 2^n.

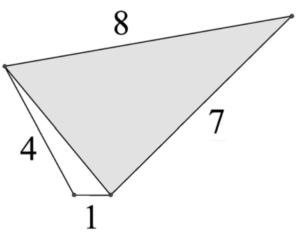
Solution. Note that

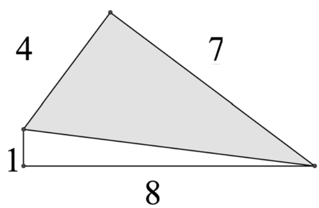
$$3^{1024} - 1 = (3^{512}+1)(3^{256}+1)(3^{128}+1)\ldots(3+1)(3-1).$$

All 11 factors are even, and $3+1$ is a multiple of 4. Clearly $3-1$ is not divisible by 4. We claim that neither is any of the other 9 factors. When the square of an odd number is divided by 4, the remainder is always 1. Adding 1 makes the remainder 2, justifying the claim. Hence the maximum value of n is 12.

8. A farmer has four straight fences, with respective lengths 1, 4, 7 and 8 metres. What is the maximum area of the quadrilateral the farmer can enclose?

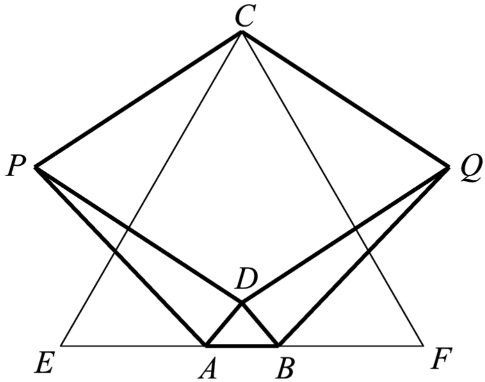
Solution. We may assume that the sides of lengths 1 and 8 are adjacent sides of the quadrilateral, as otherwise we can flip over the shaded triangle in the first diagram. Now the quadrilateral may be divided into two triangles as shown in the second diagram. In each triangle, two sides have fixed length. Hence its area is maximum if these two sides are perpendicular to each other. Since $12+82 = 42+72$, both maxima can be achieved simultaneously. In that case, the area of the unshaded triangle is 4 and the area of the shaded triangle is 14. Hence the maximum area of the quadrilateral is 18 square metres.





9. In the diagram (on the next page), $PA = QB = PC = QC = PD = QD = 1$, $CE = CF = EF$ and $EA = BF = 2AB$. Determine BD.

Solution. Let M be the midpoint of EF. By symmetry, D lies on CM. Let $BM = x$. Then $FM = 5x$, $CF = 10x$, $CM = 5\sqrt{5}x$ and $BC = 2\sqrt{19}x$. It follows that $\frac{AB}{BC} = \frac{1}{\sqrt{19}}$. Now Q is the circumcentre of triangle BCD. Hence $\angle BQD = 2\angle BCD = \angle BCA$. Since both QDB and CAB are isosceles triangles, they are similar to each other. It follows that $\frac{BD}{QB} = \frac{AB}{BC} = \frac{1}{\sqrt{19}}$, so that



$BD = \frac{1}{\sqrt{19}}$.

10. Each of the numbers 2, 3, 4, 5, 6, 7, 8 and 9 is used once to fill in one of the boxes in the equation below to make it correct. Of the three fractions being added, what is the value of the largest one?

$$\frac{1}{\square\times\square}+\frac{\square}{\square\times\square}+\frac{\square}{\square\times\square}=1$$

Solution. We may assume that the second numerator is 5 and the third 7. If either 5 or 7 appears in a denominator, it can never be neutralized. Since the least common multiple of the two remaining numbers is $8 \times 9 = 72$, we use $\frac{1}{72}$ as the unit of measurement. Now one of the three fractions must be close to 1. This can only be $\frac{5}{2\times3}$ or $\frac{7}{2\times4}$. In the first case, we are short 12 units. Of this, 7 must come from the third fraction so that 5 must come from the first fraction. This is impossible because the first fraction has numerator 1 and 5 does not divide 72. In the second case, we are short 9 units. Of this, 5 must come from the second fraction so that 4 must come from the third. This can be achieved as shown in the equation below. Hence the largest of the three fractions has value $\frac{7}{8}$.

$$\frac{1}{\boxed{3}\times\boxed{6}} + \frac{\boxed{5}}{\boxed{8}\times\boxed{9}} + \frac{\boxed{7}}{\boxed{2}\times\boxed{4}} = 1$$

11. Let x be a positive number. Denote by $[x]$ the integer part of x and by $\{x\}$ the decimal part of x. Find the sum of all positive numbers satisfying $5\{x\} + 0.2[x] = 25$.

Solution. The given equation may be rewritten as $\{x\} = \frac{125-[x]}{25}$. From $0 \leq \{x\} < 1$, we have $100 < [x] \leq 125$. For each solution x, $x = [x] + \{x\} = 5 + \frac{24}{25}[x]$. It follows that the desired sum is $5(25) + (24/25)(101 + 102 + 103 + \ldots + 125) = 2837$.

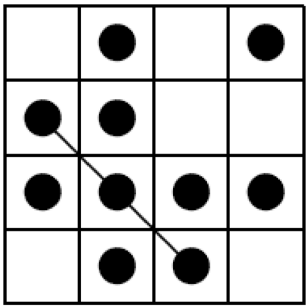
12. A positive integer n is said to be good if there exists a perfect square whose sum of digits in base 10 is equal to n. For instance, 13 is good because $7^2 = 49$ and $4 + 9 = 13$. How many good numbers are among $1, 2, 3, \ldots, 2007$?

Solution. If a positive integer is a multiple of 3, then its square is a multiple of 9, and so is the sum of the digits of the square. If a positive integer is not a multiple of 3, then its square is 1 more than a multiple of 3, and so is the sum of the digits of the square. Now the square of $9\ldots9$ with m 9s is $9\ldots980\ldots01$, with $m-1$ digits 9 and 0. Its digit sum is $9m$. Hence all multiples of 9 are good, and there are $\frac{2007}{9} = 223$ of them not exceeding 2007. On the other hand, the square of $3\ldots35$ with m 3s is 1212..1225 with m sets of 12. Its digit sum is $3m + 7$. Since 1 and 4 are also good, all numbers 1 more than a multiple are good, and there are $\frac{2007}{3} = 669$ of them. Hence there are altogether $223 + 669 = 992$ good numbers not exceeding 2007.

Section II

Answer the following 3 questions, and show your detailed solution in the space provided after each question. Each question is worth 20 points.

1. A 4×4 table has 18 lines, consisting of the 4 rows, the 4 columns, 5 diagonals running from southwest to northeast, and 5 diagonals running from northwest to southeast. A diagonal may have 2, 3 or 4 squares. Ten counters are to be placed, one in each of ten of the sixteen cells. Each line which contains an even number of counters scores a point. What is the largest possible score?



The maximum score is 17, as shown in the placement in the diagram below. The only line not scoring a point is marked.

We now prove that a perfect score of 18 points leads to a contradiction. Note that the five diagonals in the same direction cover all but two opposite corner cells. These two cells must either be both vacant or both occupied. Note also that we must have a completely filled row, and a completely filled column. We consider three cases.

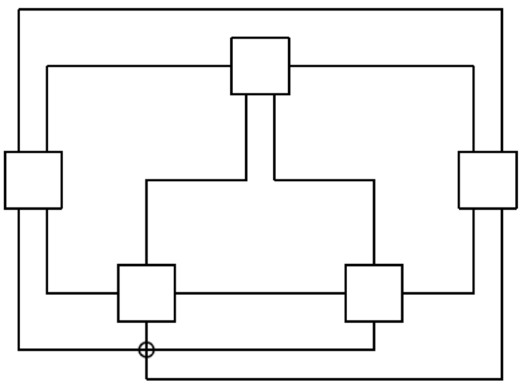
Case 1. All four corner cells are vacant.
We may assume by symmetry that the second row and the second column are completely filled. Then we must fill the remaining inner cells of the first row, the fourth row, the first column and the fourth column. These requires eleven counters.

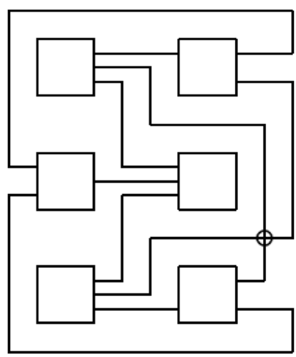
Case 2. Exactly two opposite corner cells are vacant.
By symmetry, we may assume that one of them is on the first row and first column, and the other is on the fourth row and fourth column. Then we must have exactly one more occupied inner cell on each of the first row, the first column, the fourth row and the fourth column. This means that all four cells in the interior of the table are filled. By symmetry, we may assume that the completely

filled row is the second. It is impossible to score both the diagonals of length 2 which intersect the second row.

Case 3. All four corner cells are occupied.
We claim that the completely filled row must be either the first or the fourth. Suppose to the contrary it is the second. Then we must fill the first column and the fourth column, thus using up all ten counters. Now there are several diagonals which do not yield scores. This justifies the claim. By symmetry, we may assume that the first row and the first column are completely filled. To score all rows and columns, the remaining two counters must be in the four interior cells. Again, some of the diagonals will not yield scores.

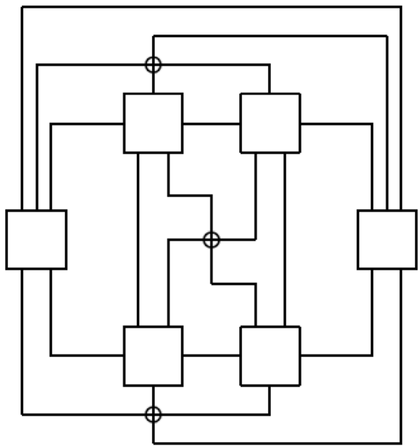
2. There are ten roads linking all possible pairs of five cities. It is known that there is at least one crossing of two roads, as illustrated in the diagram below on the left. There are nine roads linking each of three cities to each of three towns. It is known that there is also at least one crossing of two roads, as illustrated in the diagram below on the right. Of the fifteen roads linking all possible pairs of six cities, what is the minimum number of crossings of two roads?





Solution. The minimum number of crossings of two roads is three, as illustrated in the diagram below.

Suppose at most two crossings of two roads are needed. If we close one road from each crossing, the remaining ones can be drawn without any crossing. We consider two cases.



Case 1. The two roads closed meet at a city.
Consider the other five cities linked pairwise by ten roads, none of which has been closed. It is given that there must be a crossing of two roads, which is a contradiction.

Case 2. The two roads closed do not meet at a city.
Choose the two cities linked by one of the closed roads, and a third city not served by the other closed road. Call these three cities towns. Each is linked to each of the remaining three cities by a road. It is given that there must be a crossing of two roads, which is a contradiction.

