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MATHEMATICS COMPETITIONS

JOURNAL OF THE

WORLD FEDERATION OF NATIONAL MATHEMATICS COMPETITIONS

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Please send articles to:

The Editor
Mathematics Competitions
World Federation of National Mathematics Competitions
University of Canberra ACT 2601
Australia

Email:warrena@amt.canberra.edu.au

Fax:+61-2-6201-5052

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North America: Professor George Berzsenyi
13226 Piute Trail
Pine CO 80470-9557
USA

Oceania: Professor Derek Holton
Department of Mathematics
and Statistics
University of Otago
PO Box 56
Dunedin
NEW ZEALAND

South America: Professor Patricia Fauring
Department of Mathematics
Buenos Aires University
Buenos Aires
ARGENTINA

The aims of the Federation are:–

1. ***to promote excellence in, and research associated with, mathematics education through the use of school mathematics competitions;***
2. ***to promote meetings and conferences where persons interested in mathematics contests can exchange and develop ideas for use in their countries;***
3. ***to provide opportunities for the exchanging of information for mathematics education through published material, notably through the Journal of the Federation;***
4. ***to recognize through the WFNMC Awards system persons who have made notable contributions to mathematics education through mathematical challenge around the world;***
5. ***to organize assistance provided by countries with developed systems for competitions for countries attempting to develop competitions;***
6. ***to promote mathematics and to encourage young mathematicians.***

From the Editor

Welcome to *Mathematics Competitions* Vol 15, No 2.

The WFNMC held a highly successful Congress-4 in Melbourne from 4 to 11 August. It was again a pleasure to meet up with friends and colleagues from around the world. The Executive also agreed to strengthen the editorial structure of the journal *Mathematics Competitions* with the appointment of two Associate Editors, Gareth Griffith from Canada and Jaroslav Svrcek from the Czech Republic.

Again, I would like to thank the Australian Mathematics Trust for its continued support, without which the journal could not be published, and in particular Heather Sommariva, Sally Bakker and Richard Bollard for their assistance in the preparation of the journal.

Submission of articles:

The journal *Mathematics Competitions* is interested in receiving articles dealing with mathematics competitions, not only at national and international level, but also at regional and primary school level. There are many readers in different countries interested in these different levels of competitions.

- The journal traditionally contains many different kinds of articles, including reports, analyses of competition problems and the presentation of interesting mathematics arising from competition problems. Potential authors are encouraged to submit articles of all kinds.
- To maintain and improve the quality of the journal and its usefulness to those involved in mathematics competitions, all articles are subject to review and comment by one or more competent referees. The precise criteria used will depend on the type of article, but can be summarised by saying that an article accepted must be correct and appropriate, the content accurate and interesting, and, where the focus is mathematical, the mathematics fresh and well presented. This editorial and refereeing

process is designed to help improve those articles which deserve to be published.

At the outset, the most important thing is that if you have anything to contribute on any aspect of mathematics competitions at any level, local, regional or national, we would welcome your contribution.

Articles should be submitted in English, with a black and white photograph and a short profile of the author. Alternatively, the article can be submitted on an IBM PC compatible disk or a Macintosh disk. The preferred format is LaTeX or TeX, but any text file will be helpful.

Articles, and correspondence, can also be forwarded to the editor by mail to

The Editor, *Mathematics Competitions*
Australian Mathematics Trust
University of Canberra ACT 2601
AUSTRALIA

or by email to the address <warrena@amt.canberra.edu.au> or by fax to the Australian Mathematics Trust office, + 61 2 6201 5052,
(02 6201 5052 within Australia).

Warren Atkins,
December 2002

* * *

From the President

Since last time our major conference in Melbourne has come and gone. As an Australian host we were very excited about the prospects of having many of you in Australia for the very first time. For many others who had been to Australia before it may have been an opportunity to visit the city of Melbourne for the first time.

In respect to the Conference itself, the highlight during the week was the presence of John Conway, von Neumann Professor at Princeton University, who gave a memorable plenary lecture and contributed throughout the week to the various sessions.

In respect to WFNMC business it was a real landmark for us to be able to formally adopt a policy statement in which for the first time we define the scope of what we mean by the word 'competition' and relate it to the enhancement of the mathematics teaching and learning processes.

The Conference also gave us the chance to develop our task force's work on the involvement of teachers. The sessions led by Tony Gardiner made much progress in this area.

With respect to our relations with ICMI, I believe that our relations with the Executive are excellent and I am very happy with the positive lines of communication in both directions.

However, I am very disappointed with what appears to be a downgraded role for competitions discussions at the coming ICME-10 Conference in Copenhagen. In the most recent ICMEs, people involved with competitions have been very enthusiastic supporters. Competitions had been the theme of Topic areas which had been very well attended. The program for the last ICME in Tokyo still shows on the WFNMC web site that this had a very packed program with only a little time available to each speaker. It is very disappointing that at ICME-10, competitions have been relegated to a discussion group in which no oral communications can be made.

Competitions have a broadening role in the mathematics teaching and learning process and it was quite obvious after the considerable interest shown at the recent Melbourne conference that more, rather than less, time could have been well occupied in Copenhagen. I can assure you that Tony Gardiner and I had lobbied very hard for increased availability of time.

Given the situation, and given that I am sure many of us will still be attending ICME-10, we need to resolve how we can find time ourselves. I would very much appreciate your suggestions.

One solution may be for us to organize an extra day or so to convene our own mini-conference in Copenhagen immediately after ICME-10. This might in fact be a good permanent solution as ICMEs do have a very crowded agenda with many attendees and we cannot rely on the time we need being available.

Peter Taylor
December 2002

* * *

The Erdös Awards
Call for Nominations

The Awards Committee of the WFNMC calls for nominations for the Erdös Awards. As described in the formal nomination procedures (see page 84 this issue), nominations should be sent to the chair of the committee to the address below by 1 May 2003 for consideration for 2004, and must include a description of the nominee's achievements together with the names and addresses of (preferably) four persons who can act as referees.

Committee Chair:

R G Dunkley
Centre for Education in Mathematics and Computing
Faculty of Mathematics
University of Waterloo
Waterloo, ON
CANADA N2L G1

The Erdös Awards

Bogoljub Marinkovich, Yugoslavia

Bogoljub Marinkovich has had a lengthy career in mathematics education, has served as teacher, educator of teachers, and curriculum developer. He is currently Counsellor for Mathematics at the Ministry of Education, where he is responsible for the advancement of teaching mathematics in the schools. His work has resulted in significant reforms in the study of mathematics. He initiated, and has for twenty-five years, been Chair of a continuing seminar for advanced training of teachers.

Beginning in 1967, he became involved in competitions in primary and secondary schools. Since then, he has maintained a continued involvement in competitions at all levels, including the International Mathematical Olympiad.

He was founder of Archimedes, the National Mathematics Competition in Serbia, a comprehensive program aimed at identifying bright young students and then training them for potential IMO competitions and for university studies.

As an extension of the activity, in 1998 the Arhimedes organization brought the Tournament of Towns to Serbia.

He has lectured internationally on the training of teachers, is the editor of two popular mathematical journals, and has authored more than six hundred publications.

* * *

Harold Braun Reiter, USA

Harold Reiter (centre) receives his Erdös Award from WFNMC President Peter Taylor (left) and WFNMC Awards Committee Chairman, Ron Dunkley (right). The presentation took place on Saturday 10 August 2002, at the Ibis Hotel, Melbourne, as part of WFNMC's 4th conference.

For thirty years, Harold Reiter has provided competitive academic opportunities for students. Through workshops, conferences and articles, he has spread the good word about mathematics competitions. He has given generously of his time and energy in creating and improving competitions at the local, national and international levels.

A listing of his activities includes the following. At one time or another he has been:

- founder of the Charlotte Mathematics Club
- founder of the Mecklenburg Mathematics Club
- founder of the University of North Carolina at Charlotte Mathematics Contest
- Chair of the North Carolina High School Mathematics Contest

These are local activities. At the national level he has been:

- Chair of the MAA Committee on Local and Regional Competitions
- member of the Board of Advisors for the COMAP Math Modeling Contest
- member of the American Junior High School Mathematics Exam, the American Invitational Mathematics Exam, and the United States Mathematical Olympiad.
- Vice President of the International Tournament of Towns
- member of the Committee for the Canadian Mathematics Competition
- question writer of the Mathematics Foundation Middle School competition

In addition to outstanding committee administrative skills, it is estimated that he has authored some 2,000 problems for competitions at all levels from early junior level to Olympiad level.

For many years he has offered workshops locally, nationally and internationally.

In addition to this devotion to mathematics competitions, he is also an outstanding educator. In recent years, he has been awarded distinguished teaching awards by his university, by the North Carolina Council of Teachers and by the Southeastern Section Mathematics Association.

Wen-Hsien Sun, Taiwan

Executive Director, Chiu Chang Mathematics Foundation, Taipei

Wen-Hsien Sun (centre) receives his Erdös Award from WFNMC President Peter Taylor (left) and WFNMC Awards Committee Chairman, Ron Dunkley (right). The presentation took place on Saturday 10 August 2002, at the Ibis Hotel, Melbourne, as part of WFNMC's 4th conference.

Wen-Hsien Sun completed an undergraduate degree in mathematics education, but did not become a teacher because of unhappiness with an examination-driven culture. Instead, he became a businessman supplying stationery to the schools. In 1978, he created Chiu Chang Mathematics Publishing Company, aimed at making good enrichment materials available to schools. On many occasions, he subsidized publications personally in order to increase their availability.

In 1988, he was instrumental in introducing the IMO to Taiwan and since that time has played a significant role in the Taiwan IMO experience, organizing, training and leading their team, often at his own expense.

In other areas, he has created a bookstore in Beijing, through which Chinese mathematicians have had access to Western publications, has introduced the Tournament of Towns to Taiwan, and has encouraged the enrolment of Taiwan schools in the Australian Mathematics Competition.

He has been a major reason for the enrolment of Taiwan students in elementary and intermediate competitions and has ensured that enrichment materials are available for study. As an offshoot of this activity, selected students are able to attend the Chiu Chang-University of Alberta Summer Camp, learning Mathematics, English and Canadian Culture.

In 2000, he founded the Chiu Chang Mathematics Foundation, which sponsors the exchange program, and also supports local activities and puzzle competitions.

* * *

Numerical Polyhedron Problems[1]

Robert Geretschläger

Robert Geretschläger gained his Dr Phil in Functional Analysis from Karl Franzens University. He has taught at various schools in Austria. He has been actively involved with the Austrian Mathematical Olympiad since 1985 and with the national team since 1990. He is head of the Austrian committees for the Kangaroo competition, the Tournament of Towns and the Mediterranean Mathematics Competition.

1 Introduction

Problems in solid geometry have become more and more rare in competitions over the years, reflecting developments in schools in most countries. Occasionally, we do still see problems in solid geometry, that tend to be either metric in nature (requiring the calculation of some volume, angle or distance) or combinatorial (counting vertices or faces, or involving coloring, for instance).

In this paper I would like to present some slightly different problems involving "numerical" properties of certain polyhedra, i.e. problems asking about relationships between the numbers of vertices, edges and faces of specific polyhedra. Some of these questions will ask for the smallest or largest such numbers possible under certain conditions, and many will ask about the existence of polyhedra with certain properties. For simplicity's sake, we will assume that all problems are restricted to

[1]This paper was a keynote address at the WFNMC Congress 4 in Melbourne in August 2002.

convex polyhedra, although most can be stated just as easily with more general underlying conditions. The problems range in difficulty from very elementary (and accessible to very young students) to research level. While a few of these problems have actually been posed in competitions or journals, most are not too well known.

This paper was in fact inspired by the fact that a few interesting problems of this type had caught my eye in the last few years. Specifically, the following problems were either recent competition problems somewhere, or were posed in the problems section of an international mathematical journal.

Problem 1

One face of a polyhedron is a pentagon. What is the smallest number of faces the polyhedron can have?

A) 5 B) 6 C) 7 D) 8 E) 10

(Kangaroo competition 2002, Junior and Étudiant)

Problem 2

A prism has 2002 vertices. How many edges does the prism have?

A) 3003 B) 1001 C) 2002 D) 4002 E) 2001

(Kangaroo competition 2002, Étudiant)

Problem 3

Suppose we want to construct a solid polyhedron using just n pentagons and some unknown number of hexagons (none of which need be regular), so that exactly three faces meet at every vertex on the polyhedron. For what values of n is this feasible?

(*Crux Mathematicorum with Mathematical Mayhem*, April 2001, Problem H287, [1])

Problem 4

Find all bounded convex polyhedra such that no three faces have the same number of edges.

(*The American Mathematical Monthly*, Feb. 2001, Problem 10856, proposed by Andrei Jorza, [4]) These four problems illustrate the levels of difficulty quite well, since the first two are quite elementary, the third easy enough, but requiring some previous knowledge, and the fourth quite difficult. Their solutions will be given in the appropriate sections to come.

2 Elementary Problems

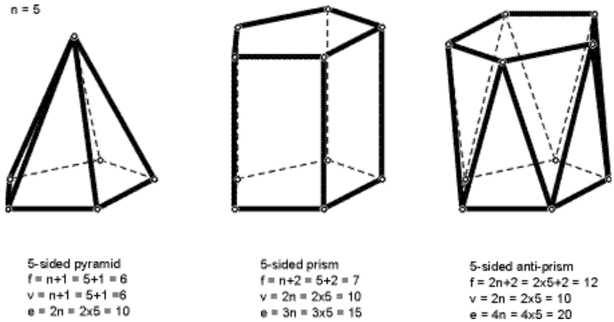
Let us consider the first two of these problems. They represent what is probably the easiest type of polyhedra problem, since they only require knowledge of those types of polyhedra that tend to be best known to students, namely pyramids and prisms.

An n-sided pyramid is, of course, a polyhedron with an n-gon as one face (the base) and n triangles as faces that all have a common vertex (the apex, Figure 1a). An n-sided pyramid has $n+1$ faces, $n+1$ vertices and $2n$ edges.

Similarly, an n-sided prism is a polyhedron with two congruent n-gons as faces (the lower and upper bases) and n parallelograms as faces joining these two (Figure 1b). An n-sided prism has $n+2$ faces, $2n$ vertices and $3n$ edges.

A further useful type of polyhedron in this context is the slightly less well known anti-prism. An n-sided anti-prism is a polyhedron with two congruent n-gons as faces (the lower and upper bases) and $2n$ triangles as faces, each of which has two corners in common with one of the bases and the third in common with the other. (Figure 1c). An n-sided anti-prism has $2n+2$ faces, $2n$ vertices and $4n$ edges.

We also have some additional information at our disposal concerning the faces of these special types of polyhedra. An n-sided pyramid ($n > 3$) has one n-sided face and n triangular faces. An n-sided prism ($n \geq 3$, $n \neq 4$) has two n-sided faces and n 4-sided faces. Finally, an n-sided anti-prism ($n > 3$) has two n-sided faces and $2n$ triangular faces.



a) b) c)

Figure 1

In the special case $n = 3$, the pyramid is a tetrahedron (4 triangular faces) and the anti-prism is an octahedron (8 triangular faces). For $n = 4$ the prism is a parallelepiped (6 four-sided faces).

