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MATHEMATICS COMPETITIONS

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The aims of the Federation are:–

1. ***to promote excellence in, and research associated with, mathematics education through the use of school mathematics competitions;***
2. ***to promote meetings and conferences where persons interested in mathematics contests can exchange and develop ideas for use in their countries;***
3. ***to provide opportunities for the exchanging of information for mathematics education through published material, notably through the Journal of the Federation;***
4. ***to recognize through the WFNMC Awards system persons who have made notable contributions to mathematics education through mathematical challenge around the world;***
5. ***to organize assistance provided by countries with developed systems for competitions for countries attempting to develop competitions;***
6. ***to promote mathematics and to encourage young mathematicians.***

From the Editor

Welcome to *Mathematics Competitions* Vol 17, No 1.

This issue is the last one where I am the editor, the next issue will be under the editorship of Dr Jaroslav Švrček. Jaroslav is in the Department of Algebra and Geometry at the Palacky University of Olomouc in the Czech Republic. He is vice-chair of the Czech Mathematical Olympiad and a member of its problem committee. He is an editor of the Czech journal *Mathematics - Physics - Informatics.* He is one of the founders and organizers of the international mathematical competition DUEL for the students of several high schools in Austria, Poland and Czech Republic.

I have thoroughly enjoyed being involved with this organisation where I have made many long lasting friends from many countries. It has been a pleasure to be associated with so many people who have made such a significant contribution to mathematics enrichment for school students and the wider community.

Again, I would like to thank the Australian Mathematics Trust for its continued support, without which the journal could not be published, and in particular Heather Sommariva and Bernadette Webster for their assistance in the preparation of the journal.

Submission of articles:

The journal *Mathematics Competitions* is interested in receiving articles dealing with mathematics competitions, not only at national and international level, but also at regional and primary school level. There are many readers in different countries interested in these different levels of competitions.

- The journal traditionally contains many different kinds of articles, including reports, analyses of competition problems and the presentation of interesting mathematics arising from competition problems. Potential authors are encouraged to submit articles of all kinds.

- To maintain and improve the quality of the journal and its usefulness to those involved in mathematics competitions, all articles are subject to review and comment by one or more competent referees. The precise criteria used will depend on the type of article, but can be summarised by saying that an accepted article must be correct and appropriate, the content accurate and interesting, and, where the focus is mathematical, the mathematics fresh and well presented. This editorial and refereeing process is designed to help improve those articles which deserve to be published.

At the outset, the most important thing is that if you have anything to contribute on any aspect of mathematics competitions at any level, local, regional or national, we would welcome your contribution.

Articles should be submitted in English, with a black and white photograph and a short profile of the author. Alternatively, the article can be submitted on an IBM PC compatible disk or a Macintosh disk. The preferred format is LaTeX or TeX, but any text file will be helpful.

Articles, and correspondence, can also be forwarded to the editor by mail to

The Editor, *Mathematics Competitions*
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or to

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or by email to the address Jaroslav Svrcek <svrcek@inf.upol.cz>

Warren Atkins,
June 2004

From the President

This edition of the Journal marks the end of an era. Warren Atkins produced the first Newsletter of WFNMC in 1984, before it had become a Journal and this will be Warren's last edition as Editor, marking 20 years of service. What started out as a rather unknown venture has developed into a full refereed Journal with a panel of one Editor and two Associate Editors and it has a very high reputation, not only as a good scholarly journal, but also for the information it has been able to provide mathematicians and teachers in developing countries. As the main form of communication within WFNMC the Journal has done an outstanding service and we have been lucky to have Warren.

In farewelling Warren, I also welcome Jaroslav Švrček, who will take over immediately. His first issue will be dated December 2004. Jaroslav, whose contact details are on page 5, will welcome your contribution. Jaroslav is the Deputy Leader of the Czech Republic IMO team and because of the clash between IMO and ICME this year he will not be able to attend ICME, however Warren will see him in Prague shortly after ICME and hand over the Journal. Members who went to the WFNMC Conference in Melbourne in 2002 will certainly remember Jaroslav from there.

The Journal will continue to be produced from the Australian Mathematics Trust office in Canberra, but with Jaroslav handing over the edited copy for each edition.

This is also the end of an era for Ron Dunkley, with Warren, another original 1984 member of the WFNMC Executive, who also served a significant term as President. Latterly Ron has finished a term as Chairman of the Awards Committee and he has resigned as it is getting more difficult for Dorothy to travel. Unfortunately Ron will not be at ICME so we will not be able to pass on our wishes personally to him but I know we all wish him the best.

I will be standing aside as President as I believe a position like this must be rotated if it is to be a truly international organisation. Alexander

Soifer is preparing changes to the Constitution to limit the term of Presidency to one term of 4 years, and for the Immediate Past President to chair the Awards Committee for four years and I am happy to do this.

I understand that due to the expense of Copenhagen there will not be as many people at ICME as usual, but WFNMC will have some important meetings and good lectures and I look forward to meeting those of you who can attend.

I encourage those of you who can attend tto participate in Discussion Group 16 on the Role of Competitions in Mathematics Education. Also I remind you of ICMI Study 16. The Discussion Document will be released in Copenhagen and I urge you to read it and consider an active role in the Study. This can be achieved by applying to attend the Study Conference, which will be held in Trondheim, Norway, from 28 June to 2 July 2006. Attendance will require an active role and the Discussion Document will be able to lead you to determining such a role. The Discussion Document will be also posted on the WFNMC web site after ICME.

Peter Taylor
June 2004

An ‘Inverse’ Pascal’s Triangle Rule

Vasile Berinde

Vasile Berinde is Professor of Mathematics at North University of Baia Mare and Head of Department of Mathematics and Computer Science. He is a member of the National Commission of the Romanian Mathematical Olympiad. He is the Editor-in-Chief of the journal Creative Mathematics.

At the 9^{th} edition of the International Mathematics Competition for university students (IMC-9, Warsaw, July 19-25, 2002; for more details see the site http://www.imc-math.org/), the author proposed the following problem (it was given in the first day of the competition as Problem 3).

Problem A. *Let n be a positive integer and let*

$$a_k = \frac{1}{\binom{n}{k}}, \quad b_k = 2^{k-n}, \quad \textit{for} \quad k = 1, 2, ..., n.$$

Show that

$$\frac{a_1 - b_1}{1} + \frac{a_2 - b_2}{2} + \ldots + \frac{a_n - b_n}{n} = 0. \tag{1}$$

This problem had been originally published in the Romanian journal *Gazeta Matematica*, no. 8/1985, as Problem O:449 in the frame of the column *Preparatory problems for IMO and BMO*, but its statement was slightly altered by a printing error: $\binom{n}{k-1}$ instead of $\binom{n}{k}$. No solution to this problem was given in *Gazeta Matematica* or elsewhere so far, so the author considered it as a quasi unpublished problem.

When grading the papers of more than 45 students (of the total of 182 participants at IMC-9), we found very interesting and diversified solutions, most of them being different versions of the same main idea.

Therefore, the aim of this paper is to present some of the most interesting solutions which together show that the problem was relatively easy to approach.

Solution 1. (Official IMC-9 Jury Solution). Since

$$k\binom{n}{k} = n\binom{n-1}{k-1}, \quad \text{for all} \quad k \geq 1\ , \tag{2}$$

the equality (1) is equivalent to

$$\frac{2^n}{n}\left[\frac{1}{\binom{n-1}{0}} + \frac{1}{\binom{n-1}{1}} + \cdots + \frac{1}{\binom{n-1}{n-1}}\right] = \frac{2^1}{1} + \frac{2^2}{2} + \cdots + \frac{2^n}{n}. \tag{3}$$

We shall prove (3) by induction. For $n = 1$, both sides are equal to 2. Assume that (3) holds for some n and let x_n denote the left hand-side of (3).

Then

$$x_{n+1} = \frac{2^{n+1}}{n+1} \cdot \sum_{k=0}^{n} \frac{1}{\binom{n}{k}} = \frac{2^n}{n+1} \cdot \left(2\sum_{k=0}^{n} \frac{1}{\binom{n}{k}}\right)$$

$$\frac{2^n}{n+1}\left[\left(\frac{1}{\binom{n}{0}} + \frac{1}{\binom{n}{1}} + \cdots + \frac{1}{\binom{n}{n-1}}\right) + \frac{1}{\binom{n}{n}} + \frac{1}{\binom{n}{0}} + \left(\frac{1}{\binom{n}{1}} + \cdots + \frac{1}{\binom{n}{n}}\right)\right]$$

$$= \frac{2^n}{n+1}\left[2 + \sum_{k=0}^{n-1}\left(\frac{1}{\binom{n}{k}} + \frac{1}{\binom{n}{k+1}}\right)\right] = \frac{2^{n+1}}{n+1} + \frac{2^n}{n+1} \cdot \sum_{k=0}^{n-1} \frac{\frac{n-k}{n} + \frac{k+1}{n}}{\binom{n-1}{k}}$$

$$= \frac{2^n}{n+1}\sum_{k=0}^{n-1}\frac{n+1}{n\binom{n-1}{k}} + \frac{2^{n+1}}{n+1} = \frac{2^n}{n}\sum_{k=0}^{n-1}\frac{1}{\binom{n-1}{k}} + \frac{2^{n+1}}{n+1} = x_n + \frac{2^{n+1}}{n+1}.$$

So, (3) is true for $n+1$ and the problem is completely solved.

Remark 1. The main idea in solving the problem was to double the sum of the inverses of binomial combinations, that is to consider

$$2 \cdot \sum_{k=0}^{n} \frac{1}{\binom{n}{k}}$$

and then to shift one half to the left and the other half to the right, respectively, and then to apply the formula

$$\frac{1}{\binom{n}{k}} + \frac{1}{\binom{n}{k+1}} = \frac{n+1}{n} \cdot \frac{1}{\binom{n-1}{k}}. \tag{4}$$

Formula (4), which involves the inverse of three "successive" combinations: $\binom{n-1}{k}$, $\binom{n}{k}$ and $\binom{n}{k+1}$, is very similar to the well known Pascal's triangle rule that also involves three "successive" combinations:

$$\binom{n}{k+1} = \binom{n-1}{k+1} + \binom{n-1}{k}. \tag{5}$$

This suggests that we call (4) *Inverse Pascal's Rule*. In fact, (4) reduces to (5). Indeed, from (4) we get

$$\binom{n}{k+1} + \binom{n}{k} = \frac{n+1}{n} \cdot \frac{\binom{n}{k}\binom{n}{k+1}}{\binom{n-1}{k}} \Longleftrightarrow \binom{n}{k+1} + \binom{n}{k} = \binom{n+1}{k+1}$$

since

$$\frac{n+1}{n} \cdot \frac{n!}{k!(n-k)!} \cdot \frac{n!}{(k+1)!(n-k-1)!} \cdot \frac{k!(n-k-1)!}{(n-1)!}$$

$$= \frac{(n+1)!}{(k+1)!(n-k)!} = \tbinom{n+1}{k+1}.$$

Remark 2. The author's solution is basically similar to solution 1.

Solution 2. Let us first rewrite (3) as

$$\frac{1}{n}\cdot\sum_{k=0}^{n-1}\frac{1}{\binom{n-1}{k}}=\frac{1}{2^{n-1}}\cdot\sum_{k=0}^{n-1}\frac{2^k}{k+1}\ , \tag{6}$$

and denote by x_n and y_n its left hand side and right hand side, respectively.

Then it is easy to see that

$$2y_{n+1}=\frac{1}{2^{n-1}}\left(\frac{1}{1}+\frac{2^1}{2}+\cdots+\frac{2^{n-1}}{n}+\frac{2^n}{n+1}\right)=y_n+\frac{2}{n+1}.$$

So (y_n) satisfies the following recurrence relation

$$2y_{n+1}=y_n+\frac{2}{n+1},\quad n\geq 1 \tag{7}$$

with $y_1=1$. Having in view that $x_1=1$ as well, it suffices to show that (x_n) satisfies the same recurrence relation (7), which follows immediately by the computations involved in Solution 1:

$$\begin{aligned}2x_{n+1}&=\frac{1}{n+1}\cdot\sum_{k=0}^{n}\frac{1}{\binom{n}{k}}=\frac{1}{n+1}\left[2+\frac{n+1}{n}\sum_{k=0}^{n-1}\frac{1}{\binom{n-1}{k}}\right]\\&=\frac{1}{n}\cdot\sum_{k=0}^{n-1}\frac{1}{\binom{n-1}{k}}+\frac{2}{n+1}.\end{aligned}$$

This completes the solution.

