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Please send articles to:

The Editor
Mathematics Competitions
World Federation of National Mathematics Competitions
University of Canberra ACT 2601
Australia

Fax:+61-2-6201-5052

or

Dr Jaroslav Švrček
Dept. of Algebra and Geometry
Palacký University of Olomouc
Tomkova 40
779 00 Olomouc
Czech Republic

Email: svrcek@inf.upol.cz

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Department of Mathematics
University of North Carolina at Charlotte
9201 University City Blvd.
Charlotte, NC 28223-0001
USA

Oceania: Professor Derek Holton
Department of Mathematics and Statistics
University of Otago
PO Box 56
Dunedin
NEW ZEALAND

South America: Professor Patricia Fauring
Department of Mathematics
Buenos Aires University
Buenos Aires
ARGENTINA

The aims of the Federation are:–

1. ***to promote excellence in, and research associated with, mathematics education through the use of school mathematics competitions;***
2. ***to promote meetings and conferences where persons interested in mathematics contests can exchange and develop ideas for use in their countries;***
3. ***to provide opportunities for the exchanging of information for mathematics education through published material, notably through the Journal of the Federation;***
4. ***to recognize through the WFNMC Awards system persons who have made notable contributions to mathematics education through mathematical challenge around the world;***
5. ***to organize assistance provided by countries with developed systems for competitions for countries attempting to develop competitions;***
6. ***to promote mathematics and to encourage young mathematicians.***

From the Editor

Welcome to *Mathematics Competitions* Vol. 18, No 1.

I would like to thank again the Australian Mathematics Trust for continued support, without which each issue of the journal could not be published, and in particular Heather Sommariva, Bernadette Webster and Pavel Calábek for their assistance in the preparation of this issue.

Submission of articles:
The journal *Mathematics Competitions* is interested in receiving articles dealing with mathematics competitions, not only at national and international level, but also at regional and primary school level. There are many readers in different countries interested in these different levels of competitions.

- The journal traditionally contains many different kinds of articles, including reports, analyses of competition problems and the presentation of interesting mathematics arising from competition problems. Potential authors are encouraged to submit articles of all kinds.
- To maintain and improve the quality of the journal and its usefulness to those involved in mathematics competitions, all articles are subject to review and comment by one or more competent referees. The precise criteria used will depend on the type of article, but can be summarised by saying that an article accepted must be correct and appropriate, the content accurate and interesting, and, where the focus is mathematical, the mathematics fresh and well presented. This editorial and refereeing process is designed to help improve those articles which deserve to be published.

At the outset, the most important thing is that if you have anything to contribute on any aspect of mathematics competitions at any level, local, regional or national, we would welcome your contribution.

Articles should be submitted in English, with a black and white photograph and a short profile of the author. Alternatively, the article can be submitted on an IBM PC compatible disk or a Macintosh disk. We prefere LaTeX or TeX format of contributions, but any text file will be helpful.

Articles, and correspondence, can also be forwarded to the editor by mail to

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Australian Mathematics Trust
University of Canberra ACT 2601
AUSTRALIA

or to

Dr Jaroslav Švrček
Dept. of Algebra and Geometry
Palacky University of Olomouc
Tomkova 40
779 00 OLOMOUC
CZECH REPUBLIC

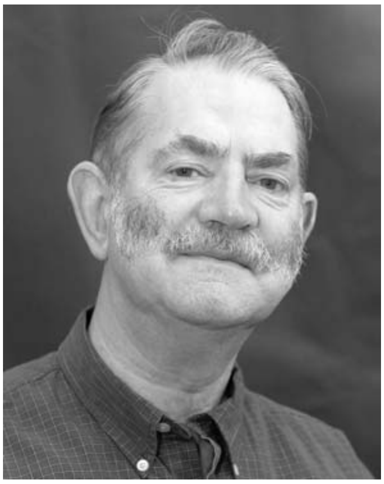
svrcek@inf.upol.cz

Note. On the website www.amt.edu.au/icmis16.html you can find further information, including the Discussion Document to the ICMI Study 16 (see our journal, Vol. 17, No 2) in three languages.

Jaroslav Švrček,
June 2005

An Elementary Method for Treating Constrained Optimisation Problems

Finbarr Holland



Finbarr Holland worked for more than forty years as Associate Professor of Mathematics at University College Cork. He taught mainly foundation and advanced courses on Real, Functional and Complex Analysis. He attended his first IMO in Cuba in 1987 as an Observer, and the following year he led the Irish MO team in Australia. Since 1987 he has run Mathematical Enrichment Classes for able secondary school children, and attended several more IMOs with the Irish team, either as Leader, Deputy Leader or Observer. Over the years he has submitted possible IMO problems, many of which have been short-listed, and two being used, in 1988 and 2004.

1 Introduction

The existence theorem that a real-valued continuous function attains its extreme values on a compact set is extremely important and hugely significant: after all, it helps to know in advance that a function achieves its maximum value, say, before searching for it! Not surprisingly, then, this theorem plays a major role in optimisation problems. The purpose of this note is to present a method which can be used to solve some constrained extremum problems in several variables by combining this fact with some elementary *one-variable* calculus recipes. Our method can therefore be taught to pre-university students who have been exposed to elementary real variable calculus, and should appeal especially to those who teach mathematical enrichment programmes and prepare candidates for mathematical competitions. It is another tool for solving homogeneous inequalities that frequently appear in such competitions.

The method that we propose can be regarded as a substitute for the one designed by Lagrange, a treatment of which can be found in books on Advanced Calculus. However, his method requires an understanding of differentiation of functions of several variables, a knowledge of differential geometry and the introduction of "multipliers" as extra variables, which are not easy to motivate. While this is often the preferred way of treating constrained optimisation problems in several variables, the level of sophistication needed to apply it successfully is beyond the capabilities of most pre-university students. By contrast, the one we advance is easier to grasp and simpler to apply than Lagrange's, and can be introduced to mathematically talented secondary students.

We present an outline of the key steps that the method entails in the next section, and illustrate it in the succeeding section by working a few examples.

2 Outline of the method

To convey the general idea of the method, and to keep the exposition as simple as possible, we apply it to determine the extreme values of some real polynomials in three variables subject to a single constraint. The interested reader will readily see how to adapt it to cope with more general functions.

Denote by S the unit sphere $\{(x, y, z)\colon x^2 + y^2 + z^2 = 1\}$ in $\mathbb{R}^3$. Noting that the projection maps

$$(x, y, z) \mapsto x, \quad (x, y, z) \mapsto y, \quad (x, y, z) \mapsto z,$$

are continuous on $\mathbb{R}^3$ to $\mathbb{R}$, it's clear that any polynomial with real coefficients is continuous on $\mathbb{R}^3$ to $\mathbb{R}$. Let

$$p(x, y, z) = \sum a_{rst} x^r y^s z^t$$

be such a polynomial, so that it assumes its maximum at a point $(a, b, c) \in S$. For brevity, denote by u, v, w the complex numbers $u = a + ib, v = b + ic, w = c + ia$, and consider, in turn, the three 2π-periodic infinitely differentiable functions f, g, h defined on $\mathbb{R}$ by

$$f(t) = p(\Re(ue^{it}), \Im(ue^{it}), c),$$

$$g(t) = p(a, \Re(ve^{it}), \Im(ve^{it})),$$
$$h(t) = p(\Im(we^{it}), b, \Re(we^{it})).$$

Since, for all real t, the points $(\Re(ue^{it}), \Im(ue^{it}), c)$, $(a, \Re(ve^{it}), \Im(ve^{it}))$, $(\Im(we^{it}), b, \Re(we^{it}))$ belong to S, and

$$f(0) = g(0) = h(0) = p(a, b, c),$$

each of these functions takes its maximum at $t = 0$. Hence

$$0 = f'(0) = g'(0) = h'(0),$$

and

$$f''(0) \leq 0, \quad g''(0) \leq 0, \quad h''(0) \leq 0.$$

The task now is to solve these equations for a, b, c subject to the constraint $a^2 + b^2 + c^2 = 1$. This is the heart of the problem, which may yet require some ingenuity to solve. In this respect it doesn't differ from Lagrange's method, but we get to this point much quicker and with much less fuss.

3 Some illustrations of the method

In each of the examples that follow, the function under consideration is a real polynomial and so achieves its extreme values on the compact set S. We'll exploit this fact and use elementary calculus to determine information about the points where the extrema are attained, and hence determine the extreme values.

Example 1

$$|x + y + z| \leq \sqrt{3}, \quad \forall (x, y, z) \in S.$$

Solution. By the foregoing, the polynomial $x + y + z$ attains its maximum, say, at some point $(a, b, c) \in S$. Let $u = a + ib$ and define the real function f on $[-\pi, \pi]$ by

$$f(t) = \Re\{ue^{it}(1 - i)\} + c.$$

Clearly, f is continuous on $[-\pi, \pi]$ and infinitely differentiable on $(-\pi, \pi)$. Moreover,

$$f(t) = a(t) + b(t) + c, \quad a(t) + ib(t) = ue^{it}, \quad (a(t), b(t), c) \in S.$$

Hence, f attains its maximum at 0, and so $f'(0) = 0$, $f''(0) \le 0$. Thus

$$0 = \Re\{iu(1-i)\} = a - b, \quad 0 \ge f''(0) = -\Re\{u(1-i)\} = -(a+b).$$

Hence $a = b \ge 0$. By symmetry, $b = c \ge 0$, $c = a$. Hence, $a = b = c = 1/\sqrt{3}$ and so $a + b + c = \sqrt{3}$. It follows that

$$x + y + z \le \sqrt{3}, \quad \forall (x, y, z) \in S.$$

In the same way, it's easy to establish that $-\sqrt{3}$ is the minimum of the polynomial on S. The result follows.

Corollary 1

$$|x + y + z| \le \sqrt{3}\sqrt{x^2 + y^2 + z^2}, \quad \forall (x, y, z) \in \mathbb{R}^3.$$

Example 2

$$|xyz| \le \frac{1}{3\sqrt{3}}, \quad \forall (x, y, z) \in S.$$

Solution. This time we apply the same idea to the polynomial xyz. It takes its maximum at $(a, b, c) \in S$, say, and, clearly, $abc \ne 0$. In the same notation, consider the auxiliary function

$$g(t) = \frac{1}{2} c \Im\{u^2 e^{2it}\}, \quad t \in [-\pi, \pi].$$

As before, g is continuous on $[-\pi, \pi]$, infinitely differentiable on $(-\pi, \pi)$ and attains its maximum abc at 0. Hence, $g'(0) = 0$, $g''(0) \le 0$. Calculating the derivatives, we infer that

$$0 = g'(0) = c\Im\{iu^2\} = c(a^2 - b^2), \quad 0 \ge g''(0) = -2c\Im\{u^2\} = -4abc.$$

Hence $a^2 = b^2$ and $abc > 0$. Similarly, $b^2 = c^2$, and so $a^2 = b^2 = c^2 = 1/3$, whence $abc = 1/(3\sqrt{3})$. In other words,

$$xyz \le \frac{1}{3\sqrt{3}}, \quad \forall (x, y, z) \in S.$$

Similarly,

$$-\frac{1}{3\sqrt{3}} \le xyz, \quad \forall (x,y,z) \in S.$$

The result follows.

Corollary 2

$$|xyz| \le \frac{(\sqrt{x^2+y^2+z^2})^3}{3\sqrt{3}}, \quad \forall (x,y,z) \in \mathbb{R}^3.$$

Corollary 3 If $x, y, z > 0$, then

$$\sqrt[3]{xyz} \le \frac{x+y+z}{3},$$

with equality iff $x = y = z$.

Example 3

$$|x+y+z-2xyz| \le \sqrt{2}, \quad \forall (x,y,z) \in S.$$

Solution. The polynomial $x+y+z-2xyz$ takes its maximum on S at (a,b,c), say. To locate this point, consider

$$h(t) = \Re\{ue^{it}(1-i)\} + c - c\Im\{u^2e^{2it}\},$$

where $u = a+ib$. Arguing as before, this function is continuous on $[-\pi,\pi]$ and takes its maximum $a+b+c-2abc$ at 0. Since h is also at least twice differentiable there, $0 = h'(0)$, $h''(0) \le 0$. From this information we deduce that

$$0 = \Re\{u(1+i)\} - 2c\Im\{iu^2\} = a-b-2c(a^2-b^2) = (a-b)(1-2c(a+b)),$$

and

$$0 \ge -\Re\{u(1-i)\} + 4c\Im\{u^2\} = -a-b+8abc.$$

Thus, by symmetry,

$$0 = (a-b)(1-2c(a+b)) = (b-c)(1-2a(b+c)) = (c-a)(1-2b(c+a)),$$

and

$$8abc \le \min\{a+b, b+c, c+a\}.$$

Suppose that none of the differences $a-b$, $b-c$, $c-a$ is zero. Then

$$1 = 2a(b+c) = 2b(c+a) = 2c(a+b),$$

so that $abc \neq 0$ and

$$a + \frac{2}{a} = b + \frac{2}{b} = c + \frac{2}{c} = 2(a+b+c).$$

But no horizontal line intersects the graph of the function

$$t \mapsto t + \frac{2}{t}, \quad t \neq 0,$$

more than twice. Hence at least one of $a-b$, $b-c$, $c-a$ is zero. But it's clear that not all are zero. Suppose $a = b$ and $c \neq a$. The latter implies that $1 = 2a^2 + c^2$ and $1 = 2b(c+a) = 2a^2 + 2ac$. Hence, either (i) $2a^2 = 1$, $c = 0$, or (ii) $c = 2a$, $6a^2 = 1$. If (ii) occurs, then the condition $8abc \le a + b$, i.e. $4a^2c \le a$, tells us that $a \le 0$ and so all components of $(a, b, c) = (a, a, 2a)$ are negative and $6a^2 = 1$. Clearly, this doesn't yield the maximum value. It holds from (i) $\sqrt{2}$ is the maximum value which occurs at $(1/\sqrt{2}, 1/\sqrt{2}, 0)$. In other words, if x, y, z are real numbers and $x^2 + y^2 + z^2 = 1$, then

$$x + y + z - 2xyz \le \sqrt{2},$$

with equality iff two of the variables are $1/\sqrt{2}$ and the third is zero. In the same way, it can be established that the minimum of this polynomial on S is $-\sqrt{2}$.

Example 4 Suppose $x, y, z \ge 0$ and $x + y + z = 1$. Then

$$0 \le xy + yz + zx - xyz \le \frac{8}{27}.$$

Solution. The function $x^2y^2 + y^2z^2 + z^2x^2 - x^2y^2z^2$ takes its maximum at a point $(a, b, c) \in S$. Noting that

$$\Re\{(a+ib)^4\} = (a^2+b^2)^2 - 8a^2b^2,$$

consider the auxiliary function

$$k(t) = (1-c^2)\frac{1}{8}[|u|^4 - \Re\{u^4 e^{4it}\}] + c^2|u|^2, \quad u = a+ib.$$

This infinitely differentiable function takes its maximum at 0, and so

$$0 = k'(0) = -(1-c^2)\frac{1}{2}\Re\{iu^4\} = -2(1-c^2)(a^3b - ab^3) =$$
$$= -2(1-c^2)ab(a^2-b^2),$$

and

$$0 \geq k''(0) = 2(1-c^2)\Re\{u^4\} = 2(1-c^2)(a^4 - 6a^2b^2 + b^4).$$

Since 0 is not the maximum value, $1 \neq c^2$. So $ab(a^2-b^2) = 0$. But if $ab = 0$, then the last displayed inequality forces $a = b = 0$, which has been ruled out. Hence $a^2 = b^2$. By symmetry $b^2 = c^2$. Hence $a^2 = b^2 = c^2 = 1/3$ and the maximum value is $8/27$ as claimed. In other words, if $x^2 + y^2 + z^2 = 1$, then

$$x^2y^2 + y^2z^2 + z^2x^2 - x^2y^2z^2 \leq \frac{8}{27},$$

whence the stated result follows easily.