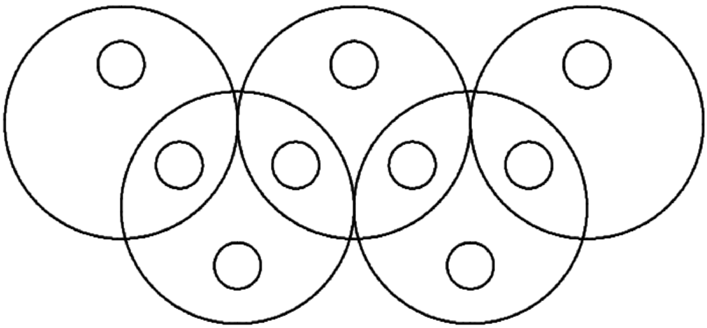
3. A prime number is called an absolute prime if every permutation of its digits in base 10 is also a prime number. For example: 2, 3, 5, 7, 11, 13 (31), 17 (71), 37 (73), 79 (97), 113 (131, 311), 199 (919, 991) and 337 (373, 733) are absolute primes. Prove that no absolute prime contains all of the digits 1, 3, 7 and 9 in base 10.

Solution. Let N be an absolute prime which contains all of the digits 1, 3, 7 and 9 in base 10. Let L be any number formed from the remaining digits. Consider the following seven permutations of N: $10000L+7931$, $10000L+1793$, $10000L+9137$, $10000L+7913$, $10000L+7193$, $10000L+1973$ and $10000L+7139$. They have different remainders when divided by 7. Therefore one of them is a multiple of 7, and is not a prime. Hence N is not an absolute prime.

2 Team Contest

2007/7/23 Changchun, China

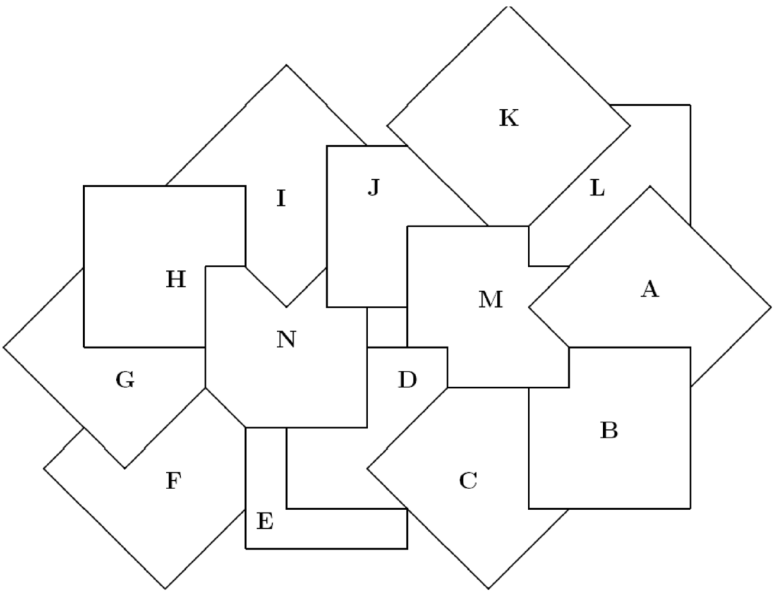
1. Use each of the numbers 1, 2, 3, 4, 5, 6, 7, 8 and 9 exactly once to fill in the nine small circles in the Olympic symbol below, so that the sum of the numbers inside each large circle is 14.



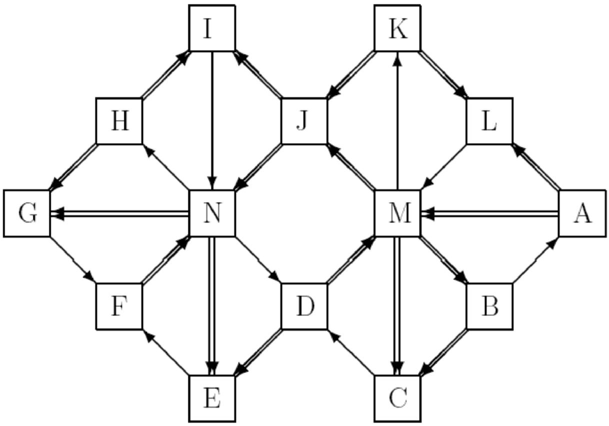
Solution. The sum of the nine numbers is $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 = 45$. The total sum of the numbers in the five large circles is $14 \cdot 5 = 70$. The difference $70 - 45 = 25$ is the sum of the four numbers in the middle row, because each appears in two large circles. The two numbers at one end must be 9 and 5 while the two numbers at the other end must be 8 and 6. Consider the two numbers at the end of the middle row. Clearly they cannot be 5 and 6. If they are 5 and 8, the other two numbers must sum to 12. With 5 and 8 gone, the only possibility is 9 and 3, but 9 cannot be in the inner part of the middle row. If they are 9 and 8, the other two numbers must sum to 8. Since neither 5 nor 6 can appear in the inner part of the middle row, the only possibility is 7 and 1. However, 7 cannot be in the same large circle with either 9 or 8. It follows that the two numbers at the end of the middle row are 9 and 6, and the other two numbers sum to 10. The only possibility is 7 and 3, and 7 must be in the same large circle with 6.

The remaining numbers can now be filled in easily, and the 9-digit number formed is 861743295.

2. The diagram below shows fourteen pieces of paper stacked on top of one another. Beginning on the piece marked B, move from piece to adjacent piece in order to finish at the piece marked F. The path must alternately climb up to a piece of paper stacked higher and come down to a piece of paper stacked lower. The same piece may be visited more than once, and it is not necessary to visit every piece. List the pieces of paper in the order visited.



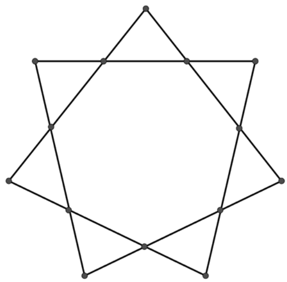
Solution. We construct below a diagram which is easier to use. An arrow from one piece of paper to another represents coming down from the first to the second. Note that each of M and N is connected to 7 other pieces, each of D and J is connected to 4 other pieces, while each of the others is connected to 3 other pieces. The path we seek consists of alternately going along with the arrow and going against it. Of the three arrows at A, the one from B cannot be used as otherwise we would be stuck at A. Equally useless are the arrows from L to M, from M to K, from I to N, from N to H, from G to F, from E to F, and from C to D. In the diagram below, they are drawn as single arrows while the useable ones are drawn as double arrows.



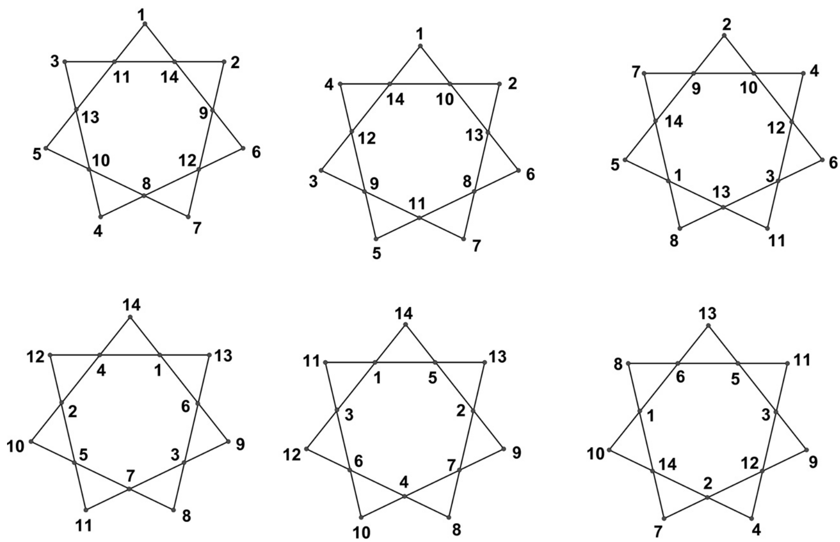
From B, if we climb down to C, then we will go up to M, but we could have gone up to M directly. Once in M, we have a choice of C or J, but C leads back to B, and we will get stuck there. From J, we have to move onto K, L, A, M, D, E, N, G, H, I, J, N and F. Thus the path is $BMJKLAMDENGHIJNF$.

3. There are 14 points of intersection in the seven-pointed star in the diagram (on the next page). Label these points with the numbers $1, 2, 3, \ldots, 14$ such that the sum of the labels of the four points on each line is the same. Give one solution.

Solution. Each of the points of intersection lies on exactly two lines. Hence the common sum is given by $2(1+2+3+\ldots+14)/7 = 30$. We claim that the smallest label on the same line with 14 is 1 or 2. Otherwise, the sum of the labels of the two lines is at least $14 + 14 + 3 + 4 + 5 + 6 + 7 + 8 = 61 > 30 + 30$, which is a contradiction. Similarly, the smallest label on the same line with 13 is 1, 2 or 3. In view of these observations, we put 14 on a line with 1 and on another line with 2, and 13 on a line with 1 and on another line with 2 or 3. This leads to the three labeling on



the top row. The labeling on the bottom row are the complements of the corresponding ones in the top row, that is, each label k is replaced by $15 - k$.



4. Mary found a 3-digit number that, when multiplied by itself, produced a number which ended in her 3-digit number. What is the sum of the numbers which have this property?

Solution. Since $1 \times 1 = 1$, $5 \times 5 = 25$ and $6 \times 6 = 36$, the last digit of the 3-digit number must be either 1, 5 or 6.

There is no 2-digit number with units digit 1 whose square ends with that number.

There is only one 2-digit number with units digit 5 whose square ends with that number and that is 25.

There is only one 2-digit number with units digit 6 whose square ends with that number and that is 76.

There is only one 3-digit number which ends in 25 whose square ends with that number and that is 625.

There is only one 3-digit number which ends in 76 whose square ends with that number and that is 376.

The sum of these two numbers is 625+376=1001.

5. Determine all positive integers m and n such that m^2+1 is a prime number and $10(m^2+1) = n^2+1$.

Solution. From the given condition, $9(m^2+1) = (n+m)(n-m)$. Note that m^2+1 is a prime number not equal to 3. Hence there are four cases.

a) $n-m = 1$, $n+m = 9(m^2+1)$.
Subtraction yields $9m^2+8 = 2m$, which is impossible.

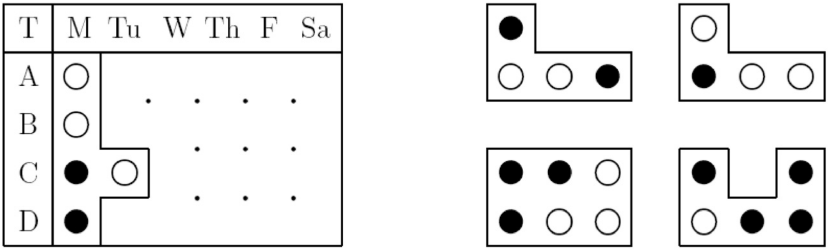
b) $n-m = 3$, $n+m = 3(m^2+1)$.
Subtraction yields $3m^2 = 2m$, which is impossible.

c) $n-m = 9$, $n+m = m^2+1$.
Subtraction yields $2m = m^2-8$, so that $m = 4$ and $n = 13$.
Note that $m^2+1 = 17$ is indeed a prime number.