Solution 3. We prove (1) directly by induction. For $n=1$ it reduces to $a_1=b_1$, obviously true. Assume it is valid for a certain n, that is

$$\sum_{k=1}^{n}\frac{a_k-b_k}{k}=0$$

and prove that the previous relation is also true for $n+1$. We have

$$\sum_{k=1}^{n+1} \frac{a_k - b_k}{k}$$

$$= \sum_{k=1}^{n+1} \frac{\frac{1}{\binom{n+1}{k}} - 2^{k-n-1}}{k}$$

$$= \sum_{k=1}^{n} \frac{\frac{n-k+1}{n+1} \cdot \frac{1}{\binom{n}{k}} - \frac{2^{k-n}}{2}}{k} + \frac{1-1}{n+1}$$

$$= \sum_{k=1}^{n} \frac{\left(1 - \frac{k}{n+1}\right) \cdot \frac{1}{\binom{n}{k}} - \frac{2^{k-n}}{2}}{k}$$

$$= \sum_{k=1}^{n} \frac{\frac{1}{\binom{n}{k}} - \frac{2^{k-n}}{2}}{k} - \frac{1}{n+1} \sum_{k=1}^{n} \frac{1}{\binom{n}{k}}$$

$$= \frac{1}{2} \cdot \sum_{k=1}^{n} \left(\frac{1}{\binom{n}{k}} - 2^{k-n}\right) + \frac{1}{2} \cdot \sum_{k=1}^{n} \frac{1}{k\binom{n}{k}} - \frac{1}{n+1} \cdot \sum_{k=1}^{n} \frac{1}{\binom{n}{k}}$$

$$= \frac{1}{2} \cdot 0 + \frac{1}{2} \sum_{k=1}^{n} \frac{1}{k\binom{n}{k}} - \frac{1}{n+1} \cdot \sum_{k=1}^{n} \frac{1}{\binom{n}{k}}.$$

Denote by S_n the result obtained above and show (see also Solution 4) that $S_n = 0$, which completes the solution.

Solution 4. It is in fact a shorter variant of the previous solution. Assume that (1) is true for some n, that is

$$\sum_{k=1}^{n} \frac{\frac{(n-k)!\,k!}{n!} - \frac{1}{2^{n-k}}}{k} = 0 \Longleftrightarrow \sum_{k=1}^{n} \frac{(n-k)\,!\,(k-1)!}{n!} = \sum_{k=1}^{n} \frac{1}{2^{n-k}k}$$

and prove that

$$\sum_{k=1}^{n+1} \frac{(n+1-k)\,!\,(k-1)!}{(n+1)!} = \sum_{k=1}^{n+1} \frac{1}{2^{n+1-k}k}$$

also holds. By cancelling the terms corresponding to the $(n+1)$-th term in each sum, we get

$$\sum_{k=1}^{n} \frac{(n+1-k)\,!\,(k-1)!}{(n+1)!} = \sum_{k=1}^{n} \frac{1}{2^{n+1-k}k}.$$

Since, by the induction hypothesis, we have

$$\sum_{k=1}^{n} \frac{1}{2^{n+1-k}k} = \frac{1}{2} \cdot \sum_{k=1}^{n} \frac{1}{2^{n-k} \cdot k} = \frac{1}{2} \sum_{k=1}^{n} \frac{(n-k)\,!\,(k-1)\,!}{n!}$$

we must prove that

$$\sum_{k=1}^{n} \left[2(n+1-k)\,!\,(k-1)\,! - (n+1)(n-k)\,!\,(k-1)\,!\right] = 0. \qquad (8)$$

It is easy to show that the generic term in (8) can be written as

$$2(n-k+1)\,!\,(k-1)\,! - (n-k+1+k)(n-k)\,!\,(k-1)\,!$$

$$= 2(n-k+1)\,!\,(k-1)\,! - (n-k+1)\,!\,(k-1)\,! - k(n-k)\,!\,(k-1)\,!$$

$$= (n+1-k)\,!\,(k-1)\,! - (n-k)\,!\,k\,!$$

and so (8) is equivalent to

$$\sum_{k=1}^{n}\left[(n+1-k)\,!\,(k-1)\,!-(n-k)\,!\,k\,!\right]=0. \tag{9}$$

To prove (9) it is convenient to split it as follows

$$\sum_{k=1}^{n}(n+1-k)\,!\,(k-1)\,!=\sum_{k=1}^{n}(n-k)\,!\,k\,!$$

$$\Longleftrightarrow n!\,0!+\sum_{k=2}^{n}(n+1-k)\,!\,(k-1)\,!=\sum_{k=1}^{n-1}(n-k)\,!\,k\,!\,+0\,!\,n\,!$$

which reduces to

$$\sum_{k=2}^{n}(n+1-k)\,!\,(k-1)\,!=\sum_{k=1}^{n-1}(n-k)\,!\,k\,!\,,$$

obviously true, since by denoting $l=k-1$, the sum of the left hand side is just $\sum\limits_{l=1}^{n-1}(n-l)\,!\,l\,!$, that is, the sum of the right hand side.

Solution 5. We take successively $k=0,1,...,n-1$ in formula (4)

$$\frac{1}{\binom{n}{k}}+\frac{1}{\binom{n}{k+1}}=\frac{n+1}{n}\cdot\frac{1}{\binom{n-1}{k}}$$

and sum all obtained equalities to obtain

$$1+2\cdot\sum_{k=1}^{n-1}\frac{1}{\binom{n}{k}}+1=\frac{n+1}{n}\cdot\sum_{k=0}^{n-1}\frac{1}{\binom{n-1}{k}}.$$

Since $1 = \frac{1}{\binom{n}{0}}$, the previous relation becomes

$$2 \cdot \sum_{k=0}^{n-1} \frac{1}{\binom{n}{k}} = \frac{n+1}{n} \cdot \left[\sum_{k=0}^{n-2} \frac{1}{\binom{n-1}{k}} + 1 \right]$$

and multiplying its both sides by $\frac{2^n}{n+1}$ we deduce that the sequence (x_n), given by

$$x_n = \frac{2^n}{n} \cdot \sum_{k=0}^{n-1} \frac{1}{\binom{n-1}{k}}, \quad n \geq 1$$

(see Solution 1) which satisfies the following recurrence relation

$$x_{n+1} = x_n + \frac{2^n}{n}, \quad n \geq 1. \tag{10}$$

By summing now (10) for $n := 1, 2, ..., n-1$ we get exactly

$$x_n = \sum_{k=1}^{n-1} \frac{2^k}{k} \qquad \text{i.e.} \qquad \sum_{k=0}^{n-1} \frac{1}{n\binom{n-1}{k}} = \sum_{k=1}^{n-1} \frac{2^{k-n}}{k}$$

which, in view of (2), gives just the required relation (1).

The solution is complete.

Remarks.

1) Despite the fact that this was the simpler idea in solving problem 3, no student has proved completely the equality (8);

2) Moreover, noticing that the terms in (9) have a combinatorial interpretation, that is

$$f_n = \sum_{k=0}^{n-1} k!\,(n-1-k)! \tag{11}$$

does represent the cardinal of the following set of permutations

$$F_n = \{\pi \in S_n \,/\, \pi[1, 2, ..., k] = \{1, 2, ..., 3\}\,,\ \pi(k+1) = k+1,$$

and $\quad \pi[k+2, ..., n] = \{k+2, ..., n\}\,,\ k = 0, 1, ..., n-1\}$, one student tried to exploit very complicated combinatorial arguments in his solution, in order to prove a recurrence relation for the sequence given by (11);

3) Many other solutions were actually more or less different versions of solutions 1 - 4;

4) If we factor the generic term in the sum (8) we find the following re-statement of our problem which is itself a new interesting problem.

Problem A. *Prove the identity*

$$\sum_{k=1}^{n} (n+1-2k)(n-k)\,!\,(k-1)\,! = 0\,.$$

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* * *

A Measure of the Effectiveness of Multiple Choice Tests in the Presence of Guessing : Part 1, Crisp Knowledge

David Clark & Graham Pollard

Dr. David Clark is a Senior Lecturer in Computing at the University of Canberra. He is a committee member for the Australian Mathematics Competition and a member of the Australian Informatics Committee. He is also a committee member of the University of Canberra Maths Day and Deputy Leader the Australian Informatics Olympiad team.

Professor Graham Pollard was Pro Vice-Chancellor, Division of Management and Technology, University of Canberra. He enjoys teaching probability and statistics, and has a particular research interest in the application of probability and statistics to a range of situations including quantum electronics, mathematical education and scoring systems in sport and assessment.

Abstract:

Multiple choice test questions have advantages of ease of administration and prompt feedback. However the effect of guessing can mitigate against their use in assessment. This paper develops a measure of the effectiveness of tests, which allows the test designer to know the consequences of design decisions such as the number of questions in the test, the number of alternatives per question, and whether there are penalties for incorrect responses. An assumption in this paper

is that the students have no partial knowledge in any of the questions.

1 Introduction

Multiple choice questions have long been popular in both academic and recreational contexts. Quizzes are best known for their entertainment value in such popular shows and games as "Sale of the Century™', "Trivial Pursuit™" and "Do You Want to be a Millionaire?™". Whilst this popular use tends to trivialise their educational value, they can be also used to test higher cognitive skills [5]. Their ease of administration and prompt feedback balances the difficulty of constructing questions which go beyond simple knowledge.

One disadvantage of multiple choice tests when they are used for academic assessment is the effect of guessing. Even in the Australian Mathematics Competition, for example, a single lucky guess can make a considerable difference in the percentile points for some students. Further, if there are no penalties for incorrect answers, it is in the students' interests to guess. This random factor in the final score reduces the effectiveness of the test. A further factor is that risk taking can differ with gender and age [1, 2]. Tests have been devised to reduce the tendency to guess [4, 10, 11]. These generally give partial marks for partial knowledge, with adjustments to marks and penalties. Despite this, little has been written about measuring the effectiveness of multiple choice tests, and in particular about how the structure of a multiple choice test impacts on its effectiveness. In this paper we develop a measure of the effectiveness of multiple choice tests. We then use it to answer questions such as the following for multiple choice tests.

- What is the effect of varying the number of questions?
- What is the effect of varying the number of alternatives in each question?
- What should the pass mark be?
- What is the effect of having penalties for incorrect answers?

In this paper we assume "crisp" knowledge. That is, for each question each student either knows the answer or does not know the answer. They are not able to eliminate some of the distracters for questions whose answer they do not know. We discuss the conditions under which this assumption is valid later in the paper. In a subsequent paper this 'crisp knowledge' assumption will be relaxed.

2 The aim of the test

If we do not know the aim of the test we cannot develop a measure of its effectiveness. Multiple choice tests have been used to distinguish between passing and failing students, to give students one of a number of grades, to give students a finer grained mark such as a percentage, and to rank students. In this paper we shall assume that the aim of the test is to distinguish between passing and failing students. In particular, we shall assume that Ovi knows 50% of the questions and should pass, whilst Capri knows 25% of the questions and should fail. These assumptions are made to illustrate the discussion, and the measure we develop can be used for any other assumptions.

3 The measure of effectiveness

Our test should minimise the chance of Ovi failing (type 1 error) and at the same time minimise the chance of Capri passing (type 2 error). Consider a test where there are 16 questions, each with 4 alternatives. Ovi knows the answers to 8 of them, Capri knows the answers to 4. Assuming there is no penalty for incorrect answers, both students will guess. Ovi will guess 8 times, and Capri will guess 12 times. If the pass mark was 10, Ovi would fail (type 1 error) if he guessed correctly fewer than twice, whilst Capri would pass (type 2 error) if he guessed correctly 6 or more times. The table below gives the type 1 and type 2 errors where the pass mark is 8, 9 and 10. The errors are computed using the binomial distribution.

Pass mark	Type 1 error(Ovi fails)	Type 2 error (Capri passes)
8	0%	35%
9	10%	16%
10	38%	5%

Table 1: Type 1 and 2 errors for tests with no penalties, 16 questions and 4 alternatives per question.

From table 1 it can be seen that if the pass mark was 8, Ovi would indeed pass, but that 35% of the time Capri would also pass, whist if the pass mark was 10 the situations would be reversed. If the pass mark was 9, then about 10% of the time Ovi would fail and 16% of the time Capri would pass.