Armed with this elementary information, we can now take a closer look at the solutions of the first two problems.

Solution to Problem 1: An edge of a polyhedron is always a common side of two of its faces. If one face of a polyhedron is a pentagon, it must have at least one more face sharing each of the five sides of the pentagon. No two of these can be the same, since that would mean that the plane of such a face would pass through at least three of the corners of the pentagon, and must therefore be identical to the plane of the pentagon, contradicting the fact that two faces of a convex polyhedron cannot lie in the same plane. The polyhedron in question must therefore have at least 6 faces. Since we know that a 5-sided pyramid is indeed a polyhedron with 6 faces and a pentagonal face, the answer to the problem is B). qed

We note at this point, that it is not sufficient to know that a polyhedron with the required property must have at least 6 faces. We must demonstrate the existence of a specific polyhedron with exactly 6 faces in order to complete the proof. As we shall see, this aspect of problems of this sort tends to be the more difficult. In this case, our knowledge of pyramids helped us with the existence aspect of the proof.

Solution to Problem 2: The solution to this problem merely requires the information about prisms we recalled earlier. We know that an n-sided prism has $2n$ vertices and $3n$ edges. If a prism has 2002 vertices, we have $n = 1001$, and the prism therefore has $3 \cdot 1001 = 3003$ edges. The answer to the problem is therefore A). qed

Many similar problems can easily be stated simply by changing the numbers in these problems. (In fact the same will hold for many of the following problems.) Another way to find similar, or at least analogous, problems is to exchange "faces" for "edges" or "vertices". In this paper, I will present a number of alternative problems, not always with solutions, but an implicit challenge to the reader is always present to find more similar problems to those stated. An example of such an analogous problem to Problem 1 is the following:

Problem 5

A polyhedron has a 6-sided face. What is the smallest number of edges the polyhedron can have?

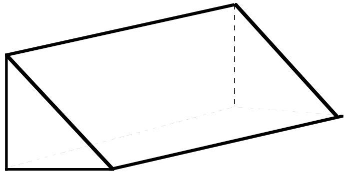
A) 6 B) 7 C) 9 D) 12 E) 18

(The answer is D, of course.)

An interesting, if easy, problem of a similar type was posed as problem 14 of the UK Senior Mathematical Challenge 2002:

Problem 6

Which shape cannot be obtained as the cross-section (in any direction) of this solid, which is a triangular prism with three rectangular faces?



A) triangle B) rectangle C) trapezium D) pentagon
E) hexagon

Solution: The sides of the polygon resulting as a cross-section of the polyhedron must be the lines that the intersecting plane has in common with the planes of the faces of the polyhedron. Since the prism only has five faces, the cross-section cannot have more than five sides. The answer is therefore E). qed

Note that actually finding planes that yield the other four shapes as cross-sections is not necessarily easy. Trying to do so is quite a valuable exercise in spatial reasoning.

A number of problems can be derived by combining the elementary numerical properties of pyramids, prisms and anti-prisms. For instance, we can obtain new polyhedra by "gluing" together two of these elementary building blocks if we choose them such that they have a common k-sided face (Figure 2). The result is a polyhedron, the number of whose vertices, edges and sides can easily be stated. If the parameters for the original two polyhedra are v_1, e_1, f_1 and v_2, e_2, f_2 respectively, the resulting "glued" polyhedron has

$$v = v_1 + v_2 - k, \quad e = e_1 + e_2 - k \quad \text{and} \quad f = f_1 + f_2 - 2.$$

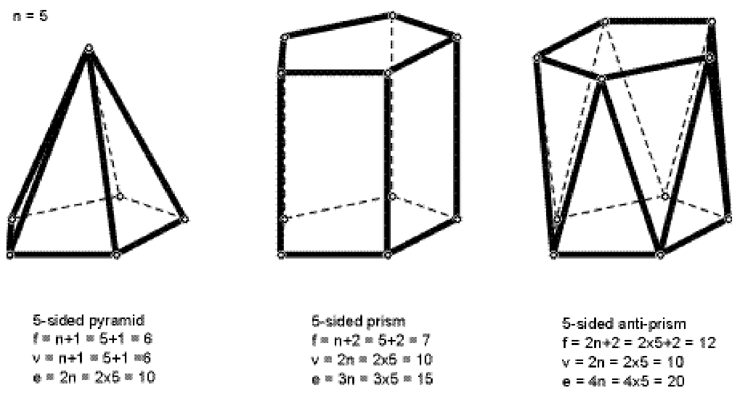


Figure 2

One problem using this idea in the proof is the following:

Problem 7

A polyhedron has two n-sided faces ($n > 3$) and t triangular faces. How many of the following values of t are possible?

$$1, 2, 3, 4, 5, 6, 7, 8$$

A) 2 B) 3 C) 4 D) 5 E) 6

Solution: The answer is B), since t can be equal to 4, 6 or 8, but not 1, 2, 3, 5 or 7. The proof of this is included in the following more general (and more difficult) version of the problem.

Problem 8

A polyhedron has two n-sided faces ($n > 3$) and t triangular faces. How many of the positive integers i with $1 \leq i \leq k$ are possible values for t for any given positive integer k?

Solution: If a polyhedron has a face with at least four sides, it must have at least five faces altogether by the reasoning used in Problem 1. We therefore have $f \geq 5$, and therefore $t = f - 2 \geq 3$.

t can never be an odd number. If we assume that a polyhedron has 2 n-sided faces and t triangular faces, the fact that each edge of the polyhedron is common to two faces means

$$e = \frac{1}{2} \cdot (2n + 3t) = n + \frac{3}{2} \cdot t,$$

but this is not an integer if t is odd.

t can, however, assume any even integer value greater than three. A polyhedron with two 4-sided faces and four triangular faces can be obtained as shown in Figure 3.

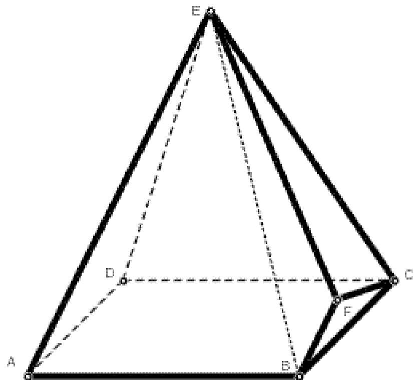


Figure 3

Starting with a regular quadratic pyramid $ABCDE$, a point F is chosen as shown, such that $BF \| AE$. The point F is then certainly in the plane determined by A, B and E, and the polyhedron $ABCDEF$ has two 4-sided faces and four triangular faces as required.

If we are given a convex polyhedron P with t triangular faces ($t > 1$), we can construct a convex polyhedron $\overline{P}$ with $t + 2$ triangular faces in the following manner.

Let A, B and C be the corners of a triangular face of P, and D be the centroid of $\triangle ABC$. Further, let X, Y and Z be the points in which the line perpendicular to the plane of $\triangle ABC$ intersects the planes of the faces of P having edges AB, BC and CA respectively in common with $\triangle ABC$. (We assume that X, Y and Z are all finite points and all are on the opposite side of P with respect to the plane of $\triangle ABC$. If this is not the case for one or more of the points, we can replace the point or points in question by any random point on the appropriate side of the plane of $\triangle ABC$ without it affecting the validity of the construction.) If we chose E as the mid-point of the line segment joining D to the point among X, Y and Z closest to D, E is certainly on the opposite side of P with respect to $\triangle ABC$. Furthermore, P must lie completely on one side of the plane joining A, B and E, since this is true for the planes of both faces of P through AB and E was chosen between D and X, Y

and Z. The same holds for the planes joining B, C and E and C, A and E respectively.

We define a polyhedron $\overline{P}$, whose faces are all identical to those of P with the exception of $\triangle ABC$, which is replaced by the new faces $\triangle ABE$, $\triangle BCE$ and $\triangle CAE$. $\overline{P}$ is then certainly convex, and has $t+2$ triangular faces as required.

We see that all even positive integer values for i greater than 3 are possible values for t, and the answer to the question is therefore

$$0 \quad \text{if} \quad 1 \le k \le 3 \quad \text{and} \quad \left[\frac{k}{2}\right] - 1 \quad \text{if} \quad k > 3.$$

qed

A somewhat surprising result related to this idea is the following. (Thanks go to Ingmar Lehmann for communicating this problem to me.)

Problem 9

We are given a quadratic pyramid and a triangular pyramid. All edges of both pyramids are the same length a. We glue one of the faces of the triangular pyramid completely onto one of the triangular faces of the quadratic pyramid. How many vertices, edges and faces does the resulting "glued" polyhedron have?

Solution: For the triangular pyramid (tetrahedron), we have $v = 4$, $e = 6$ and $f = 4$, and for the quadratic pyramid we have $v = 5$, $e = 8$ and $f = 5$. By the ideas explained above, we would expect the solution to this problem to be

$$v = 4 + 5 - 3 = 6, \quad e = 6 + 8 - 3 = 11 \quad \text{and} \quad f = 4 + 5 - 2.$$

The result for the number of vertices $v = 6$ is indeed correct by the reasoning stated above, but the results for e and f are not.

The somewhat surprising reason for this is the fact that two pairs of triangular faces of the two pyramids end up in the same plane after "gluing", resulting in two rhombic faces of the "glued" polyhedron (see Figure 4).

We can see this by adding a line segment EF to the quadratic pyramid $ABCDE$ (with square face $ABCD$), such that $EF \| AB \| CD$ and $|EF| = |AB| = |CD|$. Since $EF \| AB$, all points A, B, E and F lie in a common plane, and we have

$$\angle EBA = \angle BEF = 60^o.$$

Since $|EF| = |AB| = |EB|$, we see that $\triangle EFB$ is equilateral. The same holds for $\triangle EFC$, and since $\triangle EBC$ must therefore also be equilateral, $EFBC$ is a regular tetrahedron.

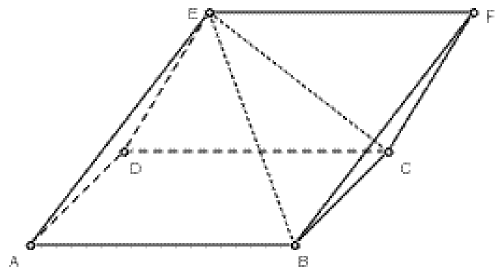


Figure 4

Gluing the tetrahedron $EFBC$ onto the quadratic pyramid $ABCDE$ therefore yields a polyhedron with triangular faces ADE and BCF and 4-sided faces $ABCD$ (square) and $ABFE$ and $DCFE$ (rhombic). The polyhedron therefore, somewhat surprisingly, has 5 faces and 9 edges ($f + v = e + 2 \ \rightarrow \ 5 + 6 = 9 + 2$). qed

We can note that this proof also includes the following result.

Problem 10

We are given a quadratic pyramid with all eight edges of equal length and a regular tetrahedron. Prove that the angle between an edge and a face of the tetrahedron is equal to the dihedral angle of the square pyramid at any of the edges of the square face.

We now come to a slightly different type of problem, the first of which is the following.

Problem 11

A polyhedron P has a 5-sided face and a 4-sided face. These two faces do not have a common edge. What is the smallest number of edges P can have?

Solution: The smallest number of edges is 14. Since the 4- and 5-sided faces do not have a common edge, there must be at least one additional edge through each of the 5 vertices of the pentagonal side. The polyhedron therefore cannot have less than $5 + 4 + 5 = 14$ edges.

That such a polyhedron is possible can be shown in many ways, but two such polyhedra can be derived from a cube or a regular 5-sided prism as shown in Figure 5.

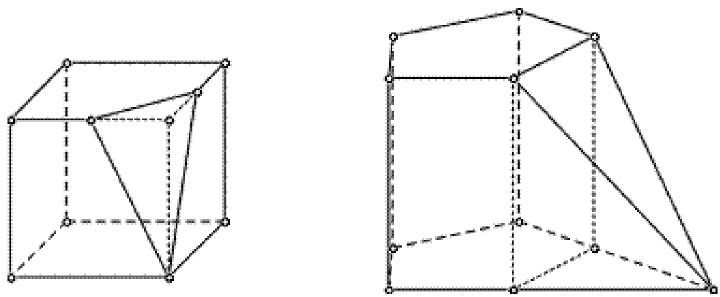


Figure 5

As already mentioned, further problems can be derived from problems of this type by exchanging the variables we ask to minimize. An example is the following:

Problem 12

A polyhedron P has a 5-sided face and a 4-sided face. These two faces do not have a common edge. What is the smallest number of faces P can have?

The answer here is 7. The solution is essentially the same as that for the preceding problem.

A very similar problem is also the following:

Problem 13

A polyhedron P has a 5-sided face and a 6-sided face. What is the smallest number of faces P can have?

Solution: Since one of the faces of P is 6-sided, P must have at least 7 sides. One 7-sided polyhedron with the required property is pictured in Figure 6.

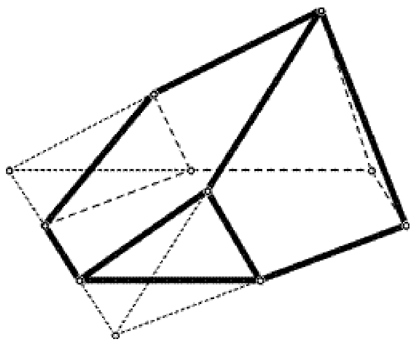


Figure 6

The answer is therefore 7. qed

A useful concept for this type of problem is that of a "k-hedral vertex". Still restricting ourselves to convex polyhedra for simplicity's sake, we say that a vertex V of a polyhedron P is "k-hedral" if exactly k faces of P share a common corner in V. Since any two neighboring faces among these determine one edge of P with an end-point in V, k is also the number of edges of P with V as an end-point.

We can immediately use this concept to formulate "dual" problems to problems involving k-sided faces. The "dual" of any problem results from the original by exchanging the concepts of "face" and "vertex" (and simultaneously "common edge of two faces" and "edge joining two vertices"). The dual problem to Problem 1, for instance, is the following:

Problem 14

One vertex of a polyhedron is pentahedral. What is the smallest number of vertices the polyhedron can have?

The answer to this problem, as to its dual, is again 6. (Note that alternative expressions for "k-hedral" are used for small values of k. We generally use "trihedral" for "3-hedral", "tetrahedral" for "4-hedral", and similarly "pentahedral" and "hexahedral". Note also that the concept of "duality" as expressed here is neither very general nor very precise. It is however sufficient for the purpose of developing problems under the limited conditions we have imposed here.)

Not only does the concept of the k-hedral vertex help us formulate dual problems, we can also combine conditions on k-hedral vertices and n-sided faces to produce new problems. A few examples of this kind are as follows:

Problem 15

Determine the smallest number of edges a polyhedron P can have if it is known to have a 5-sided face and a pentahedral vertex.