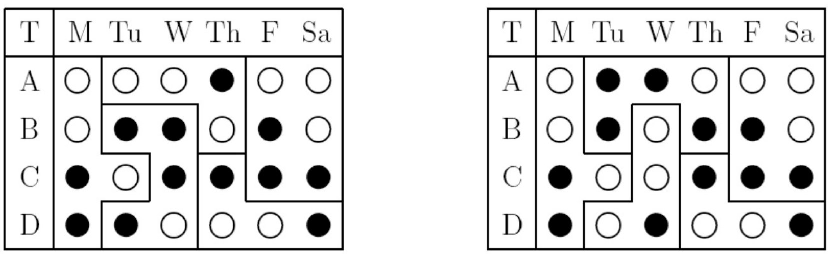
d) $n-m = m^2+1$, $n+m = 9$.
Subtraction yields $-2m = m^2-8$, so that $m = 2$ and $n = 7$.
Note that $m^2+1 = 5$ is indeed a prime number.

In summary, there are two solutions, $(m,n) = (2,7)$ or $(4,13)$.

6. Four teams take part in a week-long tournament in which every team plays every other team twice, and each team plays one game per day. The diagram below on the left shows the final scoreboard, part of which has broken off into four pieces, as shown on the diagram below on the right. These pieces are printed only on one side. A black circle indicates a victory and a white circle indicates a defeat. Which team wins the tournament?



Solution. When reconstructing the broken scoreboard, there are two positions where the U-shaped piece can be placed, so as to leave room for the 3×2 rectangle. Once it is in place, the positions for the remaining pieces are determined. They are placed so that there are two black circles and two white circles in each column. There are two possibilities, as shown in the diagrams below, but in either case, the winner of the tournament is Team C.



7. Let A be a 3 by 3 array consisting of the numbers $1, 2, 3, \dots, 9$. Compute the sum of the three numbers on the i-th row of A and the sum of the three numbers on the j-th column of A. The number at the intersection of the i-th row and the j-th column of a 3 by 3 array B is equal to the absolute difference of these two sums. For example, $b_{12} = |(a_{11} + a_{12} + a_{13}) - (a_{12} + a_{22} + a_{32})|$.

Is it possible to arrange the numbers in A so that the numbers in B are also $1, 2, 3, \dots, 9$?

a_{11}	a_{12}	a_{13}
a_{21}	a_{22}	a_{23}
a_{31}	a_{32}	a_{33}

A

b_{11}	b_{12}	b_{13}
b_{21}	b_{22}	b_{23}
b_{31}	b_{32}	b_{33}

B

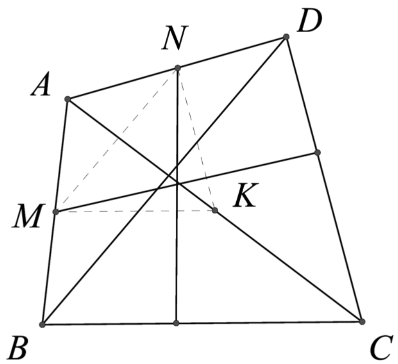
Solution. Let C be defined just like B, except that we use the actual difference instead of the absolute difference. Compute the sum of the nine numbers in C. Each number in A appears twice in this sum, once with a positive sign and once with a negative sign. Hence this sum is 0. It follows that among the nine numbers in C, the number of those which are odd is even. The same is true of the nine numbers in B, since taking the absolute value does not affect parity. Thus it is not possible for the nine numbers in B be $1, 2, 3, \ldots, 9$.

c_{11}	c_{12}	c_{13}
c_{21}	c_{22}	c_{23}
c_{31}	c_{32}	c_{33}

C

8. The diagonals AC and BD of a convex quadrilateral are perpendicular to each other. Draw a line that passes through point M, the midpoint of AB and perpendicular to CD, draw another line through point N, the midpoint of AD and perpendicular to CB. Prove that the point of intersection of these two lines lies on the line AC.

Solution. Let M, K and N be the respective midpoints of AB, AC and AD. Then MN is parallel to BD, MK is parallel to BC and NK is parallel to CD. Hence AC and the two lines in question are the altitudes of triangle MNK, and are therefore concurrent.



9. The positive integers from 1 to n (where $n > 1$) are arranged in a line such that the sum of any two adjacent numbers is a square. What is the minimum value of n?

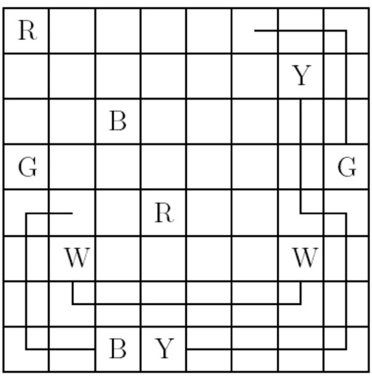
Solution. The minimum value of n is 15. Since $n > 1$, we must include 2, so that $n \geq 7$ because $2+7 = 9$. For $n = 7$, we have three separate lines $(1, 3, 6)$, $(2, 7)$ and $(4, 5)$. Adding 8 only lengthens the first to $(8, 1, 3, 6)$. Adding 9 now only lengthens the second to $(2, 7, 9)$. Hence $n \geq 10$. Now 8, 9 and 10 all have to be at the end if we have a single line, because we can only have $8 + 1 = 9$, $9 + 7 = 16$ and $10 + 6 = 16$. The next options are $8 + 17 = 25$, $9+16 = 25$ and $10+15 = 25$. Hence $n \geq 15$. For $n = 15$, we have the arrangement 8, 1, 15, 10, 6, 3, 13, 12, 4, 5, 11, 14, 2, 7 and 9.

10. Use one of the five colours (R represents red, Y represents yellow, B represents blue, G represents green and W represents white) to paint each square of an 8×8 chessboard, as shown in the diagram below. Then paint the rest of the squares so that all the squares of the same colour are connected to one another edge-to-edge. What is the largest number of squares of the same colour as compared to the other colours?

Solution. While it may tempting to colour the entire fourth row green, this will divide the red squares, the yellow squares and the blue squares into two disconnected parts. Obviously, the northeast corner is to be used to allow the green path to get around the yellow path. Similarly, the southwest corner is to be used to allow the blue

R							
						Y	
		B					
G							G
			R				
	W					W	
		B	Y				

path to get around the white path, and the southeast corner is to be used to allow the yellow path to get around the white path. In fact, we can complete the entire yellow path. Also, the white path may as well make full use of the seventh row. This brings us to the configuration as shown in the diagram below on the left. It is now not hard to complete the entire configuration, which is shown in the diagram below on the right. The longest path is the green one, and the number of green squares is 24.



R	R	R	R	R	G	G	G
G	G	G	G	R	G	Y	G
G	B	B	G	R	G	Y	G
G	B	G	G	R	G	Y	G
B	B	G	R	R	G	Y	Y
B	W	G	G	G	G	W	Y
B	W	W	W	W	W	W	Y
B	B	B	Y	Y	Y	Y	Y

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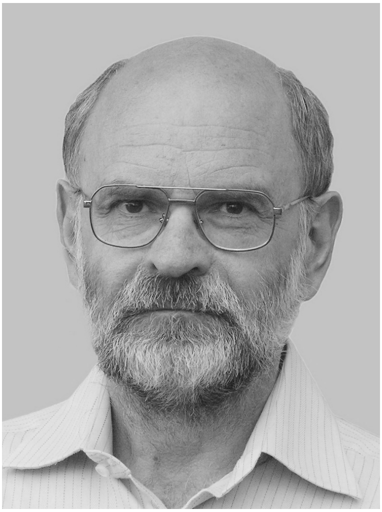
The Similarity Coefficients Method

Pavel Leischner



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1 Introduction

An application of similarity is a strong and old tool in mathematics. A Greek philosopher, Thales of Miletus (circa 580 BCE) was probably first man who used that knowledge in practical ways. Stories are told that he discovered how to obtain the height of pyramids (and the other similar objects) by measuring the shadow of the object at the time when the body and its shadow were equal in length. He also allegedly showed, by similarity, how to find the distances of ships at sea. The other great Greek, Eratosthenes of Alexandria (276–194 BCE), used similarity when he took the first measurements of the circumference of the globe.

Unfortunately, the teaching of this part of mathematics at Czech high and secondary schools is a bit neglected these days. It is not sufficient for secondary school students who take part in mathematical olympiads and similar competitions. To get better results in work with such students, we recommend to acquaint them with the similarity coefficients method:

Suppose figures $\mathcal{M}_i$ $(i = 1, 2, \ldots, n)$ are given and each of them is similar to a given figure $\mathcal{M}$. Similarity coefficients k_i are

$$k_i = \frac{x_i}{x}, \tag{1}$$

where $x_i \in \mathcal{M}_i$ and $x \in \mathcal{M}$ denote related lengths in the corresponding similarity. If some relation is true for the similarity coefficients, then applying some substitutions from (1) to this relation we can get a lot of useful relations for lengths of figures $\mathcal{M}_i$ and corresponding lengths of $\mathcal{M}$. A similarity coefficient is namely the ratio of two related *arbitrary* (non-zero) lengths of given similar forms, and according to a choice of these lengths, we can get different metric relations in the set of figures $\mathcal{M}_1, \mathcal{M}_2, \ldots, \mathcal{M}_n, \mathcal{M}$.

2 Triangles and tetrahedra

For better understanding of the similarity coefficient method some problems are solved in this part.

Problem 1

In a triangle ABC with $|AB| = c, |BC| = a$ and $|CA| = b$, there are drawn three lines tangent to the inscribed circle and parallel to the sides, cutting three small triangles off the corners of the given triangle as shown in Figure 1. These small triangles have inscribed circles too. Find the sum of the radii of all four inscribed circles.

(Similar problems: 6th IMO–1964 and 35th CZE–SVK MO–1985/86, B–I–1.)