We define the effectiveness E of a multiple choice test as

$E(N, P, X, Y, prob)$ = type 1 error, type 2 error

where N =number of questions in the test

P = pass mark on the test

X = Ovi's knowledge

Y = Capri's knowledge

prob = $1/A$, the probability of guessing correctly on a question

A = number of alternatives on each question

type 1 error = CumBin($P - X - 1, N - X$, prob), the probability of Ovi failing

type 2 error = CumBin(P - X -1, N - X, prob) , the probability of Capri passing

CumBin = cumulative binomial probability

Using the above example $E(16,\ 9,\ 8,\ 4,\ 0.2) = \{0.10,\ 0.16\}$. The effectiveness of a test ranges from {0,0} to {1,1}, with {0,0} being perfect.

The effectiveness is a pair rather than a single number. The authors believe that including both types of errors conveys more information and allows a more informed decision to be made when deciding the structure of a test.

4 Multiple choice tests with no penalties

In this section we explore the trade-off between number of questions and number of alternatives in multiple choice tests with no penalties.

4.1 Pass mark = 50%

We first keep the pass mark at 50%. This will ensure that Ovi passes (type 1 error = 0). Table 2 shows the effect of varying the number of questions per test.

Test size	Pass mark	Ovi's knowledge	Capri's knowledge	Effectiveness
8	4	4	2	{0, 0.47}
12	6	6	3	{0, 0.40}
16	8	8	4	{0, 0.35}
20	10	10	5	{0, 0.31}
24	12	12	6	{0, 0.28}
28	14	14	7	{0, 0.26}
32	18	16	8	{0, 0.23}

Table 2: Effectiveness of tests with no penalties, pass mark = 50% and 4 alternatives per question.

The information in table 2 gives some useful insights. Firstly the effectiveness of the test improves with increasing size. This is hardly surprising, but the table does quantify the effect of increasing the test size. Further, the figures are not very good even for test size of 32. If the aim of the test is to pass Ovi and fail Capri, the test does not fulfil it's aim very well. These figures are too high. More than 20% of students who only know $\frac{1}{4}$ of the answers will pass. And if the test had a more manageable 20 questions, nearly 1 in 3 of those students would pass.

We now turn our attention to tests with differing numbers of alternatives. Table 3 extends table 2, by including 2, 3, 4 and 5 alternatives per question. Again, the pass mark is 50%, so that the type 1 error is 0.

	Number of alternatives per question			
Test size	2	3	4	5
8	{0,0.89}	{0, 0.65}	{0, 0.47}	{0, 0.34}
12	{0, 0.91}	{0, 0.62}	{0,0.40}	{0, 0.26}
16	{0, 0.93}	{0, 0.61}	{0, 0.35}	{0, 0.21}
20	{0, 0.94}	{0, 0.60}	{0,0.31}	{0, 0.16}
24	{0, 0.95}	{0, 0.59}	{0,0.28}	{0, 0.13}
28	{0, 0.96}	{0, 0.58}	{0,0.26}	{0, 0.11}
32	{0, 0.97}	{0, 0.58}	{0, 0.23}	{0, 0.09}

Table 3: Effectiveness of tests with no penalties and pass mark = 50%.

Table 3 shows how the effectiveness varies with the number of alternatives. If the test size is 20, questions with 2 (true / false) alternatives pass almost all students who know 25% of the answers. This figure drops to 16% when the number of alternatives is increased to 5. From table 3 it can be seen that if there are fewer than 4 alternatives, the figures are so poor that the tests are ineffective, even when the number of questions is increased. The table also shows the improvement in increasing the number of alternatives to 5. It is the authors' opinion that these figures are still too high for test sizes of 20 or fewer.

4.2 Pass mark > 50%

Type 2 errors may be decreased by setting the pass mark at more than 50%. However, this comes at the expense of increasing the type 1 errors. Table 4 shows this trade-off. The pass marks used in Table 4 are those which minimise the maximum of type 1 and type 2 errors.

	Number of alternatives per question			
Test size	2	3	4	5
8	6{0.31, 0.33}	5{0.20, 0.32}	5{0.32, 0.17}	4{0.00, 0.34}
12	9{0.34,0.25}	5{0.09, 0.35}	7{0.18, 0.17}	7{0.26, 0.09}
16	11{0.14,0.39}	10{0.20, 0.18}	9{0.10, 0.16}	9{0.17, 0.07}
20	14{0.17, 0.30}	12{0.10, 0.20}	11{0.06, 0.15}	11 {0.11, 0.06}
24	17{0.19, 0.24}	15{0.18, 0.11}	14 {0.03, 0.14}	13 {0.07, 0.05}
28	20 {0.21,0.19}	17{0.11, 0.12}	16 {0.10, 0.06}	15 {0.04, 0.04}
32	23 {0.23,0.15}	19 {0.06, 0.14}	18{0.06, 0.06}	17 {0.03, 0.04}

Table 4: Effectiveness of tests with no penalties, pass mark $> 50\%$. The cells show the pass mark and the effectiveness.

A comparison of table 4 with table 3 shows that type 2 errors can indeed be reduced if type 1 errors are allowed to increase. Whether this is acceptable in the context of a particular assessment item is outside the scope of this paper. What the measure of effectiveness does is to allow the test designer to know the consequences of the design decisions made. For instance, if the test designer decided that both type 1 and type 2 errors were allowed to be up to 10% but no more, then for 4 alternatives per question there would need to be at least 28 questions, and for 5 alternatives per question there would need to be at least 24 questions.

One of the advantages of using this measure of effectiveness is that it requires the test designer to be clear about the aims of the test. In this case, that it should fail students with 25% knowledge and pass students with 50% knowledge. The measure of effectiveness is not limited to these particular design decisions. The formula for effectiveness can be used with other designs. For example, E(16, 13, 12, 8, 0.25) would measure the effectiveness of a test with 16 questions, pass mark 13, Ovi knows 12, Capri knows 8 and there are 4 alternatives per question.

5 Multiple choice tests with penalties

In this section we explore the effectiveness of tests where there are penalties for incorrect answers. We will assume that the penalty is $1/(A-1)$. If, for example, A = 5, a student who got the question wrong would have $\frac{1}{4}$ of a mark deducted from the total. We first leave the pass mark at 50%, and assume that Capri will guess. A confident Ovi has no need to guess, but may choose to do so. We explore both situations.

5.1 Ovi does not guess: pass mark 50%

It is in Capri's interests to guess. But in some cases, Capri's chance of passing may be improved by not guessing all questions. For instance, if there were 12 questions and 4 alternatives, Capri's chance of passing increase from 5% to 7% by guessing 7 questions instead of 9. The figures in table 5 assume optimal strategy on Capri's part. Whilst it is unlikely that a student of Capri's assumed ability would be able to work out the best strategy, the figures do give a bound on each test's effectiveness.

Table 5 is the equivalent of table 3, but with penalties.

	Number of alternatives per question			
Test size	2	3	4	5
8	{0, 0.344}	{0, 0.210}	{0, 0.169}	{0, 0.099}
12	{0, 0.254}	{0, 0.145}	{0, 0.071}	{0, 0.056}
16	{0, 0.194}	{0, 0.077}	{0, 0.054}	{0, 0.020}
20	{0, 0.151}	{0, 0.058}	{0, 0.024}	{0, 0.018}
24	{0, 0.119}	{0, 0.043}	{0, 0.019}	{0, 0.007}
28	{0, 0.095}	{0, 0.024}	{0, 0.006}	{0, 0.004}
32	{0, 0.076}	{0, 0.019}	{0, 0.007}	{0, 0.001}

Table 5: Effectiveness of tests with penalties, pass mark = 50%. It is assumed that Ovi does not guess, and Capri makes the optimal choice of how many questions to guess.

Table 5 shows that when penalties are used in scoring, the effectiveness improves as the number of questions and the number of alternatives increases. Further, a comparison of tables 3 and 5 shows a dramatic improvement in the effectiveness of the tests when penalties are used. These results are shown graphically in figure 1 below.

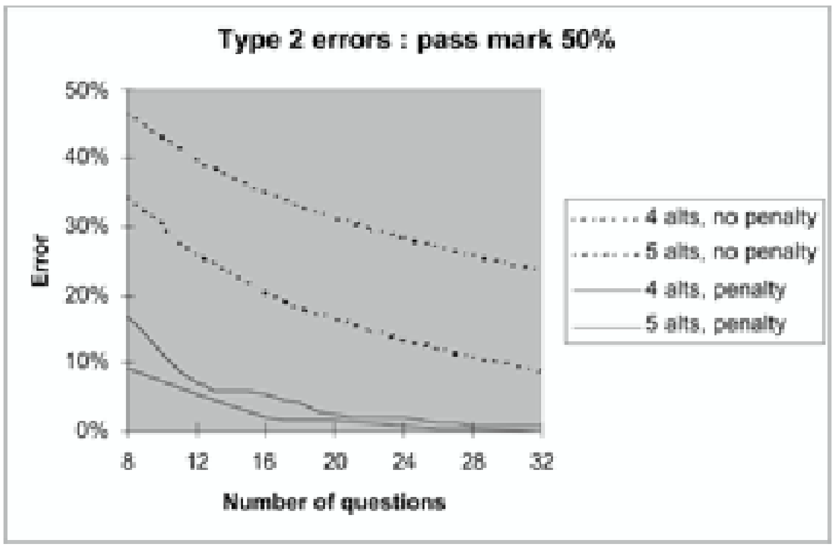

Figure 1: Type 2 errors for tests with pass mark = 50%, 4 and 5 alternatives, without and with penalties.

5.2 Ovi guesses: pass mark 50%

Ovi is assumed to know 50% of the questions. Under the crisp knowledge assumption, Ovi should be confident of his answers. He has worked through the problems and checked that his answer is on the list of alternatives. He has no need to guess. However, it would be a very brave Ovi who could answer half of the questions and omit the remainder. Ovi is likely to guess some of the unanswered questions. With penalties, this means that Ovi's score could drop below 50%, and so type 1 errors would come back in. The next question, then, is how much guessing Ovi does. Guessing all unanswered questions is not Ovi's best strategy. Instead, if Ovi is to guess, he minimises his chance of failing by guessing A questions, and this minimization does not depend on the number of questions in the test. The chance of Ovi failing is then $(1-\frac{1}{A})^A$. The actual figures are given in Table 6. A somewhat unfortunate consequence is that, in contrast with the type 2 errors, the type 1 errors increase with increasing A. (The limit is $e^{-1} = 37\%$.)

Number of alternatives	Number of questions guessed	Type 1 error (Ovi fails)
2	2	25%
3	3	30%
4	4	32%
5	5	33%

Table 6: Type 1 errors for tests with penalties, pass mark = 50%, Ovi guesses A questions.

5.3 Ovi guesses: pass mark 50% − 1

A comparison of tables 3 and 5 shows a dramatic reduction in type 2 errors when penalties are introduced. However if, as is likely, Ovi guesses, a high level of type 1 errors is introduced. These errors can be avoided completely, at the expense of an increase in type 2 errors, by dropping the pass mark by 1. So for 32 questions the pass mark would be 15, for 20 questions the pass mark would be 9, etc. Ovi can now guess $A-1$ of his unanswered questions with complete freedom from failing. Even if all of his guesses are incorrect, his score is only reduced by 1 and he still passes. This comes at the expense of an increase in type 2 errors, as shown in table 7.

		Number of alternatives per question (A)			
Test size (N)	Pass mark	2	3	4	5
8	3	{0, 0.500}	{0, 0.407}	{0, 0.367}	{0, 0.345}
12	5	{0, 0.363}	{0,0.259}	{0, 0.169}	{0, 0.148}
16	7	{0, 0.274}	{0,0.178}	{0, 0.115}	{0, 0.073}
20	9	{0, 0.212}	{0, 0.104}	{0, 0.057}	{0, 0.044}
24	11	{0, 0.166}	{0,0.076}	{0, 0.040}	{0, 0.018}

Table 7: Effectiveness of tests with penalties, with pass mark $= N/2-1$. It is assumed that Ovi guesses $A-1$ questions, and Capri makes the optimal choice of how many to guess.

Whilst the authors do not like strategy playing a part in the outcome of a test, the advice given to Ovi is very simple and does have the effect of reducing the amount of guessing.