In the previous examples the polynomials were symmetric and this facilitated the search for the extrema. We end by applying the method to two non-symmetric polynomials.

Example 5 Let (p,q,r) be a non-zero vector in $\mathbb{R}^3$. Then

$$|px+qy+rz| \leq \sqrt{p^2+q^2+r^2}, \quad \forall (x,y,z) \in S.$$

Solution. To begin with, $px+qy+rz$ achieves its maximum at some $(a,b,c) \in S$. By considering

$$t \mapsto \Re\{ue^{it}(p-iq)\} + rc, \quad u = a+ib,$$

and treating it as in Example 1—which is a special case—we find that

$$aq - bp = 0, \quad ap + qb \geq 0.$$

In much the same way, we see that

$$0 = br - cq, \quad bq + cr \geq 0; \quad 0 = ar - cp, \quad ap + cr \geq 0.$$

Using the equations, together with the fact that $a^2 + b^2 + c^2 = 1$, it's easy to see that $(ap+bq+cr)^2 = p^2+q^2+r^2$, and using the inequalities we see that $ap+bq+cr \geq 0$. Hence, the set $\{(a,b,c),(p,q,r)\}$ is linearly dependent and

$$\max\{px+qy+rz : (x,y,z) \in S\} = ap+bq+cr = \sqrt{p^2+q^2+r^2}.$$

A similar argument shows that the minimum is $-\sqrt{p^2+q^2+r^2}$.

Corollary 4

$$|px+qy+rz| \leq \sqrt{x^2+y^2+z^2}\,\sqrt{p^2+q^2+r^2}, \quad \forall (x,y,z) \in \mathbb{R}^3.$$

In each of the previous examples, the polynomial involved is homogeneous. As a final example, we consider an inhomogeneous non-symmetric polynomial.

Example 6

$$xy^3 + yz^2 \leq \frac{16}{25}\sqrt{\frac{2}{5}} \quad \forall (x,y,z) \in S.$$

SOLUTION. Assume that the given polynomial takes its maximum at $(a,b,c) \in S$, so that a,b,c are positive. Consider, in turn, the three 2π-periodic functions defined by

$$f(t) = (a\cos t - b\sin t)(a\sin t + b\cos t)^3 + (a\sin t + b\cos t)c^2,$$

$$g(t) = a(b\cos t - c\sin t)^3 + (b\cos t - c\sin t)(b\sin t + c\cos t)^2,$$

and

$$h(t) = (a\cos t - c\sin t)b^3 + b(a\sin t + c\cos t)^2.$$

Since, for all real t, the points $(a\cos t - b\sin t, a\sin t + b\cos t, c)$, $(a, b\cos t - c\sin t, b\sin t + c\cos t)$, $(a\cos t - c\sin t, b, a\sin t + c\cos t)$ belong to S, and

$$f(0) = g(0) = h(0) = ab^3 + bc^2,$$

each of these functions takes its maximum at $t = 0$. Hence

$$0 = f'(0) = g'(0) = h'(0),$$

i.e.

$$0 = -b^4 + 3a^2b^2 + ac^2 = -3ab^2c - c^3 + 3b^2c = -cb^3 + 2bca.$$

Bearing in mind that $a, b, c > 0$ and $(a, b, c) \in S$ it's easy to see that $b^2 = 2a$, $c^2 = 2a(2 - 3a)$, $a = 1/5$, whence

$$\max\{xy^3 + yz^2 : (x, y, z) \in S\} = \frac{16}{25}\sqrt{\frac{2}{5}},$$

as claimed.

Commentary

- In each of the above examples we've introduced functions of a single real variable t by substituting a, b in the given expressions by the real and imaginary parts of $(a + ib)e^{it}$. This was done merely to help reduce the algebra and keep things as tidy as possible. Alternatively, to achieve the same effect, one could replace a, b by the functions

  $$a(t) = a\cos t - b\sin t, \quad b(t) = a\sin t + b\cos t,$$

  and note that $(a(t), b(t), c) \in S$ whenever $(a, b, c) \in S$. This was the approach adopted in the last example.

- Again, people not too familiar with trigonometric functions may prefer to work with

  $$a(t) = a\sqrt{1 - t^2} - bt, \quad b(t) = at + b\sqrt{1 - t^2}, \quad -1 \le t \le 1.$$

- The method can be applied to polynomials of several variables and, in particular, used to establish extensions of the examples given here.

- All of the examples discussed above can be shown to be true without calculus, and without appealing to the theorem stated at the beginning.

- Readers will recognise that Corollaries 3 and 4 are special cases of the Arithmetic Mean Geometric Mean Inequality and the Cauchy-Schwarz Inequality, respectively.

- Examples 3 and 4 are IMO-level problems. In fact, Example 3 is a very slight modification of a problem that, as far as I can recall, was proposed by the UK for the 1987 IMO, while Example 4 is a variant of the following one that was used at the 1984 IMO:

  Prove that
  $$0 \leq xy + yz + zx - 2xyz \leq \frac{7}{27},$$
  where x, y, z are non-negative real numbers for which $x+y+z = 1$.

Prof. Finbarr Holland
School of Mathematics, Applied Mathematics, and Statistics
University College
Cork
IRELAND
email: f.holland@ucc.ie

The Mathematical Olympiad of Central America and the Caribbean

José H. Nieto Said & Rafael Sánchez Lamoneda

MSc. José H. Nieto. Professor, Universidad del Zulia, Faculty of Sciences, Department of Mathematics. OMCC: Venezuelan Team Leader (2001–2002), Deputy Leader (2000), Academic Support (2003). Member of the Venezuelan Association for Mathematical Competitions. Research interests: Enumerative Combinatorics. Algebraic combinatorics.

Dr. Rafael Sanchez. Associate Professor, Universidad Central de Venezuela, Faculty of Sciences, Dep. of Mathematics. President of the Venezuelan Association for Mathematical Competitions. IMO Venezuelan Team Leader in 1989 and since 1998. Deputy leader in 1981 and 1982. Member of the Venezuelan Mathematical Olympiads Commission for 25 years. Research interests: Characteristic-Free Representation Theory of $GL(n)$, Schur and Weyl modules. Homological Algebra.

1 Introduction

The *Mathematical Olympiad of Central America and the Caribbean* (OMCC) was born in 1999 to foster participation in international olympic mathematical competitions in countries in the region. This contest is addressed to young high school students, up to sixteen years old.

The OMCC takes place annually in a different country, under the auspices of the *Iberoamerican States Organization for Education, Science and Culture* (OEI) and with the support of diverse public and private organizations.

Each country participates with a delegation consisting of a Leader, a Deputy Leader and no more than three students. Other teachers may assist as observers, or attend a seminar aimed to elevate the mathematical level of the participants, preparing them to be olympic promoters and trainers in their own countries.

The development of the OMCC is the responsibility of an International Jury, formed by the delegation leaders and a president, designated by the Organization Committee of the host country. This Jury selects the problems, establishes the evaluation criteria and awards medals and other prizes. There is also a small committee with similar duties as the IMOAB for the International Mathematical Olympiad. In the past two years the responsibility for selecting the problems has been given to an external committee and the Leaders know about the exams a couple of hours before they take place. It is done with the idea to decrease costs and to permit countries with no previous olympic experience to organize the OMCC.

The exam is taken over two consecutive days. Each day the participants are allowed four and a half hours for solving three problems, each one with a value of seven points. During the subsequent couple of days the contestants engage in several recreational activities, while their papers are being evaluated.

One of the recreational activities is a mathematical rally or a team contest. The students work together in groups of four on several mathematical problems with a recreational flavor. The teams are selected under the hypothesis that all the members of each group belong to different countries. The results of this contest are independent of the OMCC results and the winning team receives a special prize. The idea is to foster the relationship between the olympic contestants and to give them the opportunity to work together on mathematical problems.

Medals are awarded to no more than one half of the participants, approximately in the proportion 1:2:3 between gold, silver and bronze. The students who do not obtain medals but solve a problem perfectly, receive honorable mentions. There is also a special prize for the country which shows the best improvement on its performance during two consecutive years. It is a cup given by El Salvador and its name is *Copa El Salvador*. This prize was awarded the first time to Cuba in the year 2000, during the second OMCC held in El Salvador. Afterwards the winning countries were Puerto Rico (2001), Puerto Rico (2002), Colombia (2003) and Venezuela (2004).

The OMCC has been held in the following countries: Costa Rica, El Salvador, Colombia, Mexico and Nicaragua. In Costa Rica it has been organized twice, the second one with the academic support of Venezuela. Next OMCC will be in El Salvador, in June 2005.

2 Proposed Problems

1$^{\text{st}}$ OMCC (Costa Rica 1999)

1. Five people know different partial information about a certain matter. When person A makes a phone call to person B, A gives to B all the information that he knows at that moment, while B says nothing to A. What is the minimum number of calls necessary in order that all five persons know all the information about the concerned matter? How many calls are necessary if there are n persons?

2. Find a positive integer n with 1000 digits, none of them 0, such that the digits may be grouped in 500 pairs in such a way that, if the two digits of each pair are multiplied and the 500 products are added, the result divides n.

3. In a calculator the number keys (except 0) are arranged as shown:

7	8	9	
4	5	6	+
1	2	3	

Player A turns on the calculator, presses a digit key and then presses the + key. Then a second player B presses a digit key in the same row or column of the last digit key pressed by A, except the same key pressed by A, then presses the + key. The game proceeds with the two players taking turns alternately. The first player who reaches a sum greater than 30 loses. Which player has a winning strategy? Describe the strategy.

4. Let $ABCD$ be a trapezoid with sides AB and CD parallel. Let M be the midpoint of AD and assume that angle $MCB = 150°$. Let $a = BC$ and $b = MC$. Express the area of $ABCD$ as a function of a and b.

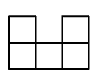
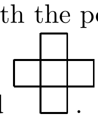
5. Let a be a positive odd integer greater than 17, such that $3a-2$ is a perfect square. Show that there are different positive integers b and c, such that $a+b$, $a+c$, $b+c$ and $a+b+c$ are perfect squares.

6. Let S be a subset of $\{1, 2, 3, \ldots, 1000\}$ such that the sum of two different elements of S does not belong to S. Which is the maximum possible number of elements in S?

2$^{\text{nd}}$ OMCC (El Salvador, 2000)

1. Find all three digits integers abc (a different from 0) such that $a^2 + b^2 + c^2$ is a divisor of 26.

2. Determine all the integers $n > 1$ for which it is possible to build a rectangle with sides 15 and n with the pentominoes

 and .

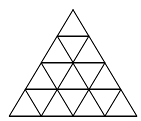
Notes:

a) The pieces cannot overlap or leave holes.

b) Each pentomino is formed by 5 little squares with side 1.

3. Let $ABCDE$ be a convex pentagon and P, Q, R and S the centroids of triangles ABE, BCE, CDE and DAE, respectively. Prove that $PQRS$ is a parallelogram and its area is $2/9$ the area of $ABCD$.

4. Write an integer inside each little triangle in such a way that the number in a triangle with at least two neighbouring triangles equals the difference of the numbers inside a couple of its neighbours.



Note: two triangles are *neighbours* if they have a common side.

5. Let ABC be an acute triangle and C_1 and C_2 the circles with sides AB and CA as diameters, respectively. C_2 intersects AB at points A and F, C_1 intersects CA at points A and E, BE intersects C_2 at P and CF intersects C_1 at Q. Prove that $AP = AQ$.

6. A *nice* representation of a positive integer n is a decomposition of n as a sum of powers of 2 where each power appears at most twice.

a) Write down all the *nice* representations of 5.

b) Which positive integers admit an even number of *nice* representations?

3^{rd} OMCC (Colombia, 2001)

1. Two players X and Y, and another 2001 persons stand in a circle, with X separated from Y by at least one person. X begins the game touching one of the two persons next to him, who must then leave the circle. Then Y touches one of his two neighbours, who must leave the circle. X and Y continue playing alternately. A player wins if he manages to touch his opponent (ousting him from the circle). Prove that one of the players has a winning strategy and describe it.

2. Let AB be the diameter of a circumference S with center O and radius 1. Let C and D be two points on S such that AC and BD intersect at a point Q in the interior of S and $\angle AQB = 2\angle COD$. Let P be the intersection point of the tangents to S that pass through C and D. Determine the length of segment OP.

3. Find all positive integers n satisfying the following two conditions:
- n has only two nonzero digits and one of them is 3.
- n is a perfect square.

4. Determine the least positive integer n such that there exist positive integers (not necessarily different) $a_1, a_2, \ldots, a_n$, all of them less or equal to 15, such that the last four digits of the sum $a_1! + a_2! + \cdots + a_n!$ are 2001.

5. Let a, b and c be real numbers such that the equation $ax^2 + bx + c = 0$ has two different real solutions p_1, p_2, and the equation $cx^2 + bx + a = 0$ has two different real solutions q_1, q_2. If the numbers p_1, q_1, p_2, q_2 (in that order) are an arithmetic progression, show that $a + c = 0$.

6. 10000 points are marked on a circumference and numbered clockwise from 1 to 10000. Then 5000 segments are drawn fulfilling the following three conditions:
- The endpoints of each segment are a couple of marked points.
- Each marked point belongs to exactly one segment.
- Each segment intersects exactly one of the other segments.

Label each segment with the product of the numbers of its endpoints. Prove that the sum of the labels of the 5000 segments is a multiple of 4.

4$^{\text{th}}$ OMCC (Mexico 2002)

1. For which integers $n > 2$ is it possible to arrange the numbers $1, 2, \ldots, n$ on a circumference (in some order) in such a way that each number divides the sum of the next two (clockwise) following numbers?

2. Let ABC be an acute triangle and let D and E be the feet of the altitudes from vertices A and B, respectively. Assuming that

$$\text{area}(BDE) \leq \text{area}(DEA) \leq \text{area}(EAB) \leq \text{area}(ABD)$$

prove that ABC is isosceles.

3. For each integer $a > 1$ an infinite sequence $L(a)$ is defined as follows:

(i) a is the first term in $L(a)$.

(ii) Given a term b in $L(a)$, the next term is $b + c$, where c is the greatest divisor of b which is less than b.

Find all the integers $a > 1$ such that 2002 appears in $L(a)$.

4. Let ABC be a triangle, D the midpoint of BC, E a point on the segment AC such that $BE = 2AD$ and F the intersection point of AD with BE. If $\angle DAC = 60°$, find the measures of the angles of triangle FEA.

5. Find an infinite set S of positive integers such that the sum of any (finite) number of (distinct) elements of S is not a perfect square.

6. Let n be an integer greater than one. Consider the $n \times n$ grid in the cartesian plane whose vertices are the points (x, y) with integer coordinates x, y such that $0 \le x \le n$ and $0 \le y \le n$. Consider the paths from $(0, 0)$ to (n, n) on the grid, made up of unit moves rightward or upward. A path is called *balanced* if the sum of the first coordinates of all its vertices equals the sum of the second coordinates of all its vertices. Show that a *balanced* path divides the square with vertices $(0, 0)$ $(n, 0)$ (n, n) $(0, n)$ in two pieces with equal area.

5^{th} OMCC (Costa Rica 2003)

1. There are 2003 stones in a pile. Two players A and B play as follows: A selects a divisor of 2003 and retires that number of stones from the pile. Then B selects a divisor of the number of stones remaining in the pile and retires that number of stones from the pile. They continue playing alternately in this way. The player who retires the last stone loses the game. Prove that one of the players has a winning strategy and describe it.

