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MATHEMATICS COMPETITIONS

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The aims of the Federation are:–

1. ***to promote excellence in, and research associated with, mathematics education through the use of school mathematics competitions;***
2. ***to promote meetings and conferences where persons interested in mathematics contests can exchange and develop ideas for use in their countries;***
3. ***to provide opportunities for the exchanging of information for mathematics education through published material, notably through the Journal of the Federation;***
4. ***to recognize through the WFNMC Awards system persons who have made notable contributions to mathematics education through mathematical challenge around the world;***
5. ***to organize assistance provided by countries with developed systems for competitions for countries attempting to develop competitions;***
6. ***to promote mathematics and to encourage young mathematicians.***

From the President

The most notable event for the WFNMC (World Federation of National Mathematics Competitions) since the previous issue of this journal was the 5$^{\text{th}}$ Conference of the Federation which took place in Cambridge, England, from July 22 to July 28, 2006. It was very successful and a great experience as well. Robinson College in Cambridge provided an excellent atmosphere for participants and a good venue for academic activities. The program included interesting plenary lectures delivered by Maria de Losada, Ben Green, Jozsef Pelikan, Robin Wilson and Simon Singh. There were parallel sessions in order to accommodate all those wishing to speak. In the mornings the participants worked in small groups on 'problem creation and improvement'. The small tours around the historical heart of Cambridge and the Trinity College Reception provided additional inspiration for the participants. Social events and the Accompanying Persons Program also left us with valuable memories. Most important is the fact that there were many new (and young) persons who liked participation at the conference. I use this occasion to thank the organizers for their countless efforts and desire to make the conference a success: Tony (and Gwyn!) Gardiner, Adam McBride, Bill Richardson and Howard Groves. For more information the reader could visit the web-site of the conference at http://www.wpr3.co.uk/wfnmc/ Traces of the atmosphere and the spirit of the Conference could be seen at the Photo Gallery posted by the Australian Mathematics Trust at http://www.amt.canberra.edu.au/wfnmcpho3.html. In particular, some of the pictures there capture the moments when the Erdös Award winners *Simon Chua* (Philippines) and *Alexander Soifer* (USA) received their awards during the Opening Session of the Conference. Unfortunately, the third winner, *Ali Rejali* (Iran), could not participate in the Conference and did not receive his award.

During the Conference a meeting of the Executive Committee of WFNMC took place at which Prof. Agnis Andzans (accompanied by two more Latvian colleagues Dace Bonka and Inese Bersina) reiterated the proposal for hosting the next Conference of WFNMC. The Executive Committee accepted the proposal. The 6$^{\text{th}}$ Conference of WFNMC will be held in Riga, Latvia, in 2010. The organizers will be supported by the University of Latvia and the Latvian Academy of Sciences. Before

2010 the WFNMC community will meet in Monterey, Mexico, during ICME-11 (International Congress on Mathematical Education). The latter will take place from July 6 to July 13, 2008. There will be 48 Topic Study Groups and one of them will deal with Activities and Programs for Gifted Students. There will also be a Discussion Group related to activities which are of importance for WFNMC. As an Affiliated Study Group of ICMI, the body that organizes ICME's, the WFNMC will hold an organizational meeting at which the new Executive Committee of the Federation will be elected. One of the issues to be discussed at that meeting is the 'membership in WFNMC'. Should the latter be made more formal or do we stay with the model which has worked successfully for more than 20 years?

Petar S. Kenderov
President of WFNMC
December 2006

From the Editor

Welcome to *Mathematics Competitions* Vol. 19, No 2.

At first I would like to thank the Australian Mathematics Trust for continued support, without which each issue of the journal could not be published, and in particular Heather Sommariva, Bernadette Webster and Pavel Calábek for their assistance in the preparation of this issue.

Submission of articles:
The journal *Mathematics Competitions* is interested in receiving articles dealing with mathematics competitions, not only at national and international level, but also at regional and primary school level. There are many readers in different countries interested in these different levels of competitions.

- The journal traditionally contains many different kinds of articles, including reports, analyses of competition problems and the presentation of interesting mathematics arising from competition problems. Potential authors are encouraged to submit articles of all kinds.
- To maintain and improve the quality of the journal and its usefulness to those involved in mathematics competitions, all articles are subject to review and comment by one or more competent referees. The precise criteria used will depend on the type of article, but can be summarised by saying that an article accepted must be correct and appropriate, the content accurate and interesting, and, where the focus is mathematical, the mathematics fresh and well presented. This editorial and refereeing process is designed to help improve those articles which deserve to be published.

At the outset, the most important thing is that if you have anything to contribute on any aspect of mathematics competitions at any level, local, regional or national, we would welcome your contribution.

Articles should be submitted in English, with a black and white photograph and a short profile of the author. Alternatively, the article can be submitted on an IBM PC compatible disk or a Macintosh disk. We prefere LaTeX or TeX format of contributions, but any text file will be helpful.

Articles, and correspondence, can also be forwarded to the editor by mail to

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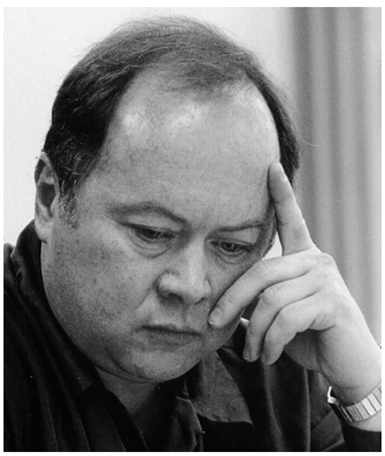
Jaroslav Švrček,
December 2006

Imagining the Real, Realizing the Imaginary: The Choice in Mathematics[1]

Alexander Soifer

Dedicated to Paul Erdös
on the 10th Anniversary of His Passing on



Alexander Soifer is a professor of Math, Art & Film History, University of Colorado at Colorado Springs, USA & Member, 2004–2007, DIMACS. He is Chair and founder of the Colorado Mathematical Olympiad, *which is in its twenty-fourth year, a member of* USA Mathematics Olympiad Subcommittee *(1996–present), Secretary of the* World Federation of National Mathematics Competitions *(1996–2008), and Editor and Publisher of the research Quarterly* Geombinatorics *(1991–present).*

More at http://www.uccs.edu/~asoifer

1 Prologue: What if We Had No Choice?

In October-November 1950, the 18-year-old youngster Ed Nelson created the following problem:

Chromatic Number of the Plane (CNP)**.** Find the smallest positive integer χ with the property that the plane can be colored in χ colors without any pair of points of the same color being distance 1 apart.

[1]This is the text of a talk at the Cambridge University July 2006 Congress of the World Federation of National Mathematics Competitions. An updated version will appear in the author's book [Soi4].

Right then, in 1950, Ed proved the lower bound $4 \leq \chi$, and his 20-year-old fellow student John Isbell added the upper bound $\chi \leq 7$ (see more on the history of this problem in [24]). Much has been proven since then *under additional conditions*. For example, the British mathematician Kenneth Falconer proved the following result.

Falconer's Theorem ([13]). Chromatic number of the plane is at least 5 if monochromatic sets are Lebesgue measurable.

A great variety of tools was used from graph theory, topology, measure theory, abstract algebra, discrete and combinatorial geometry—and yet no improvement was attained in the general case. The range for CNP remained wide open:

$$\chi = 4, 5, 6, \text{ or } 7.$$

I felt—and wrote a number of years ago—that such a wide range was an embarrassment for mathematicians. The 4-Color Map-Coloring Theorem, for example, from its beginning in 1852 had a conjecture: 4 colors suffice. Since 1890, thanks to Percy John Heawood [16], we knew that the answer was 4 or else 5. The CNP problem is an entirely different matter. After over half-a-century of very active work on the problem, we have not even been able to confidently conjecture the answer. Have mathematicians been so poor, or has the problem been so hard? Have we been missing something in our assault on the CNP?

These were the questions that occupied me as I was flying from Colorado Springs to Rutgers University in October 2002 for a week of joint research with Saharon Shelah, a genius of problem solving and a very quick learner. Per Saharon's request, I compiled a list of problems we would be interested in working on together, and numbered them according to set-theorists taste, from 0 to 12. Problem 0 read as follows:

What if we had no choice?

This was a natural question for someone who came from the Soviet Union, for there was not much choice there: we were offered to vote for one candidate per office, ate whatever food was sold at the moment, and lived wherever we were allowed to live. Yet, of course, I meant something else that made mathematical sense. So, let me explain.

2 De Bruijn–Erdös Theorem and Its History

They were both young. On August 4, 1947 the thirty-four year old Paul Erdös, in a letter to the twenty-nine year old Nicolaas Govert de Bruijn of Delft, The Netherlands, formulated the following conjecture [10]:

Conjecture. Let G be an infinite graph. Any finite subset of it is the sum of k independent sets (two vertices are independent if they are not connected). Then G is the sum of k independent sets.

Paul added in parentheses, *I can only prove it if* $k = 2$. In his August 18, 1947 reply [3, page 2 of 5], de Bruijn reformulated the Erdös conjecture in a way very familiar to us today:

Theorem. Let G be an infinite graph, any finite subgraph of which can be k-colored (that means that the nodes are coloured with k different colours, such that the two connected nodes have different colours). Then G can be k-coloured.

Following a nearly three page-long transfinite induction proof of the "Theorem", de Bruijn observed:

I am sorry that this proof takes so much paper; its idea, however, is simple. Perhaps, you do not call it a proof at all, because it contains 'Wellordering', but we can hardly expect to get along without that.

This was a valid observation, for de Bruijn and Erdös relied on the Axiom of Choice or equivalent (like Well-Ordering Principle or Zorn's Lemma) very heavily. Following receipt of the Shelah–Soifer 2003 paper, which analyzed chromatic number under varying versions of the Axiom of Choice (discussed in section 5 later), de Bruijn replied on January 27, 2004 as follows [Bru5]:

About the axiom of choice I remember a conversation with Erdös, during a walk around 1954. I told him that I hated the axiom of choice, and that I wanted to do analysis without it, maybe except for the countable case. He was surprised, and said, 'But you were always so good at it.' Indeed, I had loved transfinite induction, just because it worked exactly the same way as ordinary induction.

This de Bruijn e-mail also conveyed to me the conclusion of the de Bruijn-

Erdös Theorem's story:

Erdös and I did not take any steps to publish the k-coloring theorem. In 1951 I met Erdös in London, and from there we went together by train to Aberdeen, which took a full day. It was during that train ride that he told me about the topological proof of the k-coloring theorem. Not long after that, he wrote it up and submitted it for publication. I do not think I had substantial influence on that version.

Now you know the story of this celebrated theorem.

De Bruijn–Erdös Compactness Theorem ([4], 1951)**.** An infinite graph G is k-colorable[2] iff every finite subgraph of G is k-colorable.

Consequently, we get

De Bruijn–Erdös Theorem for the Plane. The chromatic number of the plane is equal to the maximum chromatic number of its *finite* unit distance[3] subset.

The chief promoter of CNP problem, Paul Erdös, was a loyal fan of Georg Cantor and of the axiom of choice. Because of that he never mentioned—or at least never mentioned in my presence—that De Bruijn–Erdös Theorem required the axiom of choice. Let me quote, for example, Erdös presenting this theorem to the audience of some 250 mathematicians at the XXV Southeastern International Conference on Combinatorics, Graph Theory, and Computing in Boca Raton, Florida on March 10, 1994:

De Bruijn and I showed that if there is an infinite graph which has chromatic number k, it always has a finite subgraph which has chromatic number k. So this problem is really a finite problem, not an infinite problem.[4]

[2] k-coloring is an assignment of k colors to vertices so that adjacent vertices do not get the same color. Chromatic number of a graph is the minimum number k of colors over all k-colorings.

[3] To color unit distance set means to assign colors in such a way that two vertices are colored differently iff they are unit distance apart.

[4] Do not attribute such a precise quotation to my memory, which is average at best—I simply have a video recording of this talk of Paul's.

Thus, De Bruijn–Erdös Theorem and Erdös's problem talks and papers naturally channeled research on CNP into the realm of the finite. Much of research was dedicated to finding (so far unsuccessfully) a 5-chromatic unit-distance graph. Paul O'Donnell once even advertised in his abstract of a conference talk that he would present such a graph. Surely we all came to Paul's talk. He started, however, by announcing that the required graph had not yet been found, but he was working on it!

The proof of De Bruijn–Erdös Theorem very essentially used the *Axiom of Choice*. Without it, or a near-equivalent assumption, the statement of the theorem would not be true. Now is the right time to formulate the Axiom of Choice and other related axioms of which we will make use later.

3 The Axiom of Choice and Its Relatives

To choose one sock from each of infinitely many pairs of socks requires the Axiom of Choice, but for shoes the Axiom is not needed.
— Bertrand Russell

The Axiom of Choice had been used implicitly throughout the nineteenth century. A careful observation would uncover that it was needed for proving even such a classic result as the sequential Bolzano–Weierstrass Theorem (every infinite bounded subset of reals has a sequential limit point). In 1904, while proving the Well-Ordering Principle, Ernst Friedrich Ferdinand Zermelo (1871–1953) formalized and for the first time explicitly used the Axiom of Choice [30]:

The Axiom of Choice (AC). Every family Φ of nonempty sets has a choice function, i.e. there is a function f such that $f(S) \in S$ for every S from Φ.

The newborn axiom prompted quite a debate in the mathematical world. In trying to defend the axiom, in a series of 1908–1909 papers Zermelo developed a system of axioms for set theory, which was improved by Adolf Abraham Halevi Fraenkel (1891–1965) in his 1922 and 1925 papers, and became known as the Zermelo–Fraenkel system of axioms, or ZF. ZF with an addition of the Axiom of Choice was naturally denoted by ZFC.

The historian of the Axiom of Choice, Gregory H. Moore, opened his remarkable book about it as follows [18]:

David Hilbert once wrote that Zermello's Axiom of Choice was the axiom "most attacked up to the present [1926] in mathematical literature..." To this Abraham Fraenkel later [1958] added that "the axiom of choice is probably the most interesting and, in spite of its late appearance, the most discussed axiom of mathematics, second only to Euclid's axiom of parallels which was introduced more than two thousand years ago."

The Axiom postulated the existence of the choice function, without giving any clue of how to find it. It was therefore no surprise that the Axiom was opposed by constructivists, intuitionists, and other mathematicians, who viewed non-constructive existence results with great suspicion. Moore [18] observes that "Despite this initial widespread distrust, today the vast majority of mathematicians accept the axiom without hesitation and utilize it in algebra, analysis, logic, set theory, and topology." Yes, I agree: the vast majority accept the Axiom of Choice, and consequently ZFC is the standard foundation of set theory—but is it a good thing? A majority—any majority—political, social, mathematical often loses sensitivity that is so naturally preserved among a minority. We will later look into the consequences of the near universal acceptance of the Axiom of Choice as a part of the foundation of mathematics. Here I will introduce other axioms.

Many results in mathematics really need just a countable version of choice:

The Countable Axiom of Choice ($AC_{\aleph_0}$). Every countable family of nonempty sets has a choice function.

Much later, in 1942, Paul Isaac Bernays (1888–1977) introduced the following axiom [2]:

The Principle of Dependent Choices (DC). If E is a binary relation on a nonempty set A, and for every $a \in A$ there exists $b \in A$ with $a\,E\,b$, then there is a sequence $a_1, a_2, \ldots, a_n, \ldots$ such that $a_n\,E\,a_{n+1}$ for every $n < \omega$.

AC implies DC (see, for example, theorem 8.2 in [19]), but not conversely. In turn, DC implies $AC_{\aleph_0}$, but not conversely. DC is a weak form of AC and is sufficient for the classical theory of Lebesgue measure. We observe that, in particular, DC is sufficient for Falconer's result [13] formulated above.