6 Multiple choice questions with crisp knowledge

A basic assumption in this paper (to be relaxed in a subsequent paper) is that the students' knowledge is crisp. That is, a student either knows the correct alternative or does not know anything. Although that is clearly not always a valid assumption, it certainly can be in some types of questions. Table 8 shows two simple questions displaying crisp knowledge and non-crisp or partial knowledge.

Question 1	Question 2
What is the capital of Australia? a) Sydney b) Melbourne c) Canberra d) Tokyo	What is $24 - 12/2 * 3$? a) 2 b) 6 c) 9 d) 22

Table 8: Examples of multiple choice questions with partial and crisp knowledge.

Question 1 is a typical multiple choice question where students have partial knowledge, enabling them to eliminate one or more distracters. All Japanese and many Australians would be suspicious of the final alternative. For Question 2, however, there is no partial knowledge. There are just different reasons for getting a wrong answer. Students cannot eliminate any of the alternatives without doing the problem, by which time they (should) know the correct alternative.

Both types of questions have their place, and a test may include crisp questions, partial knowledge questions, or a mixture. An example of a test where the questions are predominately crisp is the Australian Mathematics Competition. There is little scope for eliminating a distracter before working out the answer to the question.

It is useful at this stage to place the questions in a pedagogical framework.

6.1 Bloom's taxonomy

Bloom's [3] well known taxonomy identifies six levels of cognitive skills, namely Knowledge, Comprehension, Application, Analysis, Synthesis and Evaluation. Imrie's RECAP model [9] adapts Bloom's taxonomy by dividing cognitive skills into two tiers. The first tier consists of the first three levels of Bloom's taxonomy namely Knowledge, Comprehension and Application but expresses them as the skills of Recall, Comprehension and Application. Bloom's final three levels of Analysis, Synthesis and Design are combined into a Problem-solving skills category. Imrie's simplified taxonomy can be more useful because of the close integration of all these skills in many computer-based problem-solving activities. Problem-solving often requires more than one skill and this broader classification removes the need for unnecessary distinctions in measuring skill performance. Achieving problem solving skills is the ultimate goal. Whilst they can be taught and assessed at an early stage [5], they are still dependent on the tier 1 skills in a particular subject.

6.2 The "crisp knowledge" assumption and Bloom's taxonomy

Almost all quizzes in popular magazines are at the level of knowledge in Bloom's taxonomy. In university courses, however, the emphasis is on the higher cognitive skills. One of the earliest skills required is comprehension. It is not difficult to construct multiple choice questions which test students' comprehension. Applying a formula in Engineering, tracing code in Computing and reading a diagram in Economics are rich sources of comprehension questions. Although multiple choice questions which test problem solving skills can be constructed [5] it is not easy to do so. Table 9 shows an example of a question testing comprehension and one testing analysis.

Question 3	Question 4

Consider the following code.
`y = 1`
`Do While (y <= x)`
`y = y * 2`
`Loop`
If x is 8, what is the value of y after the code is executed?
 	
(a) 8
(b) 10
(c) 12
(d) 16
	Consider the following code.
	`y = 1`
	`Do While (y <= x)`
	`y = y * 2`
	`Loop`
	Assuming that x is ≥ 0, which of the statements is **false** after the code is executed?
	
	(a) y **may** be odd
	(b) y **may** be equal to x
	(C) y **may** be equal to $x + 1$
	(D) y **must** be a power of 2

Table 9: Examples of multiple choice questions testing comprehension and analysis.

Question 3 is a typical comprehension question. Questions such as these are not difficult to devise, and varying the initial value(s) yields a series of questions. Question 4 is an analysis question. For the code given there may only be one such question.

Comprehension questions such as question 3 above typically require students to work through a problem and then match their answer with one of the alternatives. They cannot apply some knowledge and eliminate one or more distracters. And although it may sometimes be possible to do some elimination by analysing the problem, analysis takes longer and it is simpler and quicker to simply solve the problem. The crisp knowledge assumption is valid for many questions which test comprehension.

7 The role of multiple choice tests in assessment

7.1 Summative and formative assessment

Assessment is a crucial component in any course of instruction. It has been argued [6, 7, 8] that summative assessment (whose main purpose is grading) is the most important factor in determining the final learning outcomes for students. Students adapt their learning strategies to achieve good grades. Lecturers evaluate the success of their courses through the answers students give in examinations.

Summative assessment should be supported and integrated with a continuous regime of formative assessment (whose only purpose is to help learning) throughout a whole course of study. Continuous formative assessment is useful to students because it helps them understand their own learning progression. It gives them feedback on progress. The type of feedback will reinforce their learning behaviour and so it is important to give students feedback that rewards all levels of learning. The authors firmly believe that the use of this assessment must be integrated with summative assessment. There is little value to formative assessment if students know that they are not going to be summatively assessed at these cognitive levels.

Continuous formative assessment is of value to staff because it enables them to see how students are progressing and to adjust course materials and activities to counter difficulties that might arise. For this to be useful the feedback must be timely and meaningful.

7.2 The role of multiple choice questions

If multiple choice tests are to be used summatively, then effectiveness is important. This means that there should be sufficient questions, and at least four and preferably more alternatives. Tests with more than 20 questions are often not practicable, although greater effectiveness can be achieved by combining several tests. For example 4 tests of 16 questions each can be considered as a single 64 question test.

As indicated in Tables 5 and 7, questions with 5 alternatives are considerably more effective than 4 alternative questions. At first glance this seems to require more effort on the part of the question constructor. It can be difficult enough to think of 4 reasonable alternatives. However, for questions with crisp knowledge, not all distracters need be the consequence of an identifiable error or misconception. Some distracters can be random provided they look reasonable. For example, in question 3 above, any of '2', '4', '6' and '14' could be added to the list of alternatives. An advantage of adding alternatives such as these is that they can give a measure of the amount of guessing taking place. So if an alternative that students get from a particular misconception attracts no more responses than the extra alternative, the class does not have a problem with that misconception.

It is the authors' opinion, however, that the main benefit of multiple choice tests is the feedback they can give. The ability to mark automatically gives very fast turn-around and therefore feedback. Automatic marking also means that instructors can rapidly analyse results and keep track of any problems which arise. An implication of this is that there should be a large stock of questions. This is possible for comprehension type questions, but is much more difficult for questions which test the higher cognitive skills.

Multiple choice questions can be incorporated into summative assessment in a number of ways. They can form one component of the test – but there should be several alternatives, and/or penalties for incorrect responses. They can be made part of a question by asking students 'which alternative?' and 'why?' with most of the marks being for the 'why' part. Also they can be used in comprehension/analysis pairs such as questions 3 and 4 above. A good balance is to give students several multiple choice tests formatively which can be used as practice for the summative exams. If students know that the test questions have relevance to the exams it will address the issues raised by Crookes [6] and others about summative assessment driving students' learning strategies.

8 Conclusions

This paper has described a measure of the effectiveness of tests with multiple choice questions and a crisp knowledge assumption. For illustrative purposes, we have assumed that the aim of the test is to distinguish between a student who knows half of the knowledge and should pass, and a student who only knows one quarter of the knowledge and should fail. A representative set of the findings is given below.

- Extra alternatives can easily be added to ‘crisp knowledge’ questions, as they cannot be eliminated without solving the problem.
- Tests with questions with fewer than 4 alternatives have a high level of type 2 and/or type 1 errors.
- For tests with 4 alternatives per question and no penalties, the type 2 errors are high if the pass mark is 50% even with 32 questions in the test. They may be reduced by increasing the pass mark, at the expense of an increase in the type 1 errors.
- For tests with 5 alternatives per question, no penalties and a pass mark of 50%, the type 2 errors are lower than in tests with 4 alternatives per question, but are still high for test sizes of 20 or fewer. They may be reduced by increasing the pass mark, at the expense of an increase in the type 1 errors.
- If a scoring system is used which penalises incorrect answers, both type 1 and type 2 errors may be kept low by reducing the pass mark to 1 fewer question than 50%.

The measure of the effectiveness described in this paper can be used with other assumptions, such as the number of alternatives per question, the amount of knowledge assumed, the pass mark and whether there are penalties in the scoring system. It can also be adapted to measure the effectiveness of a test with other aims, such as in giving equal rank to students with the same knowledge or distinguishing between distinction and non-distinction students. It does require that the designer of the test be explicit about the aim of the test and the assumptions made, which in the authors’ opinion is a desirable precondition of its use.

9 Recommendations

The authors’ intention in this paper has been to give a measure of effectiveness of multiple choice tests in the presence of guessing, rather than to recommend a particular test structure. However, based on the findings described above, the authors are prepared to make some general recommendations. The first is to identify the types of questions in the test. For crisp questions (and comprehension and application questions are typically crisp) aim for as many alternatives as possible. There should be at least four alternatives, and preferably more. Remember that distracters do not have to result from students making a specific mistake or having a particular misconception. Provided they are in range, random answers are equally effective in mitigating against the effect of guessing. If the technology supports numeric answers, use them. This effectively gives a dramatic increase in the number of alternatives. Aim for as many questions as possible. One way of doing this is to combine several tests. Know the purpose of the test, and where possible analyse its structure to measure the effect of guessing. If the effect of guessing is still too high, consider using penalties.

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A Measure of the Effectiveness of Multiple Choice Tests in the Presence of Guessing : Part 2, Partial Knowledge

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Abstract

Tests using multiple choice questions have advantages of ease of administration and prompt feedback. However, the effect of guessing can mitigate against their use in assessment. In a previous paper the authors developed a measure of the effectiveness of such tests, which allows the test designer to know the consequences of design decisions such as the number of questions in the test, the number of alternatives per question, and whether there are penalties for incorrect responses. An assumption in that paper was that the students have no partial knowledge for any of the questions. This paper extends the results of the previous paper to questions where the students may have partial knowledge, and includes a scoring system which rewards partial knowledge with partial marks.

1 Introduction

In our earlier paper [2] we addressed the issue of guessing in multiple choice tests under the 'crisp knowledge' assumption. Under this assumption, students either know the answer to a question or know nothing about the question. They cannot eliminate some of the distracters. The conditions under which this assumption is valid were discussed, and the implications for reducing the effects of guessing were explored.

In this paper we relax the crisp knowledge assumption. Instead, we assume that a student who does not know the answer to a question may have partial knowledge. Partial knowledge in a multiple choice question takes the form of being able to eliminate some but not all of the distracters.

As in the previous paper we will use a measure of effectiveness based on type 1 and type 2 errors to answer questions such as the following:

- What is the effect of varying the number of questions?
- What is the effect of varying the number of alternatives in each question?
- What should the pass mark be?
- What is the effect of having penalties for incorrect answers?

2 The aim of the test

We shall again assume that the aim of the test is to distinguish between passing and failing students. In particular, we shall assume that Ovi knows 50% of the questions and should pass, whilst Capri knows 25% of the questions and should fail. By 'knowing 50% of the questions' we mean 'being awarded 50% of the marks under a partial knowledge scoring system'. This is discussed in greater detail below.

3 The measure of effectiveness

The measure of effectiveness is again the pair $\{e_1, e_2\}$ where e_1 is the probability of Ovi failing (type 1 error) and e_2 is the probability of Capri passing (type 2 error). With crisp knowledge the effectiveness can be expressed as a pair of cumulative binomial probability expressions. With partial knowledge the expressions have to be adapted, but in a fairly straightforward manner.

4 Scoring systems which reward partial knowledge

Scoring systems which reward partial knowledge have been developed by Pollard, et al.[3, 6]. The simplest of these is a 'two-tick' scoring system which allows students to tick either one or two of the alternatives. If a student ticks two alternatives and one of them is correct, he gets $\frac{1}{2}$ mark for that question. Of course, if he just ticks one alternative he gets 1 mark if it is correct.

The two-tick structure and scoring system does not cover all situations. For instance a student may have enough partial knowledge to eliminate only one distracter in a 4 alternative question. Table 8 in [2] has a simple example. More complete structures and scoring systems cater for more fine-grained knowledge. However they are also more complex, and the experience of the authors [3] has been that students are more comfortable with a simpler system. The two-tick scoring system, on the other hand, is easily understood and students like to have the option of double-ticking, even if they only double-tick a minority of questions. In this paper we confine ourselves to a two-tick structure and scoring system.