Solution: If P has a 5-sided face, there must be at least three edges with end-points in each of the corners of the 5-sided face, two of which can be sides of the pentagonal face. There must therefore be at least one edge with an end-point in each corner of the pentagonal face beside the sides of the face, and P must therefore have at least $5 + 5 = 10$ edges. A 5-sided pyramid does indeed have a 5-sided face and 10 edges, and since its apex is a pentahedral vertex, such a pyramid is indeed a polyhedron with the required properties. 10 is therefore the smallest number of edges of such a polyhedron. qed

Problem 16

Determine the smallest number of edges a polyhedron P can have, if it is known to have a 5-sided face and a tetrahedral vertex.

Solution: Since the tetrahedral vertex can either be one of the corners of the 5-sided face or not, we must consider both of these cases.

If the tetrahedral vertex V_4 is a corner of the 5-sided face, P must have at least 7 vertices, since the 5-sided face has 5 corners and two of the edges with end-points in V_4 must also have end-points which are not corners of the 5-sided face. If V_4 is not a corner of the 5-sided face, P must also have a seventh vertex, since there must be an edge through each of the 5 corners of the 5-sided face which is not a side of that face, and not all 5 of these can have end-points in V_4, since V_4 is tetrahedral. In either case, we see that P must have at least 7 vertices.

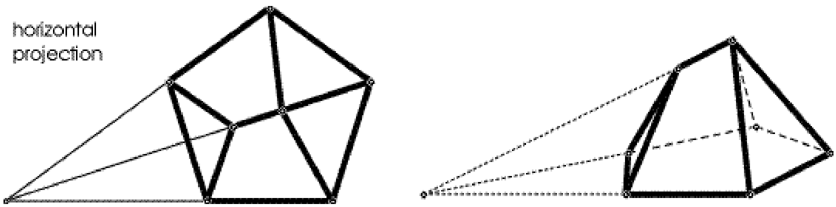


Figure 7

A well-known inequality of convex polyhedra (which I will discuss in more detail in section 3) states that

$$e \geq \frac{3}{2}v$$

must hold. Since $v \geq 7$, we have $e \geq \frac{21}{2}$, and therefore $e \geq 11$. There is indeed a polyhedron with the required properties with $e = 11$ as we see in Figure 7, and it follows that the required number is 11. qed

In this problem, the existence of a polyhedron with the required properties was shown by constructing one in various views. As was stated before, the explicit construction of a polyhedron with required properties is often the most difficult part of solving this type of problem. Perhaps this is a reason that problems like this are not more popular, since constructions of solids are not commonly studied in any depth in most countries.

One way to avoid this, which is however much more advanced from a theoretical standpoint, is to allow Schlegel diagrams in such proofs. Schlegel diagrams are graphs associated with polyhedra. The vertices and edges of the graphs correspond to the vertices and edges of the polyhedra. There is a fairly deep theorem in polyhedron theory stating that any 3-connected graph can be realized as a convex polyhedron and that the graph associated with a polyhedron is always 3-connected. 3-connectedness can be naively described as the property of any two vertices being connected by three distinct paths. It can be shown that this is equivalent to the property that any two vertices of the graph can be removed, along with all edges having either as an end-point, without disturbing the connectedness of the graph. (Figure 8)

Figure 8

It follows that all Schlegel diagrams are 3-connected graphs and vice versa. (For more information, see [3].)

In Figure 7, we can interpret the horizontal projection of P as its Schlegel diagram, since the projections of the two vertices that are not corners of the 5-sided face lie inside the projection of this face.

In this paper, I will mostly show projections of polyhedra to illustrate their existence, since I believe this to be more elementary, but also more fun. While there are many complex results in graph theory relevant to the study of polyhedra, I am attempting to keep them out of this discussion as far as possible. In the next problem, I would however like to give an example of how a graph-theoretical approach can lead to a solution of this type of problem.

Problem 17

Determine the smallest number of edges a polyhedron P can have if it is known to have a 5-sided face and a pentahedral vertex in one of the corners of the 5-sided face.

Solution: As stated, we shall consider the Schlegel diagram for this solution. In Figure 9a we see the 5-sided face as the outer polygon of the diagram. One of the corners A is pentahedral, so there are another 3 edges passing through this corner, apart from the sides of the pentagon. We name the other end-points of these edges F, G and H as shown. P must have at least 8 vertices, and due to the inequality

$$e \geq \frac{3}{2} \cdot v \geq 12,$$

we see that P must have at least 12 edges. In fact, P must have more than 12 edges.

If P had exactly 12 edges, it could not have more than 8 vertices, since this would mean $e \geq \frac{3}{2} \cdot 9 = \frac{27}{2}$. In this case, P would have at least 14 edges. If we assume that P has the 8 vertices shown, each of the vertices (other than A) must be at least trihedral, and since each edge has two end-points, this means $e \geq \frac{1}{2} \cdot (5 + 3 \cdot 7) = 13$. P must therefore have at least 13 edges.

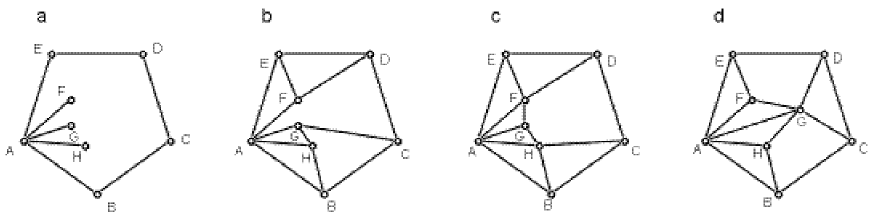


Figure 9

This is not possible either, however. Let us assume that P has 13 edges. We shall try to add the missing 5 edges to the 8 already in Figure 9a. One each must pass through each of the trihedral vertices B, C, D and E. E must be joined with F, since joining E with G or H would leave nothing to join F with. (The edges may not cross each other if they do not have a common vertex in their common point.) Similarly, B must be joined with H. This would leave us with 3 more edges, two of which must originate in C and D, and two of which must pass through G. One of C and D must be joined to G, and one to one of the other two inner points, say F.

The final edge must then join G and H in order for both to be trihedral, but that means that A, F, D, C and G must lie in a common plane, which is not possible if P is to be a polyhedron (Figure 9b). The only way to solve this problem is by adding another edge, which can be done in a number of ways (Figure 9c, 9d), and we see that the smallest number of edges fulfilling all stated requirements is 14. qed

3 Some Properties of Polyhedra

In order to deal with the solutions to more complex problems, it will help to remind ourselves of some important relationships pertaining to numerical properties of polyhedra. In order to keep things simple, we will continue to restrict ourselves to convex polyhedra, although it would be enough here to restrict ourselves to polyhedra with genus 0 (i.e. no “holes”).

For such polyhedra, perhaps the most important numerical relationship is

$$\textbf{EULER}'\textbf{s formula}: \; v + f = e + 2,$$

whereby v denotes the number of vertices of the polyhedron, f the number of faces and e the number of edges.

There are many proofs of this result, for instance in [2] or [5].

There are also a number of slightly less well known relationships between v, f and e that can make interesting problems in their own right, assuming that they are not already well known to students. (They can be found in the literature, for instance in [5]).

Problem 18

Prove that

$$3f \leq 2e \quad \text{and} \quad 3v \leq 2e$$

hold for all convex polyhedra.

Solution: Every face of a polyhedron P has at least 3 sides. Let f_k be the number of faces of P with k sides. Since each edge of P is a common side of two of the faces of P, we have

$$e = \frac{1}{2} \cdot (3f_3 + 4f_4 + 5f_5 + \ldots),$$

and therefore

$$\begin{aligned} 2e &= 3f_3 + 4f_4 + 5f_5 + \ldots \\ &\geq 3f_3 + 3f_4 + 3f_5 + \ldots \\ &= 3f, \end{aligned}$$

proving the first inequality. The other can be proven analogously, substituting the number of edges through each vertex for the number of sides of each face. qed

Problem 19

Prove that

$$e + 6 \leq 3f \quad \text{and} \quad e + 6 \leq 3v$$

hold for all convex polyhedra.

Solution: In the preceding problem, we saw that

$$3v \leq 2e$$

holds. Euler's formula states

$$v = e - f + 2,$$

and substituting for v yields

$$3 \cdot (e - f + 2) \leq 2e,$$

or

$$3e - 3f + 6 \leq 2e,$$

which is equivalent to

$$e + 6 \leq 3f,$$

proving the first inequality. The second is obtained by analogously substituting

$$f = e - v + 2$$

in the inequality

$$3f \leq 2e.$$

qed

Problem 20

Prove

$$v + 4 \leq 2f \leq 4v - 8 \quad \text{and} \quad f + 4 \leq 2v \leq 4f - 8.$$

Solution: By Euler's formula, we have

$$2v + 2f = 2e + 4.$$

Since $3f \leq 2e$ and $3v \leq 2e$ hold, we have

$$2v + 2f \geq 3f + 4 \quad \Rightarrow \quad 2v \geq f + 4$$

and

$$2v + 2f \geq 3v + 4 \quad \Rightarrow \quad 2f \geq v + 4.$$

Multiplying by 2 yields

$$2f \leq 4v - 8 \quad \text{and} \quad 2v \leq 4f - 8.$$

qed

Armed with this knowledge, we can now readily solve Problem 3.

Solution to Problem 3: Let m be the number of hexagonal faces of polyhedron P (recalling that n denotes the number of pentagonal faces). P then has

$$f = m + n$$

faces. Since each edge is shared by two faces, the number of edges of P is

$$e = \frac{6m + 5n}{2},$$

and since each edge joins two vertices and three edges pass through each vertex, the number of vertices of P is

$$v = \frac{2}{3} \cdot e = \frac{6m + 5n}{3}.$$

By Euler's formula, we have

$$v + f = e + 2,$$

and substituting yields

$$\frac{6m + 5n}{3} + (m + n) = \frac{6m + 5n}{2} + 2,$$

which simplifies to

$$n = 12.$$

qed

We can note that the regular dodecahedron and the "soccer ball" (truncated icosahedron) are polyhedra of this type, with $m = 0$ and $m = 20$ respectively.

Polyhedra with "Very Different" Faces

Let us recall that problem 4 in Section 1 asks us to "find all bounded convex polyhedra such that no three faces have the same number of edges". This is, of course, a very interesting problem in itself, and we shall consider its solution in a moment. First, however, we note that this problem suggests a whole category of numerical polyhedra problems, namely problems concerning polyhedra with "very different" faces. We can take this to (loosely) mean polyhedra with as many faces with different numbers of sides as possible. (To clarify the terminology, we shall use the term "edge" when it applies to a bounding line segment of a polyhedron, but "side" when it applies to a bounding line segment of a face of the polyhedron.)

A few such problems are as follows:

Problem 21

Does a polyhedron exist, no two of whose faces have the same number of sides?

Solution: No such polyhedron can exist. In order to see this, we assume that one does exist, and then note that one specific face of the polyhedron must have more sides than any other, since each face has a different number of sides. Let n_{max} be this maximum number. Since some other face of the polyhedron must have an edge in common with the n_{max}-sided face in each of its sides, and no two faces of the polyhedron can have more than one side in common, we see that $f \geq n_{max} + 1$ must hold. On the other hand, the number of sides of each of the faces of the polyhedron must be no less than 3 and no more than n_{max}. Since no two faces have the same number of sides, it follows that $f \leq n_{max} - 2$ holds, and we have a contradiction. qed

This result immediately implies its dual:

Problem 22

Does a polyhedron exist, no two of whose vertices are the end-points of the same number of edges?

(This was problem number 4 in round 38 of the International Mathematical Talent Search. There, it was stated as follows: Prove that every polyhedron has two vertices at which the same number of edges meet.)

Solution: Since the existence of such a polyhedron would imply the existence of its dual, which was proven not to exist in the previous problem, there can be no such polyhedron. Another way to see this is to retrace the steps of the previous solution, replacing the term "face" by "vertex" and f by v. qed

We can note that these proofs also show us that there can be no polyhedra with exactly one pair (or triple) of faces with the same number of sides, and otherwise no such pair of sides. Nor can there exist a polyhedron with exactly two such pairs of faces. (The analogous claims hold for the vertices.) Some further problems suggested by these results are therefore the following:

Problem 23

Does a polyhedron exist with exactly one pair of faces with an equal number of sides?

Problem 24

What is the smallest number m such that there exists a polyhedron with $f = k + m$ faces, k of which can be chosen such that no two of these k have an equal number of sides?

Solution: The contradiction used in Problem 21 will always hold for $m < 3$. We see that $m \geq 3$ must hold, and there are many examples for polyhedra with $m = 3$, such as those shown in Figure 10. qed

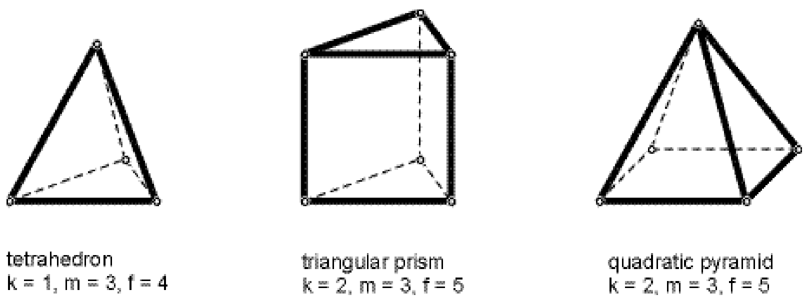


Figure 10

Note that $f \geq 2k$ holds for all these polyhedra. The following question arises:

Problem 25

Let f be the number of faces of a polyhedron and k be the largest number of faces that can be chosen such that no two of the chosen faces have an equal number of sides. Is it always true that $f \geq 2k$ must hold?

The somewhat surprising (for me) answer is no, as we can see by taking a look at the following example (due to Gottfried Perz).

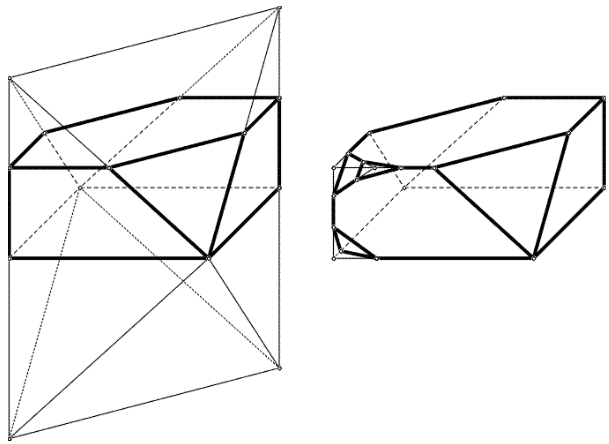


Figure 11

Starting with half of an octahedron, we cut off the top, parallel to the 4-sided face as shown in the left part of Figure 11. The result is a polyhedron with 5 four-sided faces, 2 triangular faces and a hexagonal face. Successively cutting off three vertices as shown in the right part of Figure 11 yields a polyhedron with 11 faces, one each of which has $5, 6, 7$ and 8 sides, 3 of which have 4 sides, and 4 of which have 3 sides. For this polyhedron, we have $f = 11$ and $k = 6$, and since $11 < 2 \cdot 6$, we see that this example contradicts $f \geq 2k$. qed

The next question that naturally arises from this is the following:

Problem 26

Let f be the number of faces of a polyhedron and k be the largest number of faces that can be chosen such that no two of the chosen faces have an equal number of sides. Determine the largest possible lower bound for the parameter $a = \frac{f}{k}$.