One (unfortunate in my opinion) consequence of the Axiom of Choice is the existence of sets on the line that have no length (I mean, no Lebesgue measure). This regret must have given birth to the following axiom:

Axiom (LM**).** Every set of real numbers is Lebesgue measurable.

Assuming the existence of an inaccessible cardinal[5], Robert M. Solovay (nowadays Professor Emeritus at Berkley) constructed in 1964 (published in 1970) a model that proved the following, which is critical for our discussion consistency result [28]:

Solovay's Theorem. The system of axioms ZF+DC+LM is consistent.

This is a remarkable result. Here is how Paul Joseph Cohen, the man who completed Kurt Gödel's work, described it in 1966 [7, p. 142]:

One of the most interesting results [concerning the relationship of various forms of AC] is due to R. Solovay (as yet unpublished) which says that models N can be constructed in which the countable AC holds and yet every set of real numbers is Lebesgue measurable.

Continuum Hypothesis (CH**).** It states that there is no cardinal κ such that $\aleph_0 < \kappa < 2^{\aleph_0}$.

Generalized Continuum Hypothesis (GCH**).** It states that for any infinite cardinal λ there is no cardinal κ such that $\lambda < \kappa < 2^\lambda$.

The Axiom of Constructibility (V=L**).** Introduced by Gödel in 1940 [15], asserts that every set is constructible, i.e. that every set belongs to the constructible universe L.

Kurt Gödel (1906–1978) in 1940 [15] and Paul Cohen in 1963–1964 [5] proved independence of AC (as well as of the Continuum Hypothesis, CH, and the Generalized Continuum Hypothesis, GCH) from the rest of the axioms of set theory, ZF.

[5]A cardinal κ is called *inaccessible* if $\kappa > \aleph_0$, is regular, and κ is strong limit. An infinite cardinal $\aleph_\alpha$ is *regular*, if $\mathrm{cf}\,\omega_\alpha = \omega_\alpha$. A cardinal κ is a *strong limit* cardinal if for every cardinal λ, $\lambda < \kappa$ implies $2^\lambda < \kappa$.

As Saharon Shelah playfully summarized in his 2003 "Logical Dreams" [21],

In short: The Continuum Problem asks:
How many real numbers are there?

G. Cantor proved: There are more reals than rationals. (In a technical sense: "uncountable", "there is no bijection from R into Q").
The Continuum Hypothesis (CH) says: yes, more, but barely so. Every set $A \subseteq R$ is either countable or equinumerous with R.

K. Gödel proved: Perhaps CH holds.
P. Cohen proved: Perhaps CH does not hold.

Kurt Gödel also showed that ZF + V=L implies GCH; while the founder of the famous Warsaw school of set theory and topology Wacław Franciszek Sierpiński (1882–1969) proved that ZF+GCH implies AC.

4 What Do Leading Set Theorists Think about Foundations?

In the beginning (if there was such a thing) God created Newton's laws of motion together with the necessary masses and forces. This is all; everything beyond this follows from the development of appropriate mathematical methods by means of deduction.
— Albert Einstein, 1946 [12, p. 19]

Gödel and Cohen believed that we would eventually identify all the axioms of set theory and when we have done so, we will no longer be able to choose between CH and ¬CH (or, similarly, between AC and DC+LM) because the additional axioms would exclude one of the options. Cohen shared his thoughts on the subject in 1966 [7, pp. 150–151]:

One can feel that our intuition about sets is inexhaustible and that eventually an intuitively clear axiom will be presented which decides CH. One possibility is V=L, but this is almost universally rejected... A point of view which the author feels may eventually come to be accepted is that CH is obviously *false.*

Saharon Shelah disagreed with this Platonic view in his 2003 "Logical Dreams" [21]:

Some believe that compelling, additional axioms for set theory which settle problems of real interest will be found or even have been found. It is hard to argue with hope and problematic to consider arguments which have not yet been suggested. However, I do not agree with the pure Platonic view that the interesting problems in set theory can be decided, we just have to discover the additional axiom. My mental picture is that we have many possible set theories, all conforming to ZFC.

In preparing these remarks, I asked some of the leading set theorists, the great contributors to the axiomatics of sets, the following questions[6]:

(i) Has AC been good for mathematics?

(ii) Ought AC to be a part of a "standard" system of axioms for set theory?

(iii) What do you think of the Solovay system of axioms?

(iv) How do you view the standard system of axioms for set theory?

My opinions shift and I have no obvious candidates (for the standard system of axioms). Solovay's result on LM is very nice, but hardly an axiom, Paul Cohen replied. Saharon Shelah saw a certain value in using the Solovay systems and systems weaker than ZFC:

The major question is what is true, i.e. when existence tells you something more if you give an explicit construction. Now, working in ZF*,* ZF+DC *and also* ZF+DC+LM *and many other systems are ways to explicate the word "construct".*

In Shelah's opinion, AC has been "definitely" good for mathematics, AC is true and *should be in our standard system (of axioms).* Robert Solovay too believes that ZFC *is true* (and therefore his system, which I admire so much, ZF+DC+LM is false).

This prompted my question: But... what is "true"? Shelah answered as follows:

[6] June 11–20, 2006 author's e-mail exchanges with Paul Cohen, Saharon Shelah, and Robert Solovay.

This is a meta-mathematical question. I will say (it) fits our image of set theory.
You may say this is circularly, but this is unavoidable.
You may be Platonist like Cantor then the meaning is clear.
You may say (it is) what mathematicians who have not been interested in the question will accept.
You may be a formalist and then this is a definition of ZFC.

Shelah is clear on what isn't true enough, in his 2003 paper [21]:

Generally I do not think that the fact that a statement solves everything really nicely, even deeply, even being the best semi-axiom (if there is such a thing, which I doubt) is a sufficient reason to say it is a "true axiom".

Not surprisingly, there is no rigorous definition of "truth", that this elusive notion is subjective, and all we can hope for is to recognize a true axiom when we encounter one. For Shelah, a true axiom is "what I feel/think is self-evident". This is a high bar to clear, and only ZFC seems to have cleared it for most creators of set theory. Even such serious candidates as CH, ¬CH, GCH, V=L are termed by Shelah as "semi-axioms"—because they are not sufficiently "representative" of all possibilities. Shelah elaborates in [21]:

Still most mathematicians, even those who have worked with GCH (and with other semi-axioms, A. S.) do it because they like to prove theorems and they could not otherwise solve their problems (or get a reasonable picture), i.e. they have no alternative in the short run…

What are our criteria for semi-axioms? First of all, having many consequences, rich, deep beautiful theory is important. Second, it is preferable that it is reasonable and "has positive measure". Third, it is preferred to be sure it leads to no contradiction…

The French mathematician Jean-Paul Delahaye [9] believes that the Shelah–Soifer results may have put a new emphasis on the task of finding which world of sets we think we live in:

It turns out that knowing if the world of sets satisfies the axiom of choice or a competing axiom is a determining factor in the solution of problems that no one had imagined depended on them. The questions raised by *the new results are tied to the fundamental nature of the world of sets. Is it reasonable to believe that the mathematical world of sets is real? If it exists, does the true world of sets—the one in which we think we live—allow the coloring of S. Shelah and A. Soifer in two colors or does it require an infinity of colors?*

A series of results concerning the theory of graphs, published in 2003 and 2004 by Alexander Soifer of Princeton University and Saharon Shelah, of the University of Jerusalem, should temper our attitude and invite us to greater curiosity for the alternatives offered by the axiom of choice. The observation demonstrated by A. Soifer and S. Shelah should force mathematicians to reflect on the problems of foundations: what axioms must be retained to form the basis of mathematics for physicists and for mathematicians?

So, let us proceed to these results.

5 The First Shelah–Soifer Graph

Theories come and go. Examples stay forever.
— I. M. Gelfand

We define a graph G as follows: the set $\mathbb{R}$ of real numbers serves as the vertex set, and the set of edges is $\{(s,t); s-t-\sqrt{2}\in\mathbb{Q}\}$.

Claim 1. In ZFC the chromatic number of G is equal to 2.

Proof. Let $S=\{q+n\sqrt{2}; q\in\mathbb{Q}, n\in\mathbb{Z}\}$. We define an equivalence relation E on R as follows: $s\,E\,t \Leftrightarrow s-t\in S$.

Let Y be a set of representatives for E. For $t\in\mathbb{R}$ let $y(t)\in Y$ be such that $t\,E\,y(t)$. We define a 2-coloring $c(t)$ as follows: $c(t)=l$, $l=0,1$ iff there is $n\in\mathbb{Z}$ such that $t-y(t)-2n\sqrt{2}-l\sqrt{2}\in\mathbb{Q}$. □

Without AC the chromatic situation changes dramatically:

Claim 2. In ZF+$AC_{\aleph_0}$+LM the chromatic number of the graph G cannot be equal to any positive integer n nor even to $\aleph_0$.

Proof of Claim 2 immediately follows from the first of the following two statements:

Statement 1. If $A_1, \dots, A_n, \dots$ are measurable subsets of R and $\cup_{n<\omega} A_n = [0,1)$, then at least one set A_n contains two adjacent vertices of the graph G.

Statement 2. If $A \subseteq [0,1)$ and A contains no pair of adjacent vertices of G, then A is null (of Lebesgue measure zero).

Indeed, assume there is a partition of the reals R into countably many monochromatic sets C_n, $n < \omega$, such that no subset C_n contains two vertices of the graph G. Denote $A_n = C_n \cap [0,1)$. By Statement 2, for every $n < \omega$, $\mu(A_n) = 0$. Since Lebesgue measure is a countably-additive function in $\mathrm{AC}_{\aleph_0}$, we get:

$$1 = \mu\big([0,1)\big) = \mu\left(\cup_{n<\omega} A_n\right) = \sum_{n<\omega} \mu(A_n) = 0,$$

which is absurd. □

Proof of Statements 1 & 2. We start with the proof of Statement 2. Assume to the contrary that A contains no pair of adjacent vertices of G yet A has positive measure. Then there is an interval I such that

$$\frac{\mu(A \cap I)}{\mu(I)} > \frac{9}{10}. \tag{1}$$

Choose $q \in \mathbb{Q}$ such that $\sqrt{2} < q < \sqrt{2} + \frac{\mu(I)}{10}$. Let $B = A - (q - \sqrt{2}) = \{x - q + \sqrt{2}; x \in A\}$. Then

$$\frac{\mu(B \cap I)}{\mu(I)} > \frac{8}{10}. \tag{2}$$

Inequalities (1) and (2) imply that there is $x \in I \cap A \cap B$. As $x \in B$, we have $y = x + \big(q - \sqrt{2}\big) \in A$. So, we have $x, y \in A$ and $x - y - \sqrt{2} = -q \in \mathbb{Q}$. Thus, $\{x, y\}$ is an edge of the graph G with both endpoints in A, which is the desired contradiction.

The proof of the Statement 1 is now obvious. Since $\cup_{n<\omega} A_n \supseteq [0,1)$ and Lebesgue measure is a countably-additive function in $\mathrm{AC}_{\aleph_0}$, there is a positive integer n such that A_n is a non-null set of reals. By Statement 2, A_n contains a pair of adjacent vertices of G as required. □

6 Further Shelah–Soifer Examples

In the previous section I presented the 2003 Shelah–Soifer example of a distance graph on the real line R whose chromatic number depends upon the system of axioms we choose for set theory. Ideas developed there are extended in this section to construct a distance graph G_2 on the plane R^2, thus coming much closer to the setting of the chromatic number of the plane problem. The chromatic number of G_2 is 4 in the Zermelo–Fraenkel-Choice system of axioms, and is not countable (if it exists) in a consistent system of axioms with limited choice, studied by Robert M. Solovay [28]. This example first appeared in the Soifer–Shelah 2004 paper [23], where three examples were constructed. Let us look at one of them.

Let $\mathbb{Q}$ denote the set of all rational numbers, so that $\mathbb{Q}^2$ is the "rational plane". Let $\mathbb{Z}$ denote the set of all integers. We define a graph G_2 as follows: the set $\mathbb{R}^2$ of points on the plane serves as the vertex set, and the set of edges is the union of the four sets $\{(s,t); s,t \in \mathbb{R}^2, s-t-\varepsilon \in \mathbb{Q}^2\}$ for $\varepsilon = (\sqrt{2}, 0)$, $\varepsilon = (0, \sqrt{2})$, $\varepsilon = (\sqrt{2}, \sqrt{2})$, $\varepsilon = (-\sqrt{2}, \sqrt{2})$ respectively.

Claim 1:. In ZFC the chromatic number of G_2 is equal to 4.

Proof. Let $S = \{(q_1 + n_1\sqrt{2}, q_2 + n_2\sqrt{2})\,; q_i \in \mathbb{Q}, n_i \in \mathbb{Z}\}$. We define an equivalence relation E on $\mathbb{R}^2$ as follows: $s\,E\,t \Leftrightarrow s - t \in S$.

Let Y be a set of representatives for E. For $t \in \mathbb{R}^2$ let $y(t) \in Y$ be such that $t\,E\,y(t)$. We define a 4-coloring $c(t)$ as follows: $c(t) = (l_1, l_2)$, $l_i \in \{0, 1\}$ iff there is a pair $(n_1, n_2) \in \mathbb{Z}^2$ such that $t - y(t) - 2\sqrt{2}(n_1, n_2) - \sqrt{2}(l_1, l_2) \in \mathbb{Q}^2$. □

Claim 2. In ZF+AC$_{\aleph_0}$+LM the chromatic number of the graph G_2 cannot be equal to any positive integer n nor even to $\aleph_0$.

The proof of Claim 2 is similar to the proof presented in section 5 above. The paper [23] contains two other examples of graphs, whose chromatic numbers in ZFC are 2 and 3 respectively, and infinitely uncountable (if they exist) in the Solovay system of axioms.

In my 2005 paper [26], I generalized ideas from [22] and [23] and constructed *difference graphs* on the real space R^n, whose chromatic number is a positive integer in ZFC and is not countable (if it exists) in the Solovay system of axioms. These examples illuminate how heavily combinatorial results can depend upon the underlying set theory, help appreciate the potential complexity of the chromatic number of n-space problem, and suggest that the chromatic number of n-space may depend upon the system of axioms chosen for set theory.

Example in $\mathbb{R}^n$

Let $\mathbb{Z}$ and $\mathbb{Q}$ denote the set of all integers and the set of all rational numbers respectively, so that $\mathbb{Z}^n$ is a set of integral n-tuples and $\mathbb{Q}^n$ is the "rational n-space". We define a graph G as follows: the set $\mathbb{R}^n$ of points of the n-space serves as the vertex set, and the set of edges is $\cup_{i=1}^{n}\left\{(s,t); s,t \in R^n, s-t-\sqrt{2}\varepsilon_i \in \mathbb{Q}^n\right\}$ where ε_i are the n unit vectors on coordinate axes forming a standard basis of $\mathbb{R}^n$. For example, $\varepsilon_1 = (1,0,\ldots,0)$.

Claim 1. In ZFC the chromatic number of the graph G is equal to 2.

Proof. Let $S = \left\{q + m\sqrt{2}; q \in \mathbb{Q}^n, m \in \mathbb{Z}^2\right\}$. We define an equivalence relation E on $\mathbb{R}^n$ as follows: $s\,E\,t \Leftrightarrow s - t \in S$. Let Y be a set of representatives for E. For $t \in R^n$ let $y(t) \in Y$ be such a representative that $t\,E\,y(t)$. We define a 2-coloring c as follows: $c(t) = ||k||_{\text{ mod } 2}$ iff there is $k \in \mathbb{Z}^n$ such that $t - y(t) - \sqrt{2}k \in \mathbb{Q}^n$, where $||k||$ denotes the sum of all n coordinates of k. □

Claim 2. In ZF+AC$_{\aleph_0}$+LM the chromatic number of the graph G cannot be equal to any positive integer n nor even to $\aleph_0$.