5 Multiple choice tests with no penalties and a two-tick scoring system

In this section we explore the two-tick scoring system where there are no penalties for incorrect responses. We will only consider questions with 4 or 5 alternatives. It makes no sense to allow students to tick both alternatives when there are only two, and very little to tick two out of three. We will, however, vary the number of questions in the test.

Two factors affect the effectiveness of multiple choice tests using a partial knowledge scoring system in the presence of guessing. The first is how the marks have been earned. We assume that from the questions that have been correctly answered with a single or double tick, Ovi has earned $\frac{1}{2}$ of the test marks and Capri has earned $\frac{1}{4}$ of the test marks. For example, consider an 8 question test where Capri has earned $\frac{1}{4}$ of the marks. He could have done this by single ticking 2 questions, double ticking 2 questions and single ticking one, or by double ticking 4 questions. Earning marks by double ticking means that there are fewer unanswered questions which can be guessed. We explore this in more detail below.

The second factor is how Ovi and Capri will guess their unanswered questions. Will they tick one alternative or two in the unanswered questions? It can be shown that for questions with 4 or more alternatives, it is to their advantage to guess by ticking one alternative rather than two. This is discussed in more detail in Appendix 1.

5.1 Pass mark = 50%

We first keep the pass mark at 50%. This will ensure that Ovi passes (type 1 error = 0). We will vary the number of questions per test and the proportion of Capri's earned marks that are the result of two ticks. This can vary from 0% to 100%. For instance, if the test size is 8 then Capri has earned 2 marks. He could have done this by single ticking 2 questions (0%), double ticking 2 questions and single ticking one (50%), or by double ticking 4 questions (100%). The number of unanswered questions available for guessing would then be 6, 5 and 4 respectively. Tables 1 and 2 show the type 2 errors (Capri passes) for tests whose questions have 4 and 5 alternatives respectively. The empty cells correspond to cases where the number of unanswered cells is not integral.

	Percentage of Capris marks from two ticks				
Test size	0%	25%	50%	75%	100%
8	0.466		0.367		0.262
12	0.399				0.169
16	0.351	0.287	0.224	0.160	0.114
20	0.314				0.078
24	0.283		0.148		0.054
28	0.256				0.038
32	0.234	0.162	0.102	0.057	0.027

Table 1: Type 2 errors for 4 alternative questions, varying the number of questions and the proportion of marks earned by ticking two alternatives.

	Percentage of Capris marks from two ticks				
Test size	0%	25%	50%	75%	100%
8	0.345		0.263		0.181
12	0.262				0.099
16	0.205	0.161	0.121	0.086	0.056
20	0.164				0.033
24	0.133		0.061		0.019
28	0.109				0.012
32	0.089	0.056	0.032	0.016	0.007

Table 2: Type 2 errors for 5 alternative questions, varying the number of questions and the proportion of marks earned by ticking two alternatives.

The reduction in errors as the proportion of marks earned by double ticking increases is expected. (All errors are type 2.) Capri has fewer unanswered questions to guess, so less opportunity to make lucky guesses. Tables 1 and 2 quantify this.

It is the authors' experience with two-tick scoring systems that the weaker students will get 25% to 50% of their marks from double ticking. Stronger students, on the other hand, typically double-tick only one or two questions in a 15–20 question test.

5.2 Pass mark > 50%

Type 2 errors may be decreased by setting the pass mark at more than 50%. However, this comes at the expense of increasing the type 1 errors. Table 3 shows this trade-off by varying the pass mark for 20 question tests with 4 and 5 alternatives. Table 3 gives the effectiveness of the test assuming that Ovi's 10 marks come from 10, 9 and 8 single ticks, and Capri's 5 marks come from 3 and 2 single ticks. These figures are used because they are most consistent with students' behaviour when a two-tick scoring system was used.

		Ovi (10 marks)			Capri (5 marks)	
		Marks from single ticks			Marks from single ticks	
Alternatives	Pass mark	10	9	8	3	2
4	10	0.000	0.000	0.000	0.206	0.158
4	11	0.056	0.075	0.100	0.080	0.054
4	12	0.244	0.300	0.367	0.024	0.014
5	10	0.000	0.000	0.000	0.099	0.073
5	11	0.107	0.134	0.168	0.030	0.019
5	12	0.376	0.436	0.503	0.007	0.004

Table 3: Type 1 (Ovi) and type 2 (Capri) errors for test size 20, with 4 and 5 alternative questions, varying the pass mark and the number of marks from single ticking.

A likely scenario is that Ovi will get 9 of his marks from single ticks, and one from double ticks, while Capri will get 3 of his marks from single ticks and two from double ticks. Under this assumption, the data in table 3 shows that for 20 question tests with 4 alternatives per question, the effectiveness for pass marks of 10, 11 and 12 are {0.000, 0.206}, {0.075, 0.080} and {0.300, 0.024} respectively. The test designer would then have to choose between a 20% type 2 error (pass mark 10) and a 7.5% type 1 error (pass mark 11). Choosing a pass mark of 12 does not seem a reasonable option. If there were 5 alternatives per question, the effectiveness for pass marks of 10, 11 and 12 are {0.000, 0.099}, {0.134, 0.030} and {0.436, 0.007} respectively. The test designer would then have to choose between a 10% type 2 error (pass mark 10) and an 13% type 1 error (pass mark 11). It is not the authors' purpose in this paper to suggest which design to choose, but merely to point out that if the implicit assumptions are made explicit, then the consequences of the design decisions can be calculated.

6 Multiple choice tests with penalties and a two-tick scoring system

In this section we explore the effectiveness of tests with a two-tick scoring system and penalties. If the student gets a single tick wrong, we will assume that the penalty is $1/(A-1)$, where A is the number of alternatives. In earlier papers [3, 6] this penalty was shown to be optimal. If, for example, $A = 5$, a student who got the question wrong would have $\frac{1}{4}$ of a mark deducted from the total. For double-ticking, the earlier papers showed that the optimal scoring system with penalties was $\frac{1}{2}$ mark for correct double tick and $-\frac{1}{2}$ half mark for an incorrect double tick. However, as is demonstrated in Appendix 2, students optimise their chances of passing by guessing by single ticking rather than double ticking, and we will assume this below. We first leave the pass mark at 50%, and then reduce it by 1.

6.1 Pass mark = 50%

If Ovi is confident of his 50% he does not have to guess, but in practice is likely to do so. If he does guess, he must not lose marks in total if he is to pass. That is, he must get at least 1 in every A correct. It can be shown that his chances are maximised by guessing exactly A questions. Capri, on the other hand, must guess. He should guess all or nearly all of his unanswered questions if he is to maximise his chance of passing. (He can sometimes slightly improve his chances by leaving one or two questions not guessed. Consider a Capri who has earned his 5 marks from single ticking 2 questions, and double ticking 6 questions. He has 12 questions unanswered.) If he guesses all of them, he needs to guess at least 7 correctly, for which his chances are 0.39% for 5 alternative questions. If he guesses 10, he only needs to get 6 or more right, and his chances increase to 0.64%.) Table 4 is adapted from Table 3, with a scoring system with penalties. Capri is assumed to have chosen the optimal number of questions to guess.

		Ovi (10 marks)			Capri (5 marks)	
		Marks from single ticks			Marks from single ticks	
Alternatives	Pass mark	10	9	8	3	2
4	10	0.316	0.316	0.316	0.024	0.014
5	10	0.328	0.328	0.328	0.007	0.006

Table 4: Type 1 (Ovi) and type 2 (Capri) errors using a scoring system with penalties, for test size 20, with 4 and 5 alternative questions.
It is assumed Ovi guesses A questions, and Capri makes the optimal choice of how many to guess.

A comparison of Table 4 with the 'pass mark 10' rows of Table 3 shows that when penalties are used in scoring, the type 2 errors reduce dramatically. However, if Ovi guesses, there is a large increase in the type 1 errors.

6.2 Pass mark = 50% − 1

The high level of type 1 errors shown in Table 4 can be reduced to 0, at the expense of a small increase in type 2 errors by dropping the pass mark by 1. Table 5 shows the type 2 errors for a test of size 20, for 4 and 5 alternative questions.

		Capri (5 marks)	
		Marks from single ticks	
Alternatives	Pass mark	3	2
4	9	0.054	0.054
5	9	0.030	0.020

Table 5: Type 2 (Capri) errors using a scoring system with penalties, for test size 20, with 4 and 5 alternative questions. It is assumed Capri makes the optimal choice of how many to guess.

The figures in table 5 show that by reducing the pass mark by 1, the type 1 errors can be eliminated at the expense of increasing the type 2 errors to 5% (4 alternative questions) or 2% or 3% (5 alternative questions).

7 Multiple choice questions with partial knowledge

Consider the following rather delightful question from the 2002 Australian Mathematics Competition [5], where it was Question 21 in both the Junior and Intermediate papers.

$$\begin{array}{rcccc} & \mathrm{P} & \mathrm{Q} & \mathrm{R} & \mathrm{S} \\ \times & & & & 9 \\ \hline & \mathrm{S} & \mathrm{R} & \mathrm{Q} & \mathrm{P} \end{array}$$

In this multiplication, P Q R S is a four digit number, and P, Q, R and S stand for four different digits. Which of the following statements is *not* true?'

a) P Q R S is divisible by 9
b) P = 1
c) Q = 0
d) R = 7
e) S = 9

The first step in solving this problem is to note that S must equal 9 and P must equal 1. This eliminates alternatives e) and b). The next deduction is that Q must be 0, which eliminates alternative c). At this stage it would not be hard to complete the solution, but a student who was short of time or unconfident of going further could opt to tick alternatives a) and d).

In our previous paper [2] we made the strong link between 'crisp' knowledge, where the student could not eliminate distracters, and 'comprehension' in Bloom's [1] taxonomy of cognitive skills. Questions set at the other cognitive levels are predominately partial knowledge questions. Knowledge questions are less relevant at the tertiary level, but multiple choice questions can be set at the comprehension and analysis levels [4]. The example above is a typical analysis question from mathematics. The examples below are taken from an introductory programming course. They illustrate how partial knowledge may be applied to eliminate some of the distracters. The first is an application question.

'The purpose of the following code is to count the number of unique items in an ordered array data. There are n elements in the array.

```
count = ?
start = ?  For i = start to n
    If (data(i) <> data(i-1)) Then
      count = count + 1
    End If
Next i
```

What initial values should be given to count and start in order for the code to execute correctly?'

a) `count = 0 start = 0`

c) `count = 1 start = 0`

b) `count = 0 start = 1`

d) `count = 1 start = 1`

In programming, application consists of writing code. Whilst writing code is not possible in a multiple choice question, it can be simulated by 'complete the code' questions such as the one above. Questions such as these often require the student to make two decisions, resulting in four (or sometimes six) alternatives. Knowing the correct initial value for either count or start would allow two of the alternatives to be eliminated.

In that case the $\frac{1}{2}$ mark given in a two-tick scoring system would be appropriate. Our second example is an analysis question. It also exhibits a natural four alternative structure. The purpose of both algorithms below is to sum the squares of the numbers from 1 to n.

Algorithm 1

```
sum = 0
For i = 1 to n
   sum = sum + i * i
Next i
```

Algorithm 2

```
sum=0
i = 1
While (i <= n)
   sum = sum + i * i
   i = i + 1
Loop
```

Which of the alternatives below is correct?'

a) Neither algorithm is correct.

b) Only algorithm 1 is correct.

c) Only algorithm 2 is correct.

d) Both algorithms are correct.

A student who was able to decide on the correctness of either algorithm would be able to eliminate two alternatives, and again the $\frac{1}{2}$ mark given in a two-tick scoring system would be appropriate. Not all analysis questions will have two sub-questions as does the above example, but it is common that a partial analysis will enable some of the distracters to be eliminated, and a partial knowledge scoring system will be appropriate.

8 Conclusions

This paper has described a measure of the effectiveness of tests with multiple choice questions and a partial knowledge scoring system. We have assumed that the aim of the test is to distinguish between a student who knows half of the knowledge and should pass, and a student who knows only one quarter of the marks and should fail. A representative set of the findings is given below.

- Using a system that rewards partial knowledge reduces the effect of guessing, as there are fewer unanswered questions to guess.