Solution. The notation is given in Figure 1. The perimeter of the triangle $A_1B_1C_1$ is

$$p_1 = |A_1B_1| + |B_1T_1| + |T_1C_1| + |C_1A_1|.$$

According to equality of length of tangent line segments we can rewrite it in the form

$$p_1 = (|A_1B_1| + |B_1K|) + (|A_1C_1| + |C_1M|) = |AK| + |AM| = 2|AK|.$$

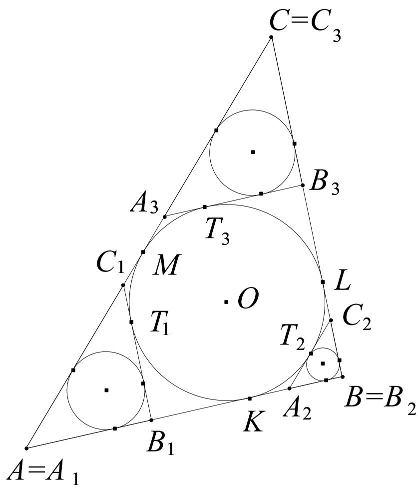

Figure 1

In the same way

$$p_2 = |BK| + |BL| = 2|BL| \qquad \text{and} \qquad p_3 = |CL| + |CM| = 2|CM|.$$

Now we can express the perimeter p of the given triangle ABC in terms p_i $(i = 1, 2, 3)$. We have

$$p_1 + p_2 + p_3 = 2|AK| + 2|BL| + 2|CM| = p$$

or, after we divide both sides by p and use (1),

$$k_1 + k_2 + k_3 = 1. \tag{2}$$

This is our desired relation. Denoting $k_i = r_i/r$ where r_1, r_2, r_3 and r are radii of the inscribed circles we obtain

$$r_1 + r_2 + r_3 = r. \tag{3}$$

From two different formulas for the area of the triangle ABC we obtain

$$r = \frac{\sqrt{s(s-a)(s-b)(s-c)}}{s} = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}},$$

where

$$s = \frac{a+b+c}{2}.$$

Now we can easily express the sum of the radii:

$$r_1 + r_2 + r_3 + r = 2r = 2\sqrt{\frac{(s-a)(s-b)(s-c)}{s}}.$$

There are some different ways to derive equality (2). Two of them follow:

1. Consider the symmetry of triangles ABC and $A'B'C'$ (see Figure 2) with the centre O (incentre of triangle ABC). The segments A_3B_3 and A_2B_1 are symmetric, therefore $|A_2B_1| = |A_3B_3| = k_3 \cdot |AB|$.

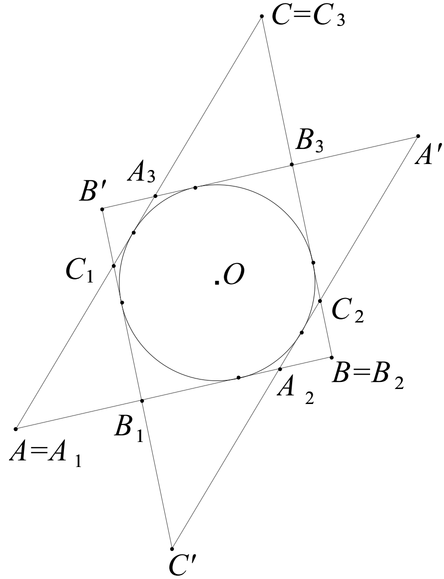

Figure 2

Hence

$$k_1|AB| + k_3|AB| + k_2|AB| = |AB_1| + |B_1A_2| + |A_2B| = |AB|,$$

whence follows (2).

2. Obviously

$$k_1 = \frac{v_{a_1}}{v_a} = \frac{v_a - 2r}{v_a} = 1 - \frac{2r}{v_a} = 1 - \frac{a}{s},$$

because the formula for the area S of the triangle ABC

$$S = \frac{1}{2} a v_a = sr$$

holds. Analogously

$$k_2 = 1 - b/s \qquad \text{and} \qquad k_3 = 1 - c/s.$$

Adding all three expressions we get $k_1 + k_2 + k_3 = 3 - (a+b+c)/s = 1$, as desired.

Remark. It must be strongly emphasized that the equality (2) gives much more than (3). If, for example, R_1, R_2, R_3, R and S_1, S_2, S_3, S are circumradii and areas of triangles $A_1B_1C_1$, $A_2B_2C_2$, $A_3B_3C_3$ and ABC respectively, we can obtain

$$R_1 + R_2 + R_3 = R \qquad \text{and} \qquad \sqrt{S_1} + \sqrt{S_2} + \sqrt{S_3} = \sqrt{S} \qquad \text{etc.}$$

In a similar way we can rewrite (2) as

$$\sqrt[3]{k_1 \cdot k_1 \cdot k_1} + \sqrt[3]{k_2 \cdot k_2 \cdot k_2} + \sqrt[3]{k_3 \cdot k_3 \cdot k_3} = 1,$$

and we get

$$\sqrt[3]{\frac{h_{a_1}}{h_a} \cdot \frac{h_{b_1}}{h_b} \cdot \frac{h_{c_1}}{h_c}} + \sqrt[3]{\frac{h_{a_2}}{h_a} \cdot \frac{h_{b_2}}{h_b} \cdot \frac{h_{c_2}}{h_c}} + \sqrt[3]{\frac{h_{a_3}}{h_a} \cdot \frac{h_{b_3}}{h_b} \cdot \frac{h_{c_3}}{h_c}} = 1,$$

where h_a, h_b, h_c are altitudes of triangle ABC and h_{a_i}, h_{b_i}, h_{c_i} are altitudes of triangle $A_iB_iC_i$. The last equation can be rewritten in the form

$$\sqrt[3]{h_{a_1} \cdot h_{b_1} \cdot h_{c_1}} + \sqrt[3]{h_{a_2} \cdot h_{b_2} \cdot h_{c_2}} + \sqrt[3]{h_{a_3} \cdot h_{b_3} \cdot h_{c_3}} = \sqrt[3]{h_a \cdot h_b \cdot h_c}.$$

Problem 2

In a tetrahedron $A_1A_2A_3A_4$ four planes are drawn parallel to the faces of the tetrahedron touching the inscribed sphere, cutting four small tetrahedra off the corners of the given tetrahedron. Find the sum of the inradii of all four small tetrahedra.

Solution. Let O, r and S be the incentre, inradius and surface area of the given tetrahedron. Denote P_1, P_2, P_3 and P_4 as the areas of triangles $A_2A_3A_4$, $A_1A_3A_4$, $A_1A_2A_4$ and $A_1A_2A_3$ respectively. Obviously

$$P_1 + P_2 + P_3 + P_4 = S, \quad \text{or} \quad \sum_{i=1}^{4} \frac{P_i}{S} = 1. \tag{4}$$

The sum the the volumes of tetrahedra $A_1A_2A_3O$, $A_2A_3A_4O$, $A_3A_4A_1O$ and $A_4A_1A_2O$ is equal to volume V of tetrahedron $A_1A_2A_3A_4$. Therefore

$$V = \frac{r}{3}(P_1 + P_2 + P_3 + P_4) = \frac{r}{3}S \tag{5}$$

The small tetrahedron, which has vertex A_1 and altitude h_1 from this vertex, is similar to the given tetrahedron. The similarity coefficient is $k_1 = \frac{h_1}{v_1}$, where v_1 is the corresponding altitude of the tetrahedron $A_1A_2A_3A_4$. Analogically, the small tetrahedra with the vertices B, C and D have similarity coefficients k_2, k_3 nad k_4 respectively, and the pairs of the corresponding altitudes are denoted by the same way. It follows from (5) and formula $V = \frac{1}{3}P_iv_i$, that

$$\frac{r}{v_i} = \frac{\frac{1}{3}P_ir}{\frac{1}{3}P_iv_i} = \frac{\frac{1}{3}P_ir}{\frac{1}{3}Sr} = \frac{P_i}{S},$$

so analogically, as in problem 1, we have

$$k_i = \frac{h_i}{v_i} = 1 - \frac{2r}{v_i} = 1 - \frac{2P_i}{S}.$$

Hence

$$\sum_{i=1}^{4} k_i = 4 - 2\sum_{i=1}^{4} \frac{P_i}{S} = 4 - 2,$$

whence

$$\sum_{i=1}^{4} k_i = 2. \tag{6}$$

Remark. The last method leads to the following n-dimensional generalization:

$$\sum_{i=1}^{n} k_i = n-1, \tag{7}$$

where k_i are similarity coefficients of the small n-dimensional simplexes cut off from an n-dimensional simplex by planes parallel to the faces of the simplex and tangent to its inscribed sphere. It is also possible to use the known relation $\sum_{i=1}^{n} \frac{1}{h_i} = \frac{1}{r}$, where h_i are the altitudes of the simplex and r is the inradius.

Problem 3

Let ABC be a triangle with circumradius R and M let be an interior point of ABC. The lines through M and parallel to the sides AB, BC and CA determine triangles $A_1B_1C_1$, $A_2B_2C_2$ and $A_3B_3C_3$, similar to triangle ABC, as shown in Figure 3 ($A_1 = B_2 = C_3 = M$). Let the circumradii of these triangles be R_1, R_2 and R_3. Show that

$$R_1 + R_2 + R_3 = R.$$

Solution. Obviously (see Figure 3) $|CC_2| = |C_1M| = k_1b$, where $b = |AC|$ and k_1 is a corresponding similarity coefficient. In the same way $|C_2A_2| = k_2b$ and $|A_2A| = k_3b$. Thus we have

$$|CC_2| + |C_2A_2| + |A_2A| = k_1b + k_2b + k_3b = |AC| = b,$$

hence

$$k_1 + k_2 + k_3 = 1. \tag{8}$$

Substituting $k_i = R_i/R$, ($i = 1, 2, 3$) we obtain the desired equality.

Remark. The relation (8) again facilitates finding a lot of other equalities (not only $R_1 + R_2 + R_3 = R$). We next show a generalization of (8) for tetrahedra. The following solution is suitable also for triangles.

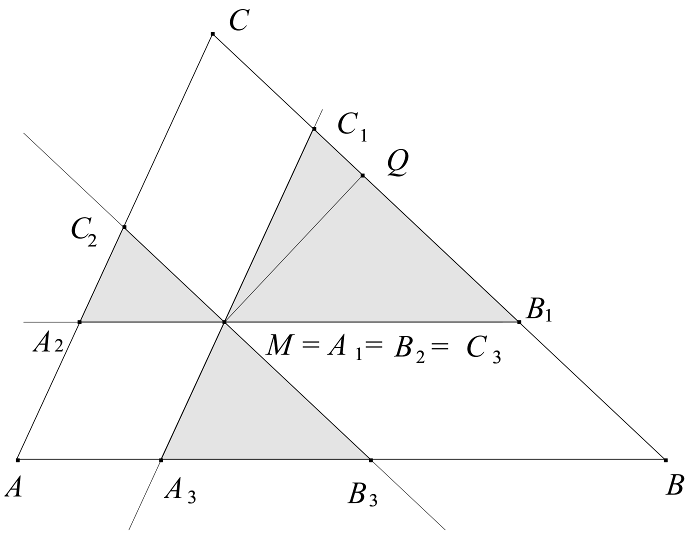

Figure 3

Problem 4

Let $ABCD$ be an arbitrary tetrahedron with an interior point M. The planes drawn through M and parallel to the faces ABC, BCD, CAD and DBA determine tetrahedra I, II, III *and* IV, *which are similar to the tetrahedron $ABCD$ (see Figure 4). Denote k_1, k_2, k_3 and k_4 similarity coefficients, respectively. Show that*

$$k_1 + k_2 + k_3 + k_4 = 1.$$

Solution. Let the altitude of the given tetrahedron $ABCD$ from vertex X or the altitudes of tetrahedra I, II, III, IV from vertex X be v_X or h_{1_X}, h_{2_X}, h_{3_X}, h_{4_X}, respectively. We can see in Figure 4 that the volume of tetrahedron $BCDM$ is

$$V_{BCDM} = \frac{1}{3} S_{BCD} h_{1_M} = k_1 \cdot \frac{1}{3} S_{BCD} v_A = k_1 V.$$