The proof of Claim 2 is similar to the one given in section 5.

7 The Conditional Chromatic Number Theorem

The importance of particular axioms being used makes a surprising difference for the question of determining the chromatic number of the plane, as recently shown by Shelah and Soifer.
— Ronald L. Graham [14]

Is AC relevant to the problem of the chromatic number χ of the plane? The answer depends upon the value of χ which we, of course, do not yet know. However, in 2003 Shelah and Soifer published the following conditional result.

Conditional Chromatic Number of the Plane Theorem[7] (Shelah-Soifer [22]). Assume that any finite unit distance plane graph has chromatic number not exceeding 4. Then:

(i) In ZFC the chromatic number of the plane is 4;

(ii) In ZF+DC+LM the chromatic number of the plane is 5, 6 or 7.

Proof. The claim (i) is true due to [4]. In Solovay's system of axioms ZF+DC+LM, every subset S of the plane R^2 is Lebesgue measurable. Indeed, S is measurable iff there is a Borel set B such that the symmetric difference $S \triangle B$ is null. Thus, every plane set differs from a Borel set by a null set. We can think of a unit segment $I = [0, 1]$ as a set of infinite binary fractions and observe that the bijection $I \to I^2$ defined as $0.a_1a_2 \ldots a_n \ldots \mapsto (0.a_1a_3 \ldots ; 0.a_2a_4 \ldots)$ preserves null sets. Due to the Falconer theorem [13], formulated in section 1 above, we can now conclude that the chromatic number of the plane is at least 5. □

It is worth mentioning, that I believe that the chromatic number of Euclidean space R^n also depends upon the system of axioms we choose for set theory.

This conditional theorem allows for a certain historical insight. Perhaps, the problem of finding the chromatic number of the plane has withstood for 56 years all assaults in the general case, leaving us with a wide range for being 4, 5, 6 or 7, precisely because the answer depends upon the system of axioms we choose for set theory.

This also begs a question: was the choice of the mathematical standard ZFC inevitable? Was this choice the best possible?

8 So, What Does It All Mean?

> *I know of mathematicians who hold that the axiom of choice has the same character of intuitive self-evidence that belongs to the*

[7]Due to the use of the Solovay's Theorem, we assume the existence of an inaccessible cardinal.

most elementary laws of logic on which mathematics depends.
It has never seemed so to me.
— Alonzo Church, 1966 [6]

The Shelah–Soifer results seem surprising and even strange. How can the presence of the axiom of choice or its version affect whether we need 2 colors or an infinity of colors for coloring a particular, easily-understood graph? How can the chromatic number of the plane be 4 in ZFC and 5, 6, or 7 in ZF+DC+LM? What *do* these results mean?

The French mathematician Jean-Paul Delahaye opens his article about these results in *Pour la Science*, the French edition of *Scientific American*, as follows [9]:

The axiom of choice, a benign matter for the non-logician, puzzles mathematicians. Today, it manifests itself in a strange way: it takes, depending on the axiom's variants, either two or infinity of colors to resolve a coloring problem. Just as the parallels postulate seemed obvious, the axiom of choice has often been considered true and beyond discussion. The inventor of set theory, Georg Cantor (1845–1918), had used it several times without realizing it; Giuseppe Peano (1858–1932) used it in 1890, in working to solve a differential equation problem, consciously; but it was Ernst Zermelo (1871–1953), at the beginning of the 20th *century, who identified it clearly and studied it.*

When Gödel and Cohen proved independence of AC from the rest of the axioms ZF of set theory, they created a parallel, so to speak, between AC and the parallels postulate. As so, when Shelah–Soifer came out, it showed that various buildings of mathematics can be constructed. Delahaye observes, *These (Shelah–Soifer) results mean, as with the parallels postulate, that several different universes can be considered.* Delahaye continues:

In the case of geometry, the independence of the parallels postulate proved that non-Euclidian geometries deserved to be studied and that they could even be used in physics: Albert Einstein took advantage of these when, between 1907 and 1915, he worked out his general theory of relativity.

Regarding the axiom of choice, a similar logical conclusion was *warranted; the universes where the axiom of choice is not satisfied must be explored and could be useful in physics.*

Jean Alexandre Eugène Dieudonné (1906–1992), one of the founding members of Nicolas Bourbaki, described the state of the foundations in 1976 as follows:

Beyond classical analysis (based on the Zermelo–Fraenkel axioms supplemented by the Denumerable Axiom of Choice), there is an infinity *of different possible mathematics, and for the time being no definitive reason compels us to choose one of them rather than another.*[8]

The Solovay system of axioms is stronger than the system referred to by Dieudonné. It allows us to build classical analysis, including the complete Lebesgue measure theory; and it eliminates such counter-intuitive objects, existing in ZFC, as non-measurable sets on the line.

Using the axiom of choice in their 1924 paper [1] two Polish mathematicians Stefan Banach (1892–1945) and Alfred Tarski (1902–1983) decomposed the three-dimensional closed unit ball into finitely many pieces, and moved those pieces through rotations and translations (pieces were allowed to move through one another) in such a way, that the pieces formed two copies of the original ball. Since the measure of the union of two disjoint sets is the sum of their measures, and measure is invariant under translations and rotations, we can conclude that there is a piece in Banach–Tarski decomposition that has no measure (i.e. volume). Having LM in the system of axioms for set theory would eliminate this and a number of other paradoxes.

ZFC allows us to create imaginary objects—or shall I say unimagined objects—such as sets on the line that have no length, sets on the plane that have no area, etc. Are we not paying a high price for the comfort of having a powerful tool?

Having lived most of my life in ZFC and having enjoyed transfinite induction, in the course of my work with Shelah I came to a realization that I prefer ZF+DC+LM over ZFC: LM assures that every set of reals is measurable (which is consistent with my intuition: every set of reals ought to have length), while DC gives us as much of choice as is possible.

[8] Quoted from [18, p. 4]

Of course, by downgrading AC to DC we would lose such tools as transfinite induction and well-ordering of uncountable sets, as would lose some important theorems, such as the existence of a basis for a vector space. But new theorems would be found when mathematicians spend as much time building on the Solovay foundation as they have on ZFC.

9 Imagining the Real vs. Realizing the Imaginary

As far as the laws of mathematics refer to reality, they are not certain; and as far as they are certain, they do not refer to reality.
— Albert Einstein, 1921 [11]

We all know that Art is not truth.
Art is a lie that makes us realize the truth.
— Pablo Picasso, 1923 [20]

The mathematician is an inventor, not a discoverer.
— Ludwig Wittgenstein, 1937–1944 [29, p. 47e]

Einstein, Picasso and Wittgenstein spelled out my views in the above epigraphs, and spelled them precisely and completely.

Undoubtedly, the vast majority of mathematicians are Platonists. They believe that mathematical objects exist out there independently of the human mind, and mathematicians merely discover them. Platonists believe that a mathematical statement, such as AC, is objectively either true or false—we simply do not know yet which it is (although in a poll, "AC is true" would win hands down). Likewise, a question, what is the chromatic number of the Shelah–Soifer graph G?, surely, must have a single answer; it cannot be "2 or infinity". Therefore, for Platonists either ZFC or ZF+DC+LM is true, we just do not know which. Platonists *imagine the real.*

How does one describe those who hold the view of mathematics that is dual to Plato's? I propose to call them *Artists*, especially since Picasso is this group. To paraphrasing Picasso, *Mathematics is an invention that makes us realize reality.*

The mathematician is an inventor, not a discoverer, confirms Wittgenstein. Mathematics is certain only as an invention, or as Einstein put it, *As far as the laws of mathematics refer to reality, they are not certain; and as far as they are certain, they do not refer to reality.*

I believe that mathematicians do not only *imagine the real*, but moreover *realize the imaginary*. Just like artists, mathematicians create objects that challenge reality in every aspect: beauty, simplicity, complexity, intuitiveness, and counter-intuitiveness. The real is but one of many inspirations for creating mathematics.

Mathematics of ZFC is the house that Jack built. Has he built the only possible house? Has he built the best possible house? Must we give up a village for a house, as Richard III offered his kingdom for a horse?

I believe that mathematicians put all their eggs in one ZFC basket, and thus missed out on alternative mathematical universes that can be built on many alternative foundations, one of which is the Solovay's ZF+DC+LM. Saharon Shelah [21] thinks too that we ought to build on many foundations, but he puts main emphasis on building *up* from ZFC. It seems, we have been too comfortable, too nonchalant about seeing problems with ZFC and doing nothing about them. Mathematical results presented here may be not important by themselves, but they illuminate the many mathematics foundations that could be built, and serve—I hope—as a wake-up call. Delahaye [9] concludes his analysis of Shelah–Soifer series with the possibility of the emerging "set-ist revolution":

In set theory, as in geometry, all axiomatic systems are not equal. Thinking carefully about their meaning and the consequences of each one of them, and asking ourselves (as it is done in geometry) what the particular usefulness of this or that axiom is in expressing and addressing issues of mathematical physics, may be relevant once again and could lead—why not?—to a revolution of set theories, similar to the revolution in non-Euclidian geometries.

Starting a revolution? Who, me? All right, as long as the revolution is imaginary, in mathematics, and not a "real" revolution that causes destruction—I am with The Beatles on this:[9]

You say you want a revolution
Well, you know
We all want to change the world.
...
You say you got a real solution
Well, you know
We'd all love to see the plan.

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TEACHER

by Alexander Soifer

To Paul Erdös

Remember the one who came through to your inner self first
That elderly man or a woman of wit and vocation
And dropped I my ball and I followed him as he paused
And sweet was the instant and lasting was this convocation

I carry that spark, that baton that he placed on my palm
His hope and his gaze at the Book that he thought was transfinite
And feel I my way to the Book with heart threatening calm
And lives he in me and continues this infinite drama

March 7, 1998, 1 PM

The "Baltic Way" Contest

Agnis Andžāns & Inese Bērziņa & Dace Bonka

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Dace Bonka is a Ph.D. student at the University of Latvia, lecturer at the University of Latvia and an educational director at A. Liepas Correspondence Mathematics School. While at school Dace Bonka was inspired by math olympiads. Now she herself has been a member of the orgcommittee and jury of Latvian mathematics olympiads for 10 years. Her research area and main activities are correspondence mathematics contests for junior students in Latvia.

1 Introduction

The great majority of mathematical competitions held over the world are scheduled for individual contestants. Another form—team contests—are gradually becoming more popular. Some of them have a long history already. Such a contest is a part of Austrian–Polish Mathematical Competitions held since 1978 [1]; various years it has been included in the Hungary–Israel Binational Mathematical Competition [2]. In Lithuania, team competitions (e.g. [3]) are even more popular than the traditional ones; in 2006, the 21$^{\text{th}}$ was held. The International mathematical olympiads for students under 14.5 also include a team competition (e.g. [4]). For more than 30 years a special form of team competitions, "mathematical battles", have been very popular in the area of former Soviet Union (e.g. [5]). The magazine *Kvant* organizes a team competition as a final stage of its annual problem solving contest by correspondence [6]. The summer conferences of *The Tournament of Towns* include team competitions in research at collegiate level, etc.

This paper describes the history, lessons and influences of the team contest *Baltic Way*, which has been held since 1990.

2 The history

During the reforms that took place in the former Soviet Union in the period 1985–1991, various activities burst out in a large number: the long oppressed activity of people demonstrated its power not only in politics, but in all areas. In Latvia, Lithuania and Estonia all people showed also a strong desire to get rid of Soviet occupation and, in the first order, to show that we are capable of managing anything without the "help" of the "Big Brother". This resulted in establishing very close contacts between these three republics and reducing contact with the other ones whenever it was possible.

Though the mathematical community of the former Soviet Union, especially the part of it involved in organizing mathematical olympiads, had always been friendly and supportive to the strive for independence of the Baltic States, the above-mentioned processes appeared also in this area. In 1990 they resulted in the establishing the team competition in mathematics called Baltic Way. The initiators of the whole thing were Prof. Algirdas Zabulionis (Vilnius, Lithuania) and Agnis Andžāns. It gained its name from the mass demonstration for the freedom of the Baltic States in August 1989 where over a million people stood hand in hand along the road Tallin–Riga–Vilnius.

23 August, 1989.

In the first two years (1990 in Riga, 1991 in Tallin) only Latvia, Lithuania and Estonia were participating. Today the contest has all the countries around the Baltic Sea and Iceland as regular participants. The participation of Iceland is due to the fact that it was the first country in the world that recognized the independence of the Baltic States in 1991.

The competition takes place annually in the autumn, with regular participating countries taking turns to host the event. Besides these 11 regular participating countries, in recent years a guest country is invited sometimes by the organizers (Israel in 2001, Belarus in 2004 and Belgium in 2005).

3 The format of the competition

The competition is traditionally scheduled for 5 days:

- arrival
- problem selection/translation/excursion for students
- contest/evaluation/excursion for students
- common excursion/closing
- departure.

The problem set is composed and the evaluation is done in the same manner as on IMO, with the difference that "shortlist" is the same as "longlist".

Each team in the competition consists of five students, and they are allowed to cooperate in any way they like during four and a half hours. The team presents one solution only for each of the twenty problems. Each solution is evaluated with 0–5 points. The regulations of the contest practically haven't changed since 2000. They can be found on the website [7].

4 The problems

The problems of the contest are easier than those of IMO but generally harder than the problems of "average" national competitions. There is an approximate balance between algebra, geometry, number theory and combinatorics with a stress on the last area. Some problems of algorithmic nature are almost always included in the set.

Generally speaking, problems don't stress elaborated technique but rather nice ideas; problems with short and unexpected solutions are especially welcome.

The problems are normally selected by a (multi-round, if needed) popularity vote. After the selection the whole set is reconsidered and some "balancing" changes are made, if needed.

As an example we give the problems from the 17th Baltic Way which was held in Turku, Finland, 1st–5th November, 2006.

5 Baltic Way 2006

1. For a sequence $a_1, a_2, a_3, \ldots, a_i, \ldots$ of real numbers it is known that

$$a_n = a_{n-1} + a_{n+2} \qquad \text{for} \quad n = 2, 3, 4, \ldots$$

What is the largest number of its consecutive elements that can all be positive?

2. Suppose that for the real numbers $a_i \in [-2, 17]$; $i = 1, 2, \ldots, 59$,

$$a_1 + a_2 + \cdots + a_{59} = 0.$$

Prove that $a_1^2 + a_2^2 + \cdots + a_{59}^2 \leq 2006$.

3. Prove that for every polynomial $P(x)$ with real coefficients there exists a positive integer m and polynomials $P_1(x)$, $P_2(x), \ldots,$ $P_m(x)$ with real coefficients such that

$$P(x) = \big(P_1(x)\big)^3 + \big(P_2(x)\big)^3 + \cdots + \big(P_m(x)\big)^3.$$

4. Let a, b, c, d, e, f be non-negative real numbers satisfying $a + b + c + d + e + f = 6$. Find the maximal possible value of $abc + bcd + cde + def + efa + fab$ and determine all 6-tuples (a, b, c, d, e, f) for which this maximal value is achieved.

5. An occasionally unreliable professor has devoted his last book to a certain binary operation $*$. When this operation is applied to any two integers, the result is again an integer. The operation is known to satisfy the following axioms:

a) $x * (x * y) = y$ for all $x, y \in \mathbb{Z}$;

b) $(x * y) * y = x$ for all $x, y \in \mathbb{Z}$.

The professor claims in his book that:

1. the operation $*$ is commutative: $x * y = y * x$ for all $x, y \in \mathbb{Z}$.
2. the operation $*$ is associative: $x * (y * z) = x * (y * z)$ for all $x, y \in \mathbb{Z}$.

Which of these claims follow from the stated axioms?

6. Determine the maximal size of a set of positive integers with the following properties:

1. The integers consist of digits from the set $\{1, 2, 3, 4, 5, 6\}$.
2. No digit occurs more than once in the same integer.
3. The digits in each integer are in increasing order.
4. Any two integers have at least one digit in common (possibly at different positions).
5. There is no digit which appears in all the integers.