2. Let S be a circumference and AB a diameter of S. Let t be the tangent to S at B and consider two points C and D on t such that B is between C and D. Let E and F be the intersections of S with AC and AD, and G and H the intersections of S with CF and DE. Prove that $AH = AG$.

3. Let a and b be positive integers, with $a > 1$ and $b > 2$. Prove that $a^b + 1 \geq b(a+1)$ and determine when the equality is attained.

4. Let S_1 and S_2 be two circumferences that intersect at two different points P and Q. Let l_1 and l_2 be two parallel straight lines such that

(i) l_1 passes by P, intersects S_1 at a point A_1 different from P and intersects S_2 at a point A_2 different from P.

(ii) l_2 passes by Q, intersects S_1 at a point B_1 different from Q and intersects S_2 at a point B_2 different from Q.

Prove that the triangles A_1QA_2 and B_1PB_2 have equal perimeters.

5. Each 1×1 square of an 8×8 chessboard may be painted red or blue. Find the number of ways of painting the board such that each 2×2 *big square* formed by four 1×1 squares with a common vertex has two red squares and two blue squares.

6. Let us say that a positive integer is *tico* if the sum of its digits (in base 10) is a multiple of 2003.

(i) Prove that there exists a positive integer N such that its first 2003 multiples N, $2N$, $3N$,..., $2003N$ are all *tico*.

(ii) Does there exist some positive integer N such that all its multiples are *tico*?

(note: *tico* is a nickname for Costa Ricans)

6th OMCC (Nicaragua 2004)

1. The numbers 1, 2, 3, 4, 5, 6, 7, 8 and 9 are written on a blackboard. Two players A and B play alternately. Each player in turn chooses one of the numbers remaining on the blackboard and erases it, along with all its multiples (if any). The player who erases the last number loses. A plays first. Determine if A or B has a winning strategy and explain that strategy.

Note: A player has a *winning strategy* if he can assure his victory (no matter how his rival plays).

2. A sequence $a_0, a_1, a_2, \ldots$ is defined as follows: $a_0 = a_1 = 1$ and for $k > 1$, $a_k = a_{k-1} + a_{k-2} + 1$.

Determine how many integers from 1 to 2004 may be expressed in the form $a_m + a_n$ with m and n positive integers and m different from n.

3. Given a triangle ABC let E and F be points on segments BC and CA, respectively, such that $CE/CB + CF/CA = 1$ and $\angle CEF = \angle CAB$. Let M be the midpoint of EF and G the intersection of the straight line CM with segment AB. Prove that the triangles FEG and ABC are similar.

4. Fifty 1×1 squares of a 10×10 square board are painted black, and the other fifty are painted white. A side common to two 1×1 squares painted with different colors is called a *frontier*. Find the maximum and minimum possible number of frontiers. Justify your answer.

5. Let $ABCD$ be a trapezoid such that $AB \parallel CD$ and $AB+CD = AD$. Let P be the point on AD such that $AP = AB$ and $PD = CD$.

a) Prove that $\angle BPC = 90°$.

b) Let Q be the midpoint of BC and R the intersection point of line AD with the circumference passing through points B, A and Q. Prove that B, P, R and C are concyclic.

6. Necklaces are formed with beads of different colors. A necklace is *prime* if it cannot be decomposed into two or more equal chains.

Let n and q be positive integers. Prove that the number of prime necklaces with n beads with q^n possible colors is equal to n times the number of prime necklaces with n^2 beads of q possible colors.

Note: Two necklaces are considered equal if they have the same number of beads and their colors can be matched rotating one of them.

3 Results and statistics

Tables 1 to 6 contain, for each Olympiad, the marks obtained by each student, the average mark for each problem and the medals and honorable mentions awarded.

Figures 1 to 6 are histograms which show the distribution of marks for each Olympiad.

Figure 7 shows the participation of countries and students in each Olympiad.

Figure 8 shows the participation of students separated by sex. The overall participation of female students in the six Olympiads is 16.4%.

Table 7 contains, for each Olympiad, the number of participant countries and the total marks of each country. Figure 9 is a graphic representation of the same information.

Figures 10 and 11 show the distribution of awards by year and for the six Olympiads, respectively.

Figure 12 reflects the average marks for each proposed problem.

Figure 13 shows the distribution of the proposed problems of each Olympiad in four mathematical areas: Algebra, Arithmetic, Combinatorics and Geometry.

Figure 14 shows the overall distribution for the six years. As you can see this distribution shows a balance between Geometry, Combinatorics and Arithmetic, with an emphasis in Geometry, but there is only a small percentage of Algebra problems—just 6% of the problems belong to Algebra. So problems selection committees will have to work on this in future OMCCs, in order to have better balanced exams. In any event, this Olympiad is a very young one and most of the participating countries do not have much experience. Our hope is that the problems short list, and therefore the exams, will improve in the near future.

Table 8 lists the countries that have won the *Copa El Salvador*.

1st OMCC (1999)								
Code	Pr 1	Pr 2	Pr 3	Pr 4	Pr 5	Pr 6	Tot.	Award
COL1	7	7	0	0	0	3	17	Bronze
COL2	7	2	7	5	3	1	25	Silver
COL3	7	1	7	6	6	1	28	Gold
CRC1	7	1	0	2	0	1	11	Mention
CRC2	6	1	1	0	0	0	8	
CRC3	7	2	1	2	0	1	13	Bronze
CUB1	7	0	7	7	4	1	26	Gold
CUB2	2	1	1	7	6	1	18	Bronze
CUB3	6	1	0	6	0	1	14	Bronze
SAL1	7	1	2	1	1	0	12	Mention
SAL2	6	0	0	6	0	0	12	
SAL3	7	0	7	0	0	0	14	Bronze
MEX1	7	3	2	6	3	0	21	Silver
MEX2	7	2	7	5	1	0	22	Silver
MEX3	7	6	0	6	1	1	21	Silver
NIC1	6	0	0	1	1	0	8	
NIC2	7	0	0	1	0	0	8	Mention
NIC3	6	2	0	1	0	0	9	
PAN1	7	0	0	0	0	0	7	Mention
PAN2	7	0	7	2	0	0	16	Bronze
PAN3	6	1	0	0	0	0	7	
PRC1	7	1	0	6	1	1	16	Bronze
PRC2	7	1	1	6	1	0	16	Bronze
PRC3	5	1	1	0	0	0	7	
VEN1	2	2	7	0	0	1	12	Mention
VEN2	7	1	0	0	0	1	9	Mention
VEN3	7	4	7	6	1	1	26	Gold
Average	6.33	1.52	2.41	3.04	1.07	0.56	14.93	

Table 1: 1st OMCC (Costa Rica, 1999)

2^{nd} OMCC (2000)								
Code	Pr 1	Pr 2	Pr 3	Pr 4	Pr 5	Pr 6	Tot.	Award
COL1	3	6	3	7	2	3	24	Silver
COL2	3	7	3	0	3	1	17	
COL3	5	4	2	7	2	1	21	Bronze
CRC1	7	2	7	0	4	1	21	Bronze
CRC2	7	4	7	1	0	3	22	Bronze
CRC3	7	4	0	0	2	3	16	
CUB1	6	5	7	0	7	3	28	Silver
CUB2	7	7	7	7	7	1	36	Gold
CUB3	6	7	7	7	3	1	31	Gold
SAL1	7	2	0	7	0	0	16	
SAL2	7	3	0	7	0	2	19	Bronze
SAL3	7	2	0	7	2	1	19	Bronze
HON1	2	2	0	0	0	0	4	
HON2	3	0	0	1	0	0	4	
HON3	4	4	0	7	0	3	18	Bronze
MEX1	7	7	7	3	2	3	29	Gold
MEX2	3	6	3	7	2	3	24	Silver
MEX3	4	5	6	7	3	1	26	Silver
NIC1	3	1	0	7	1	0	12	
NIC2	7	1	0	0	0	3	11	
NIC3	2	0	0	3	0	0	5	
PRC1	2	0	0	3	0	0	5	
PRC2	3	2	0	1	1	0	7	
PRC3	6	0	0	1	2	1	10	
VEN1	7	5	0	7	2	3	24	Silver
VEN2	5	3	0	1	1	0	10	
VEN3	7	1	0	7	2	1	18	Bronze
Average	5.07	3.33	2.19	3.89	1.78	1.41	17.67	

Table 2: 2^{nd} OMCC (El Salvador, 2000)

3^{rd} OMCC (2001)								
Code	Pr 1	Pr 2	Pr 3	Pr 4	Pr 5	Pr 6	Tot.	Award
COL1	7	7	4	7	3	7	35	Silver
COL2	7	3	2	7	0	7	26	Bronze
COL3	7	7	3	7	0	0	24	Bronze
CRC1	7	7	6	7	2	0	29	Bronze
CRC2	7	3	2	7	1	4	24	Bronze
CRC3	7	1	2	7	0	4	21	Mention
CUB1	7	7	4	7	5	4	34	Silver
CUB2	7	7	4	7	6	2	33	Silver
CUB3	7	7	1	1	0	0	16	Mention
SAL1	7	0	2	7	0	7	23	Bronze
SAL2	6	6	2	7	4	0	25	Bronze
SAL3	7	3	2	7	0	3	22	Mention
GUA1	1	0	1	0	0	0	2	
MEX1	7	1	3	5	0	7	23	Bronze
MEX2	7	7	4	7	6	7	38	Gold
MEX3	7	7	4	3	1	7	29	Bronze
NIC1	6	7	1	0	0	0	14	Mention
NIC2	7	3	1	1	0	0	12	Mention
PAN1	2	0	1	0	0	0	3	
PAN2	7	2	0	2	0	0	11	Mention
PAN3	0	0	0	1	0	0	1	
PRC1	7	7	2	7	2	7	32	Silver
PRC2	7	1	0	7	0	0	15	Mention
PRC3	7	0	2	5	0	0	14	Mention
VEN1	6	3	2	5	0	1	17	
VEN2	7	7	5	7	7	4	37	Silver
VEN3	7	7	6	7	6	7	40	Gold
Average	6.22	4.07	2.44	5.00	1.59	2.89	22.22	

Table 3: 3^{rd} OMCC (Colombia, 2001)

4^{th} OMCC (2002)								
Code	Pr 1	Pr 2	Pr 3	Pr 4	Pr 5	Pr 6	Tot.	Award
COL1	7	7	7	7	0	2	30	Bronze
COL2	7	1	0	7	0	1	16	Mention
COL3	5	7	7	7	7	1	34	Silver
CRC1	6	7	4	7	7	3	34	Silver
CRC2	7	2	4	7	6	3	29	Mention
CRC3	0	5	7	7	7	7	33	Bronze
CUB1	7	7	6	7	0	0	27	Mention
CUB2	7	7	7	7	0	3	31	Bronze
CUB3	7	7	5	7	7	0	33	Bronze
SAL1	1	7	7	7	7	2	31	Bronze
SAL2	7	0	6	0	1	0	14	Mention
SAL3	7	1	7	3	0	0	18	Mention
GUA1	0	0	0	0	0	0	0	
GUA2	0	0	3	0	3	0	6	
GUA3	7	0	6	0	6	0	19	Mention
MEX1	7	7	7	7	7	3	38	Gold
MEX2	7	7	7	7	7	1	36	Gold
MEX3	6	7	7	2	7	7	36	Gold
PRC1	5	7	7	7	7	1	34	Silver
PRC2	7	7	7	7	7	0	35	Silver
PRC3	0	0	1	2	0	0	3	
VEN1	1	1	0	0	0	0	2	
VEN2	7	7	2	7	0	0	23	Mention
VEN3	7	0	7	1	0	1	16	Mention
Average	5.08	4.21	5.04	4.71	3.58	1.46	24.08	

Table 4: 4^{th} OMCC (Mexico, 2002)

5th OMCC (2003)								
Code	Pr 1	Pr 2	Pr 3	Pr 4	Pr 5	Pr 6	Tot.	Award
COL1	7	7	7	7	1	1	30	Silver
COL2	7	5	6	7	7	2	34	Silver
COL3	7	7	1	7	7	2	31	Silver
CRC1	7	5	2	7	3	0	24	Bronze
CRC2	2	2	0	7	4	2	17	Bronze
CRC3	7	5	7	7	7	2	35	Gold
CUB1	7	7	1	7	7	5	34	Silver
GUA1	1	1	0	1	0	1	4	
GUA2	4	1	1	4	0	0	10	
GUA3	7	1	0	7	5	0	20	Bronze
MEX1	7	7	5	7	7	7	40	Gold
MEX2	7	7	7	7	7	2	37	Gold
MEX3	7	7	7	7	3	0	31	Silver
NIC1	4	0	0	0	0	0	4	
NIC2	4	0	0	0	0	0	4	
NIC3	7	0	0	2	1	0	10	Mention
PAN1	4	1	5	2	0	0	12	
PAN2	6	1	0	7	1	0	15	Bronze
PAN3	0	0	0	7	2	2	11	Mention
PRC1	0	7	2	5	0	0	14	Mention
PRC2	7	0	4	5	1	0	17	Bronze
PRC3	7	1	0	5	7	0	20	Bronze
DOM1	2	0	0	0	0	0	2	
DOM2	5	0	0	0	0	0	5	
DOM3	1	1	0	6	0	0	8	
SAL1	1	0	0	7	1	0	9	Mention
SAL2	2	1	1	0	1	0	5	
SAL3	2	1	0	5	7	0	15	Bronze
VEN1	7	4	1	7	2	0	21	Bronze
VEN2	7	7	2	7	6	1	30	Silver
VEN3	7	4	1	7	4	0	23	Bronze
Average	4.84	2.90	1.94	4.97	2.94	0.87	18.45	

Table 5: 5th OMCC (Costa Rica, 2003)

Table 6: 6th OMCC (Nicaragua, 2004), see the next page

6th OMCC (2004)								
Code	Pr 1	Pr 2	Pr 3	Pr 4	Pr 5	Pr 6	Tot.	Award
COL1	7	5	3	7	7	0	29	Silver
COL2	7	7	2	5	7	0	28	Silver
COL3	7	6	1	5	4	0	23	Bronze
CRC1	7	7	5	5	5	0	29	Silver
CRC2	7	1	4	5	3	0	20	Bronze
CRC3	7	6	3	5	3	0	24	Bronze
CUB1	7	5	5	7	7	0	31	Silver
CUB2	7	7	3	6	7	0	30	Silver
CUB3	7	1	3	5	7	0	23	Bronze
SAL1	7	1	1	5	7	0	21	Bronze
SAL2	7	0	1	5	4	0	17	Bronze
SAL3	7	1	0	5	3	0	16	Mention
GUA1	0	0	2	5	1	0	8	
GUA2	4	0	0	5	0	0	9	
GUA3	7	1	0	5	2	0	15	Mention
HON1	0	0	1	4	1	0	6	
HON2	2	0	0	0	0	0	2	
MEX1	7	6	7	7	7	0	34	Gold
MEX2	7	5	7	5	7	4	35	Gold
MEX3	7	7	7	7	7	0	35	Gold
NIC1	1	0	1	5	2	0	9	
NIC2	5	0	1	5	2	0	13	
NIC3	6	0	3	5	3	0	17	Bronze
PAN1	1	0	1	6	2	0	10	
PAN2	7	0	3	3	2	0	15	Mention
PAN3	3	0	1	5	2	0	11	
PRC1	7	2	3	5	3	0	20	Bronze
PRC2	6	1	3	6	3	0	19	Bronze
PRC3	7	4	0	4	7	0	22	Bronze
DOM1	3	0	0	4	1	0	8	
DOM2	6	0	0	4	0	0	10	
DOM3	0	0	0	1	0	0	1	
VEN1	7	7	4	5	5	0	28	Silver
VEN2	7	0	1	5	3	0	16	Mention
VEN3	7	5	3	5	5	0	25	Bronze
Average	5.46	2.43	2.26	4.89	3.69	0.11	18.83	