- For 20 question tests with 4 alternatives per question and no penalties, the type 2 errors are quite high if the pass mark is 50%. They may be reduced by increasing the pass mark, at the expense of an increase in the type 1 errors.
- For 20 question tests with 5 alternatives per question and no penalties, the type 2 errors may be considered low enough to leave the pass mark at 50%, thereby avoiding any type 1 errors.
- If a scoring system is used which penalises incorrect answers, both type 1 and type 2 errors may be kept low by reducing the pass mark to 1 fewer than 50%. This applies to both 4 and 5 alternative questions.

The measure of effectiveness described in this paper can be used with other assumptions, such as the number of alternatives per question, the amounts of knowledge assumed and how much of it is partial, the pass mark and whether there are penalties in the scoring system. It can also be adapted to measure the effectiveness of a test with other aims, such as in giving equal rank to students with the same knowledge. It does require that the designer of the test is explicit about the aim of the test and the assumptions made, which in the authors' opinion is a desirable precondition of its use.

9 Recommendations

The authors' intention in these papers has been to give a measure of effectiveness of multiple choice tests in the presence of guessing, rather than to recommend a particular test structure. However, based on the findings described above, the authors are prepared to make some general recommendations. Knowledge and analysis questions are both typically partial knowledge. In simple knowledge questions, it is not difficult to add extra distracters. In questions at the analysis level, however, it can be difficult to generate additional feasible sounding distracters. In this case, allowing students to tick 2 boxes for half marks can accurately reward partial knowledge or incomplete analysis without requiring students to resort to guessing. Tests with fewer than 20 questions can be combined to effectively form a single, larger, test.

Know the purpose of the test, and where possible analyse its structure to measure the effect of guessing. And, if the effect of guessing is still too high, consider using penalties.

10 References

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[4] Cox, Kevin and Clark, David. (1998) The Use of Formative Quizzes for Deep Learning in Programming Courses, *Computers in Education*, 30, 3/4, 157-167.

[5] Pederson, D.G. and Sommariva, H.M. (eds) *AMC Solutions and Statistics*, 2002, AMT Publishing.

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Appendix 1

In this appendix we compare single and double ticking with no penalties. The proof that Capri increases his chance of passing by single ticking is rather cumbersome, and does not give much insight. Instead we use an example to illustrate the point. Consider the situation in which Capri has earned 2 marks from 8 questions by double ticking 4 questions, (each with 4 alternatives and no penalties for incorrect answers), If Capri double ticks the remaining 4 questions, his probability of passing is $\left(\frac{1}{2}\right)^4 = .0625$. Alternatively, if he single ticks 4 questions his probability of passing equals the probability of getting 2 or more of these questions correct, and this is 0.2617. Clearly, it is better to use the single tick option.

Appendix 2

In this appendix we compare single and double ticking with penalties. Again, we use an illustrative example. Suppose there are 16 questions, each with 4 alternatives. Also suppose Capri has correctly answered 8 questions using the double-tick option. If Capri double ticks the 8 remaining questions his probability of passing is $\left(\frac{1}{2}\right)^8 = 0.0039$. Alternatively if he single ticks the 8 remaining questions he passes if he is correct in 5 or more of them. The probability of this is 0.0273. Again, single ticking is Capri's best guessing strategy.

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Concyclic Points and Concurrent Circles

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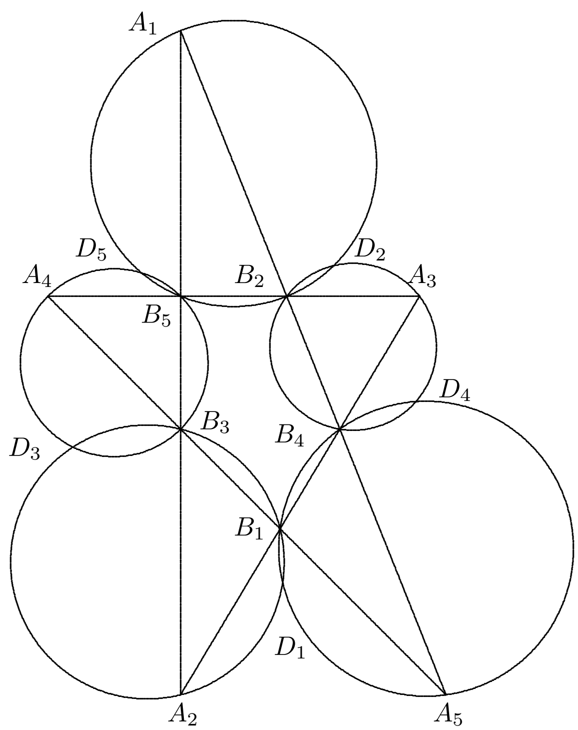
In his millenium visit to Hong Kong and Macau, President JIANG Zemin of China posed the following problem to the local mathematicians. (See J. L. Coolidge, *Treatise on the Circle and the Sphere*, Chelsea (1971) 85–95; J. W. Clawson, A chain of circles associated with the 5-line, *American Mathematical Monthly*, Vol. 61, #3 (1954) 161–166; Kin-Yin

Li, Concyclic problems, *Mathematical Excalibur*, Vol. 6, #1 (2001) 1–2; and Andy Liu, Coffee break problems, *Mathematics & Informatics Quarterly*, Vol. 11, #1 (2001) 25.)

Problem 1.
Let $A_1A_2A_3A_4A_5$ be a star pentagon, with A_1A_2 and A_3A_4 intersecting at B_5, A_2A_3 and A_4A_5 at B_1, A_3A_4 and A_5A_1 at B_2, A_4A_5 and A_1A_2 at B_3, and A_5A_1 and A_2A_3 at B_4. Let the circumcircles of $B_1A_2B_3$ and B_1, B_4A_5 intersect at D_1, those of $B_2A_3B_4$ and $B_2B_5A_1$ at D_2, those of $B_3A_4B_5$ and $B_3B_1A_2$ at D_3, those of $B_4A_5B_1$ and $B_4B_2A_3$ at D_4, and those of $B_5A_1B_2$ and $B_5B_3A_4$ at D_5. Prove that D_1, D_2, D_3, D_4 and D_5 are concyclic.

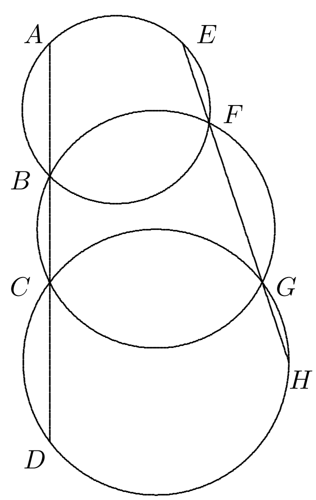


The reader is encouraged to have a go at this problem, the solution of which will be given at the end of the article. We now build towards it by first considering some simpler problems.

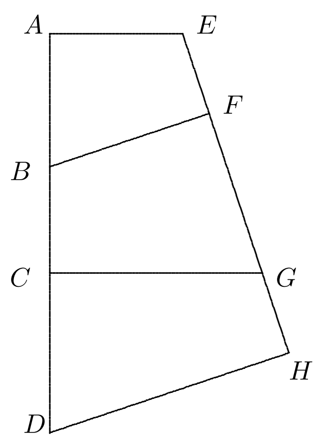
Multiple-choice questions in synthetic geometry are hard to come by. We offer an easy example.

Problem 2.
The circle C_2 intersects the circle C_1 at B and F and the circle C_3 at C and G. The line BC intersects C_1 again at A and C_3 again at D. The line FG intersects C_1 again at E and C_3 again at H. Which of the following quadrilaterals is necessarily cyclic?

(a) $ACGE$ (b) $ADEH$ (c) $BDFH$ (d) none of them

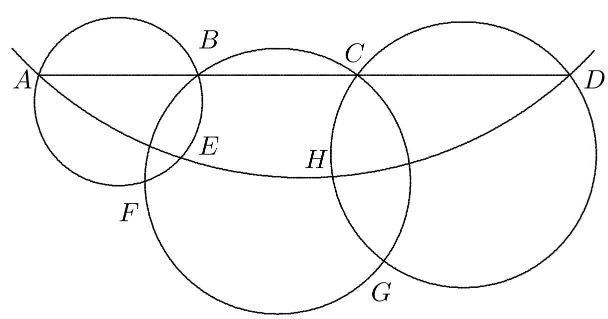


Solution:
We may start with a cyclic quadrilateral $ABFE$ in which $\angle BAE \neq \angle FEA$. Now $\angle BAE = \angle GFB$, and as $BCGF$ is cyclic, $\angle GFB = \angle DCG$. If $ACGE$ is also cyclic, then $\angle DCG = \angle FEA$, which contradicts $\angle BAE \neq \angle FEA$. Hence $ACGE$ is not necessarily cyclic. By symmetry, $BDHF$ is not necessarily cyclic either. On the other hand, $\angle DCG + \angle DHG = 180^\circ$ since $CDHG$ is cyclic. Hence $\angle DAE + \angle DHE = 180^\circ$, and $ADHE$ must be cyclic. The correct choice is (b).



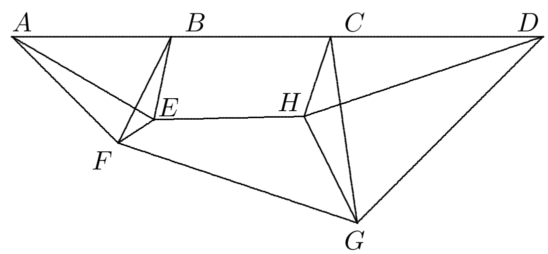
This easy problem is probably at the right level as a multiple-choice question. We now investigate some problems arising from it. If we interchange the hypothesis that E, F, G and H be collinear with the conclusion that $ADHE$ be cyclic, does this partial converse hold? It turns out that this is not quite the case. There is another possibility.

Problem 3.
The circle C_2 intersects the C_1 at B and F and the circle C_3 at C and G. The line BC intersects C_1 again at A and C_3 again at D. The circle C_4 passing through A and D intersects C_1 again at E and C_3 again at H. Prove that E, F, G and H are either collinear or concyclic.

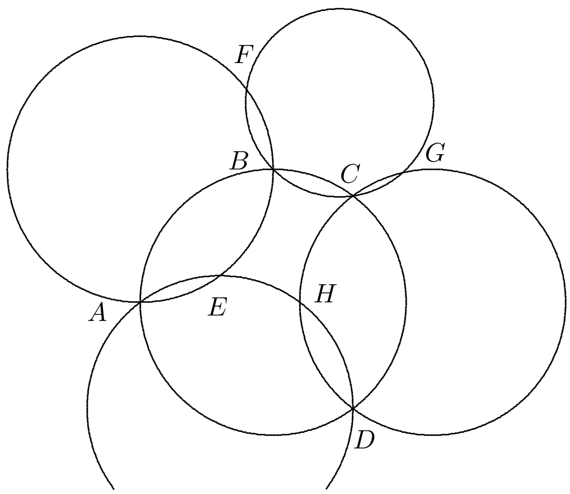


Solution:
It is not hard to prove that if three of E, F, G and H are collinear, then

they are all collinear. We only deal with the case when no three of them are collinear. Let $\angle EAD = \theta$ and $\angle GHD = \phi$. Since $ABEF$ is cyclic, $\angle EFB = \theta$. Since $ADHE$ is cyclic, $\angle EHD = 180° - \theta$. Since $CDGH$ is cyclic, $\angle GCD = \phi$. Since $BCGF$ is cyclic, $\angle BFG = \phi$. Now $\angle EFG = \phi - \theta$. On the other hand, $\angle EHG = 360° - (180° - \theta) - \phi = 180° + \theta - \phi$. It follows that $EFGH$ is indeed cyclic. Analogous arguments apply if the relative positions of the points are varied.



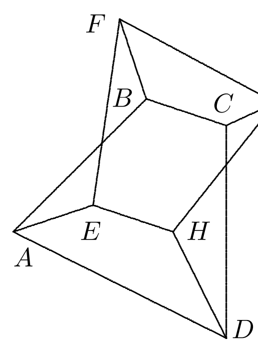
Problem 4.
The circle C_2 intersects the C_1 at B and F and the circle C_3 at C and G. A circle through B and C intersects C_1 again at A and C_3 again at D. The circle C_4 passing through A and D intersects C_1 again at E and C_3 again at H. Prove that E, F, G and H are either collinear or concyclic.