7. A photographer took some pictures at a party with 10 people. Each of the 45 possible pairs of people appears together on exactly one photo, and each photo depicts two or three people. What is the smallest possible number of photos taken?

8. The director has found out that six conspiracies have been set up in his department, each of them involving exactly 3 persons. Prove that the director can split the department in two laboratories so that none of the conspiring groups is entirely in the same laboratory.

9. To every vertex of a regular pentagon a real number is assigned. We may perform the following operation repeatedly: we choose two adjacent vertices of the pentagon and replace each of the two numbers assigned to these vertices by their arithmetic mean. Is it always possible to obtain the position in which all five numbers are zeroes, given that in the initial position the sum of all five numbers is equal to zero?

10. 162 pluses and 144 minuses are placed in a 30×30 table in such a way that each row and each column contains at most 17 signs. (No cell contains more than one sign.) For every plus we count the number of minuses in its row and for every minus we count the number of pluses in its column. Find the maximum of the sum of these numbers.

11. The altitudes of a triangle are 12, 15, and 20. What is the area of the triangle?

12. Let ABC be a triangle, let B_1 be the midpoint of the side AB and C_1 the midpoint of the side AC. Let P be the point of intersection, other than A, of the circumscribed circles around the triangles ABC_1 and AB_1C. Let P_1 be the point of intersection, other than A, of the line AP with the circumscribed circle around the triangle AB_1C_1. Prove that $2AP = 3AP_1$.

13. In a triangle ABC, points D, E lie on sides AB, AC respectively. The lines BE and CD intersect at F. Prove that if

$$BC^2 = BC \cdot BA + CE \cdot CA,$$

then the points A, D, F, E lie on a circle.

14. There are 2006 points marked on the surface of a sphere. Prove that the surface can be cut into 2006 congruent pieces so that each piece contains exactly one of these points inside it.

15. Let the medians of the triangle ABC intersect at point M. A line t through M intersects the circumcircle of ABC at X and Y so that A and C lie on the same side of t. Prove that $BX \cdot BY = AX \cdot AY + CX \cdot CY$.

16. Are there 4 distinct positive integers such that adding the product of any two of them to 2006 yields a perfect square?

17. Determine all positive integers n such that $3^n + 1$ is divisible by n^2.

18. For a positive integer n let a_n denote the last digit of $n^{(n^n)}$. Prove that the sequence (a_n) is periodic and determine the length of the minimal period.

19. Does there exist a sequence $a_1, a_2, a_3, \ldots$ of positive integers such that the sum of every n consecutive elements is divisible by n^2 for every positive integer n?

20. A 12-digit positive integer consisting only of digits 1, 5 and 9 is divisible by 37. Prove that the sum of its digits is not equal to 76.

The problems of earlier years can be found in [8] and [9]. See also http://nms.lu.lv, from which tracks to other websites can be easily followed.

6 Some pedagogical aspects

The problems, though easier than on IMO, are too hard for one or two persons to solve during $4\frac{1}{2}$ hours. So a lot of success depends on the cooperation skills and general organization of the team work. There have been situations when a team including four medallists of the former IMO remains on 6$^{\text{th}}$ or 7$^{\text{th}}$ place due to bad cooperation. The following stages appear to be essential in the team work:

- initial acquaintance with problems; identifying the known ones;
- distribution of problems among team members;
- regular short discussions on progress/difficulties;
- checking the already written solutions;
- ensuring that no problems remain without at least an attempt to solve it.

Very much depends on the captain of the team who should be (s)elected before the contest.

Team contests develop collaborative skills and so bring the students closer to some aspects of real research than the traditional Olympiads.

7 Influences

Since the Baltic Way has started, the success on IMO of the countries participating in it has generally increased. Many of the team leaders/ deputies have remained in BW since its early years; almost all "newcomers" have been participants of it. The small and friendly community of mathematicians running the competition fosters the exchange of ideas, teaching methods and materials in a more effective way than the large and quite formal IMO. Visits of teachers/students or lecturers on contest topics from one Baltic Way country into another is a common thing now.

The Baltic Way competition has given rise to other mathematical activities too. One of them is the project LAIMA (Latvian–Icelandic Mathematics Project). Its aim is to publish a series of books covering all essential topics in the arena of mathematical competitions.

Mathematical Olympiads today have become an important and essential part of the education system. In some sense they provide high standards for teaching mathematics at an advanced level. Many outstanding scientists are involved in composing problems for competitions. The "Olympiad curriculum", considered all over the world, is a good reflection of important mathematical ideas on elementary level.

It is the opinion of the publishers of the LAIMA series that there are relatively few important topics which cover almost everything that the international mathematical community has recognized as worthy to be included regularly in the search for and promotion of young talent. This (clearly subjective) opinion is reflected in the list of teaching aids which are to be prepared within the LAIMA project.

Seventeen books have been published so far in Latvian. They are also electronically available in the web page of the Latvian Education Information System (LIIS), http://liis.lv. As LAIMA is rather a process than a project, there is no idea of a final date; many of the already published teaching aids are second or third versions and they will be extended regularly.

Four LAIMA books are published in English ([10] –[13]). To obtain them, please contact us at nms@lu.lv.

Benedict Johannesson, President of the Icelandic Society of Mathematics, gave inspiration to the LAIMA project in 1996. Being a co-author of many LAIMA publications, he also was the main sponsor for many years. Now the project is sponsored also by the Scandinavian *Nord Plus Neighbours* foundation.

8 Concluding remarks

Team competitions in the format of Baltic Way have appeared to be interesting and stimulating for students. They have many descendants on a smaller scale in individual countries, bringing students, who initially aren't especially strong, into competitions and so into mathematics. The activities of this type are worth attention from the broad mathematical community.

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Michigan Autumn Take Home Challenge

Harold B. Reiter & Arthur Holshouser

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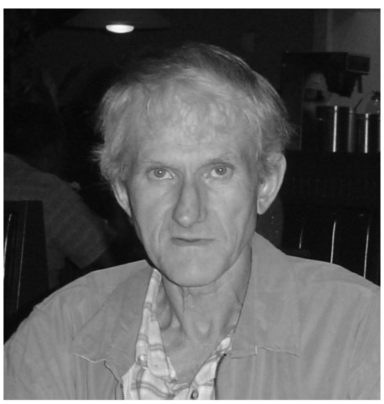

Arthur Holshouser does math all the time. He is a puzzle inventor and full-time problem solver. His special interests include combinatorial games and abstract algebraic structures.

1 History and General Information

What is the MATH Challenge?

In March 2005, we were offered the opportunity to coauthor and score a collegiate level math competition that would take place the next October.

After briefly discussing it, we accepted. Following the 2005 contest, we agreed to write and score the 2006 exam as well. The **M**ichigan **A**utumn **T**ake **H**ome Challenge is a team-oriented math competition for undergraduates. Teams of two or three students take a 3-hour exam consisting of 10 interesting problems dealing with topics and concepts found in the undergraduate mathematics curriculum. Each team takes the exam on their home campus under the supervision of a faculty advisor. This paper is the result of a presentation given by the first author at the WFNMC conference in Cambridge, England in July, 2006.

History of the MATH Challenge

In the spring of 1994, professors Mark Bollman (of Olivet College) and Timothy Sipka (of Alma College) felt the need to create a team-oriented math competition for undergraduates. That summer, with the help of Leo Schneider (of John Carroll University), the details of the competition were finalized. On November 5, 1994, 22 teams from 8 different schools participated in the first MATH Challenge.

Growth of the Challenge

During the years 1994 to 2004, the number of colleges participating grew from 8 to 16 and the number of teams from 22 into the high forties. In 2005, there were 53 teams from about 18 colleges.

Scoring

The 10 problems are scored at 10 points each for a total of 100 points. The top scoring teams have scored mostly from the mid 50s to the low 90s. The median scores have generally been in the 30s or 40s, and the lowest scores typically between 5 and 20. In 2005, the top score was 86, the median was 35 and the least score was 4.

2 The 2005 MATH Challenge

1. Suppose f, f', and f'' are continuous functions on $[0,3]$ satisfying $f(3) = 2$, $f'(3) = 1$ and $\int_0^3 f(x)\ dx = 6$. Find the value of $\int_0^3 x^2 f''(x)\ dx$.

2. (Fall 1990 O-level Tournament of Towns) Two positive real numbers are given. Their sum is less than their product. Prove that their sum is greater than 4.

3. Let S and T be finite disjoint sets of points of the plane. Prove that there exists a family L of parallel lines such that each point of S belongs to a member of L and no member of T belongs to any member of L.

4. A bug starts from the origin on the plane and crawls one unit upwards to $(0, 1)$ after one minute. During the second minute, it crawls two units to the right ending at $(2, 1)$. Then during the third minute, it crawls three units upward, arriving at $(2, 4)$. It makes another right turn and crawls four units during the fourth minute. From here it continues to crawl n units during minute n and then making a $90°$ turn, either left or right. The bug continues this until after 16 minutes, it finds itself back at the origin. Its path does not intersect itself. What is the smallest possible area of the 16-gon traced out by its path?

5. Let $f(x) = x^3 + x + 1$ and let $g(x)$ be the inverse function of f. Find $g'(3)$.

6. Let $S(n)$ denote the sum of the decimal digits of the integer n. For example $S(64) = 10$. Find the smallest integer n such that

$$S(n) + S(S(n)) + S(S(S(n))) = 2007.$$

7. Tom picks a polynomial p with nonnegative integer coefficients. Sally claims that she can ask Tom just two values of p and then tell him all the coefficients. She asks for $p(1)$ and $p(p(1) + 1)$. For example, suppose $p(1) = 10$ and $p(11) = 46{,}610$. What is the polynomial, and how did she know it?

8. Two integers are called *approximately equal* if their difference is at most 1. How many different ways are there to write 2005 as a sum of one or more positive integers which are all approximately equal to each other? The order of terms does not matter: two ways which only differ in the order of terms are not considered different.

9. An Elongated Pentagonal Orthocupolarotunda is a polyhedron with exactly 37 faces, 15 of which are squares, 7 of which are regular pentagons, and 15 of which are triangles. How many vertices does it have?

10. The bug is back! This time he crawls at a uniform rate, one unit per minute. He starts at the origin at time 0 and crawls one unit to the right, arriving at $(1, 0)$, turns $90°$ left and crawls another unit to $(1, 1)$, turns $90°$ left again, and crawls two units. He continues to make $90°$ left turns. (The path of the bug establishes a one-to-one correspondence between the non-negative integers and the integer lattice points of the plane.) Let $g(t)$ denote the position in the plane after t minutes, where t is an integer. Thus, for example, $g(0) = (0, 0)$, $g(6) = (-1, -1)$, and $g(16) = (-2, 2)$. Does there exist an integer t such that $g(t)$ and $g(t + 23)$ are exactly 17 units apart? If so, find the smallest such t.

3 The 2006 MATH Challenge

1. Let $f(x) = (x + 1)^2 e^{2x}$. The fiftieth derivative $f^{(50)}(0)$ can be expressed in the form $k2^n$ where k is an odd integer and n is a positive integer. Find k and n.

2. The three points $(4, 14, 8, 14)$, $(6, 6, 10, 8)$ and $(2, 4, 6, 8)$ are vertices of a cube in 4-space. Find the center of the cube.

3. Compute $\displaystyle\sum_{n=1}^{\infty} \frac{1}{4n^2 - 9}$.

4. Let $n \geq 1$ be fixed. Suppose n points are placed at random on a circle. Let $P(n)$ denote the probability that all n points lie on the same side of some diameter. In particular, find $P(2)$ and $P(3)$.

5. Let $D = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ denote the set of nonzero decimal digits. Note that D has $\binom{9}{4} = 126$ four-element subsets. How many of these subsets $\{a, b, c, d\}$ can be used to build a three-digit base d number $N = abc_d$ such that the difference between N and the number $\overline{N}$ obtained by reversing the digits of N is a multiple of

21? For example $123_8 - 321_8 = (8^2+2\cdot 8+3)-(3\cdot 8^1+2\cdot 8+1) = -128+2 = -126 = -6(21)$, so the set $\{1,2,3,8\}$ is one of the sets we need to count. Of course the number d used for the base must be the largest of the four numbers.

6. Suppose some faces of a large wooden cube are painted red and the rest are painted black. The cube is then cut into unit cubes. Is it possible that the number of unit cubes with some red paint is exactly $M = 2006$ larger than the number of cubes with some black paint? Find the smallest number $M \geq 2006$ for which there is such a cube and find a way to paint the faces so that the number of unit cubes with some red paint is exactly M larger than the number of cubes with some black paint.

7. There are 2005 nonzero real numbers written on a blackboard. An operation consists of choosing any two of these, a and b, erasing them, and writing $a + \frac{b}{2}$ and $b - \frac{a}{2}$ in their places. Prove that no sequence of operations can return the set of numbers to the original set.

8. Is it possible to partition the set $N = \{1, 2, \ldots\}$ of positive integers into two element subsets $\{u, v\}$ such that for each integer $n \geq 1$, there is exactly one pair $\{u, v\}$ such that $|u - v| = n$?

9. Given a triangle ABC in the plane, prove that there is a line L in the plane that cuts the triangle into two polygons of equal area and equal perimeter.

10. Suppose $(S, 0, +)$ is an Abelian group on the set S, and $(S, \cdot)$ is a binary operator on S. Also, suppose $(S, 0, +)$ distributes over $(S, \cdot)$. That is, $\forall a, b, x \in S, (a + x) \cdot (b + x) = (a \cdot b) + x$. Prove that if $(S, \cdot)$ is a group, then S is a singleton set.

4 Problems Rejected in 2005 or 2006

1. In how many different ways can a $3 \times 3 \times 3$ cube be built from nine $1 \times 1 \times 3$ blocks?

2. (London *Sunday Times*) At ABC University, the mascot does as many pushups after each ABCU score as the team has accumulated. The team always make extra points after touchdowns, so it scores only in increments of 3 and 7. For each sequence $a_1, a_2, \ldots,$ a_n where each a_k is 3 or 7, let $P(a_1, a_2, \ldots, a_n)$ denote the total number of pushups the mascot does for the scoring sequence $a_1, a_2, \ldots, a_n$. For example $P(3, 7, 3) = 3+(3+7)+(3+7+3) = 26$. Call a positive integer k *accessible* if there is a scoring sequence $a_1, a_2, \ldots, a_n$ such that $P(a_1, a_2, \ldots, a_n) = k$. Is there a number K such that for all $t \geq K$, t is accessible? If not, prove it and if so, find K.

3. What is the volume of the polyhedron Q defined by $|z - 3| + |x - y| + |x + y| + |x| + |y| \leq 6$?

4. Let C denote the 16-element set $\{(a_1, a_2, a_3, a_4) \mid a_i \in \{0, 1\}, i = 1, 2, 3, 4\}$ in Euclidean space E_4. Let T denote the set of all triangles all of whose vertices belong to C. How many members of T are acute? How many members of T are right triangles? How many members of T are obtuse?

5. You are given an unlimited supply of cubes of volumes 1, 8 and 27 and a box that measures $3 \times 3 \times 223$. What is the smallest $N \geq 100$ for which it is possible to fill the box with exactly N of the cubes?

6. The bug is back! This time he crawls at a uniform rate, one unit per minute. He starts at the origin at time 0 and crawls one unit to the right, arriving at $(1, 0)$, turns $90°$ left and crawls another unit to $(1, 1)$, turns $90°$ left again, and crawls two units. He continues to make $90°$ left turns. (The path of the bug establishes a one-to-one correspondence between the non-negative integers and the integer lattice points of the plane.) Let $g(t)$ denote the position in the plane after t minutes, where t is an integer. Thus, for example, $g(0) = (0, 0)$, $g(6) = (-1, -1)$, and $g(16) = (-2, 2)$. Suppose that the (Euclidean) distance between two lattice points A and B is an odd integer. Prove that the time required for the bug to travel between A and B is also odd.