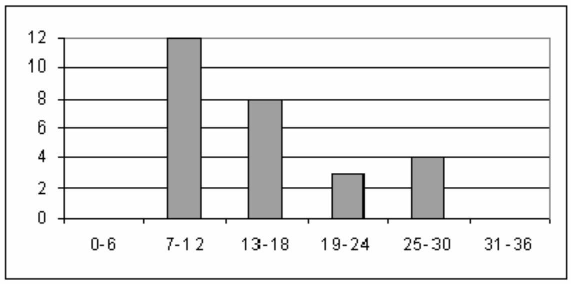


Figure 1: Distribution of Marks in 1st OMCC

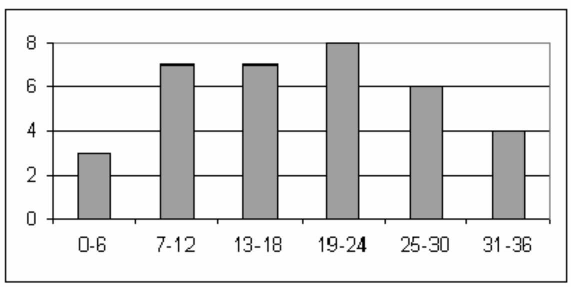


Figure 2: Distribution of Marks in 2nd OMCC

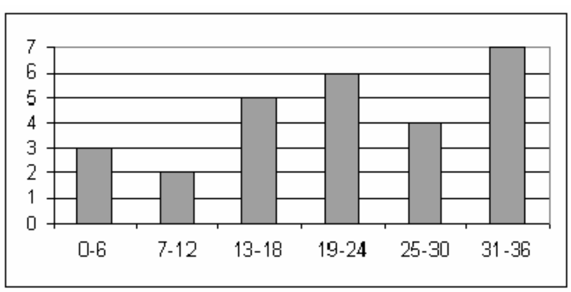


Figure 3: Distribution of Marks in 3rd OMCC

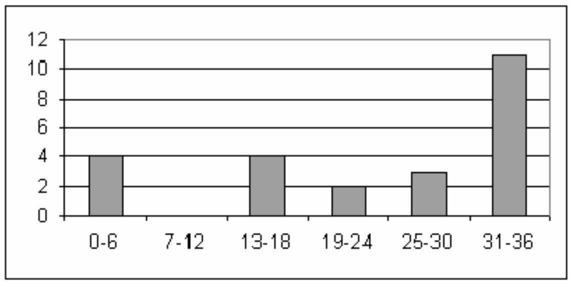


Figure 4: Distribution of Marks in 4th OMCC

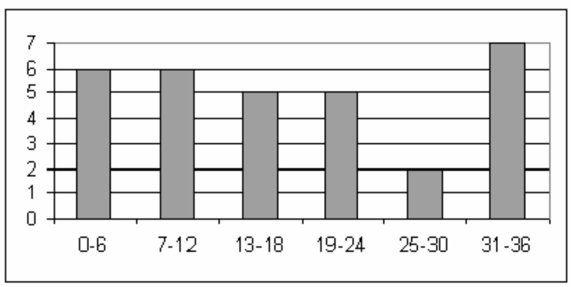


Figure 5: Distribution of Marks in 5th OMCC

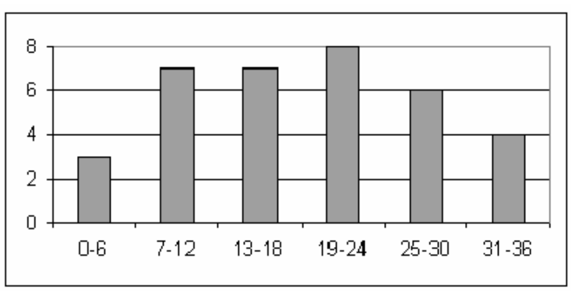


Figure 6: Distribution of Marks in 6th OMCC

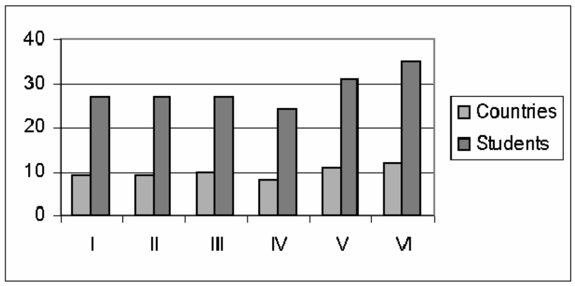


Figure 7: Participation by Year

		Olympiad					
Code	Country	1	2	3	4	5	6
COL	Colombia	70	62	85	80	95	80
CRC	Costa Rica	32	59	74	96	76	73
CUB	Cuba	58	95	83	91	34*	84
DOM	Dominican Rep.					15	19
SAL	El Salvador	38	54	70	63	29	54
GUA	Guatemala			2*	25	34	32
HON	Honduras		26				8**
MEX	Mexico	64	79	90	110	108	104
NIC	Nicaragua	25	28	26**		18	39
PAN	Panama	30		15		38	36
PRC	Puerto Rico	39	22	61	72	51	61
VEN	Venezuela	47	52	94	41	74	69
Participant countries:		9	9	10	8	11	12
* with only one student; ** with only two students							

Table 7: Total Marks of Countries by Year

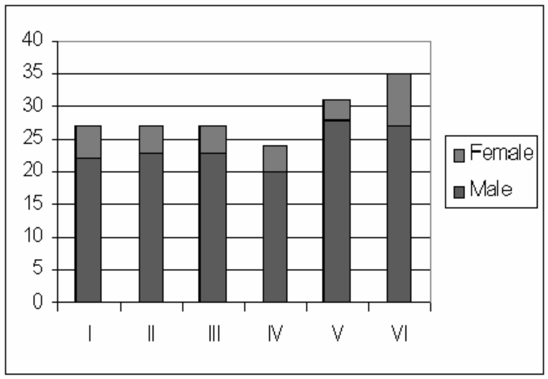


Figure 8: Students by Sex

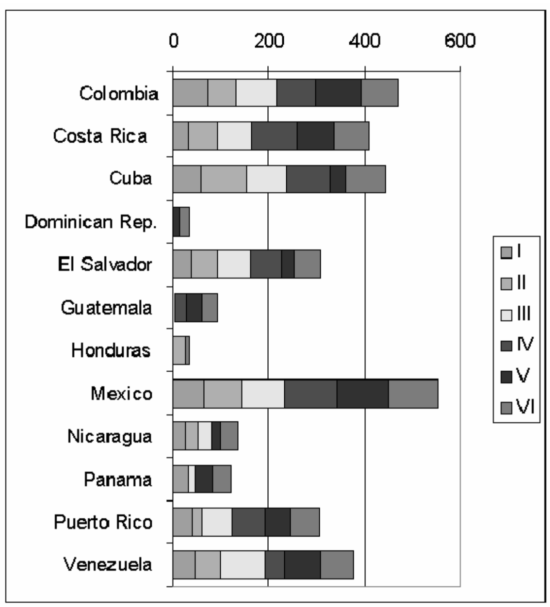


Figure 9: Marks by Country and Year

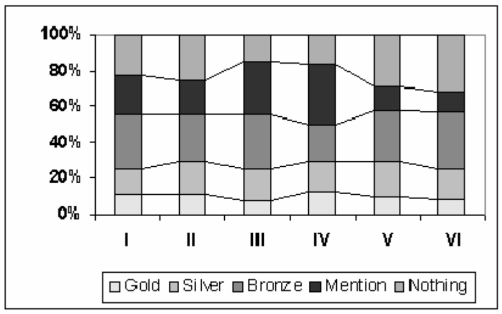


Figure 10: Distribution of Awards by Year

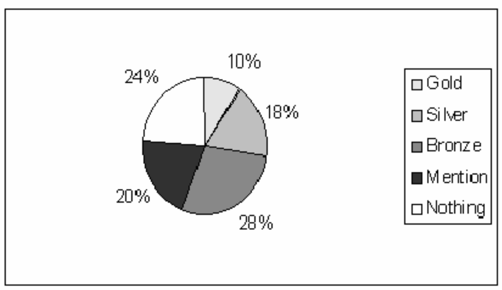


Figure 11: Distribution of Awards (average 6 years)

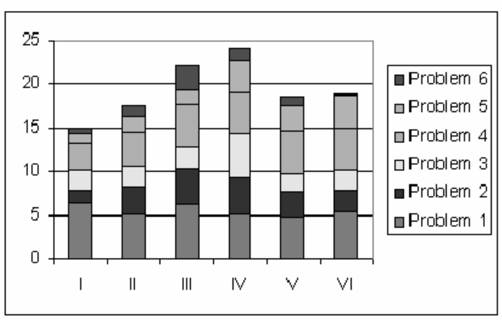


Figure 12: Average Marks by Problem and Year

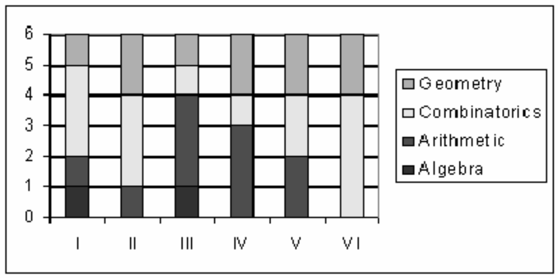


Figure 13: Distribution of Problems by Area and Year

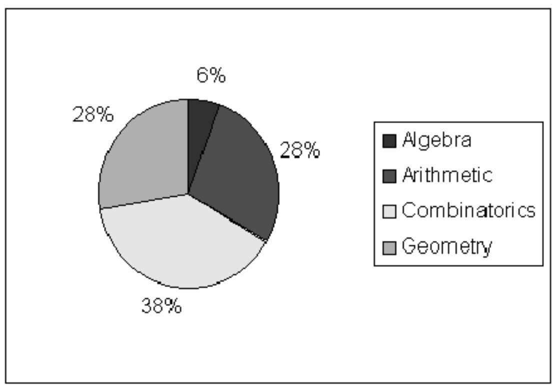


Figure 14: Problems by Area (average 6 years)

Copa El Salvador		
OMCC	Year	Winning Country
I	1999	Not awarded
II	2000	Cuba
III	2001	Puerto Rico
IV	2002	Puerto Rico
V	2003	Colombia
VI	2004	Venezuela

Table 8: Copa El Salvador

4 Conclusions

During the last six years the OMCC has proved to be an excellent way to initiate the high school students of the region to international mathematical competitions, preparing them for more demanding events such as the Iberoamerican, Asian-Pacific and International Mathematical Olympiads. It has fostered friendly relationships among students and teachers of the participant countries, creating many opportunities for the exchange of information and experiences on the teaching of mathematics in these countries with similar culture and common problems.

José H. Nieto Said
Escuela de Matemáticas
Facultad de Ciencias
Universidad del Zulia
Maracaibo. Edo. Zulia.
VENEZUELA
Email: jhnieto@cantv.net

Rafael Sánchez Lamoneda
Escuela de Matemáticas
Facultad de Ciencias
Universidad Central de Venezuela
Caracas 1020-A
VENEZUELA
Email: rsanchez@euler.ciens.ucv.ve

A Local International Mathematics Competition (Special Edition)

Robert Geretschläger & Jaroslav Švrček

Robert Geretschläger teaches Mathematics and Descriptive Geometry at BRG Kepler in Graz, Austria and at the Karl-Franzens University in Graz. He has been deputy leader of the Austrian team at the IMO for the past five years, and is also head of the organizing committees for Austria for the European Kangaroo contest, Tournament of Towns and Mediterranean Mathematics Competition.

Jaroslav Švrček is a Senior Lecturer Department of Algebra and Geometry, Palacký University of Olomouc. He is Vice-Chair of the Czech Mathematical Olympiad and member of its problems committee. He works as an editor of mathematics of the Czech journal Matematika – Fyzika – Informatika *and as an editor of our journal* Mathematics Competitions. *He is one of the founders and organizers of the local international mathematical competition DUEL.*

One of the wonderful aspects of mathematics competitions is the opportunity they offer to bring together people with a common interest. As anyone who has ever been involved with any type of international competition can attest, such a meeting of competitors from different geographical backgrounds but with a common intellectual interest gives the participants an opportunity to forge friendships across borders that often last a lifetime. For those taking part who are not quite so fortunate, the least they can hope for is a chance to gain some understanding of other cultures and other viewpoints.

Since 1993, a friendly mathematics competition named the *Mathematical Duel* has been held annually between students from two central European schools, the Mikuláš Kopernik Gymnazium in Bilovec, Czech Republic and the Juliusz Słowacki Liceum in Chorzów, Poland, and since 1997 they have been joined by the Bundesrealgymnasium (BRG) Kepler in Graz, Austria. The original inspiration for this activity came from several international competitions that were already well established at the time, such as the *Baltic Way* or the *Austrian-Polish Mathematics Competition*, but the Duel was to develop into something quite different.

The venue for the competition rotates annually between Bilovec, Chorzów and Graz. In this, the year of the 13^{th} Duel, it was slated to take place in Graz for the third time. Coincidentally, BRG Kepler was planning its second *Europe Days*, a school project meant to foster understanding between the peoples of Europe. Since last year's project had a mostly historical theme, it was decided that a scientific orientation would be appropriate for this year, coinciding with the International Physics Year as well as the Kepler Year 2005. It seemed to be an excellent idea to combine the two ideas for a special "European" Duel.

BRG Kepler has a greater than average emphasis on science, mathematics and computer science, and therefore has contacts in many schools with similar programs. It was decided to invite several, with which a long standing cooperation already exists, to participate in the Mathematical Duel. Beside those normally involved with the Duel, these were the Kilián György Gimnazium from Miskolc, Hungary, the Jakub Škoda Gymnázium from Přerov, Czech Republic and the Mathematical High School Baba Tonka from Russe, Bulgaria. All but Baba Tonka were able to accept the invitation, and so the stage was set for this special edition of the Mathematical Duel.

The competition is divided into three categories. Similarly to the Czech olympiad system, division A is for grades 11–12, B for grades 9–10 and C for grades 8 or younger. Typically, four students from each school in each of the divisions come together for a competition, making 36 participants in all, but this year both Přerov and Miskolc fielded an A team, and since there were two BRG Kepler B teams, there were 47 participants in all (one student from Bilovec was ill and could not travel to Graz).

The students write an individual competition comprising four olympiad-style problems to be solved in two hours, and a team competition comprising three problems to be solved in 90 minutes. The two competitions are completely independent of one another, and yield separate results. While the individual competition is written in supervised silence, the team competition sees one team from each school (from different divisions) placed together in a room with no supervision. The students spend the 90 minutes devising a common group answer to each problem, and only one answer sheet is accepted from each group at the end. The dynamics of the group competition are quite different from the individual work common to most competitions, and are quite fascinating to observe. The first prizes in the three team rankings all went to Chorzów this year, while top honors in the individual categories went to Marta Cieśla from Chorzów (C), Miroslav Štufka from Bilovec (B) and Zuzana Safernová from Bilovec (A).

In early years, the participants were given the questions in their own languages, but now the common neutral language English is used. The students still write the answers in their own languages, and any difficulties in understanding the problems are solved on the spot as they are handed out. This method of dealing with the language issue certainly makes for some interesting situations, but the jury has always been able to iron out any difficulties as they have arisen.

The competition is organized over the course of four days, the first and last of which are spent travelling. The greatest distance between any two participating schools is about 800 km, so travel is possible by bus or train. Also, there is always an excursion involved for the participants. This year, the group stayed in Graz, the 2003 Cultural Capital of Europe, visiting the new Kunsthaus (modern art gallery) and the historical Zeughaus (armory). Of course, there were also some small prizes, t-shirts and diplomas for all the participants.