Similarly $V_{ACDM} = k_2 V$, $V_{ABDM} = k_3 V$ and $V_{ABCM} = k_4 V$. Thus we

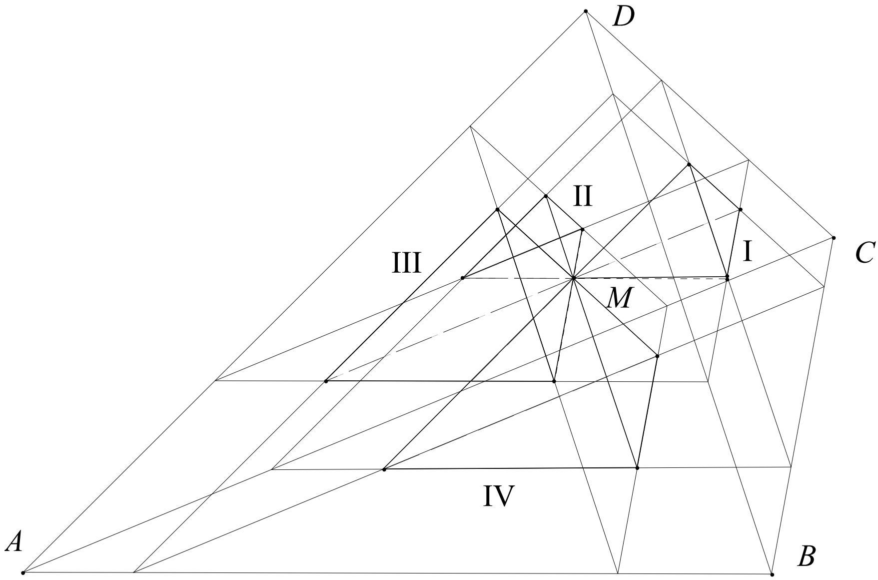

Figure 4

have

$$V_{BCDM} + V_{ACDM} + V_{ABDM} + V_{ABCM} = (k_1 + k_2 + k_3 + k_4)V = V$$

and finally

$$k_1 + k_2 + k_3 + k_4 = 1. \tag{9}$$

Remark. In the same way we can obtain next relation for an n-dimensional simplex

$$\sum_{i=1}^{n} k_i = 1. \tag{10}$$

3 A generalization of Pythagoras' theorem

Pythagoras' theorem is one of the most well known and oldest mathematical theorems. Over the course of many years it has been proved and generalized in many various ways. We are reminded of

Pappus' generalization (circa 320 ADE), the law of cosines or different spatial analogs. In spite of this, the topic of the Pythagoras' theorem hasn't been entirely exhausted yet.

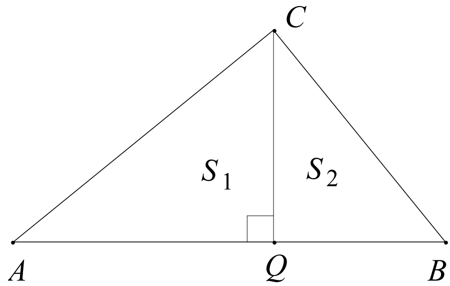

Figure 5

Consider the situation in Figure 5. The altitude CQ divides a right-angled triangle ABC into triangles ACQ and CBQ, which are similar to the original triangle ABC. Similarity coefficients k_1 and k_2 of triangles ACQ, ABC and CBQ, ABC fulfill $k_1^2 = S_1/S$ and $k_2^2 = S_2/S$, where S_1, S_2 and S are the areas of triangles ACQ, CBQ and ABC. From the equality

$$S_1 + S_2 = S$$

it follows

$$k_1^2 + k_2^2 = 1. \tag{11}$$

This relation presents a first generalization. The substitution $k_1 = \frac{b}{c}$ and $k_2 = \frac{a}{c}$ into (11) yields Pythagoras' theorem

$$a^2 + b^2 = c^2. \tag{12}$$

The substitution $k_1 = \cos\alpha$ and $k_2 = \cos\beta$ gives a basic relation of trigonometry

$$\cos^2\alpha + \cos^2\beta = 1$$

or

$$\cos^2\alpha + \sin^2\alpha = 1. \tag{13}$$

Our generalization of Pythagoras' theorem was known to Euclid, but in a geometrical form only. In his *Elementa* ([2], the proposition VI. 31) he says:

"In right-angled triangles the figure on the side subtending the right angle is equal to the similar and similarly described figures on the sides containing the right angle"

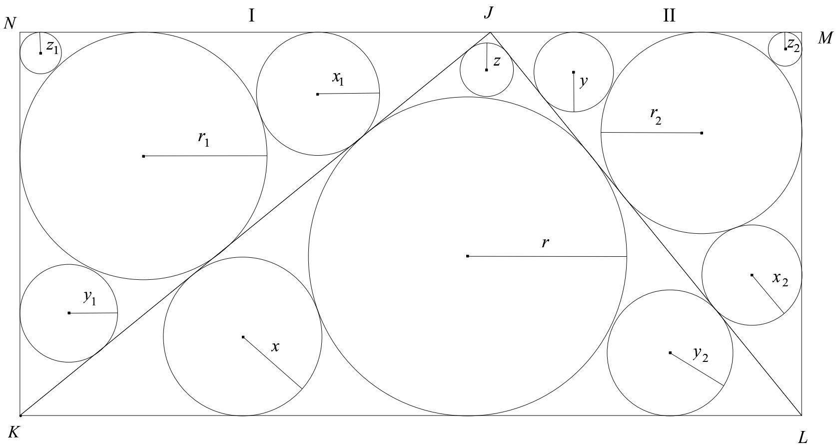

Figure 6

This Euclid theorem is clearer than the strict and abstract formula (11) but the algebraic expression gives wider applications. Considering the situation in Figure 6, for instance, where KLJ, LJM and JKN are similar rectangular triangles we can easily find using (11), the next relation probably unknown to Euclid

$$\frac{x_1y_1z_1}{r_1} + \frac{x_2y_2z_2}{r_2} = \frac{xyz}{r}.$$

Now, we shall consider the more general case. Pythagoras' Theorem (12) is a special case of the law of cosines

$$c^2 = a^2 + b^2 - 2ab\cos\gamma,$$

holding for an *arbitrary* triangle. Our idea is to find an analog of the formula (11) in the case an *arbitrary* triangle.

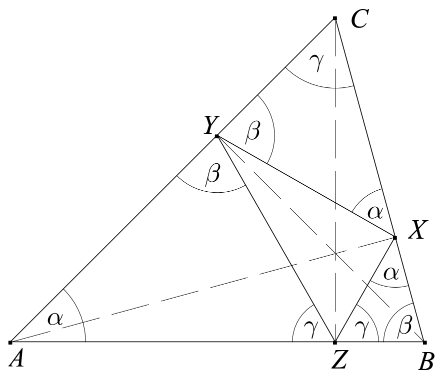

Figure 7

Let ABC be an arbitrary triangle and let AX, BY and CZ denote the altitudes. First we shall suppose that triangle ABC is acute-angled as shown in Figure 7. This is well known, that the triangles AYZ, XBZ and XYC are similar to the original triangle ABC. It is seen from the fact that the quadrilaterals $ABXY$, $BCYZ$ and $CAZX$ are inscribed into Thalet circles with diameters AB, BC and CA respectively. Therefore $|\angle ABX| + |\angle AYX| = \pi$ which implies $|\angle AYX| = \pi - \beta$. From here it follows that $|\angle CYX| = \beta$. Denote

$$\begin{aligned} k_1 &= \frac{|AZ|}{|AC|} = \frac{|YZ|}{|BC|} = \frac{|AY|}{|AB|} = \cos\alpha, \\ k_2 &= \frac{|BZ|}{|BC|} = \frac{|XZ|}{|AC|} = \frac{|BX|}{|AB|} = \cos\beta, \\ k_3 &= \frac{|CY|}{|BC|} = \frac{|XY|}{|AC|} = \frac{|CX|}{|AC|} = \cos\gamma. \end{aligned} \tag{14}$$

For the areas S_1, S_2, S_3, S_4 and S of the triangles AYZ, XBZ, XYC, XYZ and ABC the equality

$$S_1 + S_2 + S_3 + S_4 = S \tag{15}$$

holds. We have

$$S_1 = k_1^2 S, \quad S_2 = k_2^2 S, \quad S_3 = k_3^2 S. \tag{16}$$

Finally, we evalute the area S_4 of triangle XYZ. We have

$$\begin{aligned} 2S_4 &= |XY| \cdot |XZ| \cdot \sin(\pi - 2\alpha) = k_3|AB| \cdot k_2|AC| \cdot \sin 2\alpha \\ &= (k_2 k_3 \cos\alpha) \cdot |AB| \cdot |AC| \sin\alpha = 4k_1k_2k_3S. \end{aligned}$$

Hence

$$S_4 = 2k_1k_2k_3S. \tag{17}$$

By substituting (17) and (16) in the formula (15) the final result is

$$k_1^2 + k_2^2 + k_3^2 + 2k_1k_2k_3 = 1. \tag{18}$$

The relation (18) is a desired analog of the equality (11). According to (18), we get similarly

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma + 2\cos\alpha\cos\beta\cos\gamma = 1, \tag{19}$$

which is an analog of relation (13).

Now we consider an obtuse-angled triangle. In this case we can suppose that $\gamma > \pi/2$. From the Figure 8 we can see, that the expression

$$S_1 + S_2 + S_3 - S_4 = S$$

holds. The formula (19) is true in this case because $\cos\gamma < 0$. If we

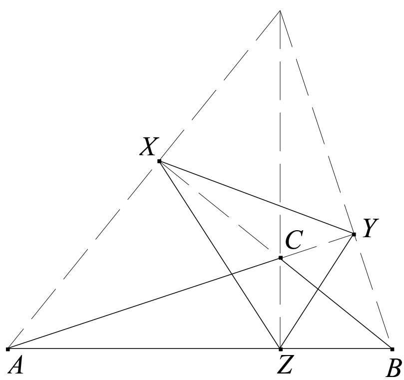

Figure 8

require also the validity of relation (12), we have to consider $k_3 < 0$. It will be useful to make a small comparison of Figures 7 and 8. We can see that there is a similarity $\varepsilon_3 : \triangle ABC \to \triangle XYC$, which is the product of both transformations, that is, the reflection φ_3 in the angle bisector of $\angle ACB$ and the homothety ψ_3 of center C and ratio h_3, where $|h_3| = k_3$. Obviously $h_3 > 0$ iff $\angle ACB$ is an acute angle and $h_3 < 0$ iff $\angle ACB$ is an obtuse angle.