7. The bug is back again!. He starts at the origin in the plane and takes steps of length $\sqrt{13}$ only, always landing at an integer lattice point. After six steps the bug path is a hexagon—none of the sides intersect one another—and every side has length $\sqrt{13}$. How many such hexagons are there? Prove that there are at least six non-congruent ones.

8. Suppose $(S, 0, +)$ is a finite Abelian group on the set S, and $(S, \cdot)$ is a commutative binary operator on S. Also, suppose $(S, 0, +)$ distributes over $(S, \cdot)$. That is, $\forall a, b, x \in S$, $(a + x) \cdot (b + x) = (a \cdot b) + x$.

(a) Show that $|S|$ is odd.

(b) Also, given $(S, 0, +)$, find all possible $(S, \cdot)$ that satisfy these conditions.

Alternate Problem. Given a finite Abelian group $(S, 0, +)$, show that a $(S, \cdot)$ satisfying the conditions of the problem exists if and only if $|S|$ is odd.

9. Let $S = \{a_1, a_2, \ldots, a_{2n+1}\}$ denote the vertex set for a complete, loopy, undirected graph. Also, suppose each vertex is adjacent to itself. Suppose each edge is labelled with a number from $1, 2, 3, \ldots, 2n+1$ in such a way that at each vertex x, all the $2n+1$ numbers are among the labels incident to x. Prove that the labels on the loops are exactly $1, 2, 3, \ldots, 2n + 1$.

10. Find the unique point (x, y) of the plane satisfying both

$$x^2 + 8x + y^2 + 4y + 11 = 0$$

and

$$x^2 - 16x + y^2 - 14y - 31 = 0.$$

11. Consider the polynomial $p(x) = x^4 - 12x^3 + 52x^2 - 93x + 59$. Find a line $L : y = mx + b$ such that $p(x)$ is tangent to L in two places.

12. How many squares in the plane have two or more vertices in the set
$S = (0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (2, 0)$?

5 Solutions, 2005 MATH Challenge

1. Use integration by parts twice to get

$$\begin{aligned}\int_0^3 x^2 f''(x)\ dx &= x^2 f'(x)|_0^3 - \int_0^3 f'(x)\cdot 2x\ dx \\ &= x^2 f'(x)|_0^3 - \left[f(x)\cdot 2x - \int_0^3 2f(x)\ dx\right] \\ &= x^2 f'(x)|_0^3 - f(x)\cdot 2x|_0^3 + 2\int_0^3 f(x)\ dx \\ &= 9f'(3) - 6f(3) + 2\cdot 6 \\ &= 9\cdot 1 - 6\cdot 2 + 2\cdot 6 = 9\end{aligned}$$

2. Let the two numbers be x and y. Then $x + y < xy$. By the Arithmetic Mean-Geometric Mean Inequality, $xy < \left(\frac{x+y}{2}\right)^2$, which implies that $x + y < (x + y)^2/4$, from which the result follows.

Or, $(x - y)^2 \geq 0$ so $x^2 + 2xy + y^2 \geq 4xy$. Thus $(x + y)/2 \geq \sqrt{xy}$. Combine this with $x + y < xy$ to get (letting $u = x + y$), $u < xy < (u/2)^2$ which implies $0 < 4u < u^2$ and the inequality follows from this.

3. Let R denote the collection of slopes of lines joining the pairs $(P, Q), P \in S, Q \in T$. (This may include ∞.) Next pick any real number r not in R. The family of lines with slope r through points of S satisfies the required conditions.

4. The answer is 384. Note that the vertical sides of the polygon are all odd lengths. Since the up sides equal the down sides, we must partition the set $\{1, 3, 5, 7, 9, 11, 13, 15\}$ into two subsets, one containing both 1 and 3 so that the sum of the members of each subset is $(1 + 3 + 5 + 7 + 9 + 11 + 13 + 15) \div 2 = 32$. Experimentation shows that this can be done in only one way: $Up = \{1, 3, 13, 15\}$ and $Down = \{5, 7, 9, 11\}$. The horizontal edges are trickier. The sum $2 + 4 + 6 + \cdots + 16 = 72$ so the partition must be into two sets each with sum 36 such that the 'right' edges include both 2 and 4. This can be done in four ways:

$\{2,4,14,16|6,8,10,12\}, \{2,4,8,10,12|6,14,16\}$, $\{2,4,6,8,16|10,12,14\}$, and $\{2,4,6,10,14|8,12,16\}$. Only two of these give rise to non-intersecting paths. The other non-intersecting path is given by the partition $Right = \{2,4,6,8,16\}$, $Left = \{10,12,14\}$, $Up = \{1,3,13,15\}$ and $Down = \{5,7,9,11\}$, which gives rise to a 16-gon whose area is 664.

5. The function g satisfies $f \circ g(x) = x$ for all x, so $\frac{d}{dx}\ f \circ g(x) = f'(g(x)) \cdot g'(x) = \frac{d}{dx}\ x = 1$ by the chain rule. Thus $g'(3) = 1/f'(g(3))$. Since $f'(x) = 3x^2 + 1$ it follows that $f'(1) = 4$. Also, $g(3) = f^{-1}(3) = 1$ since $f(1) = 3$. Finally $g'(3) = 1/f'(1) = 1/4$.

6. Since $S(n) \leq 9 \log n$, it follows that n must have at least 200 digits. In fact if n has fewer than 220 digits, then $S(n) \leq 219 \cdot 9 = 1971$ and $S(S(n)) \leq 1 + 9 + 6 + 9 = 25$ and $s(S(S(n))) \leq 10$, in which case their sum is at most 2006. Since $n \equiv S(n)(\text{mod } 9)$, and $2007 \equiv 0(\text{mod } 9)$, it follows that any solution n satisfies one of $n \equiv 0(\text{mod } 9)$, $n \equiv 3(\text{mod } 9)$, or $n \equiv 6(\text{mod } 9)$. We are left to try multiples of 3 whose sum of digits is at least 1971. Trying 1971, 1974, 1977, 1980, 1983, etc, we see that 1977 is the smallest integer that works. The other $S(n)$ values that work are 1980, 1983, and 2001. The smallest number for which $S(n) = 1977$ is a digit 6 followed by a string of two hundred and nineteen 9's. That is $n = 6 \cdot 10^{219} + 10^{219} - 1 = 7 \cdot 10^{219} - 1$.

7. His first answer 10 is the sum of the coefficients of p. She knows that $p(1)$ is at least as big as any of p's coefficients. If $p(x) = ax^4 + bx^3 + cx^2 + dx + e$, then $p(11) = a11^4 + b11^3 + c11^2 + d11 + e$ which is just the base 11 representation of $p(11)$. Expressing 46,610 in base 11 notation yields 32023_{11}, so $p(x) = 3x^4 + 2x^3 + 2x + 3$.

8. The answer is 2005. Let $G(n)$ denote the number of ways to write n as a sum of approximately equal positive integers. Trial and error produces $G(1) = 1, G(2) = 2, G(3) = 3$ and $G(4) = 4$. In fact, we can prove by mathematical induction that $G(n) = n$ for all positive integers. Suppose $n = a_1 + a_2 + \cdots a_k$ where either all the a_i are the same or there are two different values and they differ by 1. Thus we have either $n = ka + m(a-1), m > 0$ or $n = ka$. In the first case $n+1 = (k+1)a + (m-1)(a-1)$ if $m > 0$

and $n+1 = a+1+(k-1)a$. So we have a bijection between the representations of n and the representations of $n+1$ that include at least one number other that 1. In addition $n+1 = 1+1+1+\cdots+1$ represents a new representation. Thus $G(n+1) = G(n)+1$ for all integers n.

Alternatively, note that for each n, $1 \le n \le 2005$, there is exactly one sum with n terms. To see this, divide 2005 by n. If $2005 = nq+r$, then $2005 = (n-r)q + r(q+1)$. Of course q and $q+1$ are approximately equal.

9. The number of edges is $\frac{1}{2}(15 \cdot 4 + 7 \cdot 5 + 15 \cdot 3) = 70$, so by Euler's formula $e+2 = f+v$, we have $70+2 = 37+v$ and $v = 35$.

10. The path from $g(t)$ to $g(t+23)$ must involve exactly one turn. The only solutions to $u^2 + (23-u)^2 = 17^2$ are $u = 8$ and $u = 15$. The first straight edge of length 15 occurs on the left to right segment from $g(210) = (-7,-7)$ to $g(225) = (8,-7)$. Going back 8 units to $g(202) = (-7,1)$, we see that

$$D((-7,1),(8,-7)) = \sqrt{15^2 + 8^2} = 17.$$

Incidentally, the function g is given by

$$g(t) = \begin{cases} (t - 4n^2 - 3n, -n) & \text{if } 2n(2n+1) \le t < (2n+1)^2 \\ (n+1, t - 4n^2 - 5n - 1) & \text{if } (2n+1)^2 \le t < (2n+1)(2n+2) \\ (4n^2 - n - t, n) & \text{if } 0 < (2n-1)(2n+1) \le t < (2n)^2 \\ (-n, 4n^2 + n - t) & \text{if } 0 < (2n)^2 \le t < (2n)(2n+1) \end{cases}$$

6 Solutions, 2006 MATH Challenge

1. Recall the Maclaurin series for e^x is $1 + x + x^2/2! + x^3/3! + \cdots$. Therefore $e^{2x} = 1 + 2x + (2x)^2/2 + (2x)^3/3! + \cdots$. This means that

$$f(x) = (x^2 + 2x + 1)e^{2x} = (x^2 + 2x + 1)\sum_{i=0}^{\infty} \frac{(2x)^i}{i!}.$$

On one hand, the coefficient of the 50^{th} term of the series is $f^{(50)}(0)/50!$ and on the other hand, the x^{50} term is $x^2 \frac{(2x)^{48}}{48!} +$

$2x\frac{(2x)^{49}}{49!}+\frac{(2x)^{50}}{50!}$ so the coefficient is $\frac{(2)^{48}}{48!}+2\frac{(2)^{49}}{49!}+\frac{(2)^{50}}{50!}$. Now it follows that

$$\begin{aligned} f^{(50)}(0) &= 50!\left(\frac{(2)^{48}}{48!}+2\frac{(2)^{49}}{49!}+\frac{(2)^{50}}{50!}\right) \\ &= 50\cdot 49\cdot 2^{48}+50\cdot 2^{50}+2^{50} \\ &= 25\cdot 49\cdot 2^{49}+102\cdot 2^{49} \\ &= 1327\cdot 2^{49} \end{aligned}$$

2. Let $A=(4,14,8,14)$, $B=(6,6,10,8)$ and $C=(2,4,6,8)$, and let d denote the (Euclidean) distance function. Then $d(A,B)=\sqrt{54}$, $d(A,C)=\sqrt{72}$, and $d(B,C)=\sqrt{18}$, so ABC is a right triangle with hypotenuse AC, by the converse of the Pythagorean Theorem. Since the distances are in the ratio $1:\sqrt{3}:2$ the segment AC connects vertices at opposite corners of the cube. Therefore, the center of the cube is the midpoint of this segment, $(3,9,7,11)$.

3. First note that $\frac{1}{4n^2-9}=\frac{1}{6}\left(\frac{1}{2n-3}-\frac{1}{2n+3}\right)$. Note that $\displaystyle\sum_{n=1}^{\infty}\frac{1}{2n-3}=-1+1+1/3+1/5+1/7+\cdots$ while $\displaystyle\sum_{n=1}^{\infty}\frac{1}{2n+3}=1/5+1/7+\cdots$ so their difference is $1/3$. Thus, the sum of the series is $1/6\cdot 1/3=1/18$.

4. Let us first inscribe a regular $2m+1$-gon in this circle where $m\geq n-1$ and m is fixed. We will calculate $(*)$ the probability if we choose n of these $2m+1$ vertices at random then these n points will lie on the same side of some diameter.

Now the number of different ways that n vertices can be chosen from these $2m+1$ vertices so that these n vertices will lie on the same side of some diameter equals $(2m+1)\binom{n-1}{m}$. To see this, orient the circle in the counter clockwise direction. Therefore, if n points lie on the same side of a diameter of the oriented circle, then we can single out a first member of these n points and call it $\overline{0}$. Thus, the $\overline{0}$ can be chosen in $2m+1$ different ways, and once

$\overline{0}$ is chosen the other $n-1$ points can be chosen in $\binom{n-1}{m}$ different ways.

The probability required in $(*)$ equals

$$\frac{(2m+1)\binom{n-1}{m}}{\binom{n}{2m+1}} = \frac{n[m(m-1)(m-2)\cdots(m-(n-2))]}{2m(2m-1)(2m-2)\cdots(2m-(n-2))}.$$

The solution to problem (a) is

$$\lim_{m\to\infty}\frac{n[m(m-1)(m-2)\cdots(m-(n-2))]}{2m(2m-1)(2m-2)\cdots(2m-(n-2))} = \frac{n}{2^{n-1}}.$$

5. We use the notation abc_d to mean the number $a\cdot d^2+b\cdot d+c$. If $N=abc_d$, then we count the set $\{a,b,c,d\}$ if

$$\begin{aligned} N-\overline{N} &= a\cdot d^2+b\cdot d+c-(c\cdot d^2+b\cdot d+a)\\ &= (a-c)d^2+c-a\\ &= (a-c)(d-1)(d+1)=21k \end{aligned}$$

In order that $(a-c)(d-1)(d+1)$ is a multiple of 21, at least one of $a-c, d-1$, and $d+1$ must be 7. If $d=8$, then $d-1=7$ and $d+1=9$, so any a,b,c works as long as they are all less than 8. There are $\binom{7}{3}=\frac{7\cdot6\cdot5}{3\cdot2\cdot1}=35$ such sets. If $d+1=7$ then $d-1=5$ and we must have $a-c=\pm3$ (since $d=6$, we cannot have $a-c=6$). So $\{a,c\}=\{1,4\}$ or $\{a,c\}=\{2,5\}$. There are 3 sets of the form $\{1,4,6,b\}$, and 3 more of the form $\{2,5,6,b\}$, with $b<6$. Now if $a-c=\pm7$, then $\{a,c\}=\{1,8\}$ and $d=9$. In this case, $(d-1)(d+1)=80$, and $(a-c)(d-1)(d+1)$ is not a multiple of 21. Hence the number of subsets satisfying the requirements is $35+6=41$ (the two sets enumerated are disjoint).

6. The answer is $M=2046$. We paint black an opposite pair of faces of a $33\times33\times33$ cube. Note that the number r of red faces must be four, five, or six. There are two cases with four red faces: the black faces are adjacent or not. The four equations we get are $r = 6 :\ \ n^2 - 12n + 8 = M$; $r = 5 :\ \ 4(n-1)^2 = M$; $r = 4 :\ \ 2(n-1)^2 = M$ and $2n(n-2) = M$, none of which have integer solutions for $2006 \le M \le 2045$.

7. The sum of the squares of the two new numbers $(a + b/2)^2 + (b - a/2)^2 = a^2 + ab + b^2/4 + b^2 - ab + a^2/4 = 5a^2/4 + 5b^2/4$ is larger than the sum of the squares of the original numbers.

8. Yes, it is possible. Start with the pair $\{1, 2\}$. If n doubleton sets $\{u_i, v_i\}, i = 1, 2, 3, \ldots, n$, where $u_i < v_i$ have been found, define the $n + 1^{\text{st}}$ subset $\{u_{n+1}, v_{n+1}\}$ as follows. Let u_{i+1} be the smallest positive integer that does not appear in any of the first n doubletons. Then $v_{n+1} = u_{n+1} + n + 1$. Note that $u_1 = 1 < u_2 < u_3 < \cdots < u_n < u_{n+1}$, and so the same can be said about the v_is, $v_1 = 2 < v_2 < v_3 < \cdots < v_n < v_{n+1}$. Since $v_i > u_i$, it follows that the set of u_is is disjoint from the set of v_is. By Mathematical Induction, for each n there is exactly one pair $\{u, v\}$ such that $|v - u| = n$.