Perhaps this report can inspire others to try something similar in their schools. Many variations of this concept are possible. It would be interesting to try out mixed teams for the team competition, for instance, although this would only be possible where a common language can be used. For the students involved in the Duel, it has certainly proven to be a valuable experience, and the organizers hope to continue the tradition for as long as possible.

1 The Problems of the Mathematical Duel XIII, 2005

CATEGORY A

A–Ind–1

Let P be an arbitrary point in the interior of a right-angled isosceles triangle ABC with hypotenuse AB. Prove that a triangle exists with sides of the length AP, BP and $CP \cdot \sqrt{2}$.

A–Ind–2

Let x, y, z be non-zero real numbers. Prove that the inequality

$$\frac{x^2}{y^2} + \frac{y^2}{z^2} + \frac{z^2}{x^2} \geq \frac{x}{y} + \frac{y}{z} + \frac{z}{x}$$

holds. For which values does equality hold?

A–Ind–3

Determine all integer solutions of the system of equations

$$\begin{aligned} x^2z + y^2z + 4xy &= 40\,, \\ x^2 + y^2 + xyz &= 20\,. \end{aligned}$$

A–Ind–4

Let ABC be an acute-angled triangle. Let D be the mid-point of BC and O be the incenter of the triangle ABC. The line DO intersects the altitude in A at the point G. Prove that the length of AG is equal to the inradius of the triangle ABC.

A–Team–1

We are given a square $ABCD$ with sides of length a and a line segment of length d with $a \leq d < a\sqrt{2}$. Construct a quadrilateral $KLMN$ with minimal perimeter such that the points K, L, M and N lie on the sides AB, BC, CD and DA respectively and one diagonal of $KLMN$ has the length d.

A–Team–2

Let $2s$ be the length of the perimeter of a triangle ABC and let ρ, r_a, r_b and r_c be the radii of the incircle and the three excircles of ABC respectively. Prove that the following inequality holds:

$$\sqrt{\rho \cdot r_a} + \sqrt{\rho \cdot r_b} + \sqrt{\rho \cdot r_c} \leq s.$$

A–Team–3

Let V be a real function defined by the expression

$$V(x) = (x-1)(x-2) + (x-1)(x-2)(x-3)(x-4) + (x-3)(x-4).$$

a) Determine the minimum value of $V(x)$.

b) Determine all values of x for which this minimum value is assumed.

CATEGORY B

B–Ind–1

Solve the following system of equations in real numbers:

$$\begin{aligned} x + \frac{1}{x} &= 2y^2\,, \\ y + \frac{1}{y} &= 2z^2\,, \\ z + \frac{1}{z} &= 2x^2\,. \end{aligned}$$

B–Ind–2

We are given a square $ABCD$. Construct points K, L, M and N on the sides AB, BC, CD and DA respectively, such that the pentagon $KLCMN$ has maximum area with KL and MN parallel to AC and NK parallel to BD.

B–Ind–3

Determine all right-angled triangles with perimeter 180, such that the lengths of the sides are integers.

B–Ind–4

We are given a triangle $\triangle ABC$ with $\angle ABC = \beta$ and incenter I. Points A', B' and C' are symmetrical to I with respect to BC, CA and AB respectively. Prove that $\angle A'B'C' = \beta'$ is independent of the value of $\alpha = \angle BAC$ and express β' in terms of β.

B–Team–1

a) A number x can be written using only the digit a both in base 8 and in base 16, i.e.

$$x = (aa \ldots a)_8 = (aa \ldots a)_{16}.$$

Determine all possible values of x.

b) Determine as many numbers x as possible that can be written in the form $x = (11 \ldots 1)_b$ in at least two different number systems with bases b_1 and b_2.

B–Team–2

We are given an acute-angled triangle ABC with orthocenter O. Let F be the mid-point of AB and N be the point symmetric to O with respect to F. Show that $\angle ACO = \angle BCN$ must hold.

B–Team–3

Determine an integer $a < 10000$ such that $a, a+1, a+2, \ldots, a+12$ are all composite numbers.

CATEGORY C

C–Ind–1

How many right-angled triangles exist, whose sides have integer length and whose perimeter has the length 2005?

C–Ind–2

Prove that

$$x^2 + y^2 + 2 \geq (x+1)(y+1)$$

is true for all real values of x and y. For which values of x and y does equality hold?

C–Ind–3

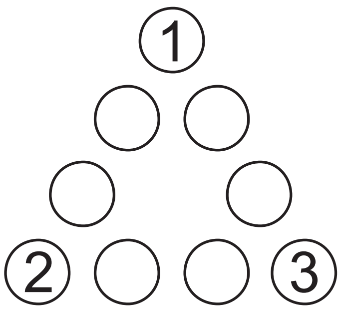
A triangle $\triangle ABC$ in the plane is given, in which $\alpha = 30°$ and $\beta = 60°$. Let S be the mid-point of AB and S_1 and S_2 be the circumcenters of triangles ASC and BSC respectively, and let r_1 and r_2 be the respective radii of these circles.

a) Prove that the triangles ABC and S_1S_2S are similar.

b) Determine the ratio $r_1 : r_2$.

C–Ind–4

The digits 1 through 9 are placed in a triangular array as shown, such that the sums of the four numbers on each side are equal. The numbers 1, 2 and 3 are placed in the corners. Where can the number 9 not be placed?



C–Team–1

We are given a regular octagon $ABCDEFGH$. Drawing the diagonals AD, BE, CF, DG, EH, FA, GB and HC yields another smaller regular octagon in the interior of the original octagon. Determine the ratio of the areas of the original octagon and the resulting octagon.

C–Team–2

Determine all positive integers z, for which positive integers m and n exist, such that

$$z = \frac{mn+1}{m+n}$$

holds.

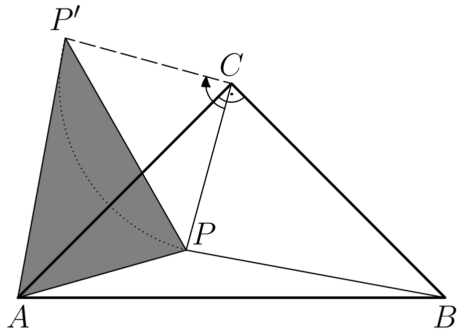
C–Team–3

Let H be the orthocenter of an acute-angled triangle ABC. The altitude in A has the length 15 cm, and this altitude divides BC into two sections of length 10 cm and 6 cm respectively. Determine the distance of H from the side BC.

2 Solutions to selected problems

Finally, we present the solutions to a few selected problems from this year's competition.

Solution to problem **A–Ind–1** by Zuzana Safernová from GMK Bilovec (this problem was proposed by *Jaroslav Švrček*):



The rotation about the center C through the oriented angle $-90°$ maps the vertex B of the given triangle ABC to its vertex A and an arbitrary point P of the interior of the triangle ABC to the point P'. This point lies in the opposite half-plane cutting by the straight line AC (see picture).

Since

$$AP' = BP \qquad \text{and} \qquad PP' = PC \cdot \sqrt{2}\,,$$

we can see that the sides of the triangle APP' have lengths AP, BP and $CP \cdot \sqrt{2}$, and thus there exists a triangle with sides of given lengths.

SOLUTION to problem **A–Ind–3** (proposed by *Robert Geretschläger*):

Since $40 = 2 \cdot 20$, the given equations imply further equivalent equations

$$\begin{aligned} x^2z + y^2z + 4xy &= 2\,(x^2 + y^2 + xyz)\,, \\ z\,(x^2 - 2xy + y^2) - 2\,(x^2 - 2xy + y^2) &= 0\,, \\ (x-y)^2\,(z-2) &= 0\,. \end{aligned}$$

From the last one follows that either $x = y$ or $z = 2$ holds. If we assume $z = 2$, substitution in either equation yields

$$x^2 + y^2 + 2xy = 20 \iff (x+y)^2 = 20,$$

which has no integer solutions. It therefore follows that $x = y$ must hold. Substitution yields

$$2x^2 + x^2z = 20 \iff x^2\,(z+2) = 20 = 2^2 \cdot 5\,.$$

Since x is an integer, it must be equal to ± 1 or ± 2, and since $x = \pm 1$ yields $z + 2 = 20$ thus $z = 18$ and $x = \pm 2$ yields $z + 2 = 5$ thus $z = 3$.

The solutions of the system of equations are $(\pm 1; \pm 1; 18)$ and $(\pm 2; \pm 2; 3)$.

SOLUTION to problem **B–Ind–1** (proposed by *Jaroslav Švrček*):

First of all, we can see that $xyz \neq 0$. Since the left sides of each of the three given equations are positive real numbers, their right sides must also be positive real numbers. Thus all three unknowns x, y, z must also be positive real numbers.

Further, for each $t > 0$ the inequality $t + 1/t \geq 2$ is fulfilled. From the third equation of the given system of equations

$$2x^2 = x + \frac{1}{x} \geq 2$$

follows. This implies $x \geq 1$. Similarly we can show that $y \geq 1$ and $z \geq 1$ also hold. From the first equation of the given system we can see that the following inequality

$$2x = x + x \geq x + \frac{1}{x} = 2y^2$$

also holds, i.e. $x \geq y^2$. Analogously, $y \geq z^2$ and $z \geq x^2$. Multiplying the last three inequalities we obtain

$$xyz \geq (xyz)^2 , \qquad \text{i.e.} \qquad 1 \geq xyz \, .$$

Since $x, y, z \geq 1$, it must be $x = y = z = 1$ fulfilled.

Thus the unique real solution of the given system of equations is the triple $(x, y, z) = (1, 1, 1)$.

Solution to problem **B–Team–3** (proposed by *Robert Geretschläger*):

If $x = 2 \cdot 3 \cdot \ldots \cdot 14$ we certainly have $2 \mid (x-2)$, $3 \mid (x-3), \ldots, 14 \mid (x-14)$, yielding 12 composite numbers in a row. Unfortunately, x is much larger than 10 000. We note that we do not require a prime factor to be present more than once in any of the numbers for them to be composite, but the number $2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 - 14 = 30016$ is still too big.

Eliminating the rather large factor 13, we can have a look at the number $y = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 - 12 = 4608$. It is immediately obvious that the numbers $y, y+1, \ldots, y+10$ and $y+12$ are all composite, and so it only remains to check the status of $y + 11 = 4619$. Since $4619 = 31 \cdot 149$, this number is also composite, and we have a number $a = y = 4608$ with the required property.

Solution to problem **C–Ind–1** (proposed by *Filip Švrček*):

Let c be the length of a hypotenuse of a right-angled triangle and a, b be the lengths of the other two sides. Using the statements given in the problem we have

$$a^2 + b^2 = c^2 \qquad \text{and} \qquad a + b + c = 2005\,.$$

From the second equation we obtain $c = 2005 - a - b$. Substituting c in the first equation we get (after easy manipulation)

$$4010(a + b) - 2ab = 2005^2\,.$$

Since both sides of the last equation are of different parity, the given problem has no integer solution.

Solution to problem **C–Ind–2** (proposed by *Jaroslav Švrček*):

The given inequality is equivalent to the following correct inequality

$$(x^2 - 2xy + y^2) + (x^2 - 2x + 1) + (y^2 - 2y + 1) \geq 0\,,$$

i.e.

$$(x - y)^2 + (x - 1)^2 + (y - 1)^2 \geq 0\,.$$

From the last inequality we can see that the equality holds if and only if $x = y = 1$.

Solution to problem **C–Team–2** (this problem was proposed by *Dimitri Dziabenko*, a 13-year-old former BRG Kepler student now living in Toronto, Canada):

All positive integers z can be expressed in this way. Let k be a positive integer. Setting $m = 2k - 1$ and $n = 2k + 1$ we obtain

$$\frac{mn + 1}{m + n} = \frac{(2k - 1)(2k + 1) + 1}{(2k - 1) + (2k + 1)} = \frac{4k^2}{4k} = k,$$

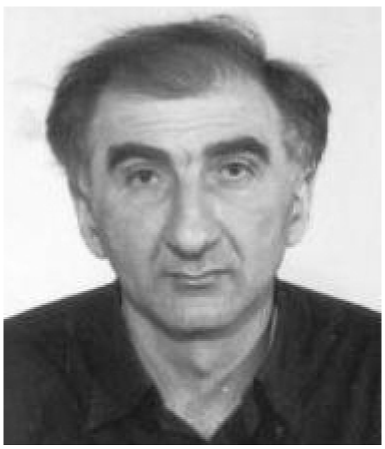
and setting $z = k$ solves the problem.

Dr. Robert Geretschläger
Bundesrealgymnasium
Keplerstrasse 1
8020 Graz
AUSTRIA
Email: robert.geretschlaeger@brgkepler.at

Dr. Jaroslav Švrček
Dept. of Algebra and Geometry
Palacký University
779 00 Olomouc
CZECH REPUBLIC
Email: svrcek@inf.upol.cz

An Interesting Inequality

Nairi M. Sedrakyan



Nairi Sedrakyan teaches mathematics in the Shahinyan High School of Physics and Mathematics in Yerevan, Armenia. From 1986 he has been a jury member of the Armenian Mathematical Olympiads. From 1997 he has been Leader and Deputy leader of the Armenian team at IMO. He is author of books Inequalities: Methods of Proving, *Fizlatlit, Moscow, 2002, 256p;* Geometrical Inequalities, *Edit Print, Yerevan, 2004 (both in Russian) 364p.*

Let us consider the following elegant problem which was proposed in [1] as a problem #112.

Problem 1 (Gabriel Dospinescu, Călin Popa)

Let $n \geq 2$ and $a_1, a_2, \ldots, a_n$ be real numbers such that $a_1 \cdot a_2 \cdot \ldots \cdot a_n = 1$ holds. Prove that

$$a_1^2 + a_2^2 + \ldots + a_n^2 - n \geq \frac{2n}{n-1} \sqrt[n]{n-1} \left(a_1 + a_2 + \ldots + a_n - n\right).$$

In the following problem there will be formulated and solved, certain questions which arose while solving the problem above.

Problem 2

Let $n \geq 2$ be positive integer and $a_1, \ldots, a_n$ arbitrary real numbers such that $a_1 \cdot \ldots \cdot a_n = 1$ holds. Further let c_n be the largest real number for which the inequality

$$a_1^2 + a_2^2 + \ldots + a_n^2 - n \geq c_n(a_1 + a_2 \ldots + a_n - n) \tag{1}$$

holds. Then the following propositions

a) $c_n = \min\limits_{(0;1]} \dfrac{(t+1)^2\,\left((t^{n-2})^2 + 2(t^{n-3})^2 + \ldots + n - 1\right)}{t(t^{n-2} + 2t^{n-3} + \ldots + n - 1)}$,

b) $c_2 = 4$,

c) $c_3 = 9 - 3\sqrt{3}$,

d) $c_2 > c_3 > \ldots > c_n > c_{n+1} > \ldots$,

e) $c_n \geq \dfrac{2n}{n - \sqrt{n-1}}$,

f) $\lim\limits_{n\to\infty} c_n = 2$.

are fulfilled. Prove.

Let us denote

$$c_n(t) = \frac{(t+1)^2\,\left((t^{n-2})^2 + 2(t^{n-3})^2 + \ldots + n - 1\right)}{t\,(t^{n-2} + 2t^{n-3} + \ldots + n - 1)},$$

where $n \geq 2$ and $t \in (0;1]$. Since $\lim\limits_{t\to 0} c_n(t) = +\infty$ there exists a real number ε_n, such that $0 < \varepsilon_n < \frac{1}{2}$ and $c_n(t) > c_n(\frac{1}{2})$, for each $t \in (0;\varepsilon_n]$. Hence

$$\min_{(0;1]} c_n(t) = \min_{[\,\varepsilon_n;1]} c_n(t) = c_n(t_n) \quad t_n \in [\varepsilon_n;1]\,.$$

First we will prove the following two lemmas.