Unfortunately, I am not yet aware of a full solution to this problem. An obvious lower bound for a is 1, but as we know from Problem 21, this value is not attainable, since all sides of a polyhedron cannot be a different number. The value of a for the example shown in Figure 11 is $\frac{11}{6}$, and the value of a for the following object (also due to Gottfried Perz) is

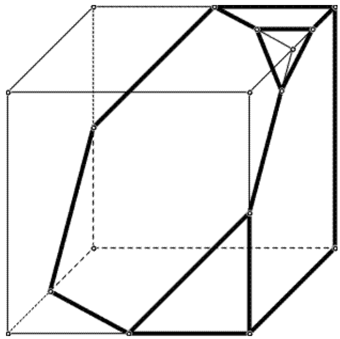


Figure 12

The object is half of a cube (the cube has been cut through its mid-point perpendicular to its diagonal), with a vertex cut off as shown. Such an object has 8 faces, of which three are 3-sided, two are 5-sided, and one each 4-,6- and 7-sided. Perhaps readers of this paper will be inspired to further improve on this lowest value for a, or even to find the largest lower bound for all possible values of a. (Experimenting with such polyhedra seems to indicate that $f \geq 2k - 2 \Leftrightarrow \frac{f}{k} \geq 2 - \frac{2}{k}$ must hold. If this is true, $\frac{8}{5}$ is indeed the smallest attainable value of a, since we can show that $k = 4$ implies $f \geq 7$ and $k = 3$ implies $f \geq 6$.)

We are now ready to turn our attention to Problem 4. A simpler version of the problem is the following:

Problem 27

Find all bounded convex polyhedra with the following property: If n_{max} is the maximum number of sides of any of the faces of the polyhedron, the polyhedron has exactly two faces with $3, 4, \ldots, n_{max}$ sides each.

Solution: To simplify notation, let $x = n_{max} - 2$. Since any polyhedron with the required property has two faces with $3, 4, \ldots, (x+2)$ sides, the total number of faces f must be equal to $2x$. The number e of edges of such a polyhedron must be equal to

$$\begin{aligned} e &= \frac{1}{2} \cdot 2 \cdot (3 + 4 + \ldots + (x+2)) \\ &= \frac{x(x+5)}{2}, \end{aligned}$$

and since each edge is bounded by two vertices and each vertex is an end-point of at least three edges, the number of vertices of the polyhedron must satisfy the condition

$$v \le \frac{2}{3} \cdot e = \frac{x(x+5)}{3}.$$

(Note that the inequality $3v \le 2e$ was part of Problem 18.)

Since all polyhedra with the required property certainly satisfy Euler's formula, we have

$$v = e + 2 - f.$$

This means that

$$v = \frac{x(x+5)}{2} + 2 - 2x \le \frac{x(x+5)}{3}$$

or

$$\begin{aligned} 3x^2 + 15x + 12 - 12x &\le 2x^2 + 10x \\ \Leftrightarrow \quad x^2 - 7x + 12 &\le 0 \\ \Leftrightarrow \quad (x-3)(x-4) &\le 0 \end{aligned}$$

must hold, and this is only possible for $x = 3$ or $x = 4$. In both cases, there do exist polyhedra with the required property (Figure 13), and these are the only two. qed

Note that the "uniqueness" of these polyhedra is meant in a vaguely topological sense, i.e. all kinds of transformations can yield different looking polyhedra, but these will have essentially the same structure with respect to the relative positions of vertices, edges and faces.

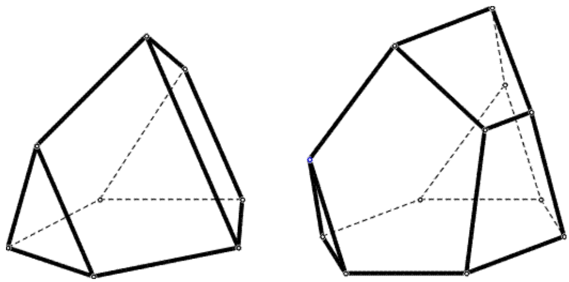


Figure 13

Solution to Problem 4: (The idea for this proof is due to Martin Windischer.) Again we let n_{max} denote the maximum number of sides of any of the faces of the polyhedron and $x = n_{max} - 2$. If f_k denotes the number of faces of P with k sides, we have

$$f = f_3 + f_4 + \ldots + f_{x+2},$$

with $f_i \in \{0, 1, 2\}$ for $3 \leq i \leq x + 1$ and $f_{x+2} \in \{1, 2\}$.

We define the number of "missing faces"

$$t := 2x - f$$

and the number of "missing edges"

$$s := \sum_{i=3}^{x+2} (2 - f_i) \cdot i.$$

For the defining values of P, we then have

$$\begin{aligned} f = &\quad 2x - t \\ e = &\quad \frac{1}{2} \cdot (x(x+5) - s) \\ \text{and} \quad v \leq &\quad \frac{1}{3} \cdot (x(x+5) - s). \end{aligned}$$

By Euler's formula, we therefore have

$$\frac{1}{2} \cdot (x^2 + 5x + 4 - s) = e + 2 = f + v \leq \frac{1}{3} \cdot (x^2 + 5x - s) + 2x - t$$

$$\Leftrightarrow 3x^2 + 15x + 12 - 3s \leq 2x^2 + 10x - 2s + 12x - 6t$$

$$\Leftrightarrow x^2 - 7x + 12 \leq s - 6t.$$

We now note that t and s are not completely independent. If $t = 0$, we must have $s = 0$. If $t = 1$, we have $s \leq x + 2$, since the number of edges of the single "missing face" is at most $x + 2$. Similarly, we have $s \leq (x+2) + (x+1) = 2x+3$ for $t = 2$, $s \leq (x+2) + 2(x+1) = 3x+4$ for $t = 3$, $s \leq (x+2) + 2(x+1) + x = 4x + 4$ for $x = 4$, and so on. In general we have

$$s \leq tx + w(t),$$

where $w(t)$ is defined by $w(0) = 0$, $w(1) = 2$ and $w(t+2) = w(t) + 3 - t$.

In order for it to be possible that

$$x^2 - 7x + 12 \leq s - 6t$$

holds for some value of x, it must be true that

$$x^2 - 7x + 12 \leq s - 6t \leq tx + w(t) - 6t$$

must hold. This is only possible if

$$x^2 - 7x + 12 - tx - w(t) + 6t = 0$$

has a real root, i.e. if its discriminant is non-negative. This is the case if

$$\begin{aligned} & (t+7)^2 - 4 \cdot (12 - w(t) + 6t) \geq 0 \\ \Leftrightarrow \quad & t^2 + 14t + 49 - 48 + 4w(t) - 24t \geq 0 \\ \Leftrightarrow \quad & t^2 - 10t + 1 + 4w(t) \geq 0. \end{aligned}$$

This inequality is certainly correct for $t = 0$ and $t = 1$, since both

$$0^2 - 10 \cdot 0 + 1 + 4 \cdot 0 = 1 \geq 0$$

and

$$1^2 - 10 \cdot 1 + 1 + 4 \cdot 2 = 0 \geq 0$$

hold. We shall now show by induction that the inequality is not correct for any $t \geq 2$.

For $t = 2$, we have

$$2^2 - 10 \cdot 2 + 1 + 4 \cdot 3 = -3 < 0$$

and for $t = 3$

$$3^2 - 10 \cdot 3 + 1 + 4 \cdot 4 = -4 < 0.$$

If we assume that

$$t^2 - 10t + 1 + 4w(t) < 0$$

holds for some t, we have

$$\begin{aligned} 4w(t) &< -t^2 + 10t - 1 \\ \Rightarrow 4w(t) &< -t^2 + 10t - 1 + 4 \\ \Rightarrow 4(w(t) + 3 - t) &< -t^2 + 10t - 1 + 4 + 12 - 4t \\ &= -t^2 - 4t - 4 + 10t + 20 - 1 \\ &= -(t+2)^2 + 10(t+2) - 1, \end{aligned}$$

or

$$(t+2)^2 - 10(t+2) + 1 + 4w(t+2) > 0,$$

and it follows that

$$t^2 - 10t + 1 + 4w(t) < 0$$

holds for all $t \geq 2$. No polyhedron with the required properties can therefore exist with $t \geq 2$. The only possible values for t are therefore 0 or 1. The case $t = 0$ was discussed in problem 27, and it only remains to consider the case $t = 1$.

For $t = 1$ we have $s \leq x + 2$, and therefore

$$\begin{aligned} & x^2 - 7x + 12 \leq s - 6t \leq x + 2 - 6 \\ \Leftrightarrow\quad & x^2 - 8x + 16 \leq 0 \\ \Rightarrow\quad & x \in \{1, 2, \ldots, 9\}. \end{aligned}$$

Since t is equal to 1 and P has a face with $x + 2$ sides, each of which has a common edge with a different face, we have $f \geq x + 3$, and due to $f = 2x - t$, this implies

$$2x - 1 \geq x + 3,$$

or $x \geq 4$. Since

$$s \leq tx + w(t) = x + 3$$

holds and s is even due to $e = \frac{1}{2} \cdot (x(x+5) - s)$ (noting that one of x and $x + 5$ must be even and that e is an integer), we must consider various possibilities. If $x = 4$, we have $s \leq 4 + 2 = 6$ on the one hand, and due to $s \geq x^2 - 7x + 12 + 6t$, $s \geq 16 - 28 + 12 + 6 = 6$ on the other hand. The only possible value for s in this case is therefore $s = 6$.

If $x = 5$, we have $s \leq 5+2 = 7$ on the one hand and $s \geq 25-35+12+6 = 8$ on the other hand, and there can be no possible value for s. Since similar contradictions are obtained for all larger values of x, we see that there is only one possible polyhedron P with $t = 1$. This polyhedron has $x = 4$, and therefore $f = 6$, and since $s = 6$, it must have two 3-sided faces, two 4-sided faces, two 5-sided faces and one 6-sided face. Such a polyhedron is shown in Figure 14. We see that there are only three possible polyhedra with the required property. qed

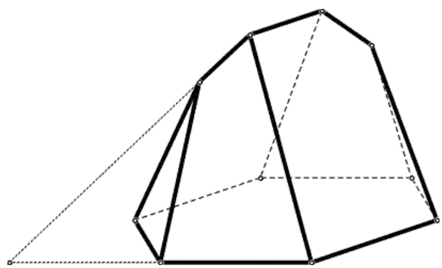


Figure 14

5 "n-faced" Polyhedra

In this section, we will consider problems pertaining to polyhedra whose faces all (or almost all) have the same number of sides. We name a convex polyhedron "n-faced", if its faces all have exactly n sides. For instance, deltahedra are examples of 3-faced polyhedra. If the faces of a convex polyhedron are all n-gons with the same value of n with the exception of a small, well-defined number of faces, with a different number of sides, we call the polyhedron "quasi n-faced".

Question 28

Prove that no n-faced polyhedron can exist for $n \geq 6$.

Solution: If a polyhedron P has f faces, all of which have $n \geq 6$ sides, the fact that each edge of P is shared by two faces means that the number of edges e of P must fulfill the inequality

$$e = \frac{1}{2} \cdot n \cdot f \geq \frac{1}{2} \cdot 6f = 3f.$$

This, however, is a contradiction to the inequality

$$e \leq 3f - 6,$$

which was established in Problem 19. qed

Problem 29

Prove that there cannot exist a 3-faced polyhedron with an odd number of faces.

Solution: (This problem was posed as Problem 3 in [8].) If such a polyhedron exists with f triangular faces, the number of edges of the polyhedron in question must be

$$e = \frac{1}{2} \cdot 3f,$$

which is not an integer if f is odd, giving us a contradiction. qed

We note that this proof will also work if we wish to prove the impossibility of 5-faced polyhedra with an odd number of faces, or more generally, of any polyhedron with an odd number of odd-sided faces.

Problem 30

Let P be a 3-faced polyhedron with f faces. Determine all possible values of f for which such a polyhedron exists.

Solution: In the preceding problem, we saw that f cannot be odd. Also, since P has a triangular face, which must have a common edge with each of three different faces, f cannot be smaller than 4. All even values for

$f \geq 4$ are possible, however. For $f = 4$, P is simply a tetrahedron, and for $f = 2k$ with $k \geq 3$, a double k-sided pyramid has exactly $f = 2k$ triangular faces (Figure 15). qed

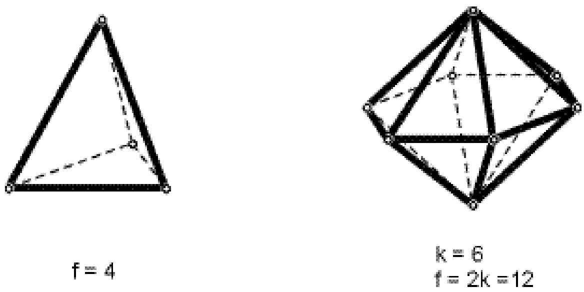


Figure 15

After considering these problems introducing us to the idea of n-faced polyhedra, we can turn our attention to the following series of questions dealing with the number of edges e of n-faced polyhedra.

Problem 31

Let P be a 3-faced polyhedron with e edges. Prove $3|e$.

Solution: P has f faces, and each of these faces has three sides. Each edge of P is common to two of the faces, and we therefore have

$$e = \frac{1}{2} \cdot 3f.$$

This number is certainly divisible by 3. qed

Since the proof is completely analogous, we also have the following:

Problem 32

Let P be a 5-faced polyhedron with e edges. Prove $5|e$.

Also, a quite similar problem is the following:

Problem 33

Let P be a 4-faced polyhedron with an even number of sides and e edges. Prove $4|e$.

Solution: Since P has an even number of faces, we can write $f = 2k$. The number of edges of P is therefore equal to

$$e = \frac{1}{2} \cdot 4f = \frac{1}{2} \cdot 4 \cdot 2k = 4k,$$

which is certainly divisible by 4. qed

If we take a closer look at the last 3 problems, it seems reasonable to ask the following question:

Problem 34

Let P be an n-faced polyhedron with e edges. Is it always true that $n|e$ must hold?

If we take a look at what we know so far, Problem 27 showed us that no n-faced polyhedron exists for $n \geq 6$, and it is quite obvious that there can be none for $n \leq 2$. For $n = 3$ and $n = 5$, we have just shown that the claim is true, as we have also shown for $n = 4$ if P has an even number of faces. Answering the question at hand is therefore closely related to answering the following question:

Problem 35

Do 4-faced polyhedra with an odd number of faces exist?