Students familiar with Wythoff's game will recognize this problem. For example, see
http://www.cut-the-knot.org/pythagoras/withoff.shtml

9. Suppose the area of the triangle is a and the perimeter is p. For each X on the boundary of $\triangle ABC$, define $f(X)$ as follows:

(a) If $X \in \overline{AB},\ f(X) = AX$,
(b) If $X \in \overline{BC},\ f(X) = AB + BX$, and
(c) If $X \in \overline{CA},\ f(X) = AB + BC + CX$.

Note the ambiguity $f(A) = 0$ and $f(A) = p$. For each $t \in [0, p/2]$, define $(X(t), Y(t))$ as follows:

(a) $X(t)$ and $Y(t)$ are points on the boundary of $\triangle ABC$,
(b) $f(X(t)) = t$, and
(c) $f(Y(t)) = t + p/2$.

Note that $X(0) = A$ and $Y(p/2) = A$. For each $t \in [0, p/2]$, the line segment $L = X(t)Y(t)$ divides $\triangle ABC$ into two parts with each part having the same perimeter. For each $t \in (0, p/2)$, define

$A(t)$ to be the area of that part which contains the point A. Also, let $A(0)$ be the area of $\triangle AY(0)C$ and let $A(p/2)$ be the area of $\triangle ABX(p/2)$. Note that $Y(0) = X(p/2)$. Now $A(t)$ is a continuous function on $t \in [0, p/2]$. Also, $(A(0)+A(p/2))/2 = a/2$. Therefore, by the Intermediate Value Theorem, there exists $t \in [0, p/2]$ such that $A(t) = a/2$.

10. For all $x \in S$, $x \cdot x = (0 + x) \cdot (0 + x) = (0 \cdot 0) + x$. If $(S, \cdot)$ is a group with identity i, then $i \cdot i = (0 \cdot 0) + i$. Since $i \cdot i = i$, it follows that $0 \cdot 0 = 0$. Therefore, for all $x \in S, x \cdot x = x$. From this it follows that for all $x \in S, (x \cdot x)x^{-1} = x = i$, proving that S is a singleton.

7 Solutions to the Rejected Problems

1. The answer is 21. Equivalent statement: how many ways can the space $[0, 3] \times [0, 3] \times [0, 3]$ be tiled with $1 \times 1 \times 3$ blocks? There are 3 ways to cover the $1 \times 1 \times 1$ cube with $(0, 0, 0)$ as a vertex. Once this is done, note that at least two other blocks must be parallel to the first one, and there are two planes in which this can occur. There is one way this can happen in *both* planes (all nine blocks are parallel). Given three parallel blocks in one plane, there are four ways to complete the tiling. Thus there are $4+4-1 = 7$ ways to complete the tiling once the first block has been determined. Hence there are 21 ways to tile the $3 \times 3 \times 3$ cube.

2. The number of pushups the mascot must do is $7x + 3(T_n - x)$ where $T_n = 1 + 2 + \cdot + n$, when the team scores n times during the game. Now the remainders when $T_n = n(n + 1)/2$ is divided by 4 are $1, 3, 2, 2, 3, 1, 0, 0$, and this sequence repeats. In other words, $T_1 \equiv 1 (\text{mod } 4)$, etc. Let $K_n = \{7x + 3(T_n - x) \mid x = 0, 1, \ldots, n\}$. Note that K_n is the number of possible pushups the mascot must complete when his team scores n times. For example $K_1 = \{3, 7\}$, $K_2 = \{9, 13, 17, 21\}$. Its easy to see that the elements of each K_n all differ by a multiple of 4 from one another. Thus, we have four problems, one for each remainder r.

For $r = 1$, we compute $K_1, K_6, K_9, K_{14}, K_{17}$, etc. Since $K_1 = \{3, 7\}$ and $K_6 = \{63, 67, \ldots, 147\}$ and $K_9 = \{135, 139, \ldots, 315\}$ and $K_{14} = \{315, 319, \ldots\}$ the largest unachievable number is 59.

For $r = 2$, we must compute K_3, K_4, K_{11}, K_{12} and K_{19}. The largest member of K_{12} is $7 \cdot 12 \cdot \frac{13}{2} = 546$ and the smallest member of K_{19} is $3 \cdot 19 \cdot \frac{20}{2} = 570$, so 566 is the largest unachievable number of this sort.

For $r = 3$, we have $K_2 = \{9, 13, 17, 21\}$, $K_5 = \{45, 49, 53, \ldots, 105\}$, $K_{10} = \{165, 169, \ldots, 385\}$, and K_{13} has smallest member 273, we can see that 161 is the largest unachievable number of pushups for which the remainder is 2.

Finally, for $r = 0$, a similar analysis shows that the largest unachievable number of this sort is 116. Therefore, 566 is the largest unachievable number.

3. Let P denote the polygonal solution to

$$* \qquad |x - y| + |x + y| + |x| + |y| = 6.$$

Note that P is symmetric with respect to each of $x = 0$, $y = 0$ and $(0, 0)$. If $0 \le y \le x$, the line segment from $(2, 0)$ to $(3/2, 3/2)$ satisfies $*$. It follows that P is a convex octagon with vertices $(\pm 2, 0), (0, \pm 2), (\pm 3/2, \pm 3/2)$ and area $16 - 4 = 12$.

Now Q is a pair of attached cones over P, attached at a copy of P three units above the x-y plane. This 10-vertex polyhedron has extreme points $(\pm 2, 0, 3), (0, \pm 2, 3)$,
$(\pm 3/2, \pm 3/2, 3), (0, 0, 9)$ and $(0, 0, -3)$. The volume V of Q is given by $V = 2 \cdot \frac{1}{3} \cdot 12 \cdot 6 = 48$.

4. Every three-element subset of C determines a triangle because no three members of C are collinear. There are $\binom{16}{3} = 560$ three-element subsets. For each triangle ABC, we define the *shape* $abc, a \le b \le c$, where a, b and c represent the number of coordinates in which two points differ. For example, the shape of the triangle with vertices $(0, 0, 0, 0), (1, 0, 0, 0)$ and $(0, 1, 1, 1)$ is 134. There are exactly six different shapes: 112, 123, 134, 224, 222, and 233. None are obtuse, by the converse of the Pythagorean theorem. The first four shapes are right triangles and the last two acute. Now each of the 16 vertices belongs to exactly 105 triangles since there are $3 \cdot 560 = 1680$ vertices among the 560 triangles and each vertex appears the same number of times as any other. Therefore, we can simply count the triangles of each shape that include the origin $O = (0, 0, 0, 0)$. In what follows, it is useful to know the number of members of C that differ in 1, 2, 3 and 4 places from O. This is easily seen to be $4, 6, 4, 1$ respectively. So, to count the number of triangles of shape 112 that include O, we note that there are two types, one in which the right angle is at O and the other where it is not. There are $\binom{4}{2} = 6$ of the former and $4 \cdot 3 = 12$ of the latter. Continuing in this way, there are 36 of the shape 123, 12 of shape 134 and 9 of shape 224. There are 12 of shape 222 and 18 of shape 233 for a total of 105. Since $12 + 18 = 30$ of the 105 triangles with a vertex O are acute, 160 of the 560 triangles are acute and the other 400 are right triangles.

5. The answer is 102. Let (a, b, c) represent the number of cubes of each type, where a is the number of cubes of volume 27, b, the number of cubes of volume 8 and c, the number of unit cubes. Then $27a + 8b + c = 2007$. We show that if $a + b + c = 100$ (case A) or if $a + b + c = 101$ (case B), there is no such collection of cubes. Before discussing these two cases, note that each $2 \times 2 \times 2$ cube requires 10 unit cubes to build out the $2 \times 3 \times 3$ section of the box occupied by the $2 \times 2 \times 2$ cube. Thus $c = 10b + 9k$ for some non-negative integer k.

Case A leads to $26a + 7b = 1907$ where we insist that $a + b \leq 100$. There are just two solutions $a = 72, b = 5$ and $a = 65, b = 31$ satisfying both these conditions: solutions are $(72, 5, 23)$ and $(65, 31, 4)$. But neither of these satisfies $c \geq 10b$.

Case B leads to $26a + 7b = 1906$. Only one solution to $26a + 7b = 1906$ satisfies $b + c \leq 100$ and that is $(69, 16, 15)$, but clearly, there are not enough unit cubes to build the big box. Finally, trying $N = 102$, success! The ordered triple $(73, 1, 28)$ works.

6. Let $T(A, B) = T((a, b), (c, d))$ denote the time between the points. Suppose $T((a, b), (c, d))$ is even. Then $a + b + c + d$ is even, so $c - a$ and $d - b$ are the same parity. If both are even, then $(c-a)^2 + (d-b)^2$ is divisible by 4, so $D(A, B)$ cannot be an odd number. If both are odd, then $(c - a)^2 + (d - b)^2$ is congruent to 2 modulo 4, in which case $\sqrt{(c - a)^2 + (d - b)^2}$ cannot be an odd number.

7. One reason we left this one off is that we didn't have a solution for it.

8. First, suppose $\forall x \in S, 0 \cdot x$ is known. Then $\forall a, b \in S, a \cdot b = (0+a)\cdot(b-a+a) = [0 \cdot (b-a)]+a$. Therefore, $(S, \cdot)$ is completely defined from the values $0{\cdot}x, x \in S$. Now since $\forall a, b \in S, a{\cdot}b = b{\cdot}a$ we know that $\forall a, b \in S, a \cdot b = [0 \cdot (b-a)] + a = [0 \cdot (a-b)] + b = b \cdot a$. Letting $b - a = x$, we see that $0 \cdot (-x) = [0 \cdot x] - x$. Therefore, $(S, \cdot)$ is commutative if and only if $\forall x \in S, 0 \cdot (-x) = (0 \cdot x) - x$. This condition determines all commutative $(S\cdot)$ such that $(S, 0, +)$ distributes over $(S\cdot)$. Now suppose $|S|$ is even. Then since $(S, 0, +)$ is an Abelian group $\exists x \in S$ such that $x \neq 0$ and $x + x = 2x = 0$. This means $x \neq 0$ and $-x = x$. But then $0 \cdot (-x) = [0\,(x)] - x$ is impossible since $0 \cdot (-x) = 0\,(x)$.

Note. If $|S|$ is odd, then $\forall x \in S \setminus \{0\}, x \neq -x$.

Alternate Problem. Given a finite Abelian group $(S, 0, +)$, show that a $(S, \cdot)$ satisfying the conditions of the problem exist if and only if $|S|$ is odd.

9. Suppose there is a number y in the set $\{1, 2, 3, \ldots, 2n+1\}$ that is not assigned to any loop. Then the edges assigned the label y represent a pairing of the vertices. This is impossible because $|S|$ is odd.

10. The first equation is a circle of radius 3 centered at $(-4, -2)$ and the second is a circle of radius 12 centered $(8, 7)$. Since the points $(-4, -2)$ and $(8, 7)$ are exactly 15 units apart, the convex combination $0.8(-4, -2) + 0.2(8, 7) = (-1.6, -0.2)$ is $1/5$ the way from $(-4, -2)$ to $(8, 7)$ and therefore is the unique point belonging to both circles.

11. Let $g(x) = p(x) - mx + c$. Note that g has two pairs of repeated roots (because of the tangency condition). Let a and b denote these roots. Thus $g(x) = (x-a)^2(x-b)^2 = x^4 - 2x^3(a+b) + (a^2+4ab+b^2)x^2 + \cdots = x^4 - 12x^3 + 52x^2 + \cdots$, and it follows that $a+b = 6$ and $a^2+4ab+b^2 = a^2+2ab+b^2+2ab = (a+b)^2+2ab = 6^2+2ab = 52$. Solving the resulting equation for a yields $a = 2$ and $a = 4$, and from this it follows that $m = 3$ and $c = -5$.

12. One way to do this is to determine the number of each size. The possible areas of the squares are $1/2, 1, 2, 4, 5,$ and 8 and there are respectively $6, 7, 8, 3, 4$ and 2 of these for a total of 30. Alternatively, there are $\binom{6}{2} = 15$ pairs of points, each of which could give rise to as many as 3 squares. Since one square has all four of its vertices in the grid, this overcounts by 5, and there are 5 squares with three vertices in the grid, each of which leads to overcounts by 2 for a total of $5 + 10 = 15$. Thus there are $45 - 15 = 30$ squares. There is yet another method: start with two points and count the squares: 3. Add a third point and count the new squares, then add a fourth point and count the new ones.

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Math Circles for Students and Teachers

Tatiana Shubin

Tatiana Shubin obtained her undergraduate mathematical education in the USSR, at Moscow State and Kazakh State Universities. In 1983 she earned her Ph.D. from University of California at Santa Barbara, and after holding a visiting lecturer position at UC Davis, she joined the faculty of San Jose State University in 1985. She is a co-founder and one of the coordinators of the Bay Area Mathematical Adventures (BAMA), the San Jose Math Circle and The Teachers Circle, and is involved in various local and national math competitions for middle and high school students. She is a co-editor of the book Mathematical Adventures for Students and Amateurs, *and in 2006 she received a Mathematical Association of America Section Award for Distinguished Teaching.*

In the United States, a great many K-12 students and teachers are starving for real mathematics as a coherent logical system, learning of which requires deep thinking above and beyond prescribed limits. For some students, competitions serve as motivation for learning advanced topics not usually found in the school curriculum. Other students do not respond well to a competitive environment. Math circles serve both types of students, and teachers, as well. In this article, I will mostly describe two math circles: the San Jose Math Circle (SJMC[1]) that is serving primarily middle school students and has been running since 1998, and The Teacher's Circle[2] that started in August of 2006.

[1] http://mathematicaladventures.org
[2] http://theteacherscircle.org

What is a math circle? The idea originated in Eastern Europe, and gradually made its way to various parts of the world, including the United States. Usually, *Mathematical Circle* means a weekly gathering of students of high school age or younger working on a problem involving complex and advanced topics, guided by mathematicians and educators. The motivation for this program is to increase the interest and enjoyment of mathematics among young students through thought-provoking problems and in studying the structure of their solutions. I believe that the best description of a math circle belongs to Mark Saul who said: "Mathematical Circles are a form of outreach that brings mathematicians and mathematical scientists into direct contact with pre-college students. One thing all math circles have in common is that the students enjoy learning mathematics, and the circle gives them a social context in which to do so."

In the San Francisco Bay Area the idea of math circles was brought forth by a young postdoc Zvezdelina (Zvezda) Stankova. Some of you might recognize the name—Zvezda used to be a Bulgarian IMO team member, and recently she served as a coach of the USA IMO team. She found energetic and able supporters such as Paul Zeitz of the University of San Francisco (USF), and Hugo Rossi of the Mathematical Sciences Research Institute (MSRI) in Berkeley, and thus the Berkley Math Circle (BMC[3]) began in the fall of 1998. At approximately the same time, and about 45 miles to the South, we started our own circle—SJMC.

From the very beginning, SJMC students tended to be pretty young—most of them have always been 7th or 8th graders (12–14 years old); in the last two years, we had an influx of even younger students, some of them as young as 8. We believe that catching them early is crucial—it is a well-known fact that it is at precisely that age the students' initial interest in and appreciation of mathematics turns into apathy, boredom, and active dislike. Math circles come to rescue them from the onset of these negative feelings and misconceptions.