Lemma 1

Let k, n be integers such that $0 \leq k < n$ holds. Then the function

$$f(t) = \frac{t^k + t^{k-1} + \cdots + 1}{t^n + t^{n-1} + \cdots + 1}$$

is decreasing on the interval $[0;1]$.

Indeed, it is enough to note that

$$f(t) = \frac{1}{\frac{1}{\left(\frac{1}{t}\right)^{k+1} + \left(\frac{1}{t}\right)^{k} + \ldots + \frac{1}{t}} + 1} \cdot$$

$$\cdot \frac{1}{\frac{1}{\left(\frac{1}{t}\right)^{k+2} + \left(\frac{1}{t}\right)^{k+1} + \ldots + \frac{1}{t}} + 1} \cdot \ldots \cdot \frac{1}{\frac{1}{\left(\frac{1}{t}\right)^{n} + \left(\frac{1}{t}\right)^{n-1} + \ldots + \frac{1}{t}} + 1}$$

in which each multiplicand is a decreasing function on the interval $(0;1]$, thus the proof of Lemma 1 is completed.

Lemma 2

For any integer $n \geq 2$ the inequality $\min\limits_{(0;1]} c_n(t) \geq \min\limits_{(0;1]} c_{n+1}(t)$ holds.

First we will prove that

$$\frac{(t^{n-2})^2 + 2(t^{n-3})^2 + \ldots + n - 1}{t^{n-2} + 2t^{n-3} + \ldots + n - 1} \geq \frac{(t^{n-1})^2 + 2(t^{n-2})^2 + \ldots + n}{t^{n-1} + 2t^{n-2} + \ldots + n}$$

for any $t \in (0;1]$. Let $p(t) = t^{n-2} + 2t^{n-3} + \cdots + n - 1$, then we must prove that the inequality

$$\frac{t^{2n-2} + t^{2n-4} + \ldots + 1 + p(t^2)}{t^{n-1} + t^{n-2} + \ldots + 1 + p(t)} \leq \frac{p(t^2)}{p(t)},$$

i.e.

$$\frac{p(t)}{t^{n-1} + t^{n-2} + \ldots + 1} \leq \frac{p(t^2)}{(t^2)^{n-1} + (t^2)^{n-2} + \ldots + 1}$$

holds. Since $t \geq t^2$, the last inequality is true, and according to Lemma 1 the function

$$\frac{p(t)}{t^{n-1} + t^{n-2} + \ldots + 1} = \frac{t^{n-2} + t^{n-3} + \ldots + 1}{t^{n-1} + t^{n-2} + \ldots + 1} +$$

$$+ \frac{t^{n-3} + t^{n-4} + \ldots + 1}{t^{n-1} + t^{n-2} + \ldots + 1} + \ldots + \frac{1}{t^{n-1} + t^{n-2} \ldots + 1}$$

is decreasing on the interval $(0;1]$.

Therefore we can see that $\min\limits_{(0;1]} c_n(t) = c_n(t_n) \geq c_{n+1}(t_n) \geq \min\limits_{(0;1]} c_{n+1}(t)$.

Solution to Problem 2.

a) Suppose the inequality (1) holds, take

$$a_1 = \ldots = a_{n-1} = \frac{1}{t}, \quad a_n = t^{n-1}, \quad t \in (0;1)$$

then

$$\frac{n-1}{t^2} + t^{2n-2} - n \geq c_n \left(\frac{n-1}{t} + t^{n-1} - n \right),$$

i.e.

$$\frac{t^{2n} - nt^2 + n - 1}{t} \geq c_n(t^n - nt + n - 1).$$

Further we have

$$\frac{(t^2-1)^2 \left((t^2)^{n-2} + 2(t^2)^{n-3} + \ldots + n - 1 \right)}{t} \geq$$
$$\geq c_n (t-1)^2 (t^{n-2} + 2t^{n-3} + \ldots + n - 1).$$

This implies

$$c_n \leq \frac{\left((t^{n-2})^2 + 2(t^{n-3})^2 + \ldots + n - 1 \right) (t+1)^2}{t(t^{n-2} + 2t^{n-3} + \ldots + n - 1)} = c_n(t),$$

hence $c_n \leq \lim\limits_{t \to t_n} c_n(t) = c_n(t_n) = \min\limits_{(0;1]} c_n(t)$.

Using mathematical induction we can prove: For any integer $n \geq 2$ and for arbitrary positive real numbers $a_1, \ldots, a_n$ such that $a_1 \cdot \ldots \cdot a_n = 1$ the following inequality

$$a_1^2 + \ldots + a_n^2 - n \geq \min_{(0;1]} c_n(t) \left(a_1 + \ldots + a_n - n \right) \qquad (2)$$

is fulfilled.

(i) For $n = 2$ we have

$$c_2(t) = \frac{(t+1)^2}{t} = 4 + \frac{(t-1)^2}{t}, \qquad \min_{(0;1]} c_2(t) = 4.$$

Therefore it is sufficient to prove the inequality

$$a_1^2 + a_2^2 - 2 \geq 4(a_1 + a_2 - 2), \qquad \text{i.e.} \qquad (a_1 + a_2 - 2)^2 \geq 0.$$

(ii) Further we suppose that the inequality (2) is fulfilled for an integer $n \geq 2$. Now, we shall prove that this inequality is true also for $n+1$.

Let $a_1 \cdot a_2 \cdot \ldots \cdot a_n = a^n$ $(a > 0)$ and $1 \geq a_{n+1} > 0$ such that the equality $a_1 \cdot a_2 \cdot \ldots \cdot a_n \cdot a_{n+1} = 1$ holds, then

$$a \geq 1, \qquad na + \frac{1}{a^n} - (n+1) \geq 0 \quad \text{and} \quad \frac{a_1}{a} + \ldots + \frac{a_n}{a} \geq n.$$

Further we have

$$\begin{aligned}
& a_1^2 + \ldots + a_n^2 + a_{n+1}^2 - n + 1 \\
&= a^2 \left(\left(\frac{a_1}{a}\right)^2 + \ldots + \left(\frac{a_n}{a}\right)^2 - n \right) + a^2 n + a_{n+1}^2 - (n+1) \\
&\geq a^2 \min_{(0;1]} c_n(t) \left(\frac{a_1}{a} + \ldots + \frac{a_n}{a} - n \right) + a^2 n + \frac{1}{a^{2n}} - (n+1) \\
&= a^2 \min_{(0;1]} c_n(t) \left(\frac{a_1}{a} + \ldots + \frac{a_n}{a} - n \right) + c_{n+1}\left(\frac{1}{a}\right) \left(na + \frac{1}{a^n} - (n+1) \right) \\
&\geq a \min_{(0;1]} c_n(t) \left(\frac{a_1}{a} + \ldots + \frac{a_n}{a} - n \right) + \min_{(0;1]} c_{n+1}(t) \left(na + a_{n+1} - (n+1) \right) \\
&\geq \min_{(0;1]} c_{n+1}(t)(a_1 + \ldots + a_n + a_{n+1} - (n+1)).
\end{aligned}$$

Now we shall prove that the inequality (1) holds for $c_n = \min\limits_{(0;1]} c_n(t)$. Indeed, we have

$$\begin{aligned}
a_1^2 + \ldots + a_n^2 - n &\geq \min_{(0;1]} c_n(t)(|a_1| + \ldots + |a_n| - n) \\
&\geq \min_{(0;1]} c_n(t)(a_1 + \ldots + a_n - n).
\end{aligned}$$

Thus the solution of part a) is completed.

b) According to part a) we obtain

$$c_2 = \min_{(0;1]} c_2(t) = \min_{(0;1]} \left(4 + \frac{(t-1)^2}{t}\right) = 4.$$

c) According to part a) we have

$$c_3 = \min_{(0;1]} c_3(t) \quad \text{where} \quad c_3(t) = \frac{(t+1)^2(t^2+2)}{t(t+2)}.$$

Note that

$$(c_3(t))' = \frac{2(t+1)(t^2+t+1)(t^2+2t-2)}{t^2(t+2)^2},$$

hence the function $c_3(t)$ attains its minimal value on the interval $(0;1]$ at the point $t = \sqrt{3}-1$. Then

$$\min_{(0;1]} c_3(t) = 9 - 3\sqrt{3}$$

which means that $c_3 = 9 - 3\sqrt{3}$.

d) According to part a) and Lemma 2 we have

$$c_n = \min_{(0;1]} c_n(t) \geq \min_{(0;1]} c_{n+1}(t) = c_{n+1}$$

for any integer $n \geq 2$.

e) Let $k_n = \dfrac{n}{n-\sqrt{n-1}}$. We must prove now that for any $t \in (0;1]$ the following inequality

$$c_n(t) \geq 2k_n\,,$$

holds, i.e. according to the proof of part a) we have

$$\frac{n-1}{t^2} + t^{2n-2} - n \geq 2k_n\left(\frac{n-1}{t} + t^{n-1} - n\right) \tag{3}$$

Let us consider a function

$$\varphi(t) = \frac{n-1}{t^2} + t^{2n-2} - n - 2k_n\left(\frac{n-1}{t} + t^{n-1} - n\right)$$

which is defined on the interval $(0;1]$. Note that $\lim_{t\to 0}\varphi(t) = +\infty$, hence there exists $t_0 \in (0;1]$ such that $\min_{(0;1]}\varphi(t) = \varphi(t_0)$. If $t_0 = 1$, then $\varphi(t) \geq \varphi(1) = 0$. If $t_0 \neq 1$, then

$$0 = \varphi'(t_0) = \frac{2(n-1)}{t_0^2}(t_0^n - 1)\left(\frac{t_0^n + 1}{t_0} - k_n\right)$$

from which we can see that $t_0^n = k_n t_0 - 1$, whence

$$\varphi(t_0) = \frac{1}{t_0^2}[(n-1+(t_0^n)^2 - nt_0^2 + 2nk_n t_0^2 - 2k_n t_0^n \cdot t_0$$
$$- 2(n-1)k_n t_0] = \frac{1}{t_0^2}[(-k_n^2 - n + 2nk_n)t_0^2 - 2k_n(n-1)t_0 + n] \geq 0,$$

since $D = 0$, where D is the discriminant of the quadratic trinomial

$$(-k_n^2 + 2nk_n - n)t^2 - 2k_n(n-1)t + n$$

in a real variable t, and $-k_n^2 + 2nk_n - n > 0$, since the latter is equivalent to

$$\frac{1}{\sqrt{n} + \sqrt{n-1}} < \frac{\sqrt{n}}{n - \sqrt{n-1}} < \frac{1}{\sqrt{n} - \sqrt{n-1}},$$

which is an evident inequality.

Thus $\varphi(t) \geq \varphi(t_0) \geq 0$, hence for $t_0 \in (0;1]$ we have $\varphi(t) \geq 0$ and consequently (3) is proved.

Remark 1. If n is an integer, $n \geq 16$, we have

$$\frac{2n}{n - \sqrt{n-1}} \geq \frac{2n\sqrt[n]{n-1}}{n-1}. \tag{4}$$

Indeed, we shall prove that for $n \geq 15$ holds the following inequality

$$\left(\frac{n}{n+1-\sqrt{n}}\right)^{n+1} \geq n.$$

For $n \geq 15$ we have

$$\begin{aligned}\left(\frac{n}{n+1-\sqrt{n}}\right)^{n+1} &= \left(1+\frac{\sqrt{n}-1}{n+1-\sqrt{n}}\right)^{n+1} \geq \left(1+\frac{1}{\sqrt{n}+\frac{1}{2}}\right)^{n+1} \\ &> 1+\frac{n+1}{\sqrt{n}+\frac{1}{2}}+\frac{(n+1)n}{2(\sqrt{n}+\frac{1}{2})^2}+\frac{(n+1)n(n-1)}{6(\sqrt{n}+\frac{1}{2})^3} \\ &= 1+\frac{(n+1)(\sqrt{n}+1)^2}{2(\sqrt{n}+\frac{1}{2})^2}+\frac{(n+1)n(n-1)}{6(\sqrt{n}+\frac{1}{2})^3} \\ &> 1+\frac{n+1}{2}+\frac{(n+1)n(n-1)}{6(\sqrt{n}+\frac{1}{2})^3} > \frac{n+3}{2}+\frac{(n+1)n(n-1)}{6(n\sqrt{n}+2n)} \\ &> \frac{n+3}{2}+\frac{n-3}{2}\cdot\frac{n+3}{3(\sqrt{n}+2)} > \frac{n+3}{2}+\frac{n-3}{2} = n.\end{aligned}$$

Since $n+3 \geq 3(\sqrt{n}+2)$, i.e. $\sqrt{n}(\sqrt{n}-3) \geq 3$, we obtain the inequality

$$\sqrt{n}(\sqrt{n}-3) \geq \sqrt{15}(\sqrt{15}-3) \geq 3$$

therefore the inequality (4) holds for all integers $n \geq 15$. Thus the inequality (4) holds also for all $n \geq 2$.

f) Let us choose $a_1 = \ldots = a_{n-1} = t^{-1}$ and $a_n = t^{n-1}$, where $t = \dfrac{1}{\sqrt[n]{n-1}}$. Then we have

$$n-1+\frac{1}{(n-1)^2}-\frac{n}{(\sqrt[n]{n-1})^2} \geq c_n\left(\frac{n-1}{\sqrt[n]{n-1}}+\frac{1}{n-1}-\frac{n}{(\sqrt[n]{n-1})^2}\right),$$

hence

$$n-\frac{n}{(\sqrt[n]{n-1})^2} \geq c_n\left(\frac{n-1}{\sqrt[n]{n-1}}-\frac{n}{(\sqrt[n]{n-1})^2}\right).$$

Thus

$$c_n \leq \frac{n\left((\sqrt[n]{n-1})^2-1\right)}{(n-1)\sqrt[n]{n-1}-n}.$$

Now let us denote $\sqrt[n]{n-1} = 1+\alpha_n$, then $\dfrac{\ln(n-1)}{n} = \ln(1+\alpha_n) \leq \alpha_n$ and

$$n-1 = (1+\alpha_n)^n > \frac{n(n-1)}{2}\,\alpha_n^2, \qquad \alpha_n < \sqrt{\frac{2}{n}}.$$

Then

$$\frac{2n}{n-\sqrt{n-1}} \le c_n \le \frac{2n\alpha_n + n\alpha_n^2}{(n-1)\alpha_n - 1},$$

thus

$$\lim_{n\to\infty} \frac{2n}{n-\sqrt{n-1}} = 2.$$

Also

$$\lim_{n\to\infty} \frac{2n\alpha_n + n\alpha_n^2}{(n-1)\alpha_n - 1} = \lim_{n\to\infty} \frac{2+\alpha_n}{\dfrac{n-1}{n} - \dfrac{1}{n\alpha_n}} = 2$$

because

$$0 < \frac{1}{n\alpha_n} \le \frac{1}{\ln(n-1)},$$

and thus

$$\lim_{n\to\infty} \frac{1}{n\alpha_n} = 0.$$

Remark 2. The largest real number c for which the inequality

$$a_1^2 + a_2^2 + \ldots + a_n^2 - n \ge c(a_1 + a_2 + \ldots + a_n - n)$$

holds for all integers $n \ge 2$ and all $a_1 \cdot \ldots \cdot a_n = 1$, where $a_1, a_2, \ldots, a_n$ are real numbers, is $c = 2$.