Solution to Problems 33 and 34: The surprising (to me) answer is that 4-faced polyhedra with an odd number of faces do indeed exist, as we can see in the examples due to Gottfried Perz and Christopher Albert in Figure 16.

The "Perz Polyhedron" is derived from a cube, as can be seen in the Figure. The resulting polyhedron has 9 faces, 18 edges and 11 vertices. The "Albert Polyhedron" results from a double truncated quadratic pyramid as shown. This polyhedron has 11 faces, 22 edges and 13 vertices. In both cases, we see that the number of edges (18 and 22 respectively) is not divisible by the number of sides of each face (4). qed

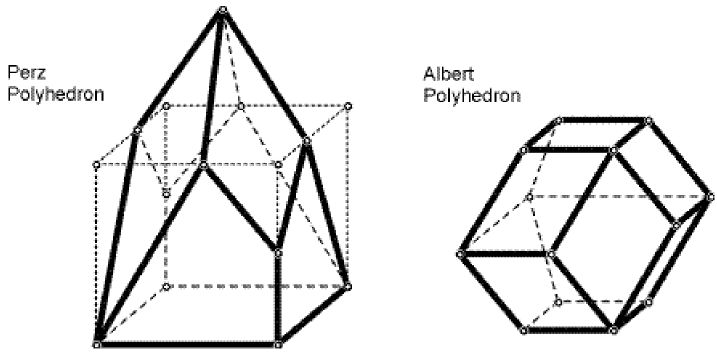


Figure 16

Having determined the existence of such 4-sided polyhedra with odd values of f, it is interesting to ask the analogous question to Problem 30 for 4-sided polyhedra.

Problem 36

Let P be a 4-faced polyhedron with f faces. Determine all possible values of f for which such a polyhedron exists.

Solution: We shall show that 4-faced polyhedra with f faces exist for all values $f \geq 8$ and for $f = 6$, but not for $f = 7$ or $f \leq 5$. In order to prove this, we shall divide the proof into the following steps:

a) 4-faced polyhedra exist for all even values of $f \geq 6$.

b) 4-faced polyhedra exist for all odd values of $f \geq 9$.

c) No 4-faced polyhedra exist for $f \leq 5$.

d) No 4-faced polyhedra exist for $f = 7$.

ad a): If $f = 2k$ with $k \geq 3$, there certainly exists a 4-sided double "k-pyramid", as illustrated in Figure 17.

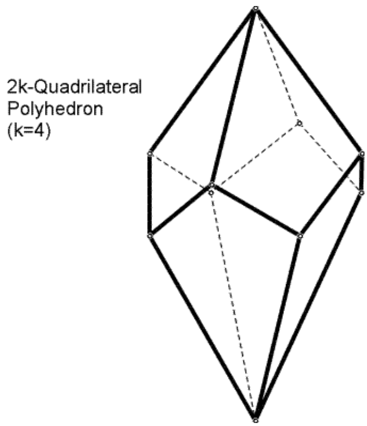


Figure 17

ad b): We already know of the existence of 4-sided polyhedra with 9 and 11 sides from the preceding problem. If we can show that the existence of a 4-sided polyhedron with f faces implies the existence of one with $f+4$ faces, this part of the proof is finished.

This is indeed the case. If P is a 4-sided polyhedron with f faces, and the vertices of one such face are A, B, C and D, we can define a new polyhedron P' with $f+4$ faces in the following manner. The planes of the faces of P divide space into a finite collection of sections, two of which border on the quadrilateral $ABCD$. P is one of these. We choose a point S in the interior of the other, and define A' as the mid-point of SA, B' as the mid-point of SB, and similarly C' and D'. We now define P' as having all faces in common with P except $ABCD$, which we replace by the five quadrilaterals $A'B'C'D'$, $ABB'A'$, $BCC'B'$, $CDD'C'$ and $DAA'D'$. If P was convex, so is P', and P' certainly has $f+4$ 4-sided faces, as required.

ad c): If a polyhedron P has a 4-sided face, it must have at least four more faces, since each side of the 4-sided face must be a common edge with a different face. We therefore certainly have $f \geq 5$. If $f = 5$, P must have $e = \frac{4 \cdot 5}{2} = 10$ edges, and therefore $v = 10 + 2 - 5 = 7$ vertices. This is not possible, since P must have a face $ABCD$ with at least three edges through each of the vertices A, B, C and D. One each of these is not a side of $ABCD$, and none of these can have a common end-point, since this would mean that P would have a triangular face. (If the edges through A and B had a common vertex E, for instance, ABE would be a face of P.) We see that $f \leq 5$ is not possible for 4-sided polyhedra.

ad d): To prove the impossibility of $f = 7$ for a 4-sided polyhedron P, it is useful to have a look at what a Schlegel diagram of P would look like, if it existed. We assume that such a P does indeed exist. Since $f = 7$, we have $e = \frac{4 \cdot 7}{2} = 14$ and $v = 14 + 2 - 7 = 9$. If each of the vertices were trihedral, the number of faces would be $f = \frac{3 \cdot 9}{4}$, which is not possible. Indeed, since we know that $f = 7$, the only possibility for the 9 vertices is that 8 of them are trihedral and one tetrahedral, since this is the only case that yields $f = \frac{3 \cdot 8 + 4}{4} = 7$.

If A is the tetrahedral vertex, P has edges AB, AC, AD and AE. Since each of the faces of P is 4-sided, P must have faces $ABFC$, $ACGD$, $ADHE$ and $AEKB$. Defining these four faces, we have already "used up" all 9 vertices and 12 edges, and the vertices F, G, H and K can therefore not be trihedral, which is a contradiction. We see that $f = 7$ is not possible for 4-sided polyhedra. qed

Having asked which values of f are possible for 3- and 4-faced polyhedra, it is interesting to ask the same of 5-faced polyhedra. The only such polyhedron known to most people is the dodecahedron, and it takes some thought to find another. One question we can therefore ask is the following:

Problem 37

We are given a polyhedron whose f faces are all pentagons. Can f have any value other than 12? Is there an upper limit to the possible values of f?

Solution: There is no upper limit to the possible values of f. One way to see this is to note that it is possible to augment any 5-sided polyhedron with more pentagonal sides in a similar fashion to that described for 4-sided polyhedra in part b) of the preceding problem.

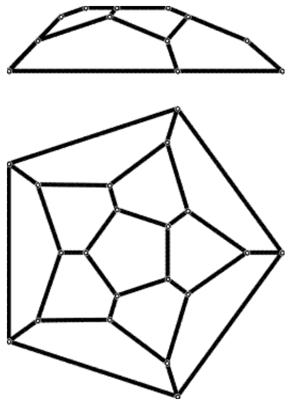


Figure 18

Figure 18 shows us the horizontal and vertical projections of such an augmentation. The horizontal projection is simultaneously the Schlegel diagram of a dodecahedron, and it is always possible to replace a pentagon in the Schlegel diagram of any 5-sided polyhedron with this graph. The full proof that such an augmentation is always possible is somewhat more convoluted, but application of the result on 3-connected graphs and polyhedra mentioned in section 2 yields one such proof.

qed

Applying the augmentation described in this problem to the dodecahedron shows us that we can successively replace any pentagonal face of a 5-sided polyhedron by 11 such faces. This means that we can "build" 5-sided polyhedra with $f = 2 + 10k$ faces for any positive integer values of k. Of course, this immediately leads us to the next question, namely:

Problem 37

Does there exist a 5-sided polyhedron with f faces and $f \neq 2 + 10k$ for all positive integer values of k?

Solution: The answer to this question is yes. An interesting class of such polyhedra is due to Gerd Baron, an example of which is shown in Figure 19.

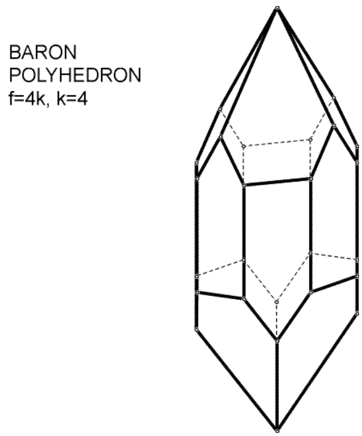


Figure 19

The middle part of such a polyhedron is made up of pentagons resulting from the rectangular sides of a regular $2k$-sided prism, augmented by points alternately attached to opposite sides. "Closing" the polyhedron with the planes joining these points with the sides of the pentagons as shown yields a polyhedron with $4k$ pentagonal sides for any value of $k \geq 3$. This means that we can obtain such values for f as $4 \cdot 4 = 16$ or $4 \cdot 5 = 20$, which are not of type $2 + 10k$. qed

Of course, we would like complete information concerning which values of f are possible for 5-sided polyhedra. We know from the comment after Problem 29 that f can never be odd. Since all values of the form $f = 2 + 10k$, $k \geq 1$ and $f = 4j$, $j \geq 3$ are possible, we can produce 5-sided polyhedra with

$$f = 12, 16, 20, 22, 24, 28, 32, \ldots.$$

The augmentation described in Problem 37 can also be applied to "Baron Polyhedra", of course, and so we can also produce 5-sided polyhedra with $f = 16 + 10k$, $f = 20 + 10k$, $f = 24 + 10k$ and $f = 28 + 10k$. This means that all even values of $f \geq 20$ are possible. Are $f = 14$ and $f = 18$

possible? I do not know the answer to this question yet. I believe that $f = 14$ may not be possible, but $f = 18$ is unclear.

What remains to be shown is the following:

Problem 39

Prove that there cannot exist a 5-sided polyhedron with f faces and $f < 12$.

I leave the proof of this to the reader (mainly because I can't think of an elegant proof, even though the result seems so obvious).

6 "Two-faced" Polyhedra

There is no fundamental way for polyhedra to be dishonest or unfaithful, and their faces cannot be 2-sided. Obviously, something else is implied in the title of this section.

We define a "two-faced" polyhedron as a convex polyhedron P, whose faces are all either k-sided or m-sided, with k and m being different positive integers. In order for P to qualify as being two-faced, we assume that f_k and f_m are both greater than 0, whereby f_i denotes the number of faces with i sides. If $f_k = p$ and $f_m = q$, we will call P a "$(p_k; q_m)$-polyhedron".

Most commonly studied polyhedra fall into this category. Consider the following table:

$$\begin{aligned}
\text{regular } n-\text{sided pyramid},\ n > 3: &\quad (n_3; 1_n)\\
\text{regular } n-\text{sided prism},\ n \geq 3,\ n \neq 4: &\quad (n_4; 2_n)\\
\text{regular } n-\text{sided anti}-\text{prism},\ n > 3: &\quad (2n_3; 2_n)\\
\text{some Archimedean solids}: &\quad (12_5; 20_6),\ (8_3; 6_4),\ (4_3; 4_6).
\end{aligned}$$

While it may seem that such polyhedra are too well known to make them well suited to this type of question, they are in fact very much so.

Problem 40

Does there exist a $(1_3; q_4)$-polyhedron for some positive integer q? If so, determine the smallest possible value of q. If not, prove why not.

Solution: There can be no such polyhedron. If one existed, it would have

$$e = \frac{1 \cdot 3 + q \cdot 4}{2} = 2q + 1 + \frac{1}{2}$$

edges, and this is not a whole number. qed

Problem 41

Does there exist a $(1_k; q_m)$-polyhedron with an even value of k and an odd value of q?

Solution: The only "standard" $(1_k; q_m)$ polyhedron is the n-sided pyramid, for which we have $k = q = n$ (and $m = 3$). The parity of k and q must be the same for any pyramid, of course. If a polyhedron with the required property exists, it cannot be a pyramid.

Since the number of edges of such a polyhedron is

$$e = \frac{k + q \cdot m}{2} = \frac{k}{2} + q \cdot \frac{m}{2},$$

m must be an even number. If both k and m are to be even (but different), it is not a good idea to search for solutions where $m > 4$ (see also Problem 42). An obvious place to start looking for a polyhedron with the required property is $k = 6$ and $m = 4$. There does indeed exist a $(1_k; q_m)$ polyhedron with $k = 6$, $m = 4$ and q odd, for instance the $(1_6; 9_4)$-polyhedron as shown in Figure 20. qed

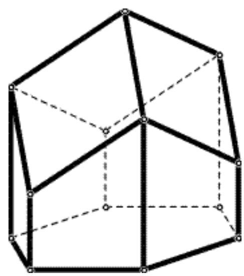


Figure 20

Problem 42

Prove that there cannot exist a $(1_k; q_m)$-polyhedron with $m \geq 6$.

Solution: If such a polyhedron did exist, it would have $f = q + 1$ faces and $e = \frac{m}{2} \cdot q + \frac{k}{2}$ edges. In Problem 19 we saw that

$$e + 6 \leq 3f$$

must hold for any convex polyhedron, and it would therefore follow that

$$\frac{m}{2} \cdot q + \frac{k}{2} + 6 \leq 3q + 3 \quad \Leftrightarrow \quad \left(\frac{m}{2} - 3\right) \cdot q + \frac{k}{2} + 3 \leq 0$$

must hold. If $m \geq 6$ holds, it follows that $\frac{m}{2} - 3 \geq 0$ also holds however, and this is then certainly not possible. qed

Problem

Prove that there cannot exist a $(1_4; q_m)$-polyhedron with even m.

Solution: If m is even and not equal to 4, we must have $m \geq 6$, since each face of a polyhedron must have at least 3 sides. The result of the preceding problem therefore shows us that such a polyhedron cannot exist. qed

Problem 44

Does there exist a $(1_k; q_m)$-polyhedron with $k < m$?

Solution: If such a polyhedron exists, there are many limitations to the possible values of k, m and q. First of all, we can not have $k \geq 5$ (and therefore $m \geq 6$), as was just shown in Problem 42. The impossibility of a $(1_3; q_4)$-polyhedron was shown in Problem 40. The only categories of such polyhedra that can possibly exist are therefore $(1_3; q_5)$ and $(1_4; q_5)$. In order for the number of edges to be an integer, the value of q must be odd in the former case, and even in the latter. Finding a polyhedron in either case is not easy. One example for a $(1_3; q_5)$-polyhedron is shown in Figure 21. (I do not know yet whether a $(1_4; q_5)$-polyhedron exists or not.)