Our circle meets weekly during the school year on the San Jose State University (SJSU) campus, Wednesday, 7:00–8:45 pm. In reality, students usually get so interested and involved that the circle rarely breaks up before 9:00 pm. The attendance varies from 10 to 35 students

[3]http://mathcircle.berkeley.edu

from different schools. Every meeting is self-contained and taught by different instructors from several universities and mathematical institutions, including SJSU, the University of California, Berkeley, Stanford University, the American Institute of Mathematics (AIM), and others. The topics discussed at the meetings vary (take a look at our current schedule on our web page) and the styles of presenting mathematics are as different as the people who teach it. What remains constant, though, is students' involvement, and the central role that is played by problem solving and deep thinking.

Since the very start of SJMC in 1998, about once a month a regular meeting of the SJMC is replaced by a talk organized by the Bay Area Mathematical Adventures (BAMA[4]). BAMA runs a series of talks whose main goal is to encourage an interest in mathematics among bright secondary school students, as well as their teachers, parents, and the general public. Six talks per school year alternate between the campuses of Santa Clara University and San Jose State University. The series is sponsored by the Departments of Mathematics and Computer Science of these universities, with support also provided by AIM, and MSRI. The presentations include a broad variety of topics, and are given by outstanding mathematicians. BAMA talks have been very successful, with enthusiastic audiences that consist mainly of students, ranging from sixth and seventh grades through high school. On the average, BAMA talks draw 100–120 students and others. Among the past BAMA speakers there have been members of the National Academy of Sciences, the American Academy of Arts and Sciences, Presidents and Vice Presidents of both the Mathematical Association of America (MAA), and the American Mathematical Society (AMS), and invited speakers of the International Congress of Mathematicians. I simply cannot refrain from listing just a few of the illustrious speakers' names: John H. Conway, Hendrick Lenstra, Jr., Ron Graham, Kenneth Ribet. You may find the list of all 50 former talks and abstracts on our web site. In 2004, a book consisting of 19 BAMA talks was published by the MAA in its *Spectrum* series.

BMC and SJMC are not the only math circles that have helped to bring mathematics in contact with young and eager minds. There are also

[4] http://mathematicaladventures.org

well known circles in other parts of the country, such as Boston, Salt Lake City and Chicago, to name a few. The need to bring together all the people working in different circles has resulted in the first National Conference on Mathematical Circles and Olympiads. The conference was held on December 16–18, 2004 at the Mathematical Sciences Research Institute, Berkeley, California. It brought together people from across the country who have developed Math Circles, and have been involved in Math Competitions and Camps, as well as those interested in developing such projects. The program included talks, panel discussions, workshops on math circles and camps, exposition of available materials, and special sessions of the Boston and Berkeley Math Circles. One of the purposes of the conference was to create a national network of mathematical circles that will serve as a resource and facilitator for new circles, and within which ideas and materials can be exchanged. MSRI has been supporting this project ever since.

Mathematical competitions are often associated with math circles. In fact, some circles are primarily aimed at training students to successfully compete in math competitions, including USAMO and IMO. Here in the Bay Area, we have our own Bay Area Olympiad (BAMO[5]). It is an annual exam given on the last Tuesday of February to students at participating secondary schools in the San Francisco Bay area. The exams are mailed to registered schools, proctored locally, and then returned to be graded by a group of math circles' instructors. The following weekend there is an awards ceremony, with prizes for individuals and schools, free lunch for everybody, and a math lecture by a distinguished mathematician as a treat. This event has been hosted by various universities—UC Berkeley, USF, Mills College, Stanford, SJSU; this year it was held at MSRI. BAMO differs from many other math competitions in that it is proof/essay-style: all the problems demand creative thinking and clear reasoning. BAMO has been held every year since 1999. The average participation has been 250 students from approximately 45 schools.

So far we've been talking about students. But what about teachers? They are often as eager as their students to learn more mathematics, and to think deeply about real mathematical problems in order to enhance

[5] http://mathcircle.berkeley.edu/BMC6/pages/BAMO/bamo.html

their teaching. At the MSRI conference in December of 2004, an idea of a math circle for teachers first came about, and after almost two years of planning and fund raising efforts, The Teacher's Circle came to being.

As stated on The Teacher's Circle web page[6], "The aim of The Teacher's Circle is to equip educators with an effective problem-solving approach to teaching mathematics. This style of learning is based on the math circle environment that has proven to be successful for students around the world. Therefore The Teacher's Circle will immerse a group of middle school math teachers in engaging mathematics and expose them to a dynamic style of classroom presentation. Participants will come away with a variety of resources, lesson modules, and a renewed sense of appreciation for the fascinating world of mathematics. Teachers will also be eligible for continuing education credit, professional development units, or college course credits."

The Teacher's Circle program started with a summer workshop held at the American Institute of Mathematics (AIM) in Palo Alto, California, during the week of August 14–18, 2006. The event was co-sponsored by MSRI. At the workshop, there were more than 20 local middle school mathematics teachers and administrators. There were also four outside observers, three of whom were professional mathematicians from St. Louis, Chicago, and Charlotte. The observer from Charlotte was Harold Reiter who had heard about the upcoming Teacher's Circle at the WFNMC conference in Cambridge, and was interested enough to come! His participation was invaluable to the workshop.

The daily Monday to Thursday curriculum included six hours of talks with audience participation, delivered by five instructors (Tom Davis, Tatiana Shubin, Sam Vandervelde, Paul Zeitz, and Joshua Zucker) and a less formal one-hour evening session. In addition, there were two hours of instruction on Friday. Topics covered during the week included an introduction to problem solving, number theory, Euclidean and combinatorial geometry, elements of topology, the fourth dimension, symmetry and visualizing algebra, and probability.

One of our goals was to demonstrate that problem solving is an effective mechanism for learning, since we believe that it provides the strongest

[6]http://theteacherscircle.org

motivation for studying mathematics. We also wanted to free teachers from the fear of tackling difficult problems. Failing to solve a problem in a very short time is a positive rather than negative thing. By thinking hard about a problem for a long time students learn material at a deeper level. The more often this occurs, the better they are able to absorb new and more complicated concepts both inside and outside of mathematics.

This notion seems to have been successfully conveyed. One of the participants stated, "I hope to introduce many of the problems I learned during the week to get my students to think more deeply about a problem rather than just calculating an answer." Another wrote, "I will focus more on problem solving and strive to keep it ever present in my classes. I am more willing to pose challenging problems, even if they may be 'too' challenging, and to share strategies with children. I will emphasize process as much as outcome." All of the participating teachers replied in the affirmative to the question "Do you anticipate changing the way you teach middle school mathematics in the upcoming year?" The various changes that they proposed included giving more open-ended problems to their students, providing more group problem solving time, replacing repeated drills with a good problem that requires patience, and illustrating different methods for approaching a problem.

A recurring theme in teachers' responses was that of collaborative learning. Teachers appreciated the collaboration that they experienced at the workshop and how this teamwork enhanced their own learning. They perceived this as the true essence of a math circle, and they are ready to implement more collaboration in their classrooms. Some of their responses were, "I will allow more thinking and collaboration time," and, "I hope that I will be able to integrate more 'student circle' opportunities for kids to be talking to each other while tackling problems."

Teachers also pointed out the positive effect of experiencing different teaching styles. They admitted that even though the instructions were fast-paced and covered many topics leaving them behind at times, this experience in itself was valuable for them. One teacher wrote, "Some of the problem solving was way over my head. Even so, I learned – if only to understand what my students' experience when lost or overwhelmed." Another said, "Some of the topics required us to struggle, but it was a good reminder of how our students might feel, so even that had value."

Mathematics is often taught at school as an unrelated collection of facts. At the workshop, teachers were able to see a more coherent picture. We hope that they will begin revealing this interconnectedness of ideas and underlying mathematical structure to their students. In their final evaluations, teachers said that they would begin "discussing a given topic from as many angles as possible."

Among the aspects of the workshop that participants found to be particularly valuable they listed the following: the variety of educators present (middle school teachers, college professors, etc.), the opportunity to work with other teachers that allowed them to weigh and solidify ideas, the time to explore and work on mathematics with a partner or small group, the developing network of teachers with whom they are now able to share ideas, the discovery of new materials and resources, and the compatibility of 'hard' mathematics with the middle school curriculum.

The workshop clearly showed that one of the greatest deficiencies in professional development for middle school teachers is the lack of solid mathematical content. All the participating teachers greatly enjoyed and valued the mathematics that was offered. It seems that the teachers are really starving for mathematics, and that math circles for teachers may help to alleviate this problem.

The summer workshop was the first component of The Teacher's Circle year-long program. An equally important component of the program consists of seven follow-up events which will occur once a month throughout the school year. Every month from September to April (except December), The Teacher's Circle will offer an evening three-hour event for middle school math teachers interested in exploring accessible, exciting topics in mathematics and learning about a problem-solving approach to teaching math. Participants in The Teacher's Circle program who attended the summer workshop and who attend at least six of these evening sessions will be eligible for course credit through San Jose State University.

We still need to wait and see what the long-term effect of The Teacher's Circle will be, even though we do believe that if a good beginning is at all indicative, the impact will be great and profound, and over time, thousands of middle school students will benefit from this circle. But we do know the very real results achieved by the SJMC in its eight years of existence. Many of the former participants have successfully competed in various math competitions and olympiads, all the way to, and beyond, USAMO. Two of our "circlers" have been on the US IMO team, and one of them, Tiankai Liu, was a three-time IMO gold medalist. But it is even more important and satisfying that so many of our students have graduated from high school with a true love of mathematics and desire to continue studying and doing it. Dozens of them do continue as undergraduate and graduate students in some of the best universities in the nation, such as MIT, Princeton, Harvard, etc.

Our circles run well into this young millennium. Will you join?

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47$^{\text{th}}$ International Mathematical Olympiad 6–18 July 2006 Ljubljana, Slovenia

The 47$^{\text{th}}$ International Mathematical Olympiad (IMO) was held during the period July 6–18 in Ljubljana, Slovenia. Teams from 90 countries were present. These contributed to a total of 498 contestants who participated in this year's IMO. Of particular note is that Slovenia is the smallest country by population to have organised an IMO so far. They did an outstanding job.

Academic preparations had been well under way many months prior to the IMO. Each country has its own system for training and selecting students for their team. Furthermore, each country is permitted to propose up to six problems to be considered for the IMO exam itself. The host country received over 100 problems this year. From these, the Problems Selection Committee selected a shortlist of 30 problems considered highly suitable for the exams. The "Jury", comprising of the Leader of each team, arrived in the Mediterranean town of Portorož on July 6. They carefully studied the shortlist over the next three days. Three problems had to be deleted as already being known in the public domain. Through a system of discussion and voting, the Jury eventually chose six problems for the competition. The shortlist was very good this year with a wide variety of problems of various difficulties. Indeed there were many good exam papers that could have been formed out of the shortlist. The final problems, in order, included (1) a very easy, perhaps too easy, geometry question, (2) a combinatorics question which turned out to be more difficult than the Jury generally thought, (3) an inequality with an unusual case for equality, (4) a number theory problem, (5) a very nice problem about fixed points of iterations of a polynomial and (6) a very, very difficult, but lovely, geometric inequality that needed some combinatorial-geometry type thinking. The problems, in order, came from (1) Korea, (2) Serbia and Montenegro, (3) Ireland, (4) United States, (5) Romania and (6) Serbia and Montenegro. The competition itself is held over two days. The exam paper for each day has 3 questions to be attempted in $4\frac{1}{2}$ hours. Of course each contestant writes the exam in his own language so the Jury must also spend time to ensure uniform translations of the exams into the required languages. After this marking schemes are discussed.

The first event for the contestants after settling in was the opening ceremony in Ljubljana. Welcoming speeches were made by the President of the 2006 IMO Organizing Committee, Zvonko Trontelj, and by the Chairman of the IMO Advisory Board, Jozsef Pelikan. Both spoke about the challenges, benefits and memories that participation in an IMO would bring to each contestant. Addresses were also made by the Lady Mayoress of Ljubljana, Danica Simšič, and the Minister of Education and Sport of Slovenia, Milan Zver, who officially opened the 47^{th} IMO. Following some folk musical entertainment there was the parade of the participating national teams. Most teams conservatively walked onto the stage, waved and walked off. Some offered something extra, such as the Australian team who flung toy koalas into the audience. The New Zealand team did likewise with kiwis as did the Colombian team with coffee. The Japanese team gave a brief demonstration of one of their martial arts. The Italian team, still jubilant after their recent World Cup victory, performed a slow motion replay of the winning penalty kick. The contestants sat the first day's exam the very next day.

After the exams the Leaders and their Deputies assess the work of the students from their own countries. They are guided by the marking schemes discussed earlier. To ensure consistency, a team of local markers called Coordinators also assess the exams. They are also guided by the marking schemes but may allow some flexibility if, for example, a Leader brings something to their attention in a contestant's exam script which is not covered by the marking scheme. The Coordinators regularly communicate with the Chief Coordinator about precedents set in this to ensure uniformness and fairness. While five of the questions ran smoothly, many Leaders ended up being rather unhappy with the marking scheme for question 2 which had been agreed on earlier. The reason was that the marking scheme was generally seen as not ideal once the students scripts were viewed and coordination was underway. This has nothing to do with the Coordinators, rather it has to do with the limitations of setting a marking scheme in advance. Despite this difficulty the Coordinators did their job very well in ensuring a consistent awarding of marks.

In the final outcome, question 1 turned out to be one of the easiest questions on an IMO, in a long time. The average mark was 5.61 and there were 358 full solutions. At the other end of the scale, question 6 turned out to be one of the most difficult IMO questions ever. The average mark was just 0.19 and there were only 8 full solutions. At the closing ceremony there were 253 (=50.8 %) medals awarded: the distributions were 122 (=24.5 %) bronze, 89 (=17.9 %) silver and 42 (=8.4 %) gold. Of those who did not get a medal, a further 139 contestants received an honourable mention for solving at least one question perfectly. The remarkable feat of a perfect score was achieved by three contestants, Zhiyu Liu of China, Iurie Boreico of Moldova and Alexander Magazinov of Russia. They had their gold medals awarded by Janez Potočnik, the European Commissioner for Science and Research.

Our thanks for this highly successful IMO go to the Slovenian Organizing Committee of the 2006 IMO. They were supported by the Ministry of Education and Sport of the Republic of Slovenia, and the Ministry of Higher Education, Science and Technology of the Republic of Slovenia.

The 2007 IMO is scheduled to be held in Hanoi, Vietnam.

1 IMO Paper

First Day

1. Let ABC be a triangle with incentre I. A point in the interior of the triangle satisfies

$$\angle PBA + \angle PCA = \angle PBC + \angle PCB.$$

Show that $AP \geq AI$, and that equality holds if and only if $P = I$.

2. Let P be a regular 2006-gon. A diagonal of P is called *good* if its endpoints divide that boundary of P into two parts, each composed of an odd number of sides of P. The sides of P are also called *good*.

Suppose P has been dissected into triangles by 2003 diagonals, no two of which have a common point in the interior of P. Find the maximum number of isosceles triangles having two good sides that could appear in such a configuration.

3. Determine the least real number M such that the equality

$$\left|ab(a^2 - b^2) + bc(b^2 - c^2) + ca(c^2 - a^2)\right| \leq M(a^2 + b^2 + c^2)^2$$

holds for all real numbers a, b, and c.

Time allowed: 4 hours 30 minutes
Each problem is worth 7 points

Second Day

4. Determine all pairs (x, y) of integers such that

$$1 + 2^x + 2^{2x+1} = y^2.$$

5. Let $P(x)$ be a polynomial of degree $n > 1$ with integer coefficients and let k be a positive integer. Consider the polynomial $Q(x) = P(P(\ldots P(P(x))\ldots))$, where P occurs k times. Prove that there are at most n integers t such that $Q(t) = t$.

6. Assign to each side b of a convex polygon P the maximum area of a triangle that has b as a side and is contained in P. Show that the sum of the areas assigned to the sides of P is at least twice the area of P.