Solution:
It is not hard to prove that if three of E, F, G and H are collinear, then they are all collinear. We only deal with the case when no three of them are collinear. Since $BCFG$ is cyclic, $\angle CBF = 180^\circ - \angle CGF$. Since $ABCD$ is cyclic, $\angle ABC = 180^\circ - \angle ADH - \angle CDH$. Since $ADHE$ is cyclic, $\angle AEH = 180^\circ - \angle ADH$. We have $\angle CGH = \angle CDH$ and $\angle ABF = \angle AEF$ since $CGDH$ and $AEBF$ are cyclic. Hence

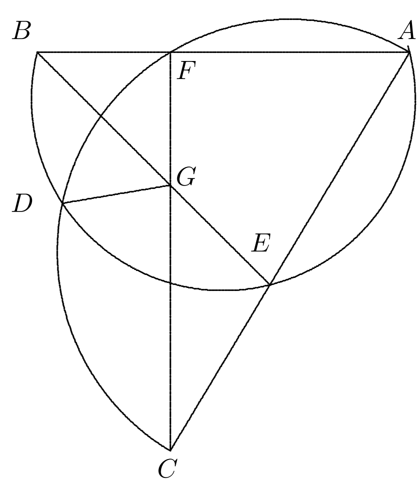
$$\begin{aligned}
\angle FEH &= 360^\circ - \angle AEH - \angle AEF \\
&= (\angle CBF + \angle ABC + \angle ABF) - (180^\circ - \angle ADH) - \angle ABF \\
&= (180^\circ - \angle CGF) + (180^\circ - \angle CDH - \angle ADH) - (180^\circ - \angle ADH) \\
&= 180^\circ - \angle CGF - \angle CGH.
\end{aligned}$$

It follows that $EFGH$ is indeed cyclic. Analogous arguments apply if the relative positions of the points are varied.



We now consider a problem at the intermediate level of difficulty.

Problem 5.
Two arcs AB and AC intersect again at D. The arc AB intersects the segment AC at E while the arc AC intersects the segment AB at F. The segments BE and CF intersect at G. Prove that $BDGF$ and $CDGE$ are cyclic quadrilaterals.

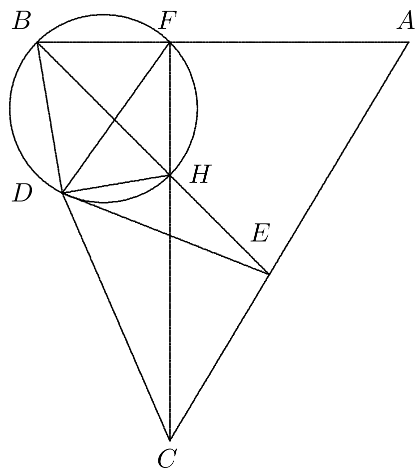


Solution:
Let the segment BE intersect the circumcircle of triangle BDF again at

H. Then $\angle BHD = \angle BFD$. Since $AFDC$ is cyclic, $\angle BFD = \angle ACD$. It follows that $CDHE$ is a cyclic quadrilateral. Since $AEDB$ is also cyclic, $\angle ABD = \angle CED$. It follows that

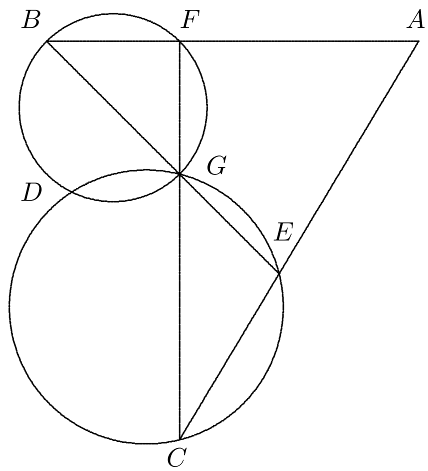
$$\angle BDF = 180^\circ - \angle FBD - \angle BFD = 180^\circ - \angle CED - \angle ECD = \angle CDE.$$

Since $BDHF$ and $CDHE$ are cyclic, we have $\angle BHF = \angle BDF = \angle CDE = \angle EHC$. Since B, H and E are collinear, so are C, H and F. It follows that $H = G$, and we have the desired conclusion.

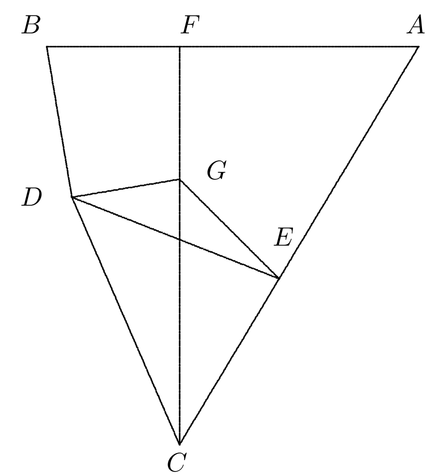


This problem is really saying that the circumcircles of the triangles ABE, ACF, BFG and CEG are concurrent. Given that two of them intersect again at D, it can be proved that the other two also pass through D. Thus there are four different ways of posing this problem. Problem 5 seems to be the only one that seems to require an indirect argument.

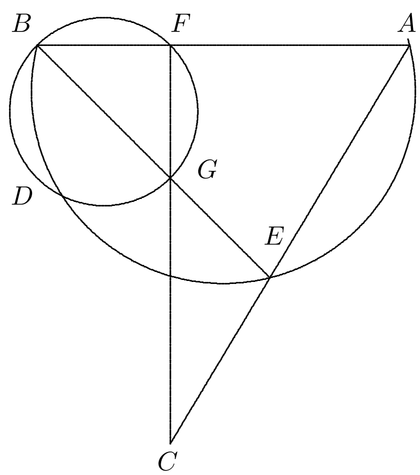
Problem 6.
Let E be a point on the segment AC and F be a point on the segment AB. Let BE and CF intersect at G. Prove that if the circumcircles of triangles BFG and CEG intersect again at D, then the circumcircles of triangles ABE and ACF also pass through D.



Solution:
Since $BDGF$ is cyclic, $\angle FBD = \angle DGC$. Since $CDGE$ is cyclic, $\angle CDG = \angle GEA$ and $\angle GED = \angle GCD$. Hence $\angle DBA + \angle DEA = \angle DGC + \angle GCD + \angle CDG = 180°$, so that $ABDE$ is a cyclic quadrilateral. Similarly, so is $ACDF$.

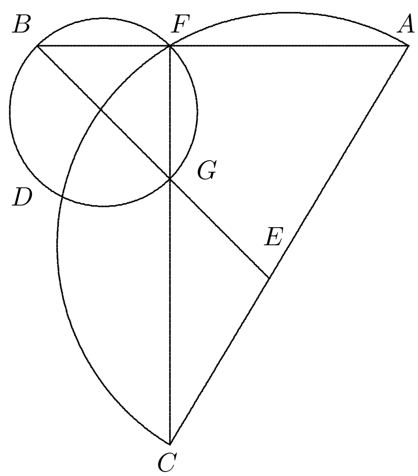


Problem 7.
Let E be a point on the segment AC and F be a point on the segment AB. Let BE and CF intersect at G. Prove that if the circumcircles of triangles BFG and ABE intersect again at D, then the circumcircles of triangles CEG and ACF also pass through D.



Solution:
Since $BDGF$ and $ABDE$ are cyclic, $\angle DGC = \angle DBF = \angle DEC$. Hence $CDGE$ is cyclic. Now $\angle DCE = \angle BGD = \angle BFD$. Hence $ACDF$ is also cyclic.

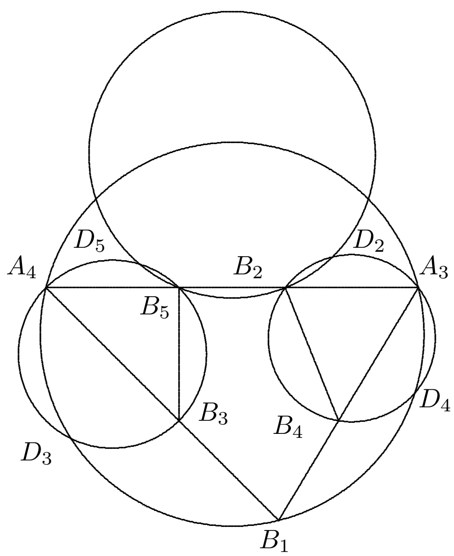
Problem 8.
Let E be a point on the segment AC and F be a point on the segment AB. Let BE and CF intersect at G. Prove that if the circumcircles of triangles BFG and ACF intersect again at D, then the circumcircles of triangles ABE and CEG also pass through D.



Solution:
Since $BDGF$ and $ACDF$ are cyclic, $\angle BGD = \angle BFD = \angle DCE$. Hence $CDGE$ is cyclic. Now $\angle CED = \angle CGD = \angle DBF$. Hence $ABDE$ is also cyclic.

We now give the solution to Problem 1.

Solution:
By symmetry, it is sufficient to prove that D_2, D_3, D_4 and D_5 are concyclic. By Problem 6, D_3 is the other point of intersection of the circumcircles of triangles $A_4B_3B_5$ and $A_4B_1A_3$ while D_4 is the other point of intersection of the circumcircles of triangles $A_3B_2B_4$ and $A_4B_1A_3$. The desired conclusion now follows from Problem 3.



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WFNMC International & National Awards

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Paul Erdös National Award

The Paul Erdös National Award was established to recognise contributions of mathematicians who have played a significant role over a number of years in the development of mathematical challenges at the national level and which have been a stimulus for the enrichment of mathematics learning.

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Past recipients have been: Luis Davidson (Cuba), Nikolay Konstantinov (Russia), John Webb (South Africa), Walter Mientka (USA), Ronald Dunkley (Canada), Peter Taylor (Australia), Sanjmyatav Urjintseren (Mongolia), Qiu Zonghu (China), Jordan Tabov (Bulgaria), George Berzsenyi (USA), Tony Gardiner (UK), Derek Holton (New Zealand), Wolfgang Engel (Germany), Agnis Andžāns (Latvia), Mark Saul (USA), Francisco Bellot Rosado (Spain), János Surányi (Hungary), Istvan

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The general meeting of the WFNMC in Melbourne agreed, from 2003, to merge the above two awards into one award titled the Paul Erdös Award.

Requirements for Nominations for the Paul Erdös Award

The following documents and additional information must be writtten in English:

- A one or two page statement which includes the achievements of the nominee and a description of the contribution by the candidate which reflects the objectives of the WFNMC.
- Candidate's present home and business address and telephone/telefax number.

Nominating Authorities

The aspirant to the Awards may be proposed through the following authorities:

- The President of the World Federation of National Mathematics Competitions.
- Members of the World Federation of National Mathematics Competitions Executive Committee or Regional Representatives.

The Federation encourages the submission of such nominations from Directors or Presidents of Institutes and Organisations, from Chancellors or Presidents of Colleges and Universities, and others.

* * *

Competitions and Concepts

W. Ramasinghe

W Ramasinghe is a Senior Lecturer of the Open University of Sri Lanka. He was a visiting Associate Professor of The Ohio State University, Ohio, USA before coming to the Open University. He has served the University of Colombo as the Head of the Mathematics earlier. He received his Ph.D. in Functional Analysis from the Ohio State University. He is the founding president of the Sri Lanka National Mathematical Olympiad Association.

Introduction

'Exploring the reality - Mathematics Competition' was held at Royal College, Colombo, Sri Lanka last November. The competition had only two problems and the students were given one week to send their solutions by post to the Open University.

Junior Problem

Saman, who is good at Mathematics, is an old boy of Royal College, Colombo. He met Vinitha, an Udarata Menike, for the first time on a Mathematics Day of Royal College several years ago. Their friendship has developed over the years and they have decided to get married with the blessings of Saman's grandfather. Grandfather has assured Saman that Vinitha would make a loving mother. However, he is not happy about the decision taken by the couple, Vinitha and Saman, to have only one baby. The ages of these three people in years are prime numbers. The difference between the ages of Saman and Vinitha is a number among 1, 2, 3, 4, 5, 6, 7, 8 and 9 which is symmetric about a vertical line. The number in the first place (of the base 10 expansion) of the product of their ages is the same as the number of members of their future family. Deletion of the number in the first place of the above product gives the age of the grandfather in years. Grandfather does not intend to live till his age becomes the number in the list of prime numbers at the position of Vinitha's age. Find the ages of Vinitha, Saman and grandfather.