Analogously for the triangles AYZ and XBZ there exist corresponding similarities ε_1 and ε_2, reflections φ_1 and φ_2 and finally homotheties ψ_1 and ψ_2. Thus we can establish a next convention:

The similarity coefficients k_1, k_2 and k_3 are real numbers and $k_i = h_i \quad (i = 1, 2, 3)$. In other words, $k_i > 0$, ($k_i < 0$) iff the corresponding angle is an acute (obtuse) angle, $k_i = 0$ iff the corresponding angle is a right angle.

Applying this convention we can use relation (18) for an *arbitrary* triangle. Notice that in the case of a right-angled triangle we get relation (11) or an analogical equality.

Remark. The equality (19) is well known in trigonometry. We can prove it directly in the following way. In the previous notation we have

$$a = c\cos\beta + b\cos\gamma, \quad b = a\cos\gamma + c\cos\alpha, \quad c = b\cos\alpha + a\cos\beta,$$

which is the same as

$$a = ck_2 + bk_3, \quad b = ak_3 + ck_1, \quad c = bk_1 + ak_2.$$

Multiplying the last three expressions we get

$$\begin{aligned} abc &= (ck_2 + bk_3)(ak_3 + ck_1)(bk_1 + ak_2) \\ &= 2abck_1k_2k_3 + abk_3^2(bk_1 + ak_2) + bck_1^2(ck_2 + bk_3) + ack_2^2(ak_3 + ck_1) \\ &= abc(2k_1k_2k_3 + k_1^2 + k_2^2 + k_3^2), \end{aligned}$$

from which follows the relation (19).

4 The Euclid extremal theorem and its generalization

> Especially interesting is Prop. VI. 27 of *Euclid Elements*, which contains, including proof, the first known in the history of mathematics maximum problem that the square encloses the largest area among all rectangles with a prescribed perimeter.
>
> Dirk J. Struik, *History of Mathematics*[1]

It could be an inexact translation or an inaccurate formulation by Struik in the sentence above from which some people presume two propositions:

1. The Proposition VI. 27 of *Euclid's Elements* is the first known in the history of mathematics maximum problem.

2. The Proposition VI. 27 says: "The square encloses the largest area among all rectangles with a prescribed perimeter."

The first statement isn't quite correct, the second isn't the statement of the Proposition VI. 27.

As for the first, it is the isoperimetric problem which is known in the history of mathematics as an oldest maximum problem, that the semicircle encloses the largest area among all figures limited by given line and a curve with prescribed length. It is well known as the problem of princess Dido from a myth about how Carthago was founded.

As for the second, let us introduce correct text of Prop. VI. 27 in accordance with Ralph H. Abraham [3]:

Proposition VI. 27
Of all the parallelograms applied to the same straight line and deficient by parallelogramic figures similar to and similarly situated to that described on the half of the straight line, that parallelogram is greatest which is applied to the half of the straight line and is similar to the defect.

The proposition is something of a muddle because of the old way of description. The following proof gives a sense of the theorem.

[1] Translation from Czech edition [1] into English.

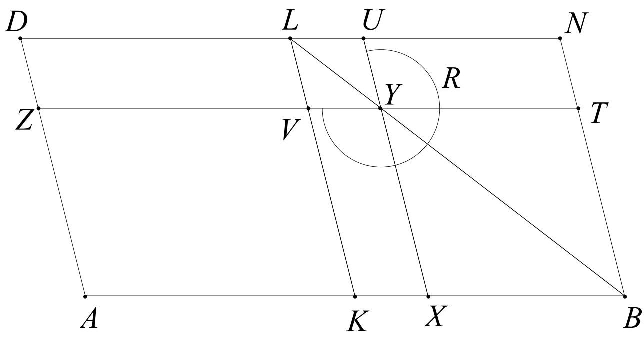

Figure 9

Proof. (By Euclid [2], [3])[2] Let AB be a straight line and let it be bisected at K. Let there be applied to the straight line AB the parallelogram AL falling short by the parallelogramic figure LB described on the half of AB, that is, KB. (See Figure 9.) I say that, of all the parallelograms applied to AB falling short by parallelogramic figures similar and similarly situated to LB, AL is greatest. Let there be applied to the straight line AV the parallelogram AY falling short by the parallelogrammic figure YB similar and similarly situated to LB. I say that AL is greater than AY.

Since the parallelogram LB is similar to the parallelogram YB, therefore they have about the same diameter. Draw their diameter LB and describe the figure. Then, since parallelogram CY equals YN, and YB is common, therefore the whole KT equals the whole XN. But KT equals CZ, since AK also equals KB. Therefore CZ also equals XN. Add CY to each. Therefore the whole AY equals the gnomon VRU,[3] so that the parallelogram LB, that is, AL, is greater than the parallelogram AY.

Therefore: *Of all the parallelograms applied to the same straight line and deficient by parallelogramic figures similar to and similarly situated to that described on the half of the straight line, that parallelogram is greatest which is applied to the half of the straight line and is similar to the defect.*

[2] Distinguish similar denotation of straight lines and parallelograms, please. A straight line XY usually means a segment XY.

[3] It means the polygon $UYVKBN$.

Probably, the theorem and its proof are not still quite obvious because of the archaic way of notation. Therefore a modern version of the Prop. VI. 27 will be introduced below (see Proposition 1), which is equivalent to the original theorem. We show how the statement in the beginning of the paper follows from it, give a proof and also a generalization of the next proposition.

Proposition 1
A parallelogram $AKLM$, inscribed in the given triangle ABC with vertices K, L and M laying inside segments AB, BC and CA respectively, has the greatest area iff its vertex L is the midpoint of the side BC.

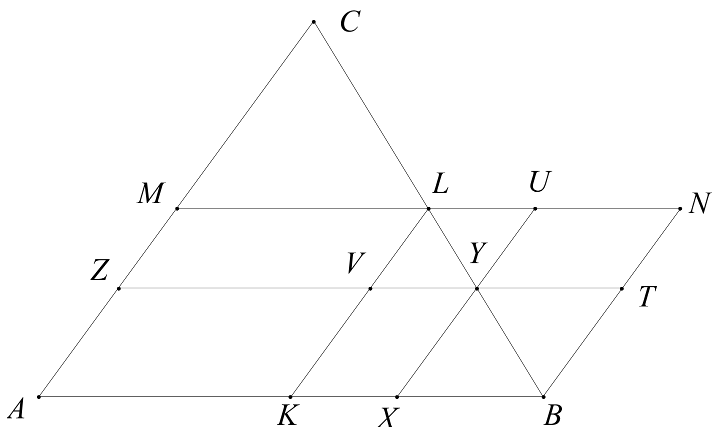

Figure 10

Proof. Let the side BC of the given triangle ABC be bisected at a vertex L of parallelogram $AKLM$ (see Figure 10). Consider another parallelogram $AXYZ$, inscribed in triangle ABC in the same way. We shall prove, if $Y \neq L$, than the area S_{AXYZ} of parallelogram $AXYZ$ is less than the area S_{AKLM} of parallelogram $AKLM$. Draw parallelograms $XBTY$ and $YTNU$ as Figure 10 shows. Denote S_{KBL} the area of triangle KBL and so on. We have $S_{KXYV} = S_{KBL} - (S_{XBY} + S_{VYL}) = S_{LBN} - (S_{YBT} + S_{LYU}) = S_{YTNU} < S_{VTNL} = S_{ZVLM}$. Hence $S_{AXYZ} = S_{AKLM} + S_{KXYV} - S_{ZVLM} < S_{AKLM}$, which was desired.

Note if $\triangle ABC$ is an *arbitrary* triangle, the segments LV and YV must not be equal. Therefore parallelograms $AKLM$ and $AXZY$ must not the same perimeter and the Proposition 1 is unisoperimetric generally. But if $\triangle ABC$ has $|AB| = |AC|$, the perimeters of parallelograms $AKLM$ and $AXZY$ are the same. Thus we have the next isoperimetric result:

Corollary 1
The rhomb encloses the largest area among all parallelograms $ABCD$ with a constant measure of the angle DAB and a prescribed perimeter.

Now if $\angle BAC$ is right in addition, we obtain the proposition which D. J. Struik showed:

Corollary 2
The square encloses the largest area among all rectangles with a prescribed perimeter.

Apparently, Propositon VI. 27 of Euclid *Elementa* is unisoperimetric and it could not be interchanged with Corollary 2. The equivalent Proposition 1 gives the maximum area of parellelogram $AKLM$ iff L is in the centre of BC, that is, iff L is the barycentrum of the side BC of triangle ABC. This evokes the next generalization:

Proposition 2
Let $ABCD$ be a tetrahedron and L a point inside the face BCD. The other faces of the tetrahedron and the planes parallel with them, which pass through L, define a parallelepiped $\mathcal{M} = AEFGJKLM$, inscribed in the tetrahedron $ABCD$, as in Figure 11.

The volume of the parallelepiped $\mathcal{M}$, is maximal iff L is the barycentrum of the triangle BCD.

Probably it will be unsuccessful to try to prove the last statement in a similar way to Proposition 1. Let us show two different proofs, easily applicable in the case of Proposition 1 too.

Proof. (Similarity coefficients method.) Let the planes JKL and EFL intersect the edge BD in points P and X, as shown in Figure 11. The planes through P and X and parallel with the faces ACD and ABC respectively, cut off the tetrahedrons $QBRP$ and $ZXYD$ in the corners of the given tetrahedron. The tetrahedra $ZXYD$, $KPLX$ and