Time allowed: 4 hours 30 minutes
Each problem is worth 7 points

2 Statistics

Some Unofficial Country Rankings					
Rank	Country	Score	Rank	Country	Score
1	China	214	17	Hungary	122
2	Russia	174	18	Slovakia	118
3	Korea	170	19	United Kingdom	117
4	Germany	157	19	Turkey	117
5	United States	154	21	Bulgaria	116
6	Romania	152	22	Ukraine	114
7	Japan	146	23	Belarus	111
8	Iran	145	24	Mexico	110
9	Moldova	140	25	Israel	109
10	Taiwan	136	26	Australia	108
11	Poland	133	27	Singapore	100
12	Italy	132	28	France	99
13	Vietnam	131	29	Brazil	96
14	Hong Kong	129	30	Switzerland	95
15	Thailand	123	30	Kazakhstan	95
15	Canada	123	30	Argentina	95

Distribution of Awards at the 2006 IMO					
Country	**Score**	Gold	Silver	Bronze	H. Men.
Albania	76	0	1	1	2
Argentina	95	0	2	2	1
Armenia	90	0	1	1	4
Australia	108	0	3	2	1
Austria	83	0	0	3	3
Azerbaijan	77	0	1	1	4
Bangladesh (4 mem.)	22	0	0	0	2
Belarus	111	0	3	2	1
Belgium	75	0	0	1	4
Bolivia (2 members)	5	0	0	0	0
Bosnia and Herzeg.	84	0	1	2	3
Brazil	96	0	0	6	0
Bulgaria	116	0	4	1	1
Canada	123	0	5	1	0
China	214	6	0	0	0
Colombia	76	0	0	2	3
Costa Rica (2 mem.)	27	0	0	1	1
Croatia	72	0	1	1	2
Cyprus	19	0	0	0	1
Czech Republic	77	0	0	3	3
Denmark	45	0	0	0	1
Ecuador	40	0	0	1	2
El Salvador (3 mem.)	27	0	0	0	2
Estonia	80	0	0	2	2
Finland	86	0	0	4	2
France	99	1	0	3	2
Georgia	94	0	1	3	2
Germany	157	4	0	2	0
Greece	69	0	0	2	3
Hong Kong	129	1	3	2	0
Hungary	122	0	5	1	0
Iceland	63	0	0	1	2
India	92	0	0	5	1
Iran	145	3	3	0	0
Ireland	49	0	0	0	4

Distribution of Awards at the 2006 IMO					
Country	**Score**	Gold	Silver	Bronze	H. Men.
Israel	109	0	3	1	2
Italy	132	2	2	0	1
Japan	146	2	3	1	0
Kazakhstan	95	0	1	4	1
Korea	179	4	2	0	0
Kuwait (4 members)	5	0	0	0	0
Kyrgyzstan	31	0	0	0	2
Latvia	75	0	0	3	2
Liechtenstein (1 mem.)	2	0	0	0	0
Lithuania	94	0	1	2	3
Luxembourg (2 mem.)	12	0	0	0	1
Macau	63	0	0	2	1
Macedonia	57	0	0	1	3
Malaysia	40	0	0	1	1
Mexico	110	1	2	1	1
Moldova	140	2	1	3	0
Mongolia	80	0	0	2	4
Morocco	55	0	0	0	4
Mozambique (3 mem.)	0	0	0	0	0
Netherlands	57	0	0	0	5
New Zealand	66	0	0	2	2
Nigeria	11	0	0	0	0
Norway	52	0	0	1	2
Pakistan (5 members)	32	0	0	0	1
Panama (4 members)	33	0	0	0	2
Paraguay (4 members)	47	0	1	0	1
Peru	85	0	1	1	4
Poland	133	1	2	3	0
Portugal	78	0	0	3	1
Puerto Rico	11	0	0	0	0
Romania	152	3	1	2	0
Russia	174	3	3	0	0
Saudi Arabia (4 mem.)	3	0	0	0	0
Serbia	88	0	0	5	1
Singapore	100	0	2	3	1

Distribution of Awards at the 2006 IMO					
Country	**Score**	Gold	Silver	Bronze	H. Men.
Slovakia	118	1	2	3	0
Slovenia	90	0	1	3	2
South Africa	57	0	0	0	5
Spain	80	0	1	2	2
Sri Lanka (5 members)	71	0	0	3	2
Sweden	82	0	0	3	3
Switzerland	95	1	1	0	4
Taiwan	136	1	5	0	0
Tajikistan	35	0	0	0	3
Thailand	123	1	3	2	0
Trinidad and Tobago	34	0	0	0	2
Turkey	117	0	4	1	1
Turkmenistan (5 m.)	59	0	1	1	1
Ukraine	114	1	2	2	1
United Kingdom	117	0	4	1	1
USA	154	2	4	0	0
Uruguay (2 members)	12	0	0	0	1
Uzbekistan	68	0	0	2	3
Venezuela (4 members)	34	0	0	0	3
Vietnam	131	2	2	2	0
Totals (498 contest.)		42	89	122	139

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Australian Mathematics Competition (AMC) Solutions and Statistics
Edited by DG Pederson

This book provides, each year, a record of the AMC questions and solutions, and details of medallists and prize winners. It also provides a unique source of information for teachers and students alike, with items such as levels of Australian response rates and analyses including discriminatory powers and difficulty factors.

Australian Mathematics Competition Book 1 (1978-1984)

Australian Mathematics Competition Book 2 (1985-1991)

Australian Mathematics Competition Book 3 (1992-1998)

Book 3 also available on CD (for PCs only).

Australian Mathematics Competition Book 4 (1999-2005)

Excellent training and learning resources, each of these extremely popular and useful books contains over 750 past AMC questions, answers and full solutions. The questions are grouped into topics and ranked in order of difficulty.

Problem Solving Via the AMC
Edited by Warren Atkins

This 210 page book consists of a development of techniques for solving approximately 150 problems that have been set in the Australian Mathematics Competition. These problems have been selected from topics such as Geometry, Motion, Diophantine Equations and Counting Techniques.

Methods of Problem Solving, Book 1
Edited by JB Tabov, PJ Taylor

This introduces the student aspiring to Olympiad competition to particular mathematical problem solving techniques. The book contains formal treatments of methods which may be familiar or introduce the student to new, sometimes powerful techniques.

Methods of Problem Solving, Book 2
JB Tabov & PJ Taylor
After the success of Book 1, the authors have written Book 2 with the same format but five new topics. These are the Pigeon-Hole Principle, Discrete Optimisation, Homothety, the AM-GM Inequality and the Extremal Element Principle.

Mathematical Toolchest
Edited by AW Plank & N Williams

This 120 page book is intended for talented or interested secondary school students, who are keen to develop their mathematical knowledge and to acquire new skills. Most of the topics are enrichment material outside the normal school syllabus, and are accessible to enthusiastic year 10 students.

Asian Pacific Mathematics Olympiads 1989-2000
H Lausch & C Bosch-Giral

With innovative regulations and procedures, the APMO has become a model for regional competitions around the world where costs and logistics are serious considerations. This 159 page book reports the first twelve years of this competition, including sections on its early history, problems, solutions and statistics.

Polish and Austrian Mathematical Olympiads 1981-1995
ME Kuczma & E Windischbacher

Poland and Austria hold some of the strongest traditions of Mathematical Olympiads in Europe even holding a joint Olympiad of high quality. This book contains some of the best problems from the national Olympiads. All problems have two or more independent solutions, indicating their richness as mathematical problems.

Seeking Solutions
JC Burns

Professor John Burns, formerly Professor of Mathematics at the Royal Military College, Duntroon and Foundation Member of the Australian Mathematical Olympiad Committee, solves the problems of the 1988, 1989 and 1990 International Mathematical Olympiads. Unlike other books in which only complete solutions are given, John Burns describes the complete thought processes he went through when solving the problems from scratch. Written in an inimitable and sensitive style, this book is a must for a student planning on developing the ability to solve advanced mathematics problems.

101 Problems in Algebra from the Training of the USA IMO Team
Edited by T Andreescu & Z Feng

This book contains one hundred and one highly rated problems used in training and testing the USA International Mathematical Olympiad team. These problems are carefully graded, ranging from quite accessible towards quite challenging. The problems have been well developed and are highly recommended to any student aspiring to participate at National or International Mathematical Olympiads.

Hungary Israel Mathematics Competition
S Gueron

This 181 page book summarizes the first 12 years of the competition (1990 to 2001) and includes the problems and complete solutions. The book is directed at mathematics lovers, problem solving enthusiasts and students who wish to improve their competition skills. No special or advanced knowledge is required beyond that of the typical IMO contestant and the book includes a glossary explaining the terms and theorems which are not standard that have been used in the book.

Bulgarian Mathematics Competition 1992-2001
BJ Lazarov, JB Tabov, PJ Taylor, AM Storozhev

The Bulgarian Mathematics Competition has become one of the most difficult and interesting competitions in the world. It is unique in structure, combining mathematics and informatics problems in a multi-choice format. This book covers the first ten years of the competition complete with answers and solutions. Students of average ability and with an interest in the subject should be able to access this book and find a challenge.

International Mathematics — Tournament of Towns (1980-1984)

International Mathematics — Tournament of Towns (1984-1989)

International Mathematics — Tournament of Towns (1989-1993)

International Mathematics — Tournament of Towns (1993-1997)

International Mathematics — Tournament of Towns (1997-2002)
Edited by PJ Taylor

The International Mathematics Tournament of Towns is a problem solving competition in which teams from different cities are handicapped according to the population of the city. Ranking only behind the International Mathematical Olympiad, this competition had its origins in Eastern Europe (as did the Olympiad) but is now open to cities throughout the world. Each book contains problems and solutions from past papers.

Challenge! 1991 – 1995
Edited by JB Henry, J Dowsey, A Edwards, L Mottershead, A Nakos, G Vardaro

The Mathematics Challenge for Young Australians attracts thousands of entries from Australian High Schools annually and involves solving six in depth problems over a 3 week period. In 1991-95, there were two versions – a Junior version for Year 7 and 8 students and an Intermediate version for Year 9 and 10 students. This book reproduces the problems from both versions which have been set over the first 5 years of the event, together with solutions and extension questions. It is a valuable resource book for the class room and the talented student.

USSR Mathematical Olympiads 1989 – 1992
Edited by AM Slinko

Arkadii Slinko, now at the University of Auckland, was one of the leading figures of the USSR Mathematical Olympiad Committee during the last years before democratisation. This book brings together the problems and solutions of the last four years of the All-Union Mathematics Olympiads. Not only are the problems and solutions highly expository but the book is worth reading alone for the fascinating history of mathematics competitions to be found in the introduction.

Australian Mathematical Olympiads 1979 – 1995
H Lausch & PJ Taylor

This book is a complete collection of all Australian Mathematical Olympiad papers since the first competition in 1979. Solutions to all problems are included and in a number of cases alternative solutions are offered.

Chinese Mathematics Competitions and Olympiads 1981-1993 and 1993-2001
A Liu

These books contain the papers and solutions of two contests, the Chinese National High School Competition and the Chinese Mathematical Olympiad. China has an outstanding record in the IMO and these books contain the problems that were used in identifying the team candidates and selecting the Chinese teams. The problems are meticulously constructed, many with distinctive flavour. They come in all levels of difficulty, from the relatively basic to the most challenging.

Mathematical Contests – Australian Scene
Edited by AM Storozhev, JB Henry & A Di Pasquale

These books provide an annual record of the Australian Mathematical Olympiad Committee's identification, testing and selection procedures for the Australian team at each International Mathematical Olympiad. The books consist of the questions, solutions, results and statistics for: Australian Intermediate Mathematics Olympiad (formerly AMOC Intermediate Olympiad), AMOC Senior Mathematics Contest, Australian Mathematics Olympiad, Asian-Pacific Mathematics Olympiad, International Mathematical Olympiad, and Maths Challenge Stage of the Mathematical Challenge for Young Australians.

WFNMC — Mathematics Competitions
Edited by Jaroslav Švrček

This is the journal of the World Federation of National Mathematics Competitions (WFNMC). With two issues each of approximately 80-100 pages per year, it consists of articles on all kinds of mathematics competitions from around the world.

Parabola incorporating Function

This Journal is published in association with the School of Mathematics, University of New South Wales. It includes articles on applied mathematics, mathematical modelling, statistics, pure mathematics and the history of mathematics that can contribute to the teaching and learning of mathematics at the senior secondary school level. The Journal's readership consists of mathematics students, teachers and researchers with interests in promoting excellence in senior secondary school mathematics education.

ENRICHMENT STUDENT NOTES

The Enrichment Stage of the Mathematics Challenge for Young Australians (sponsored by the Dept of Education, Science and Training) contains formal course work as part of a structured, in-school program. The Student Notes are supplied to students enrolled in the program along with other materials provided to their teacher. We are making these Notes available as a text book to interested parties for whom the program is not available.

Newton Enrichment Student Notes
JB Henry
Recommended for mathematics students of about Year 5 and 6 as extension material. Topics include polyominoes, arithmetricks, polyhedra, patterns and divisibility.

Dirichlet Enrichment Student Notes
JB Henry
This series has chapters on some problem solving techniques, tessellations, base five arithmetic, pattern seeking, rates and number theory. It is designed for students in Years 6 or 7.

Euler Enrichment Student Notes
MW Evans and JB Henry
Recommended for mathematics students of about Year 7 as extension material. Topics include elementary number theory and geometry, counting, pigeonhole principle.

Gauss Enrichment Student Notes
MW Evans, JB Henry and AM Storozhev
Recommended for mathematics students of about Year 8 as extension material. Topics include Pythagoras theorem, Diaphantine equations, counting, congruences.

Noether Enrichment Student Notes
AM Storozhev
Recommended for mathematics students of about Year 9 as extension material. Topics include number theory, sequences, inequalities, circle geometry.

Pólya Enrichment Student Notes
G Ball, K Hamann and AM Storozhev
Recommended for mathematics students of about Year 10 as extension material. Topics include polynomials, algebra, inequalities and geometry.

Problems to Solve in Middle School Mathematics
B Henry, L Mottershead, A Edwards, J Mcintosh, A Nakos, K Sims, A Thomas & G Vardaro.
This collection of problems is designed for use with students in Years 5 to 8. Each of the 65 problems is presented ready to be photocopied for classroom use. With each problem there are teacher's notes and fully worked solutions. Some problems have extension problems presented with the teacher's notes. The problems are arranged in topics (Number, Counting, Space and Number, Space, Measurement, Time, Logic) and are roughly in order of difficulty within each topic. There is a chart suggesting which problem-solving strategies could be used with each problem.

T-SHIRTS

T-shirts celebrating the following mathematicians are made of 100% cotton and are designed and printed in Australia. They come in white, and sizes Medium (Polya only) and XL.

Carl Friedrich Gauss T–shirt

The Carl Friedrich Gauss t-shirt celebrates Gauss' discovery of the construction of a 17-gon by straight edge and compass, depicted by a brightly coloured cartoon.

Emmy Noether T–shirt

The Emmy Noether t-shirt shows a schematic representation of her work on algebraic structures in the form of a brightly coloured cartoon.

George Pólya T–shirt

George Pólya was one of the most significant mathematicians of the 20th century, both as a researcher, where he made many significant discoveries, and as a teacher and inspiration to others. This t-shirt features one of Pólya's most famous theorems, the Necklace Theorem, which he discovered while working on mathematical aspects of chemical structure.

Peter Gustav Lejeune Dirichlet T–shirt

Dirichlet formulated the Pigeonhole Principle, often known as Dirichlet's Principle, which states: "If there are p pigeons placed in h holes and p>h then there must be at least one pigeonhole containing at least 2 pigeons." The t-shirt has a bright cartoon representation of this principle.

Alan Mathison Turing T-shirt

The Alan Mathison Turing t-shirt depicts a colourful design representing Turing's computing machines which were the first computers.

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