Indeed, we have $c \le c_n$ for all integers $n \ge 2$, hence $c \le \lim_{n\to\infty} c_n = 2$. For $c = 2$ the inequality is true because

$$a_1^2 + \cdots + a_n^2 - n - 2(a_1 + \cdots + a_n - n) = (a_1 - 1)^2 + \cdots + (a_n - 1)^2 \ge 0.$$

References

[1] Andreescu, T., Dospinescu, G., Cîrtoaje, V., Lascu, M.: *Old and New Inequalities*, GIL Publishing House, 2004.

Nairi M. Sedrakyan
Shahinyan High School Physics and Mathematics
Azatutyan Street, Second bypass 9
375 037 Yerevan
ARMENIA
Email: gagri@mail.yerphi.am

The UK Primary Mathematics Challenge

Peter Bailey

"They didn't fool me with that one!"

Peter Bailey is a retired secondary mathematics teacher who has been involved in The Mathematical Association for many years, as chair of the Publications Committee and, for the last six years, as chair of the Primary Mathematics Challenge teams.

1 Background

Back in 1999, secondary pupils in the UK had the opportunity to take part in mathematics challenges organised by the United Kingdom Mathematics Trust (UKMT). These are well organised and popular. But there was no similar challenge for primary pupils. So The Mathematical Association decided to set up the Primary Mathematics Challenge (PMC).

The structure of the PMC is based on ideas from Dr Tony Gardiner. The planning group wanted to set a challenge for as many primary (aged 10 and 11) pupils as possible, which was fun, motivating and mathematical. We also wanted it to feed pupils into the Junior Mathematics Challenge run by the UKMT for junior secondary pupils.

2 The Structure of the PMC

The PMC is aimed at the top 60% of pupils in the final two years of primary school in England and equivalent ages in Scotland, Wales and Northern Ireland. Teachers can give the PMC to their pupils at any time during November. It has 25 multiple-choice problems which are interesting to these pupils and are mathematical. The first 10 are easy, the next 10 a little harder and the last five harder still. Correct answers for problems 1–20 score one mark each; correct answers to problems 21–25 score two marks each. The idea is that the majority of pupils achieve at least 15 marks but that there is differentiation among the top marks.

The 25 problems are on mathematical material which the majority of pupils will have covered in lessons. It is not the intention to set problems on secondary topics but rather to set problems on primary topics which can lead to discovery and further work at this level. Some problems have funny names and are set in amusing situations. Highest-scoring pupils are invited to take the PMC Finals the following February.

The central aim in planning the administration of the PMC is to make it as easy as possible for the teachers! PMC papers are sold in packs of ten, with seven certificates provided in each pack (1 gold, 2 silver, 3 bronze and one photocopiable 'Took the Challenge' certificate). Teachers mark the pupils' responses themselves, using the easy-to-use mark sheet. The certificates are awarded in-school by the teacher who decides who gets what! Teachers also receive an Answers and Notes sheet which provides some discussion on problem-solving methods for the 25 problems, and ideas for further activity in classrooms following the Challenge.

Feedback Sheets for pupil and teacher comments are provided; teachers can also fill in and return the Tally Form which asks for the mark distribution so that the PMC Problems Team finds out how pupils have managed on their paper. The names of the highest-scoring pupils and their marks are also sent to the PMC office, so that invitations to take the PMC Finals can be made.

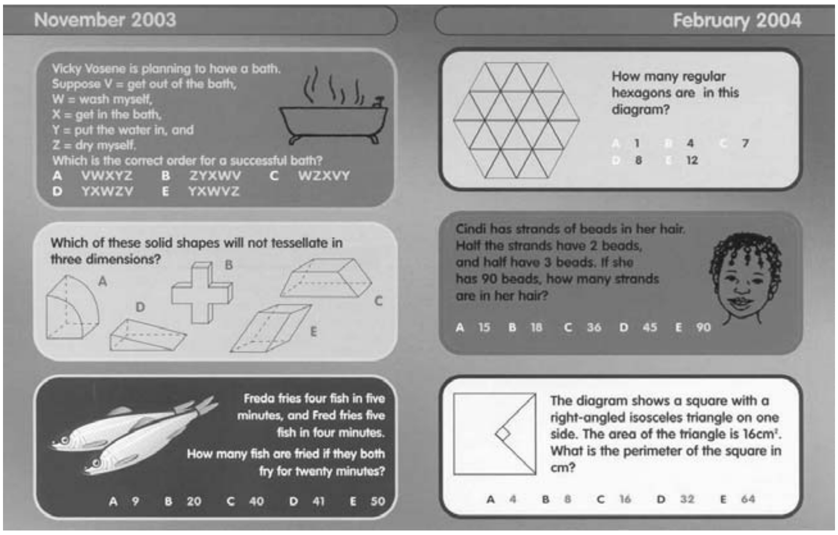
With most pupils scoring at least 15/30 on the PMC in November, there will be some pupils (the best young mathematicians in the UK) who have all scored very high marks. They are invited to take the PMC Finals in the following February. The structure of the Finals is the same as the November PMC with 25 multiple-choice problems, but the problems, still on primary mathematics topics, are more difficult. Pupils' Answer Sheets for the Finals are marked by the PMC office; certificates and medals are awarded by The Mathematical Association and sent to schools in March.

3 Growth in the PMC

The number of schools and pupils has grown rapidly each year. A grant from the Nuffield Foundation provided money to get the challenge started. Here are the figures:

	No of	Schools in					PMC papers
	schools	En	Sc	Wa	NI	Other	sold
1999	311	240	14	12	43		10930
2000	780	626	44	34	72		32010
2001	1175	936	62	98	77		45060
2002	1598	1398	83	40	73		62000
2003	2615	2277	144	85	100	9	88300
2004	3179	2790	148	86	105	50	110880

4 A few PMC problems



5 Feedback from schools

The PMC office asks for and receives lots of feedback each year. Here are a few of the comments sent in to the office:

Comments from pupils

I liked the funny names
I can't believe I got a silver
I didn't panic—I just got on and did it
Tricky but interesting
Best test I've ever done
It really made me think
I enjoyed getting my certificate
I thought of pizzas when solving no 20
In the end I just guessed
It wasn't hard but it was tricky
They didn't fool me with that one
It got tricky after the double lines

At first I thought oh no! but when I got into it I was quite excited

Comments from teachers

It got all pupils thinking
Super—easy to mark
Nice to raise the profile of maths in school
Good range of problems
All pupils could participate
Thank you for the follow-up suggestions
Great for thinking skills
Fair but challenging
Good ideas for follow-up
Found some gaps in my teaching
Generated a lot of discussion
Parents seem keen on the PMC too
The award winners were really proud of their certificates

6 Some difficulties

Not all feedback is positive. Teachers send in helpful comments about the problems and how the administration can be improved. There are always some aspects of the PMC which give the organisers food for thought. Here are a few.

- The PMC is meant to be a positive experience for all pupils, with the large majority scoring more than 15/20. The Problems Team constantly finds it difficult to provide interesting non-trivial mathematical problems which the majority of pupils can get correct!
- Many of the problems are set in real-life amusing situations. It is not easy to keep the reading age of the contexts at a reasonable level. We aim for a reading age of nine. But some comments from teachers have suggested the paper has challenged literacy rather than mathematics!
- It is not easy to provide the correct number of certificates for schools' needs. Teachers themselves award the certificates. Sometimes the certificates which they receive in the packs does not allow them to give awards as they would wish. Allowing schools to phone for more would be expensive in office time.
- The PMC Finals is provided for those pupils who score the highest marks in the November PMC. The best young mathematicians in the country are now taking the Finals. We want to avoid the PMC feeling like a government test! In the first few years, finalists could take the Finals on any day within a two-week period. The dilemma is that, whilst the PMC should be fun, the pressure is on to make sure that there are no irregularities in pupil's responses. For the first time (2005) teachers have to set the PMC Finals on a specified day in February.

7 For the future

In the coming years the Management Team will be aiming to:

- further increase the number of pupils taking the PMC each year, possibly by working with an education company to assist in promotion and sales
- provide challenge papers in which the majority of pupils score 50% or more, and which give pupils a positive mathematical experience!
- keep the PMC lively and unlike government assessment.

For further information on the Primary Mathematics Challenge, please contact the PMC office at The Mathematical Association, 259 London Road, Leicester LE2 3BE, phone 0116 221 0013, fax 0116 212 2835, email pmc@m-a.org.uk or the website www.m-a.org.uk .

Peter Bailey
Chair, PMC Teams
41 Middleton Road
Birmingham B14 7HX
UK
Email: factors@ic24.net

WFNMC Congress 5 – Cambridge July, 22–28, 2006

WFNMC is a support network for all those interested in running mathematics competitions. Every four years it holds a conference covering all aspects of mathematics competitions. The last conference was in Melbourne in 2002.

The WFNMC 2006 conference will be held in

Robinson College, Cambridge, England
from Saturday 22 to Friday 28 July 2006.

Full details and a registration form are available on the conference website www.wpr.aaugonline.net/wfnmc (or go to www.amt.edu.au, click on "Links", "WFNMC", then "WFNMC 2006").

The venue has been carefully chosen to offer:

- a historic location with all the colleges and atmosphere of Cambridge on your immediate doorstep;
- all the conference facilities together on a single site, within a short distance of each other, so as to facilitate the personal interactions which characterise WFNMC conferences;
- a college with excellent facilities (two auditoriums, numerous mid-sized and small teaching rooms, chapel, bar and grounds); and
- a very reasonable all-in price (lower than 2002 and remarkable for the UK in July).

This should provide an ideal venue for the family atmosphere of a conference (for information about the venue see www.robinson.cam.ac.uk).

The attractions of the venue are complemented by the quality of those who have agreed to give plenary talks.

- Simon Singh is not only the author of the bestsellers *Fermat's Last Theorem*, *The Code Book*, and most recently *Big Bang*, but is also an excellent presenter, who has worked tirelessly on behalf of mathematics, see www.simonsingh.com.

- Robin Wilson is the author and editor of more than 20 books, one of the most recent being *Four Colours Suffice* (Princeton University Press 2002), and is well-known for his entertaining, yet scholarly, lectures on the history of mathematics.

- A well-known Cambridge mathematician with an enviable track record as a lecturer for wider audiences will give a strictly mathematical lecture. (Insiders may well be able to fit names to this description; none of those approached have declined, but we have yet to sort out exactly which of those approached will be available!) We expect this talk will be held amidst the stunning architecture of the new "Centre for Mathematical Sciences" and the "Isaac Newton Institute".

- The WFNMC President Petar Kenderov will open the conference; the WFNMC Vice-President Maria de Losada will extract lessons for us all from a brief history of mathematics competitions in Colombia; and Jozsef Pelikan (Chair, IMO Advisory Board) will draw on his extensive experience to reflect on issues arising from the present state of mathematics competitions, and in particular on the question of what determines whether those who are successful in competitions eventually become research mathematicians.

There is nothing harder, or more important, than a good beginning.

- As soon as delegates have registered and freshened up, they will be put in teams, given a map and a set of clues, and will be set to follow a "mathematics trail" around the very compact, but mostly hidden, city of Cambridge. Cars are largely excluded from the city centre, and following this "trail" should make it clear at the outset how much there is to see, and how accessible the city, its historic colleges and beautiful grounds are on foot. It will also introduce all delegates to some of the reference points for Robin Wilson's historical lecture the next day on "Cambridge Mathematical Figures".

- In the evening Petar Kenderov (WFNMC President) will begin the serious business of the meeting.

- Thereafter each day will have a more predictable structure. Each day will begin with delegates in small working groups focussing on "problem creation and improvement" in the spirit of the first WFNMC conference in Waterloo (1990). We plan to have groups to cover most of the obvious domains including some new ones: these will distinguish
  - (i) different ages (primary (roughly Grades 3–5), Junior (Grades 6–8), Intermediate (Grades 9–10), and Senior (Grades 11–12)),
  - (ii) target groups (popular multiple choice or "Olympiad"),
  - (iii) content (traditional, or applied), and
  - (iv) formats (individual timed written, take-home, team, or student problem journal).

  Those who have accepted to act as Chairs of these groups are: Primary (Peter Bailey), Junior popular (Gregor Dolinar), Intermediate popular (Ian VanderBurgh), Senior popular (Harold Reiter), Junior Olympiad (Bruce Henry), Intermediate Olympiad (Andrew Jobbings), Senior Olympiad (Gerry Leversha), Team (Steve Mulligan), Student problem journals (John Webb). Delegates should declare their preferences on the Registration Form.

- The collection of problems that result from this exercise will be circulated after the conference, provided delegates agree to embargo publication for 12–15 months, so that the problems can be used in national competitions during the ensuing period. We hope that an ad hoc jury will award prizes (donated by Cambridge University Press) for the best problems each day.

- The session before lunch provides an opportunity for delegates to present papers in parallel sessions. Those who wish to present such a paper should indicate this on their registration form and provide a title and abstract by the date specified. We anticipate that there will be sections covering competition types (as for the problem creation sessions); interactions between competitions and ordinary classroom teaching; new developments; research related to competitions; serious work relating to practical aspects of running mathematics competitions (administration, finance, sponsorship, how to organise problem setting groups and marking weekends, etc.).

- The programme needs to be flexible to fit the requirements of delegates. However, some of the themes listed in the previous paragraph, though important to all of us, are less likely to lend themselves to formal papers. We have therefore labelled two pre-lunch sessions as "Forums", to allow for two or three parallel "open debates" on themes of this kind, in relatively small groups, with each Forum beginning with one or two short contributions from delegates with relevant experience.

- Two afternoon sessions provide further slots for parallel sessions. One slot has been allocated for a "Team competition", in which delegates can experience this kind of event at first hand. Two afternoon sessions have been allocated for "Visits", where we plan to arrange a selection of guided tours to colleges and local sights (including "punting on the river Cam").

- After the first day, the early evening session will always be used for a plenary lecture.

- The evenings (apart from the conference dinner) will remain informal.
  - (i) On two evenings we will arrange a chamber recital in the very striking College Chapel (e.g. a piano quartet, an evening of Lieder, or a wind quintet).
  - (ii) Most evenings in July one can find open-air performances of Shakespeare in various college grounds.
  - (iii) And on one evening we plan a bit of modern English culture—a pub quiz (with a mathematical bent).

See you there!

Tony Gardiner

Chair, Organising Committee WFNMC 2006

Proposed Programme of the WFNMC 5, Cambridge

July 2006	0900-1045	Break	1115-1245	Lunch	1400-1615	Break	1645-1815	1845-1945	2000-
Sat 22					Mathematical trail			Dinner	President's Address
Sun 23	Problem refinement		Parallel sessions		Team competition		Plenary lecture Robin Wilson	Dinner	Concert
Mon 24	Problem refinement		Parallel sessions		Visits		Plenary lecture Maria de Losada	Dinner	
Tues 25	Problem refinement		Forums		Parallel sessions		Plenary lecture	Dinner	Concert
Weds 26	Problem refinement		Parallel sessions		Visits		Plenary lecture Simon Singh	Conference Dinner	
Thurs 27	Problem refinement		Forums		Parallel sessions		Plenary lecture Jozsef Pelikan	Dinner	Pub quiz
Fri 28	Problem refinement		Closing session		Depart				

WFNMC International & National Awards

1 David Hilbert International Award

The David Hilbert International Award was established to recognise contributions of mathematicians who have played a significant role over a number of years in the development of mathematical challenges at the international level which have been a stimulus for mathematical learning.

Each recipient of the award is selected by the Executive and Advisory Committee of the World Federation of National Mathematics Competitions on the recommendations of the WFNMC Awards Sub-committee.

Past recipients have been: Arthur Engel (Germany), Edward Barbeau (Canada), Graham Pollard (Australia), Martin Gardner (USA), Murray Klamkin (Canada), Marcin Kuczma (Poland), Maria de Losada (Colombia), Peter O'Halloran (Australia) and Andy Liu (Canada).