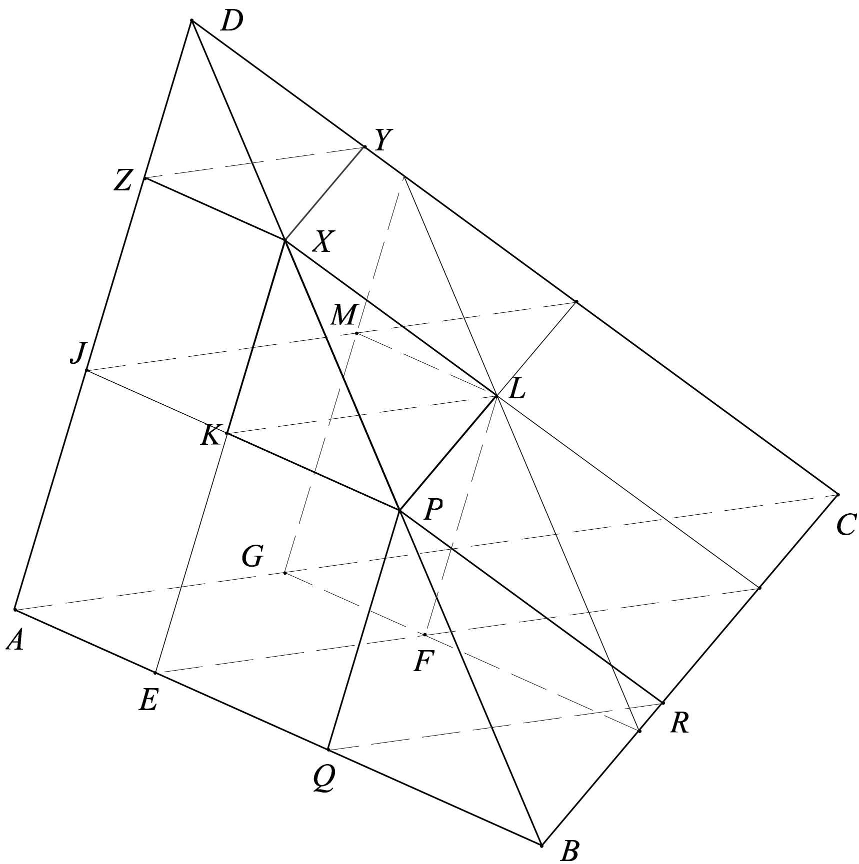

Figure 11

$QBRP$ are similar to $ABCD$. Similarity coefficients corresponding to tetrahedrons $ZXYD$, $KPLX$ and $QBRP$ denote k_1, k_2 and k_3 respectively. Obviously, (see Figure 11),

$$k_1 + k_2 + k_3 = \frac{|DX|}{|BD|} + \frac{|XP|}{|BD|} + \frac{|PB|}{|BD|} = 1. \tag{20}$$

The volume of the parallelepiped $\mathcal{M}$ is $V = |AE| \cdot |AG| \cdot |AJ| \cdot \sin\alpha \cdot \sin\varepsilon$, where $\alpha = |\angle BAC|$ and ε denotes the deviation AD from ABC. Next we have $|AE| = |ZX| = k_1 \cdot |AB|$, $|AG| = |KL| = k_2 \cdot |AC|$ and $|AJ| = |PQ| = k_3 \cdot |AD|$. Therefore $V = 6\,V_0 k_1 k_2 k_3$, where $V_0 = \frac{1}{6}\,|AB|\cdot|AC|\cdot|AD|\cdot\sin\alpha\cdot\sin\varepsilon$ is the volume of the parallelepiped $ABCD$. Now, using arithmetic mean–geometric mean (AM–GM) inequality and relation (20) we will obtain

$$V = 6\,V_0 k_1 k_2 k_3 \le 6\,V_0 \left(\frac{k_1 + k_2 + k_3}{3}\right)^3 = \frac{2}{9} V_0.$$

Hence we get maximal volume $V_{\max} = \frac{2}{9} V_0$ iff $k_1 = k_2 = k_3 = \frac{1}{3}$. In other words the volume of the parallelepiped $\mathcal{M}$ is maximal iff

$$\overrightarrow{AL} = \frac{1}{3}(\overrightarrow{AB} + \overrightarrow{AC} + \overrightarrow{AD}), \tag{21}$$

that is, iff L is situated in the barycentrum of the face BCD.

Another proof. (Method of coordinates) Let $\langle A, \vec{i}, \vec{j}, \vec{k}\,\rangle$, be a linear system of coordinates, where $\vec{i}$, $\vec{j}$ and $\vec{k}$ are unit vectors parallel and correspondingly parallel with the rays AB, AC and AD. Let $B = [b, 0, 0]$, $C = [0, c, 0]$, $D = [0, 0, d]$ and $L = [x, y, z]$. Obviously $|AB| = b$, $|AC| = c$ and $|AD| = d$. Since L belongs to the triangle BCD, applying the equation of plane BCD we get:

$$\frac{x}{b} + \frac{y}{c} + \frac{y}{d} = 1, \qquad x > 0, y > 0, y > 0. \tag{22}$$

Now, the volume of the parallepiped $\mathcal{M}$ is

$$V = 6 \cdot \frac{x}{b} \cdot \frac{y}{c} \cdot \frac{z}{d} \cdot \frac{bcd}{6} \cdot \sin\alpha \cdot \sin\varepsilon \le 6V_0 \left(\frac{\frac{x}{b} + \frac{y}{c} + \frac{z}{d}}{3}\right)^3 = \frac{2}{9} V_0,$$

where α, ε and V_0 denotes the same as in the first proof. Finally, the rest of our proof follows the same way.

Remark. The method of coordinates proof leads to the following generalization.

Let $\langle A, \vec{i}_1, \vec{i}_2, \ldots, \vec{i}_n \rangle$ be a Cartesian system of coordinates in n-dimensional Euclidean space. The origin A and points $A_r =$

$[0, 0, \ldots, 0, a_r, 0, \ldots, 0]$, $a_r > 0$ $(r = 1, 2, \ldots, n)$ are vertices of a simplex $\mathcal{A} = AA_1A_2 \ldots A_n$ which has volume

$$V_0 = \frac{1}{n!} a_1 a_2 \ldots, a_n. \tag{23}$$

One of the faces of simplex $\mathcal{A}$ is an $(n-1)$-dimensional simplex $\mathcal{S} = A_1A_2 \ldots A_n$, with a barycentrum

$$T = \frac{1}{n} \sum_{r=1}^{n} A_r = \left[\frac{a_1}{n}, \frac{a_2}{n} \ldots \frac{a_n}{n}\right]. \tag{24}$$

Let $L = [x_1, x_2, \ldots x_n]$ be a point inside $\mathcal{S}$. Obviously $x_i > 0$ $(i = 1, 2, \ldots, n)$ and L is on the hyperplane which contains $\mathcal{S}$. Thus we have

$$\sum_{r=1}^{n} \frac{x_r}{a_r} = 1. \tag{25}$$

Hyperplanes through L, which are perpendicular to the coordinate axes contain the points $L_r = [0, 0, \ldots, x_r, 0, \ldots, 0]$, $(r = 1, 2, \ldots, n)$ and define together the coordinate hyperplanes n-dimensional parallelepiped $\mathcal{M}$. Applying (23), AM–GM inequality and (25), we get the volume of $\mathcal{M}$:

$$\begin{aligned} V = x_1 x_2 \ldots x_n &= \frac{x_1}{a_1} \cdot \frac{x_2}{a_2} \ldots \frac{x_n}{a_n} \cdot V_0 \cdot n! \\ &\le \left(\frac{\frac{x_1}{a_1} + \frac{x_2}{a_2} + \ldots + \frac{x_n}{a_n}}{n} \right)^n \cdot V_0 \cdot n! = \frac{n!}{n^n} \cdot V_0. \end{aligned}$$

The last relation fulfills equality iff

$$\frac{x_r}{a_r} = \frac{1}{n} \qquad (r = 1, 2, \ldots n).$$

According to (24) it means $L = T$, as we desired.

5 Conclusion

Let us consider the last methods. If we directly use synthetic geometry, our steps are vivid, clear. In Euclid's method, for instance, we can see the proof of Proposition 1 from Figure 10 directly. The idea, in its visual form, is simpler than a corresponding algebraic entry. It is not necessary to know how to work with mathematical symbols and algebraic expressions if you have some geometric perception and you can take information from a picture, a scheme or your image.

Algebraic methods are opposed. An abstraction is characteristic for them. The same results you can find here by using an easy survey work with symbols and algebraic expressions. But you can get much more results and simultaneously you can solve a bigger range of problems.

The similarity coefficients method is something like a bridge between both methods. It is built on visual and objective ideas, but uses algebra to get results which we cannot see immediately. From this viewpoint, the similarity coefficients method is very useful for younger students of mathematics.

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[1] *Struik, D. J.*: Dějiny matematiky. (Czech translation of the *History of Mathematics*.) Orbis, Praha 1963

[2] http//aleph0.clarku.edu/˜djoyce/java/elements/elements.html

[3] http://thales.vismath.org/euclid/

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Tournament of Towns Corner

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1 Selected Problems from the Second Round of Tournament 29

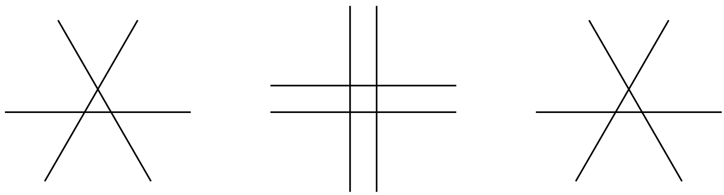
In the second round of Tournament 29, both Junior and Senior O Level papers consisted of five problems, while both Junior and Senior A Level papers were made up of seven problems. Below are selected questions with solutions from the second round of Tournament 29.

1. Can it happen that the least common multiple of $1, 2, \ldots, n$ is 2008 times the least common multiple of $1, 2, \ldots, m$ for some positive integers m and n?

Solution. Let the highest power of 2 less than or equal to m be 2^r. Since $2008 = 2^3 \cdot 251$, the highest power of 2 less than or equal to n must be 2^{r+3}. It follows that $n > 4m$. Let the highest power of 3 less than or equal to m be 3^s. Then the highest power of 3 less than or equal to n must also be 3^s since 3 does not divide 2008. However, $n > 4m \geq 4 \cdot 3^s > 3^{s+1}$, which is a contradiction. Hence no such positive integers m and n exist.

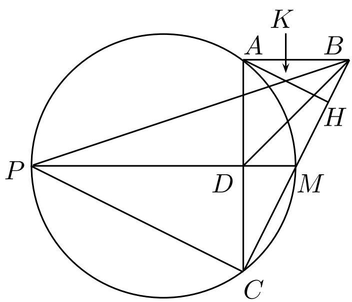
2. There are ten congruent segments on a plane. Each point of intersection divides every segment passing through it in the ratio 3:4. Find the maximum number of points of intersection.

Solution. On each segment, there are exactly two points which divide it in the ratio 3:4. Hence the total count segment by segment is at most 20. However, it takes two segments to produce a point of intersection. Hence there are at most 10 such points. The diagram below shows how this can be attained, so that 10 is indeed the maximum.



3. In triangle ABC, $\angle A = 90°$. M is the midpoint of BC and H is the foot of the altitude from A to BC. The line passing through M and perpendicular to AC meets the circumcircle of triangle AMC again at P. If BP intersects AH at K, prove that $AK = KH$.

Solution.



Since both AB and MP are perpendicular to AC and $BM = MC$, MP intersects AC at its midpoint D. It follows that MP is a diameter of the circumcircle, so that MC is perpendicular to PC. It follows that triangles MCD and MPC are similar, so that $\frac{MD}{MC} = \frac{MC}{MP}$. Hence $\frac{MD}{MB} = \frac{MB}{MP}$. Since $\angle DMB = \angle BMP$, triangles DMB and BMP are also similar. It follows that

$\angle CBD = \angle BPM = \angle ABK$. Now triangles BAH and BCA are also similar. Since $CD = DA$, we have $AK = KH$.

4. No matter how two copies of a convex polygon are placed inside a square, they always have a common point. Prove that no matter how three copies of the same polygon are placed inside this square, they also have a common point.