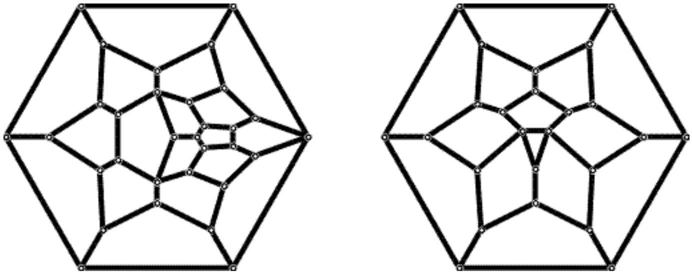


Figure 21

In this figure, we see the polyhedron cut in two. Each piece has a hexagonal periphery, and the Schlegel diagrams of each half are illustrated. One piece is made up of only 18 pentagons, and the other is made up of 13 pentagons and one triangle. The resulting polyhedron is therefore of type $(1_3; 31_5)$. qed

7 Additional Questions

In this final section, I have listed a number of additional questions suggested by the problems discussed in this paper. I have not ordered them according to difficulty. While some of these problems are quite simple, some are not at all easy to solve. Those for which I do not know the full answer are marked with an asterisk. I hope that readers will find some enjoyment in solving these problems.

Problem 45

Does a 4-faced polyhedron exist, such that an even number of faces meet at each vertex? If so, what is the smallest number of faces that such a polyhedron can have?

Problem 46

Is it true that the edges of a 4-faced polyhedron can always be colored with two colors, such that no two edges of the same color meet in a common vertex?

Problem 47

Does a 4-faced polyhedron exist, such that exactly three edges meet in each vertex and $f > 6$? For which values of f does a polyhedron with this property exist? (Note that a cube is such a polyhedron with $f = 6$.)

Problem 48*

Let v (the number of vertices of a polyhedron) be given. Determine the smallest possible values for f and e.

Problem 49*

For which values of p can we find k and m such that a $(p_k; p_m)$-polyhedron exists?

Problem 50*

For which values of q can we find k and m such that a $(1_k; q_m)$-polyhedron exists?

Problem 51

For which values of q does a $(1_4, q_3)$-polyhedron exist?

References

[1] Problem H287, *Crux Mathematicorum with Mathematical Mayhem*, April 2001

[2] Martin Aigner, Günter M. Ziegler, *Proofs form The Book*, Springer Verlag, Berlin, 2000

[3] Branko Grünbaum, *Convex Polytopes*, John Wiley and Sons, New York, 1967

[4] Andrei Jorza, Problem 10856, *The American Mathematical Monthly*, Feb. 2001

[**5**] Tiberiu Roman, *Reguläre und halbreguläre Polyeder*, VEB Deutscher Verlag der Wissenschaften, Berlin, 1987, (Original: Poliedre regulate si semiregulate, Bukarest, 1965)

[**6**] Leo J. Schneider, *The Contest Problem Book VI*, Mathematical Association of America, Washington, D.C., 2000, pp. 199-200

[**7**] Marjorie Senechal, George Fleck (eds.), *Shaping Space; A Polyhedral Approach*, Birkhäuser Verlag, Boston, 1988

[**8**] James Tanton, *A Dozen Questions about a Triangle*, Math Horizons, April 2002, pp. 23-28

Robert Geretschläger
Breunergasse 23
A-8051 Graz
AUSTRIA
email: robert.geretschlaeger@brgkepler.at

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The 43rd International Mathematical Olympiad, 19-30 July 2002, Glasgow, U.K.

The IMO experience is always a thoroughly enjoyable one. For students all around the world it is an opportunity to meet other talented young students as well as to be challenged by two exams each consisting of three first class problems. For Leaders and their Deputies being involved with other mathematicians and cooperating together in making jury decisions (Leaders only) certainly involves a great deal of work, but it is very rewarding.

This, the 43rd IMO, with 84 countries and 479 students made it the largest IMO since its inception.

At the beginning of every IMO, Leaders are geographically separated from their Deputies and the students. They form what is called the "Jury". The Jury amongst other things sets the examination papers, approves translations of the papers into various languages and decides on awards for the students.

The first task at hand was to study the shortlisted problems. Leaders had $1\frac{1}{2}$ days in which to study them without solutions, and then with solutions. Indeed it is all too easy to dismiss a problem as very easy once one has seen a solution. Trying a problem for oneself really does give one an accurate feel for its difficulty. I must say that the Problem Selection Committee under the guidance of Imre Leader (University of Cambridge) did an excellent job. There was an abundance of very nice problems of varying difficulties. The Problem Selection Committee had worked on the proposed problems for weeks, coming up with their own solutions to all of them and finally shortlisting 27 problems. Unfortunately four of them had to be deleted from consideration because they were judged as having been too similar to known problems. I must say that the IMO is inexorably getting harder and harder. Many team leaders did not even manage to solve a third of the shortlisted problems.

The problems that ended up making it on to the papers were a nice blend of many areas. They came from three different continents with two from Romania, and one each from Colombia, Korea, India and Ukraine. The easier problems being of combinatorics, number theory and classical geometry. There was a functional equation which ended up being moderately difficult. The two difficult questions turned out to be extremely difficult, one being a technical algebra question disguised as number theory and the other a combinatorial geometric inequality. Only three students, two from China and the other from Russia, managed to solve these two questions, and they achieved perfect scores for the entire IMO exams.

While all this was occurring, the Deputies with their students were getting comfortable with their surroundings. Many teams had traveled half the world to be there and were recovering from jetlag.

The Opening Ceremony provided an opportunity for each team to parade on stage. There were a few short speeches and some entertainment in the form of Scottish dancing to bagpipe accompaniment as well as some mathematical juggling. Even at the Opening Ceremony Leaders were all kept separate from everyone else in a balcony area. However, they still had occasion to wave at their students from a distance.

During the examinations students are allowed to ask for clarification of the paper for the first half hour. Their questions were faxed to the Jury and the responses returned promptly. These were all dealt with quite smoothly, although one student wanted to know where the 2005 IMO would be held, to which the Jury's response was "concentrate on the exam".

After the two exams, the Leader and his Deputy were united to assess their students scripts. Local professionals also assessed the scripts and met with the leaders of each country to decide on scores to award. This is called "coordination". The coordination process was absolutely first class. Many of the coordinators were ex-olympians and thus had an excellent feel for the process. One coordinator, Tim Gowers, was a Field's medalist in 1998. The actual coordination was very strict but consistent. Students were rewarded for complete solutions whereas part marks were very hard to come by. They were only awarded when real progress had been made.

In the final Jury meeting, cutoff scores for the various medals were decided. The jury awarded 39 golds, 73 silver and 120 bronze medals. A further 66 students received an honourable mention for solving one question perfectly.

It was a treat for some teams to meet HRH Princess Anne. The top countries from each continent had a few minutes private audience with her. She personally presented all the gold medals and spoke to all delegates at the closing ceremony.

Finally I would like to note the outstanding role played by Chairman of the Jury, Adam McBride (University of Strathclyde), who ensured that the Jury meetings ran smoothly and progressively.

Angelo Di Pasquale
Australian IMO Team Leader
email: pasqua@ms.unimelb.edu.au

* * *

Questions from the 43rd International Mathematical Olympiad

First Day

Q1 Let n be a positive integer. Let T be the set of points (x, y) in the plane where x and y are non-negative integers and $x+y < n$. Each point of T is coloured red or blue. If a point (x, y) is red, then so are all points (x', y') of T with both $x' \leq x$ and $y' \leq y$. Define an X-set to be a set of n blue points having distinct x-coordinates, and a Y-set to be a set of n blue points having distinct y-coordinates. Prove that the number of X-sets is equal to the number of Y-sets.

Q2 Let BC be a diameter of the circle Γ with centre O. Let A be a point on Γ such that $0° < \angle AOB < 120°$. Let D be the midpoint of the arc AB not containing C. The line through O parallel to DA meets the line AC at J. The perpendicular bisector of OA meets Γ at E and at F. Prove that J is the incentre of the triangle CEF.

Q3 Find all pairs of integers $m, n \geq 3$ such that there exist infinitely many positive integers a for which

$$\frac{a^m + a - 1}{a^n + a^2 - 1}$$

is an integer.

Second Day

Q4 Let n be an integer greater than 1. The positive divisors of n are $d_1, d_2, \ldots, d_k$ where

$$1 = d_1 < d_2 < \cdots < d_k = n.$$

Define $D = d_1d_2 + d_2d_3 + \cdots + d_{k-1}d_k$.
(a) Prove that $D < n^2$.
(b) Determine all n for which D is a divisor of n^2.

Q5 Find all functions f from the set R of real numbers to itself such that

$$(f(x) + f(z))(f(y) + f(t)) = f(xy - zt) + f(xt + yz)$$

for all x, y, z, t in R.

Q6 Let $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$ be circles of radius 1 in the plane, where $n \geq 3$. Denote their centres by $O_1, O_2, \ldots, O_n$ respectively. Suppose that no line meets more than two of the circles. Prove that

$$\sum_{1 \leq i < j \leq n} \frac{1}{O_iO_j} \leq \frac{(n-1)\pi}{4}.$$

Distribution of Awards

Country	Score	Gold	Silver	Bronze	Hon Men
Albania	25	-	-	1	-
Argentina	96	-	-	5	-
Armenia	33	-	-	-	1
Australia	117	1	2	1	1
Austria	50	-	-	1	3
Azerbaijan	37	-	-	1	1
Belarus	135	1	2	3	-
Belgium	58	-	-	1	3
Bosnia and Herzegovina	42	-	-	1	-
Brazil	123	-	1	5	-
Bulgaria	167	3	2	1	-
Canada	142	1	3	1	1
China	212	6	-	-	-
Colombia	81	-	-	3	3
Croatia	70	-	-	2	1
Cuba	78	-	-	2	3
Cyprus	29	-	-	-	-
Czech Republic	115	-	2	3	-
Denmark	53	-	-	-	3
Ecuador	3	-	-	-	-
Estonia	75	-	2	-	2
Finland	79	-	-	3	3
France	127	-	2	3	-
Georgia	84	-	-	2	2
Germany	144	2	1	2	1
Greece	62	-	-	2	-
Guatelmala (3 members)	4	-	-	-	-
Hong Kong	120	1	2	2	-
Hungary	142	1	2	3	-
Iceland	36	-	-	-	3
India	156	1	3	2	-
Indonesia	38	-	-	1	1
Iran	143	-	4	2	-

Country	Score	Gold	Silver	Bronze	Hon Men
Ireland	125	-	-	-	1
Israel	130	-	3	3	-
Italy	88	-	-	5	1
Japan	133	1	3	1	-
Kazakhstan	133	-	3	3	-
Korea	163	1	5	-	-
Kuwait (4 members)	2	-	-	-	-
Kyrgystan (4 members)	17	-	-	-	1
Latvia	75	-	1	2	2
Lithuania	74	-	1	2	1
Luxembourg (2 members)	12	-	-	-	1
Macau	50	-	1	3	-
FYR Macedonia	73	-	-	1	1
Malaysia	26	-	-	-	1
Mexico	67	-	-	3	-
Mongolia	82	-	-	3	-
Morocco	39	-	-	1	1
Netherlands	55	-	-	1	1
New Zealand	82	1	-	-	4
Norway	72	1	-	1	1
Paraguay (2 members)	11	-	-	-	1
Peru (5 members)	59	-	-	2	-
Philippines (5 members)	18	-	-	-	1
Poland	123	-	4	1	1
Portugal	15	-	-	-	-
Puerto Rico	17	-	-	-	-
Republic of Moldova	60	-	-	2	-
Romania	157	2	3	1	-
Russia	204	6	-	-	-
Singapore	112	-	2	2	1
Slovakia	119	-	2	4	-
Slovenia	46	-	-	1	1
South Africa	90	-	1	3	-
Spain	44	-	-	1	1
Sri Lanka (4 members)	16	-	-	-	1
Sweden	60	-	-	2	1
Switzerland	44	-	-	1	2
Taiwan	161	1	4	1	-

Country	**Score**	**Gold**	**Silver**	**Bronze**	**Hon Men**
Thailand	123	-	2	2	2
Trinidad and Tobago	22	-	-	-	-
Tunisia	22	-	-	-	1
Turkey	135	1	1	4	-
Turkmenistan	45	-	-	1	1
Ukraine	124	1	3	-	-
United Kingdom	116	-	2	2	-
United States of America	171	4	1	-	1
Uruguay (1 member)	1	-	-	-	-
Uzbekistan	60	-	-	-	2
Venezuela (5 members)	58	-	1	1	1
Vietnam	166	3	1	2	-
Yugoslavia	114	-	1	5	-
Total (479 contestants)		**39**	**73**	**120**	**66**

* * *

Abilities Tested by Mathematics Olympiads

Zhu Huawei

Zhu Huawei is one of the coaches of the Chinese National Olympiad team and has been head coach on several occasions. He is the editor of the Hua Luo-gen *magazine for teenage and middle school students. He has published many papers and monographs, including a* Course Book for the IMO, *and has translated several works and monographs. His video series on* Super Teachers *has won wide acclaim.*

The International Mathematics Olympiad (IMO) is a contest of intelligence, a major purpose of which is to discover and foster youths with mathematical talents. Consequently, question-setting in the IMO should put its emphasis on testing contestants' mathematical abilities. As Prof. Hua Luogen, a famous mathematician, pointed out, 'Mathematics contests are different from either examinations at school or the matriculation to colleges. Therefore, contestants are required not only to apply formulae and theorems, but also to grasp and employ known principles and theorems to solve practical problems, or even to generate new methods and create new principles to solve problems.' Then what abilities of contestants should be tested in IMO?

I. A general introduction to mathematical abilities

Mathematical abilities are defined as the steady psychological states and features of a person while performing mathematical activities smoothly. However, how are these abilities formed? What major components are they made up of?

Since the beginning of the last century, mathematicians from home and abroad, mathematics educators and psychologists have analyzed this question from different perspectives. No consensus has been reached though. The major ideas are as follows:

The first idea was proposed by a Russian psychologist B A Kruteskiy, who, after his systematic study on the qualities and structure of mathematical abilities based on a broad range of experiments, proposed that the components of mathematical abilities include:

1. the ability to formalize mathematical materials, to extract images out of contents, to abstract the concrete value relations and spatial forms, and to calculate by using forms and structures (the structure of relations and connections);

2. the ability to summarize mathematical materials, to search for the most important things and ignore irrelevant content, and to find out the common features among diverse objects;

3. the ability to calculate by using numbers and other signs;

4. the ability of logical reasoning in a coherent process with proper divisions, which is necessary in proofs, figurations and inductions;

5. the ability to shorten the reasoning process, and to think in a shortened structure;

6. the ability to think in a reverse psychological process (to turn a thought sequence to an inverted order);

7. flexibility of thoughts, namely, the ability to shift from one mental stance to another, and to free one's thoughts from the restraints of conventions. This feature of thoughts is very important for the creative work of a mathematician;

8. mathematical memory is a kind of memory which is fit to record generalization, figurations and logical modes, the framework of which can be presumed to come from the framework of science itself;

9. the ability to form spatial concepts, which is directly connected to a branch of mathematics—geometry, especially solid geometry.

The second idea was put forward by the famous Russian mathematician A H Kolmogorov who believes that mathematical abilities include the following parts according to the characteristics of mathematics as a discipline:

1. the ability of computational algorithms, namely, the ability to masterly transform a complicated algebraic expression so as to solve an equation which can not be solved in the normal way of using a standard equation;

2. the imaginary ability and the intuition towards geometric figures;

3. mastery of logical reasoning in a consecutive process with proper divisions.