2 Paul Erdös National Award

The Paul Erdös National Award was established to recognise contributions of mathematicians who have played a significant role over a number of years in the development of mathematical challenges at the national level which have been a stimulus for the enrichment of mathematics learning.

Each recipient of the award is selected by the Executive and Advisory Committee of the World Federation of National Mathematics Competitions on the recommendations of the WFNMC Awards Sub-committee.

Past recipients have been: Luis Davidson (Cuba), Nikolay Konstantinov (Russia), John Webb (South Africa), Walter Mientka (USA), Ronald Dunkley (Canada), Peter Taylor (Australia), Sanjmyatav Urjintseren (Mongolia), Qiu Zonghu (China), Jordan Tabov (Bulgaria), George Berzsenyi (USA), Tony Gardiner (UK), Derek Holton (New Zealand), Wolfgang Engel (Germany), Agnis Andžāns (Latvia), Mark Saul (USA), Francisco Bellot Rosado (Spain), János Surányi (Hungary), Istvan Reiman (Hungary), Bogoljub Marinkovich (Yugoslavia), Harold Reiter

(USA), Wen-Hsien Sun (Taiwan), and in the year 2004, Warren Atkins (Australia), Patricia Fauring (Argentina) and André Deledicq (France).

The general meeting of the WFNMC 4 in Melbourne agreed, from 2003, to merge the above two awards into one award titled *The Paul Erdös Award*.

Requirements for Nominations for the Paul Erdös Award

The following documents and additional information must be writtten in English:

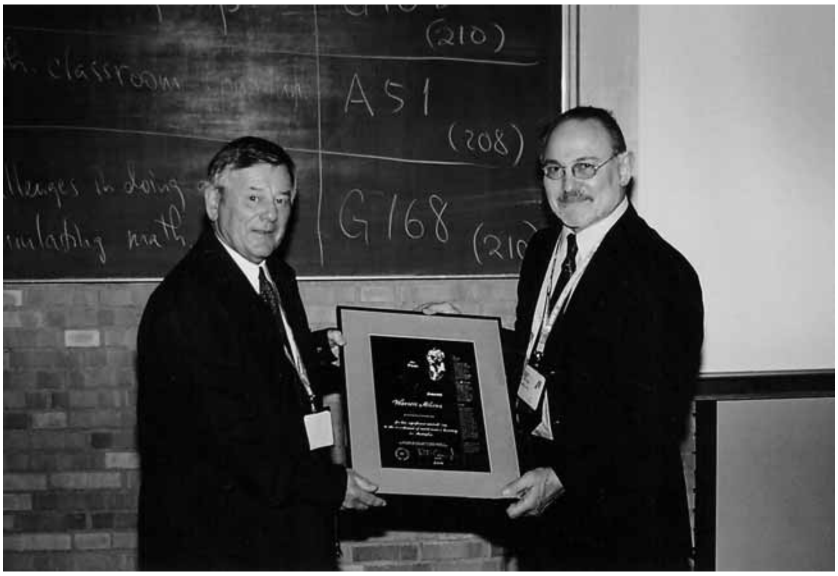
- A one or two page statement which includes the achievements of the nominee and a description of the contribution by the candidate which reflects the objectives of the WFNMC.
- Candidate's present home and business address and telephone/telefax number.

Nominating Authorities

An aspirant to the Awards may be proposed through the following authorities:

- The President of the World Federation of National Mathematics Competitions.
- Members of the World Federation of National Mathematics Competitions Executive Committee or Regional Representatives.

The Federation encourages the submission of such nominations from Directors or Presidents of Institutes and Organisations, from Chancellors or Presidents of Colleges and Universities, and others.



Warren Atkins (left) receives the Paul Erdös Award, 2004

Maria de Losada receives the P. Erdös Award on behalf of Patricia Fauring

André Deledicq (left) receives the Paul Erdös Award, 2004

W. Atkins, M. de Losada, P. Taylor and A. Deledicq

The Erdös Award
Call for Nominations

The Awards Committee of the WFNMC calls for nominations for the Erdös Award.

As described in the formal nomination procedures (see www.amt.edu.au/wfnmcaw.html or page 75 this issue), nominations should be sent to the Chair of the Committee at the address below by 1 October 2005 for consideration and presentation in 2006. Such a nomination must include a description of the nominee's achievements together with the names and addresses of (preferably) four persons who can act as referees.

Committee Chair:

Peter J Taylor
Australian Mathematics Trust
University of Canberra ACT 2601
AUSTRALIA

or directly by email at pjt@olympiad.org

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These books are a valuable resource for the school library shelf, for students wanting to improve their understanding and competence in mathematics, and for the teacher who is looking for relevant, interesting and challenging questions and enrichment material.

To attain an appropriate level of achievement in mathematics, students require talent in combination with commitment and self-discipline. The following books have been published by the AMT to provide a guide for mathematically dedicated students and teachers.

Australian Mathematics Competition (AMC) Solutions and Statistics
Edited by DG Pederson

This book provides, each year, a record of the AMC questions and solutions, and details of medallists and prize winners. It also provides a unique source of information for teachers and students alike, with items such as levels of Australian response rates and analyses including discriminatory powers and difficulty factors.

Australian Mathematics Competition Book 1 (1978-1984)

Australian Mathematics Competition Book 2 (1985-1991)

Australian Mathematics Competition Book 3 (1992-1998)

An excellent training and learning resource, each of these extremely popular and useful books contains over 750 past AMC questions, answers and full solutions. The questions are grouped into topics and ranked in order of difficulty. Book 3 also available on CD (for PCs only).

Problem Solving Via the AMC
Edited by Warren Atkins

This 210 page book consists of a development of techniques for solving approximately 150 problems that have been set in the Australian Mathematics Competition. These problems have been selected from topics such as Geometry, Motion, Diophantine Equations and Counting Techniques.

Methods of Problem Solving, Book 1
Edited by JB Tabov, PJ Taylor

This introduces the student aspiring to Olympiad competition to particular mathematical problem solving techniques. The book contains formal treatments of methods which may be familiar or introduce the student to new, sometimes powerful techniques.

Methods of Problem Solving, Book 2
JB Tabov & PJ Taylor
After the success of Book 1, the authors have written Book 2 with the same format but five new topics. These are the Pigeon-Hole Principle, Discrete Optimisation, Homothety, the AM-GM Inequality and the Extremal Element Principle.

Mathematical Toolchest
Edited by AW Plank & N Williams

This 120 page book is intended for talented or interested secondary school students, who are keen to develop their mathematical knowledge and to acquire new skills. Most of the topics are enrichment material outside the normal school syllabus, and are accessible to enthusiastic year 10 students.

International Mathematics — Tournament of Towns (1980-1984)

International Mathematics — Tournament of Towns (1984-1989)

International Mathematics — Tournament of Towns (1989-1993)

International Mathematics — Tournament of Towns (1993-1997)
Edited by PJ Taylor

The International Mathematics Tournament of Towns is a problem solving competition in which teams from different cities are handicapped according to the population of the city. Ranking only behind the International Mathematical Olympiad, this competition had its origins in Eastern Europe (as did the Olympiad) but is now open to cities throughout the world. Each book contains problems and solutions from past papers.

Challenge! 1991 – 1995
Edited by JB Henry, J Dowsey, A Edwards, L Mottershead, A Nakos, G Vardaro

The Mathematics Challenge for Young Australians attracts thousands of entries from Australian High Schools annually and involves solving six in depth problems over a 3 week period. In 1991-95, there were two versions – a Junior version for Year 7 and 8 students and an Intermediate version for Year 9 and 10 students. This book reproduces the problems from both versions which have been set over the first 5 years of the event, together with solutions and extension questions. It is a valuable resource book for the class room and the talented student.

USSR Mathematical Olympiads 1989 – 1992
Edited by AM Slinko

Arkadii Slinko, now at the University of Auckland, was one of the leading figures of the USSR Mathematical Olympiad Committee during the last years before democratisation. This book brings together the problems and solutions of the last four years of the All-Union Mathematics Olympiads. Not only are the problems and solutions highly expository but the book is worth reading alone for the fascinating history of mathematics competitions to be found in the introduction.

Australian Mathematical Olympiads 1979 – 1995
H Lausch & PJ Taylor

This book is a complete collection of all Australian Mathematical Olympiad papers since the first competition in 1979. Solutions to all problems are included and in a number of cases alternative solutions are offered.

Chinese Mathematics Competitions and Olympiads 1981-1993
A Liu

This book contains the papers and solutions of two contests, the Chinese National High School Competition from 1981-82 to 1992-93, and the Chinese Mathematical Olympiad from 1985-86 to 1992-93. China has an outstanding record in the IMO and this book contains the problems that were used in identifying the team candidates and selecting the Chinese teams. The problems are meticulously constructed, many with distinctive flavour. They come in all levels of difficulty, from the relatively basic to the most challenging.

Asian Pacific Mathematics Olympiads 1989-2000
H Lausch & C Bosch-Giral

With innovative regulations and procedures, the APMO has become a model for regional competitions around the world where costs and logistics are serious considerations. This 159 page book reports the first twelve years of this competition, including sections on its early history, problems, solutions and statistics.

Polish and Austrian Mathematical Olympiads 1981-1995
ME Kuczma & E Windischbacher

Poland and Austria hold some of the strongest traditions of Mathematical Olympiads in Europe even holding a joint Olympiad of high quality. This book contains some of the best problems from the national Olympiads. All problems have two or more independent solutions, indicating their richness as mathematical problems.

Seeking Solutions
JC Burns

Professor John Burns, formerly Professor of Mathematics at the Royal Military College, Duntroon and Foundation Member of the Australian Mathematical Olympiad Committee, solves the problems of the 1988, 1989 and 1990 International Mathematical Olympiads. Unlike other books in which only complete solutions are given, John Burns describes the complete thought processes he went through when solving the problems from scratch. Written in an inimitable and sensitive style, this book is a must for a student planning on developing the ability to solve advanced mathematics problems.

101 Problems in Algebra from the Training of the USA IMO Team
Edited by T Andreescu & Z Feng

This book contains one hundred and one highly rated problems used in training and testing the USA International Mathematical Olympiad team. These problems are carefully graded, ranging from quite accessible towards quite challenging. The problems have been well developed and are highly recommended to any student aspiring to participate at National or International Mathematical Olympiads.

Hungary Israel Mathematics Competition
S Gueron

This 181 page book summarizes the first 12 years of the competition (1990 to 2001) and includes the problems and complete solutions. The book is directed at mathematics lovers, problem solving enthusiasts and students who wish to improve their competition skills. No special or advanced knowledge is required beyond that of the typical IMO contestant and the book includes a glossary explaining the terms and theorems which are not standard that have been used in the book.

Bulgarian Mathematics Competition 1992-2001
BJ Lazarov, JB Tabov, PJ Taylor, AM Storozhev

The Bulgarian Mathematics Competition has become one of the most difficult and interesting competitions in the world. It is unique in structure, combining mathematics and informatics problems in a multi-choice format. This book covers the first ten years of the competition complete with answers and solutions. Students of average ability and with an interest in the subject should be able to access this book and find a challenge.

Mathematical Contests – Australian Scene
Edited by AM Storozhev, JB Henry & DC Hunt

These books provide an annual record of the Australian Mathematical Olympiad Committee's identification, testing and selection procedures for the Australian team at each International Mathematical Olympiad. The books consist of the questions, solutions, results and statistics for: Australian Intermediate Mathematics Olympiad (formerly AMOC Intermediate Olympiad), AMOC Senior Mathematics Contest, Australian Mathematics Olympiad, Asian-Pacific Mathematics Olympiad, International Mathematical Olympiad, and

Maths Challenge Stage of the Mathematical Challenge for Young Australians.

WFNMC – Mathematics Competitions
Edited by Warren Atkins

This is the journal of the World Federation of National Mathematics Competitions (WFNMC). With two issues each of approximately 80-100 pages per year, it consists of articles on all kinds of mathematics competitions from around the world.

Parabola

This Journal is published in association with the School of Mathematics, University of New South Wales. It includes articles on applied mathematics, mathematical modelling, statistics, and pure mathematics that can contribute to the teaching and learning of mathematics at the senior secondary school level. The Journal's readership consists of mathematics students, teachers and researchers with interests in promoting excellence in senior secondary school mathematics education.

ENRICHMENT STUDENT NOTES

The Enrichment Stage of the Mathematics Challenge for Young Australians (sponsored by the Dept of Education, Science and Training) contains formal course work as part of a structured, in-school program. The Student Notes are supplied to students enrolled in the program along with other materials provided to their teacher. We are making these Notes available as a text book to interested parties for whom the program is not available.

Newton Enrichment Student Notes
JB Henry

Recommended for mathematics students of about Year 5 and 6 as extension material. Topics include polyominoes, arithmetricks, polyhedra, patterns and divisibility.

Dirichlet Enrichment Student Notes
JB Henry

This series has chapters on some problem solving techniques, tessellations, base five arithmetic, pattern seeking, rates and number theory. It is designed for students in Years 6 or 7.

Euler Enrichment Student Notes
MW Evans and JB Henry

Recommended for mathematics students of about Year 7 as extension material. Topics include elementary number theory and geometry, counting, pigeonhole principle.

Gauss Enrichment Student Notes
MW Evans, JB Henry and AM Storozhev

Recommended for mathematics students of about Year 8 as extension material. Topics include Pythagoras theorem, Diaphantine equations, counting, congruences.

Noether Enrichment Student Notes
AM Storozhev

Recommended for mathematics students of about Year 9 as extension material. Topics include number theory, sequences, inequalities, circle geometry.

Pólya Enrichment Student Notes
G Ball, K Hamann and AM Storozhev

Recommended for mathematics students of about Year 10 as extension material. Topics include polynomials, algebra, inequalities and geometry.

T-SHIRTS

T-shirts celebrating the following mathematicians are made of 100% cotton and are designed and printed in Australia. They come in white, and sizes Medium (Polya only) and XL.

Carl Friedrich Gauss T-shirt

The Carl Friedrich Gauss t-shirt celebrates Gauss' discovery of the construction of a 17-gon by straight edge and compass, depicted by a brightly coloured cartoon.

Emmy Noether T-shirt

The Emmy Noether t-shirt shows a schematic representation of her work on algebraic structures in the form of a brightly coloured cartoon.

George Pólya T-shirt

George Pólya was one of the most significant mathematicians of the 20th century, both as a researcher, where he made many significant discoveries, and as a teacher and inspiration to others. This t-shirt features one of Pólya's most famous theorems, the Necklace Theorem, which he discovered while working on mathematical aspects of chemical structure.

Peter Gustav Lejeune Dirichlet T-shirt

Dirichlet formulated the Pigeonhole Principle, often known as Dirichlet's Principle, which states: "If there are p pigeons placed in h holes and p>h then there must be at least one pigeonhole containing at least 2 pigeons." The t-shirt has a bright cartoon representation of this principle.

Alan Mathison Turing T-shirt

The Alan Mathison Turing t-shirt depicts a colourful design representing Turing's computing machines which were the first computers.

ORDERING

All the above publications are available from AMT Publishing and can be purchased on-line at:

www.amt.edu.au/amtpub.html

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The Trust, of which the University of Canberra is Trustee, is a non-profit organisation whose mission is to enable students to achieve their full intellectual potential in mathematics. Its strengths are based upon:

- a network of dedicated mathematicians and teachers who work in a voluntary capacity supporting the activities of the Trust;
- the quality, freshness and variety of its questions in the Australian Mathematics Competition, the Mathematics Challenge for Young Australians, and other Trust contests;
- the production of valued, accessible mathematics materials;
- dedication to the concept of solidarity in education;
- credibility and acceptance by educationalists and the community in general whether locally, nationally or internationally; and
- a close association with the Australian Academy of Science and professional bodies.