Solution. Let a copy F of the convex polygon be placed anywhere inside the square. Consider the copy F' obtained from F by a half-turn about the centre O of the square. By hypothesis, F and F' must have a point in common. Let it be P. Then the point P' obtained from P by a half-turn about O is also in the intersection of F and F'. Since F is convex, O is also in F. It follows that a copy of the convex polygon placed anywhere inside the square must cover O. It follows that if three copies are placed in the square, they will have O in common.

5. We may permute the rows and the columns of the table below. How may different tables can we generate?

1	2	3	4	5	6	7
7	1	2	3	4	5	6
6	7	1	2	3	4	5
5	6	7	1	2	3	4
4	5	6	7	1	2	3
3	4	5	6	7	1	2
2	3	4	5	6	7	1

Solution. The columns may be permuted in 7! ways so that the first row is different. The remaining rows may be permuted in 6! ways so that the first column is different. Once the first row and the first column have been fixed, the remaining entries in the table are also fixed. Hence the total number of different tables we can generate is $7! \cdot 6!$.

6. Given are finitely many points in the plane, no three on a line. They are painted in four colours, with at least one point of each colour. Prove that there exist three triangles, distinct but not necessarily disjoint, such that the three vertices of each triangle have different colours, and none of them contains a coloured point in its interior.

Solution. Consider all sets of four points of different colours. Since the number of points is finite, we can choose the set whose convex hull has the smallest area. If the convex hull is a quadrilateral, then there are no coloured points in its interior, as otherwise we have a set whose convex hull has smaller area. The four vertices of the quadrilateral determine four triangles each with vertices of different colours, and any three of these four triangles will satisfy the requirement. Suppose the convex hull is a triangle ABC, say with A red, B yellow and C blue. Then only points of the fourth colour, say green, can be inside ABC, and there is at least one such point D. If there are no green points other than D, then triangles ACD, BAD and CBD satisfy the requirement. Suppose BAD contains other green points. Choose among them a point E such that triangle BAE has the smallest area. Then it cannot contain any green points in its interior, and we can replace BAD by BAE. A similar remedy can be applied if either ACD or CBD contains green points in its interior. Hence we will get three triangles which satisfy the requirement.

2 World Wide Web

Information on the Tournament, how to enter it, and its rules are on the World Wide Web. Information on the Tournament can be obtained from the Australian Mathematics Trust web site at

http://www.amt.edu.au

3 Books on Tournament Problems

There are four books on problems of the Tournament available. Information on how to order these books may be found in the Trust's advertisement elsewhere in this journal, or directly via the Trust's web page.

Please note the Tournament's postal address in Moscow:

NN Konstantinov
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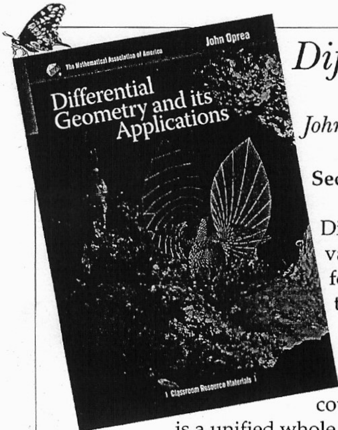

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John Oprea

Textbook

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Australian Mathematics Competition Book 3 (1992-1998)

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Methods of Problem Solving, Book 1
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This introduces the student aspiring to Olympiad competition to particular mathematical problem solving techniques. The book contains formal treatments of methods which may be familiar or introduce the student to new, sometimes powerful techniques.

Methods of Problem Solving, Book 2
JB Tabov & PJ Taylor
After the success of Book 1, the authors have written Book 2 with the same format but five new topics. These are the Pigeon-Hole Principle, Discrete Optimisation, Homothety, the AM-GM Inequality and the Extremal Element Principle.

Mathematical Toolchest
Edited by AW Plank & N Williams

This 120 page book is intended for talented or interested secondary school students, who are keen to develop their mathematical knowledge and to acquire new skills. Most of the topics are enrichment material outside the normal school syllabus, and are accessible to enthusiastic year 10 students.

International Mathematics — Tournament of Towns (1980-1984)

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International Mathematics — Tournament of Towns (1997-2002)
Edited by PJ Taylor

The International Mathematics Tournament of Towns is a problem solving competition in which teams from different cities are handicapped according to the population of the city. Ranking only behind the International Mathematical Olympiad, this competition had its origins in Eastern Europe (as did the Olympiad) but is now open to cities throughout the world. Each book contains problems and solutions from past papers.

Challenge! 1991 – 1998 Book 1
Edited by JB Henry, J Dowsey, A Edwards, L Mottershead, A Nakos, G Vardaro & PJ Taylor

This book is a major reprint of the original *Challenge!* (1991-1995) published by the Trust in 1997. It contains the problems and full solutions to all Junior and Intermediate problems set in the Mathematics Challenge for Young Australians, exactly as they were proposed at the time. It is expanded to cover the years up to 1998, has more advanced typography and makes use of colour. It is highly recommended as a resource book for classes from Years 7 to 10 and also for students who wish to develop their problem solving skills. Most of the problems are graded within to allow students to access an easier idea before developing through a few levels.

USSR Mathematical Olympiads 1989 – 1992
Edited by AM Slinko

Arkadii Slinko, now at the University of Auckland, was one of the leading figures of the USSR Mathematical Olympiad Committee during the last years before democratisation. This book brings together the problems and solutions of the last four years of the All-Union Mathematics Olympiads. Not only are the problems and solutions highly expository but the book is worth reading alone for the fascinating history of mathematics competitions to be found in the introduction.

Australian Mathematical Olympiads 1979 – 1995
H Lausch & PJ Taylor

This book is a complete collection of all Australian Mathematical Olympiad papers since the first competition in 1979. Solutions to all problems are included and in a number of cases alternative solutions are offered.

Chinese Mathematics Competitions and Olympiads 1981-1993 and 1993-2001
A Liu

These books contain the papers and solutions of two contests, the Chinese National High School Competition and the Chinese Mathematical Olympiad. China has an outstanding record in the IMO and these books contain the problems that were used in identifying the team candidates and selecting the Chinese teams. The problems are meticulously constructed, many with distinctive flavour. They come in all levels of difficulty, from the relatively basic to the most challenging.

Asian Pacific Mathematics Olympiads 1989-2000
H Lausch & C Bosch-Giral

With innovative regulations and procedures, the APMO has become a model for regional competitions around the world where costs and logistics are serious considerations. This 159 page book reports the first twelve years of this competition, including sections on its early history, problems, solutions and statistics.

Polish and Austrian Mathematical Olympiads 1981-1995
ME Kuczma & E Windischbacher

Poland and Austria hold some of the strongest traditions of Mathematical Olympiads in Europe even holding a joint Olympiad of high quality. This book contains some of the best problems from the national Olympiads. All problems have two or more independent solutions, indicating their richness as mathematical problems.

Seeking Solutions
JC Burns

Professor John Burns, formerly Professor of Mathematics at the Royal Military College, Duntroon and Foundation Member of the Australian Mathematical Olympiad Committee, solves the problems of the 1988, 1989 and 1990 International Mathematical Olympiads. Unlike other books in which only complete solutions are given, John Burns describes the complete thought processes he went through when solving the problems from scratch. Written in an inimitable and sensitive style, this book is a must for a student planning on developing the ability to solve advanced mathematics problems.

101 Problems in Algebra from the Training of the USA IMO Team
Edited by T Andreescu & Z Feng

This book contains one hundred and one highly rated problems used in training and testing the USA International Mathematical Olympiad team. These problems are carefully graded, ranging from quite accessible towards quite challenging. The problems have been well developed and are highly recommended to any student aspiring to participate at National or International Mathematical Olympiads.

Hungary Israel Mathematics Competition
S Gueron

This 181 page book summarizes the first 12 years of the competition (1990 to 2001) and includes the problems and complete solutions. The book is directed at mathematics lovers, problem solving enthusiasts and students who wish to improve their competition skills. No special or advanced knowledge is required beyond that of the typical IMO contestant and the book includes a glossary explaining the terms and theorems which are not standard that have been used in the book.

Bulgarian Mathematics Competition 1992-2001
BJ Lazarov, JB Tabov, PJ Taylor, AM Storozhev

The Bulgarian Mathematics Competition has become one of the most difficult and interesting competitions in the world. It is unique in structure, combining mathematics and informatics problems in a multi-choice format. This book covers the first ten years of the competition complete with answers and solutions. Students of average ability and with an interest in the subject should be able to access this book and find a challenge.

Mathematical Contests – Australian Scene
Edited by AM Storozhev, K McAvaney & A Di Pasquale

These books provide an annual record of the Australian Mathematical Olympiad Committee's identification, testing and selection procedures for the Australian team at each International Mathematical Olympiad. The books consist of the questions, solutions, results and statistics for: Australian Intermediate Mathematics Olympiad (formerly AMOC Intermediate Olympiad), AMOC Senior Mathematics Contest, Australian Mathematics Olympiad, Asian-Pacific Mathematics Olympiad, International Mathematical Olympiad, and Maths Challenge Stage of the Mathematical Challenge for Young Australians.

WFNMC — Mathematics Competitions
Edited by Jaroslav Švrček

This is the journal of the World Federation of National Mathematics Competitions (WFNMC). With two issues each of approximately 80-100 pages per year, it consists of articles on all kinds of mathematics competitions from around the world.

Parabola incorporating Function

This Journal is published in association with the School of Mathematics, University of New South Wales. It includes articles on applied mathematics, mathematical modelling, statistics, history and pure mathematics that can contribute to the teaching and learning of mathematics at the senior secondary school level. The Journal's readership consists of mathematics students, teachers and researchers with interests in promoting excellence in senior secondary school mathematics education.

ENRICHMENT STUDENT NOTES

The Enrichment Stage of the Mathematics Challenge for Young Australians (sponsored by the Dept of Education, Science and Training) contains formal course work as part of a structured, in-school program. The Student Notes are supplied to students enrolled in the program along with other materials provided to their teacher. We are making these Notes available as a text book to interested parties for whom the program is not available.

Newton Enrichment Student Notes
JB Henry
Recommended for mathematics students of about Year 5 and 6 as extension material. Topics include polyominoes, arithmetricks, polyhedra, patterns and divisibility.

Dirichlet Enrichment Student Notes
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This series has chapters on some problem solving techniques, tessellations, base five arithmetic, pattern seeking, rates and number theory. It is designed for students in Years 6 or 7.

Euler Enrichment Student Notes
MW Evans and JB Henry
Recommended for mathematics students of about Year 7 as extension material. Topics include elementary number theory and geometry, counting, pigeonhole principle.

Gauss Enrichment Student Notes
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Noether Enrichment Student Notes
AM Storozhev
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G Ball, K Hamann and AM Storozhev
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Alan Mathison Turing T-shirt

The Alan Mathison Turing t-shirt depicts a colourful design representing Turing's computing machines which were the first computers.

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