The third idea emerged in the report *Everyone Calculates* made by the Mathematical Educational Committee and the Mathematical Committee of the US Research Association in 1989. From the direction of educational reform in the discipline of mathematics, the report holds that as mathematical education shifts its focus from innumerable conventional exercises to developing mathematical abilities with a broad foundation, students' mathematical abilities are accordingly required to reach a level of being capable of discerning all kinds of relations and connections, of logical reasoning, of solving various unconventional problems by employing a wide range of mathematical methods. Students are also expected to have the ability of mental arithmetic as well as effective prediction, and they need to decide when a precise solution is needed and when to predict, and which kind of mathematical operation is the most suitable one under particular conditions; they are capable of forming special questions from confusing practical problems; and lastly they can select an effective strategy to solve problems.

Based on the above three ideas and the creed of IMO, the author believes that questions for IMO should test not only contestants' basic abilities but also their creative abilities.

II. Basic abilities

Basic mathematical talents mainly include abilities in observation, association, computation, summarizing, logical reasoning and expressiion. These are the basis of forming the ability to analyze and solve problems, and therefore are the basic requirements of the question.

1. The observative ability.

All man's knowledge is gained through observations. Mathematics needs observations, and Gauss even said that mathematics is an observative discipline. The observative ability in mathematics is mainly demonstrated in recognizing the 'number' and 'figure' of things quickly, in other words, the ability to find inner connections of a problem from its form and structure. In the IMO, this ability is shown in the following aspects:

(a) To find out the structural features and interrelations of mathematical relations.

(b) To recognize some special figures and relations from a geometric figure.

Example 1.

As is shown in Figure 1, a hexagon is divided into black and white triangles such that any two adjacent triangles have different colors and all the triangles sharing the sides of the hexagon are black. Prove that a decagon can not be divided in the same way.

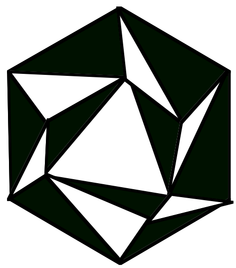


Figure 1

It seems that it is hard to start. But if you examine the figure carefully, you may find out the relationship of the number of sides of the black triangles and that of the white triangles. And this is the key point of this problem. Presume that a decagon can be divided in this way. Let m be the number of the sides of the black triangles and n that of the white triangles. Then we can know that $m - n = 10$. As $3|m$ and $3|n$, but $3 \not| 10$, this leads to a contradiction. So the problem is solved. The author has even tried this problem among some Junior One students in an IMO training class. Zhou Lu, a Junior One student from Wuhan Foreign Language School, found the key to the problem very quickly through observation. This shows that Junior One students have a very keen observative ability.

2. The Associative Ability

Association is the psychological process in which someone, when perceiving or remembering one thing, recalls some other relevant things. Association is the bridge of transferring problems. Since the connections between questions in IMO and basic knowledge are complex, neither obvious nor direct, contestants are required to be skilled in corresponding associations based on relevant knowledge so as to find a way of solving the problem.

Here is another example, which the author set for the contestants of the Beijing training team for the 1991 CMO.

Solve the set of equations:

$$\begin{cases} y = 4x^3 - 3x \\ z = 4y^3 - 3y \\ x = 4z^3 - 3z \end{cases}$$

If we want to solve this problem by the conventional method of elimination, we will get into complicated calculations. However, if we notice the regular pattern of each equation, we may associate it with triple-angle formulas. In that case, we can find an easier solution using substitution of variables.

3. The Computational Ability

At present, various objectives in mathematical education cover the topic of computational ability. Take IEA, the Investigation of International Mathematical Education, for example, whose objectives in cognitive aspects include:

(a) Computation: the ability to directly operate with the elements of a problem based on learnt principles, as well as the ability to employ particular knowledge concerning a fact or term.

(b) Comprehension: the mastery of concepts, principles, rules and general provisions, and the ability to convert a problem in various ways.

(c) Application: the ability to associate relevant knowledge, to select a proper algorithm, and to fulfil the computations; the ability to solve a problem in conventional ways.

(d) Analysis: the ability to apply unconventional methods, to find out their modes, and to form proofs and criticism, which is an advanced process of thought.

The first three items of these objectives involve the testing of computational ability. This ability is listed as a single item respectively in the three ideas mentioned above, showing the importance of the computational ability. The author thinks that questions for IMO should test the accuracy, flexibility and swiftness of contestants' computational ability. This will mainly be demonstrated as follows:

(a) The ability to comprehend abstract formalized notional language.

(b) The ability to memorize the definition, formulas and the rules of operations.

(c) The ability of transformation.

(d) The ability to simplify the operational process, namely, to perform operations in a concise and leaping way.

(e) The ability to reverse a computational process, as well as the ability to check it. There are many formulas in mathematical operations which require students to apply both a conventional order and a reverse order.

(f) The flexibility in performing operations, namely, the flexibility in applying formulas, rules and concepts and also the ability to turn to another operation as soon as one operation ends up in a blind alley.

The above six components are the basic elements to make up the computational ability, and they are very important criteria to judge a student's computational ability. However, more advanced requirements of IMO expect students to have the following abilities as well:

(g) The Predictive and Estimative Ability

It is necessary to transform inequalities while predicting and estimating, which calls for much higher abilities than transforming equations. It requires not only the ability to transform (like enlargement and narrowing) inequalities flexibly and swiftly, but also a keen insight. As a result, there are common questions that ask contestants to solve polynomials, equations, functions, geometric problems, combinations and problems about number theory in both domestic and overseas mathematical competitions in recent years, apart from conventional problems of proving inequalities.

Take the first question in the 33rd IMO for example:

Find all possible integers a, b, c $(1 < a < b < c)$, such that $(a-1)(b-1)(c-1)$ is a factor of $abc - 1$.

To solve this problem, we must first estimate the range of

$$\frac{S = (abc-1)}{(a-1)(b-1)(c-1)} (S \in N).$$

As we get $S < 4$, we then check it for S=1, 2, 3 one by one, so as to find the key to the problem.

Here is another example, the first question in the 1992 IMO.

All the coefficients a_n in equation

$$x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0$$

are real numbers and

$$0 < a_0 \leq a_1 \leq \ \leq a_{n-1} \leq 1.$$

Given that λ is the complex root of the equation, and $|\lambda| \geq 1$, prove that $\lambda^{n+1} = 1$.

This problem requires a strong ability of identical transformation on one side and an estimative method using inequalities on the other. Statistics collected by the author show, the degree of difficulty of this item is 0.248, with an average score of 5.521 points (the total score for this item is 21 points). This indicates that our contestants' abilities in transformation and estimation are not well grounded enough and need further intensive training.

(h) The ability of recurrence and induction.

This is also a kind of important computational ability often tested in IMO in the forms of sequence of numbers and functional equations. It is shown by statistics that this ability is also a often tested point in the IMOs.

4. The Abstractive Summary Ability

This ability refers to the Abstract Summary of the relationship between mathematical objects, numbers and spatial figures as well as the computations based on these relationships. It is often tested in the IMO in the following three directions:

(a) To form a mathematical problem from a practical problem, that is, to extract forms and relations with mathematical senses from the concrete materials containing numerical relations and spatial concepts.

It is often referred to as the mathematization or formalization of practical problems. There are a great number of such problems in domestic and international mathematical competitions.

For example, the fifth question in the second-round test of the Mathematical Competition among Eight Domestic Provinces and Municipalities in 1978 read:

There are ten people fetching water. Each of them has a bucket. It takes the ith person ($i = 1, 2, \ldots, 10$) T_i minutes to fill his bucket and each T_i is different from another. Question:

i. Given only one tap, how do you arrange the order of filling the buckets such that the total time is the shortest? How long will the total time be? (You need to prove your conclusion.)

ii. What will the situation be like, given two taps? (Prove your conclusion as well.)

This problem requires contestants to form a mathematical model from a real problem, and then solve it by using the rearrangement inequality.

(b) To summarize a general rule and to make a hypothesis, and then to prove it formally. For example, here is the sixth problem in the 18th IMO.

A sequence of numbers $u_0, u_1, u_2, \ldots, u_n, \ldots$ is defined as follows:

$$u_0 = 2,\ u_1 = \frac{5}{2},\ , u_{n+1} = u_n(u_{n-1}^2 - 2) - u_1$$

for $n = 1, 2, 3, \ldots$.

Prove that

$$[u_n] = 2^{\frac{2^n - (-1)^n}{3}} \text{ for } n = 1, 2, 3, \ldots$$

in which $[x]$ stands for the largest positive integer not greater than x.

As the recurrence formula in this item is very complex, it is hard for us to work out the general form for u_n. However, we can work out $u_2 = 2,\ u_3 = 8,\ u_4 = 32$ according to the recurrence formula and the initial value of u_0 and u_1. From this we can deduce that $_n == 2^{\frac{2^n-(-1)^n}{3}} + = 2^{\frac{2^n-(-1)^n}{3}}$.We then complete the proof by mathematical induction.

(c) To summarize or generalize a particular problem, generalize an abstract conclusion through the analysis and synthesis of a concrete problem, and finally to apply the conclusion to the specific problem which needs solving.

For example, here is a problem from the first selection in 26th IMO:

There are 1985 points on a circle, each of which is labelled either plus one or minus one. Now someone starts from one point and goes along the circle. If at each point he passes, the sum of all the points he has passed is positive, we call the starting point a 'good' one. Prove that there is at least one good point as long as the number of the points labelled minus one is less than 662.

It is enough to prove that there is a good point when there are 661 points labelled minus one. Since $1985 = 3 \times 661 + 2$, we can consider proving the following general proposition: If there are K points labelled minus one among an arbitrary permutation of $3K + 2$ points, there must exist a good point. We can prove it by using mathematical induction and then apply the conclusion to the original problem.

5. The Ability of Logical Reasoning

The ability of logical reasoning is the core of mathematical abilities. It is commonly manifested as:

(a) Understanding of formal expressions, as well as mastery of the relationship among formulas, principles, theorems and axioms in a conceptual system.

(b) Mastery of relevant logical knowledge (such as the four forms of propositions, sufficient condition and necessary condition, inductive reasoning, deductive reasoning, analogical reasoning, transformation between equivalence and nonequivalence of propositions), and also the ability of correct reasoning.

(c) Mastery of commonly used mathematical methods (such as analytical method, synthetic method, reduction to absurdity, and mathematical induction).

(d) Clear thoughts with a proper presentation. The ability to think in a concise pattern by skipping and simplifying the reasoning process. According to statistics, problems on plane geometry play a very important part in the IMO, which mainly test the ability of deductive reasoning strictly, concisely and flexibly. In the past decades, there have been a large number of problems on combination, graph theory, and logic. To solve these problems, contestants do not need advanced specialized knowledge, but a very solid background in logical reasoning.

For example, the fifth item in the 1992 CMO reads:

In a simple graph with 8 vertices, if there is no quadrilateral, what is the maximum number of many laterals?

This is a problem on graph theory. However, it does not require contestants to master advanced knowledge about graph theory, but only a strong ability of logical reasoning.

6. The ability of writing and expressing oneself.

A simplified proof for a difficult item in IMO usually needs three to four pages to present. This demands good writing and expressing abilities of contestants. In other words, it is quite crucial for a contestant to express ideas explicitly, strictly and fully. As a result, emphasising this ability should be carried out throughout the whole process of training.

III. Creative abilities

The famous mathematician, Professor Di Longnie said, 'The only difference between making an important scientific discovery and solving a good mathematical problem in IMO is that the latter will take you five hours while the former will take you 5,000 hours.' Thus we can say that solving a high-level IMO item and doing mathematical research only differ in degree and level, but they share the same quality. From this we know that questions set for IMO ask for higher levels of creative abilities of contestants. To be specific, mathematical creative abilities include the ability of mathematical imagination, the ability of mathematical intuition, the ability of mathematical conjecture, the ability of mathematical transformation and the ability of mathematical construction. All of these put together give the ability to solve unconventional problems.

1. The Ability of Mathematical Imagination

Imagination is the mental process in which the human brain processes the existing representations and then produces a new one. It is figurative, generalized, integral, free and flexible. As a result, it is capable of creations. Mathematical imagination, accordingly, is the process in which someone acquires and employs figurative thoughts in a mathematical cognitive activity. It calls for necessary background knowledge and the ability of figurative thinking.

In the IMO, mathematical imagination is usually manifested in the following aspects:

(a) Sophistication in associating mathematical problems with geometric images (the integration of numbers and graphs).

Take the 2nd question in 1989 IberoAmericamn Mathematical Olympiad, for example.

Let x, y, z be real numbers with $0 < x < y < z < \dfrac{\pi}{2}$.

Prove that

$$\frac{\pi}{2} + 2\sin x\cos y + 2\sin y\cos z > \sin 2x + \sin 2y + \sin 2z.$$

Analysis: To prove the original proposition, we need only prove

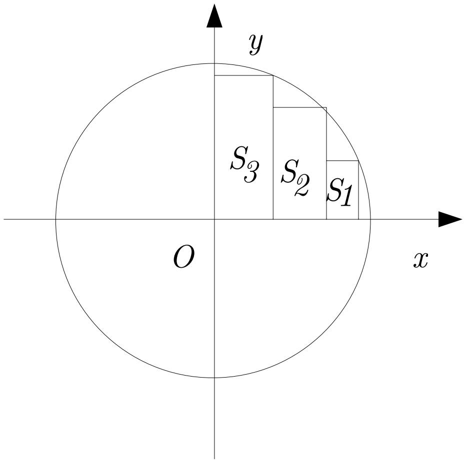
$$\frac{\pi}{4}+\sin x\cos y+\sin y\cos z > \sin x\cos x+\sin y\cos y+\sin z\cos z.$$

or

$$\frac{\pi}{4} > \sin x(\cos x - cosy) + \sin y(\cos y - \cos z) + \sin z\cos z.$$

As the right side of the last inequality contains $(\cos x, \sin x)$, $(\cos y, \sin y)$ and $(\cos z, \sin z)$, we can relate these to a unit circle in a plane with orthogonal coordinates with the origin as its center. As the figure shows, $(\cos x, \sin x)$, $(\cos y, \sin y)$

and $(\cos z, \sin z)$, are three points on the circle.



Construct the vertical lines to $x-$axis and $y-$axis through each of these points, and thus get three rectangles. In fact, the right side of the inequality stands for the total area of these three rectangles, which is obviously smaller than $\frac{\pi}{4}$.

The difficulty of this problem lies in the connection of an algebraic problem and a geometrical image. However, this connection does not come out of a void but out of the broad background knowledge and the ability of figurative thoughts of contestants.

(b) The ability to add auxiliary lines in geometric problems.

In both domestic and overseas mathematical Olympiads, there have been many problems requiring geometric proof. To prove these problems, a contestant should not only be good at logical reasoning, but also be able to form new graphs by adding auxiliary lines according to the given conditions. And these construction lines as well as their relations to the original graph are mentally visualised.