Senior Problem

$Let f : \mathcal{R} \to \mathcal{R}$ be a differentiable function satisfying

$$f(2003) = -\frac{1}{4}, \quad \text{for } f(0) < 1$$

and

$$[f(x)]^3 + 3(x-1)[f(x)]^2 + 2(x^2-4x+1)f(x) - x^6 + 4x^4 + 2x^3 - 7x^2 + 2x = 0$$

for each $x \leq \frac{1}{2}$.

Find the derivatives of f at 0 and -1.

Give an example of such a function f.

Junior Competition

The Junior Competition problem added an attraction to the competition. Although the solution to this problem is routine, there was no complete solution with justifications in my mail not only during that one week period but also after the event. However, I received the correct pairs 23, 31 and 29, 37 with no explanations from two students. Needless to say that this made my day.

Now let us take a look at the solution to this problem:

It is clear that the number of the members of this future family is 3. Also it is clear that the age gap between Saman and Vinitha is 8. The ages of Saman and Vinitha are assumed to be less than 100. Since the ages of this couple are prime numbers and the digit of the first place of the base 10 expansion of the product of their ages is 3, the possibilities for their ages are as follows:-

Vinitha	Saman
3	11
23	31
29	37
53	61
59	67
89	97

It is probably not necessary to state any reasons why the pair 3, 11 can not be their ages. The pair 89, 97 is not possible since grandfather's age can not be 863 years as $87 \times 97 = 8633$. $53 \times 61 = 3233$, $323 = 19 \times 17$, $59 \times 67 = 3953$ and $395 = 5 \times 79$ imply that the pairs 53, 61 and 59, 67 can not be their ages either.

Also, grandfather's age can not be 323 or 395 years. $23 \times 31 = 713$, 71 is prime and $71 < 83$, the 23rd prime implies that 23, 31 is a possible pair of ages for Saman and Vinitha. Similarly, $29 \times 37 = 1073$, 107 is prime and $107 < 109$, the 29th prime implies that 29, 37 is a possible pair of ages for this couple. Although I have heard of several American grandfathers as old as 107 years when I was in Ohio, I have never heard of a Sri Lankan grandfather with that age.

Senior Competition

This problem was given to test whether the students understand the concept of differentiability. There was one written solution to the 3rd part of this problem from a school teacher. The teacher had shown that the quadratic polynomial $x^2 - x$ satisfies our functional equation for $x \leq \frac{1}{2}$. Also, I got a phone call from a university teacher saying the same thing. They had computed the derivatives of $x^2 - x$ at 0 and -1. However, they had not computed the derivatives of any function f satisfying our functional equation for $x \leq \frac{1}{2}$.

J.N. Senadheera is the Divisional Secretary, Kebitigollewa who was one of my students at the Colombo University several years ago. He gave me the solution $x^2 - x$ over the phone similar to that of the two teachers mentioned earlier. Since I had not received a single correct solution by that time, I explained to him that justification of $x^2 - x$ as a possibility for f when $x \leq \frac{1}{2}$ does not lead to $f(-1)$ as asked in the qustion. He received my message and consequently, I received a correct solution in the post from him after several days. It had the same flavour as my solution, though it was bit more complicated.

Now let us look at my solution to the Senior Problem.

$$[f(x)]^3+3(x-1)[f(x)]^2+2(x^2-4x+1)f(x)-x^6+4x^4+2x^3-7x^2+2x = 0$$

for $x \leq \frac{1}{2}$.

By substituting $x = 0$ we obtain

$$[f(0)]^3 - 3[f(0)]^2 + 2f(0) = 0.$$

This leads to $f(0) = 0, 1, 2$.

Since $f(0) < 1$, we conclude that $f(0) = 0$. By differentiating with respect to x we obtain

$$3[f(x)]^2 f'(x) + 6(x - 1)f(x)f'(x) + 3[f(x)]^2 + 2(x^2 - 4x + 1)f'(x) \\ +2(2x - 4)f(x) - 6x^5 + 16x^3 + 6x^2 - 14x + 2 = 0$$

for $x \leq \frac{1}{2}$.

This leads to $f'(0) = -1$. Substitution of $x = -1$ in the original equation leads to $[f(-1) - 2]^3 = 0$. Therefore, $f(-1) = 2$.

Substitition of $x = -1$ and $f(-1) = 2$ in the above equation does not lead to the value of $f'(-1)$. It leads to $0 = 0$ only. Perhaps this is why I received some mail saying that f has no derivative at -1!

Some simplifications (there was time to do it since the competition was one week in duration) yield,

$$[f(x) - 2]^3 + 3[x - (-1)][f(x) - 2]^2 + 2[x - (-1)]^2[f(x) - 2] - x[x - (-1)]^3(x - 1)(x - 2) = 0$$

for $x \leq \frac{1}{2}$.

This implies that

$$\left[\frac{f(x)-f(-1)}{x-(-1)}\right]^3+3\left[\frac{f(x)-f(-1)}{x-(-1)}\right]^2+2\left[\frac{f(x)-f(-1)}{x-(-1)}\right]$$
$$-x(x-1)(x-2)=0$$

for $x<\frac{1}{2}$ and $x\neq -1$.

By taking the limit as x tends to -1 we obtain,

$$[f'(-1)]^3+3[f'(-1)]^2+2f'(-1)+6=0.$$

Hence,

$$f'(-1)+3[f'(-1)]^2+2=0.$$

Thus, $f'(-1)=-3$ or $f'(-1)=\pm\sqrt{2}i$.

Since f is real-valued, we conclude that $f'(-1)=-3$.

In fact all I wanted to test was whether the students could compute $f'(-1)$. I was not interested in $f'(0)$. Besides, I wrote the last part of the problem as this competition was a take home one.

It is easy to show that the only polynomial $p(x)$ of degree less than or equal to 3 satisfying $p(0)=0$, $p'(0)=-1$, $p(-1)=2$ and $p'(-1)=-3$ is $p(x)=x^2-x$. Also, it is not difficult to show that $p(x)=x^2-x$ satisfies the functional equation. Justifying the functions such as f_1 or f_2 given by

$$f_1(x)=\begin{cases} x^2-x & x\leq\frac{1}{2}\\ -\frac{1}{4} & x\geq\frac{1}{2}\end{cases}$$

or

$$f_2(x)=\begin{cases} x^2-x & x\leq\frac{1}{2}\\ -\frac{1}{4} & \frac{1}{2}\leq x\leq 2003\\ \left(x-2003+\frac{1}{2}\right)^2-\left(x-2003+\frac{1}{2}\right) & 2003\leq x\end{cases}$$

To show that this satisfies the requirements of the problem is routine.

Remarks

1. It is easy to show that if

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

for some a_n, then $a_1 = -1$, $a_2 = 1$ and $a_n = 0$ for $n \neq 1$ and $n \neq 2$.

2. The functional equation can be factored as

$$f(x) - x^2 + x[f(x)]^2 + (x^2 + 2x - 3)f(x) + x^4 + x^3 - 3x^2 - 5x + 2$$
$$= 0.$$

This leads to

$$f(x) = x^2 - x \text{ or } f(x) = \frac{-x^2 - 2x + 3 \pm (x+1)\sqrt{-3x^2 + 6x + 1}}{2}.$$

The function $f(x) = \dfrac{-x^2 - 2x + 3 \pm (x+1)\sqrt{-3x^2 + 6x + 1}}{2}$
satisfies the conditions that

$$f'(x) = -x - 1 \pm \frac{-3x^2 + 3x + 2}{-3x^2 + 6x + 1}$$

when $x \neq 1 - \dfrac{2}{\sqrt{3}}$, $f(0) = 2, 1$, $f'(0) = 1, -3$ and

$$f(-1) = \pm\sqrt{2}i.$$

In addition these functions are not real-valued for $x < 1 - \frac{2}{\sqrt{3}}$.
Notice that a function f_3, different from f_1 and f_2 of the form

$$f_3(x) = \begin{cases} x^2 - x & x \leq 1 - \frac{2}{\sqrt{3}} \\ \dfrac{-x^2 - 2x + 3 - (x+1)\sqrt{-3x^2 + 6x + 1}}{2} & 1 - \frac{1}{\sqrt{3}} \leq x \leq \frac{1}{2} \\ h(x) & \frac{1}{2} \leq x \end{cases}$$

for a suitable $h(x)$ could satisfy all the requirements except the differentiability at $1 - \frac{1}{\sqrt{3}}$.

History

This is not the only time the concept of derivative has been used in a competition. The following is the question 8(a) of the G.C.E. (Adv. Level) Examination 1998 (New Syllabus), Pure Mathematics 1.

Suppose that f is a differentiable function on $\mathbf{R}$ satisfying the condition

$$[f(x)]^3 - x[f(x)]^2 - x^2 f(x) - 2x^3 - 7x^4 + 7x^5 = 0$$

at each $x \in \mathbf{R}$. Using the definition of derivative, show that $f'(0) = 2$. Evalute $f'(1)$ [1].

Solution

It is easy to see that $f(0) = 0$. Observe that

$$\left(\frac{f(x)}{x}\right)^3 + \left(\frac{f(x)}{x}\right)^2 + \frac{f(x)}{x} - 2 - 7x + 7x^2 = 0$$

when $x \neq 0$.

Since

$$\frac{f(x) - f(0)}{x - 0} = \frac{f(x)}{x}$$

by taking the limit as x tends to 0 we obtain

$$[f'(0)]^3 - [f'(0)]^2 - f'(0) - 2 = 0.$$

Consequently,

$f'(0) = 2, \dfrac{-1 \pm \sqrt{3}i}{2}.$

Since f is real-valued, we conclude that $f'(0) = 2$.

First, by differentiating the equation with respect to x and then by substituting $x = 0$ does not lead to the value of $f'(0)$ similar to the situation in the Senior Competition at $x = -1$, though this method leads to the value of $f'(1) = 1$. This is the reason that many participants of this A. L. Examination thought that if a function f satisfies the given functional equation then f does not have a derivative at $x = 0$. I remember a visit of a Royal student, a participant of the 1998 A. L. Exam, with his father, who is an engineer, to my office in Colombo University, (then I was the Head of Math), one week after the examination. Both father and son were of the view that such a function f has no derivative at 0! They were saying that when $x = 0$ is substituted in the equation after differentiating, it ends up with $0 = 0$ because there is no derivative at $x = 0$! What a conclusion! I still wonder whether the students read the problem carefully enough to see that they were expected to use the definition of derivative to find $f'(0)$.

Observations

It seems that the students are not sufficiently interested in learning mathematical concepts these days. They like mechanical problems and they work like robots. In fact students generally do not read the problems such as our A.L. problem carefully before solving them. Perhaps, this is happening not only in Sri Lanka but also in many countries around the world. The results in the 'Exploring the reality - Mathematics Competition' seem to be disastrous. This is not a good sign as the actual Sri Lankan suicide rates seem to be high among those of the world community. Whether we like it or not that is the reality.

References

[1] General Certificate of Education (Advanced Level) Examination, August 1998 (New Syllabus), Pure Mathematics I, Department of Examinations, Sri Lanka.

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International Mathematics — Tournament of Towns (1984-1989)

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International Mathematics — Tournament of Towns (1993-1997)
Edited by PJ Taylor

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USSR Mathematical Olympiads 1989 – 1992
Edited by AM Slinko

Arkadii Slinko, now at the University of Auckland, was one of the leading figures of the USSR Mathematical Olympiad Committee during the last years before democratisation. This book brings together the problems and solutions of the last four years of the All-Union Mathematics Olympiads. Not only are the problems and solutions highly expository but the book is worth reading alone for the fascinating history of mathematics competitions to be found in the introduction.

Australian Mathematical Olympiads 1979 – 1995
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Asian Pacific Mathematics Olympiads 1989-2000
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With innovative regulations and procedures, the APMO has become a model for regional competitions around the world where costs and logistics are serious considerations. This 159 page book reports the first twelve years of this competition, including sections on its early history, problems, solutions and statistics.

Polish and Austrian Mathematical Olympiads 1981-1995
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Poland and Austria hold some of the strongest traditions of Mathematical Olympiads in Europe even holding a joint Olympiad of high quality. This book contains some of the best problems from the national Olympiads. All problems have two or more independent solutions, indicating their richness as mathematical problems.

Seeking Solutions
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WFNMC — Mathematics Competitions
Edited by Warren Atkins

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ENRICHMENT STUDENT NOTES

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Dirichlet Enrichment Student Notes
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