2. The Ability of Mathematical Intuition.

Intuition is the direct comprehension and insights into the essence of things, and mathematical intuition is the direct comprehension and insight into the essence of mathematical objects. The difficulties of IMO questions just lie in the insights into mathematical essence instead of mastering more mathematical knowledge.

In the IMO, the ability of mathematical intuition is demonstrated:

(a) To directly comprehend the essence of mathematical objects integrally.

(b) To get a clear picture of a mathematical problem, its structure and inner relations.

(c) To directly grasp the solution procedures and the result of a problem.

For example, here is the first item in the second-round test for 1990 Chinese National Mathematical Competition for Senior Students.

A quadrilateral is inscribed in a circle O, with two diagonals AC and BD intersected by P. Let O_1, O_2, O_3, and O_4 be the circumcenters of triangles ABP, BCP, CDP and DAP respectively. Prove that lines OP, O_1O_3 and O_2O_4 are concurrent.

Intuitive judgment: Once rectangles OO_1PO_3 and O_2PO_4O are parallelograms, their diagonals O_1O_3 and OP bisect at G, O_2O_4 and OP bisect at G, also. As a result, OP, O_1O_3 and O_2O_4 intersect at the same point G, the middle point of OP. So we should try to prove that rectangles OO_1PO_3 and O_2PO_4O are parallelograms. (proof omitted)

Here we guess first, and then prove our presumption. Once we intuitively guess that rectangles OO_1PO_3 and O_2PO_4O are parallelograms, we find the right direction of solution. But our guess is not purposeless. To guess precisely, we need a strong ability of mathematical intuition.

3. The Ability of Mathematical Conjecture

Conjecture refers to the plausible reasoning of unknown things and their laws and regularities based on some known facts and knowledge. Mathematical conjecture is the plausible reasoning of unknown values and their relationship according to known mathematical conditions and theories. Therefore, it is partially scientific with a great degree of assumption. It can be made not only through experiments, induction, analogy and specialization, but also by imagination, intuition, and reverse thinking. In the IMO, the ability of mathematical conjecture is mainly shown in the following two aspects:

(a) Guessing the approach to proofs.

(b) Guessing the conclusion of propositions, which is particularly important in solving 'explorative' problems.

Take the first item in 1991 the CMO for instance.

Given a convex rectangle $ABCD$ on a plane,

(a) if there exists a point P on the plane such that $\triangle ABP$, $\triangle BCP$, $\triangle CDP$ and $\triangle DAP$ are all of the same area, what qualities should the rectangle $ABCD$ have?

(b) how many points which satisfy the conditions in (a) are there at the most on the plane? Prove your conclusion.

This is an old problem with its conclusion left out. But it surprised the Administrative Committee of Contests that the level of scores was so low. However, if the contestants had been supposed to prove it with the conclusion given to them, few of them would have made mistakes. This indicates that the contestants are not good at solving explorative problems, and their ability in mathematical conjecture is not solid enough. This kind of explorative problem sets a higher requirement for contestants.

4. The Ability of Mathematical Transformation

The ability of mathematical transformation is the ability of transformation from one kind of psychological operation to another, which is similar to the flexibility and creativity of thoughts in certain degrees. In the IMO, this ability is mainly demonstrated as:

(a) Being able to smash the bonds of usual practice and to find new approaches and methods, if the problem cannot be solved in conventional approaches and modes.

(b) Being able to shift from conventional thinking to reverse thinking.

Consider this example:

Divide a convex N-sided polygon into triangles with non-intersecting diagonals such that there are an odd number of triangles around each vertex.

Prove that N is divisible by 3.

The conventional methods that students resort to are mathematical induction and the method of dissection. However, these are not suitable for this problem. On the contrary, if contestants are skillful in transformation and turn to coloring proofs, the problem is much easier to prove.

5. The Ability of Mathematical construction

In recent years, there are more and more problems in the IMO which call for constructive proofs. This type of problem require the contestants to meticulously devise the mathematical objects according to the requirements of the problems. Contestants need to observe carefully, do well in experiments, be good at association, imagine and guess boldly, transform flexibly and infer strictly while solving these problems. Thus they are good tests for contestants' comprehensive ability and creative ability. The ability of mathematical construction is mainly shown in the following ways in the IMO.

(a) From the structural features of the problem, form a mathematical model, such as a function, an equation, a graph, and an algorithm, to relate the conditions to the conclusion. For example, the 2nd item in 1989 IberoAmerican Mathematical Olympiad mentioned above can be proved concisely by constructing a geometric graph according to the structural features of the inequality which needs proving.

(b) Directly construct a mathematical object of the conclusion. Take the sixth question in the 32nd IMO for example.

Given a real number $a > 1$, construct a bounded infinite sequence of numbers $x_0, x_1, x_2, \ldots$ such that for each pair of unequal non-negative integers i and j, $|x_i - x_j|.|i - j|^a > 1$.

In this problem, contestants are supposed to directly construct a sequence of numbers satisfying the given conditions.

(c) Construct a counter example satisfying the conditions so as to negate the conclusion. For example, this is the 2nd item in the 2nd-round test for the Chinese National Mathematical Competition for Senior Students.

Is the proposition that a rectangle with a pair of opposite sides of the same length must be a parallelogram, true or false? If true, prove your conclusion. If not, construct a rectangle which is not a parallelogram but satisfies the conditions and then justify your method of construction.

The above 11 kinds of abilities are the main elements required of the questions for the IMO. As they are all related, conditioned and overlapped with each other, the requirements of question-setting in IMO are also comprehensive rather than isolated. Usually a single question calls for several abilities. Only for the purpose of of this discussion did the author sort the examples into different areas with specific emphases.

Zhu Huawei
Zhuhai Superstar Experimental School
Yinwan Rd Wanzi,
Zhuhai, Guangdong, G519030
CHINA

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WFNMC International & National Awards

David Hilbert International Award

The David Hilbert International Award was established to recognise contributions of mathematicians who have played a significant role over a number of years in the development of mathematical challenges at the international level which have been a stimulus for mathematical learning.

Each recipient of the award is selected by the Executive and Advisory Committee of the World Federation of National Mathematics Competitions on the recommendations of the WFNMC Awards Sub-committee.

Past recipients have been: Arthur Engel (Germany), Edward Barbeau (Canada), Graham Pollard (Australia), Martin Gardner (USA), Murray Klamkin (Canada), Marcin Kuczma (Poland), Maria de Losada (Colombia), Peter O'Halloran (Australia) and Andy Liu (Canada).

Paul Erdös National Award

The Paul Erdös National Award was established to recognise contributions of mathematicians which have played a significant role over a number of years in the development of mathematical challenges at the national level and which have been a stimulus for the enrichment of mathematics learning.

Each recipient of the award is selected by the Executive and Advisory Committee of the World Federation of National Mathematics Competitions on the recommendations of the WFNMC Awards Sub-committee.

Past recipients have been: Luis Davidson (Cuba), Nikolay Konstantinov (Russia), John Webb (South Africa), Walter Mientka (USA), Ronald Dunkley (Canada), Peter Taylor (Australia), Sanjmyatav Urjintseren (Mongolia), Qiu Zonghu (China), Jordan Tabov (Bulgaria), George Berzsenyi (USA), Tony Gardiner (UK), Derek Holton (New Zealand), Wolfgang Engel (Germany), Agnis Andžāns (Latvia), Mark Saul (USA), Francisco Bellot Rosado (Spain), János Surányi (Hungary) , Istvan

Reiman (Hungary), Bogoljub Marinkovich (Yugoslavia), Harold Reiter (USA) and Wen-Hsien Sun (Taiwan).

The general meeting of the WFNMC in Melbourne agreed, from 2003, to merge the above two awards into one award titled *the Paul Erdös Award.*

Requirements for Nominations for the Paul Erdös Award

The following documents and additional information must be writtten in English:

- A one or two page statement which includes the achievements of the nominee and a description of the contribution by the candidate which reflects the objectives of the WFNMC.
- Candidate's present home and business address and telephone/telefax number.

Nominating Authorities

The aspirant to the Awards may be proposed through the following authorities:

- The President of the World Federation of National Mathematics Competitions.
- Members of the World Federation of National Mathematics Competitions Executive Committee or Regional Representatives.

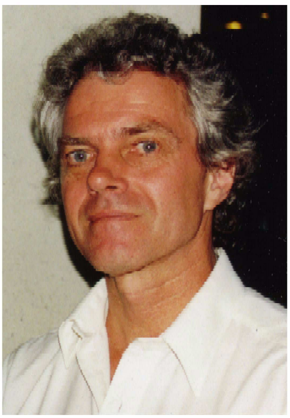
The Federation encourages the submission of such nominations from Directors or Presidents of Institutes and Organisations, from Chancellors or Presidents of Colleges and Universities, and others.

* * *

Dynamic Assessment Methods with Substantially Enhanced Reliability and Efficiency

Graham Pollard & Ken Noble

Professor Graham Pollard was Pro Vice-Chancellor, Division of Management and Technology, University of Canberra. He enjoys teaching probability and statistics, and has a particular research interest in the application of probability and statistics to a range of situations including quantum electronics, mathematical education and scoring systems in sport and assessment.



Dr Ken Noble is a Senior Analyst for ABARE, an Australian Government economic research agency in Canberra, where he designs and develops specialised software for energy system modelling. As a leisure time pursuit, he enjoys devising computer programs for solving puzzles and games.

Abstract

In a recent paper the increased reliability and efficiency of a dynamic assessment procedure using a fixed number of questions and a 1-step dependent question-allocation procedure, was demonstrated. In this paper we demonstrate that a further substantial increase in reliability and efficiency can be achieved by modifying the earlier procedure, and using a question-allocation process that is not a simple one-step dependent one, and by using a sequential stopping-rule in which the number of questions is not fixed.

1. Introduction

In a recent paper the increased reliability and efficiency of a dynamic assessment method for grading examinees into three grades (A, B and C (fail)), was demonstrated (Pollard and Noble (2001)). By using the proposed dynamic method, the better examinees attempted a higher percentage of harder questions than the weaker examinees, and the weaker examinees attempted a higher percentage of easier questions than the better examinees.

The situation considered in the paper referred to above was one in which an examiner was required to categorise each examinee into one of three categories, Pass with Merit (A), Pass (B), or Fail (C) by using two types of questions ... easy (E) questions used primarily to discriminate between examinees who should receive a B rather than a C, and harder questions (H) used to discriminate between examinees who should receive an A rather than a B. Under the proposed dynamic method, questions were allocated in pairs to the examinees using the following one-step dependent procedure:

1. The examination commences with a pair (H,E) of one H and one E question for all examinees.

2. When answering an (H,E) pair of questions, the examinee who

   (a) correctly answers both questions, is next given a pair of hard questions, (H,H).

   (b) incorrectly answers both questions, is next given a pair of easy questions, (E,E).

   (c) correctly answers exactly one of the questions, is next given an (H,E) pair.

3. When answering an (H,H) pair of questions, the examinee who

(a) correctly answers at least one of the questions, is next given an (H,H) pair.

(b) incorrectly answers both questions, is next given an (H,E) pair.

4. When answering an (E,E) pair of questions, the examinee who

(a) incorrectly answers at least one of the questions, is next given an (E,E) pair.

(b) correctly answers both questions, is next given an (H,E) pair.

Examinees were given a fixed number (n) of pairs of questions, and an examinee who obtains:

(i) greater than or equal to 50% of the hard questions correct and greater than or equal to 50% of all questions correct, receives an A.

(ii) less than 50% of all easy questions correct and less than 50% of all questions correct, receives a C.

(iii) an outcome not covered by (i) or (ii) above, receives a B.

The dynamic method using the above one-step dependent procedure was shown to be more reliable, and approximately 37% more efficient, than the static method in which examinees are given a fixed and equal number of H and E questions.

In this paper we use the same framework (the reader is referred to the earlier paper for a description of this general framework) and consider two developments or extensions of the proposed dynamic method. Firstly, we consider a sequential method of allocating questions that takes into account the examinee's performance on earlier questions as well as on the latest pair of questions. Secondly, we consider a sequential stopping-rule in which examinees can be awarded an A, B or C grade prior to answering all n pairs of questions, removing the need to answer unnecessary questions. These two extensions are then combined, and the overall effect on reliability and efficiency noted.

2. A Modified Method of Allocating Questions

We now modify the above one-step question-allocation procedure by making use of information on the examinee's performance on all questions that have been attempted so far. For example, an examinee who has 'performed sufficiently well' on the H questions so far, is given another (H,H) pair even if the last (H,H) pair were both answered incorrectly. Also, for example, following an (H,E) pair, we make use of the performance over all questions so far, to determine whether it is (probably) more appropriate to focus, at the next pair-allocation, on H questions or E questions.

The modified dynamic or sequential procedure involves the allocation of pairs of questions in the following manner:

1. The examination commences with a pair (H,E) of one H and one E question for all examinees.

2. When answering an (H,E) pair of questions, the examinee who

   (a) has correctly answered greater than or equal to 50% of all questions attempted so far, is next given an (H,H) pair if the H question was answered correctly, and an (H,E) pair if the H question was incorrect.

   (b) has correctly answered less than 50% of all questions attempted so far, is next given an (E,E) pair if the E question was incorrect, and an (H,E) pair if the E question was correct.

3. When answering an (H,H) pair of questions, the examinee who

   (a) correctly answers at least one of the questions, is next given an (H,H) pair.

   (b) incorrectly answers both questions, is next given an (H,E) pair unless the examinee has greater than or equal to 50% of H questions correct so far, in which case the examinee is next given an (H,H) pair.

4. When answering an (E,E) pair of questions, the examinee who

(a) incorrectly answers at least one of the questions, is next given an (E,E) pair.

(b) correctly answers both questions, is next given an (H,E) pair unless the examinee has less than 50% of E questions correct so far in which case the examinee is next given an (E,E) pair.

We now compare the previously proposed dynamic method with the modified dynamic method described in the previous paragraph. The number of question-pairs is fixed, as before. Using Tables 1 to 4 from the earlier paper, we can summarise the number of (relevant) misclassifications for various values of (P_H, P_E), where P_H is the probability an examinee answers an H question correctly, and P_E is the probability an examinee answers an E question correctly. These are given in Table 1, with some additional entries not previously tabled. Table 2 is the corresponding table for the modified dynamic method described in the previous paragraph.

A comparison of Tables 1 and 2 indicates that the number of misclassifications is less under the modified method. For example, with 35 pairs of question, the number of misclassifications is reduced by approximately 12% from 26433 to 23176. Table 3 shows the difference in the number of misclassifications for the two methods. It can be seen that the modified method is superior for the smaller values of n relevant in many practical situations.

Turning now to efficiency, we note that when $n = 105$ the number of misclassifications under the modified procedure (namely 1152) is less than the number of misclassifications under the previous proposal when $n = 115$ (namely 1264). So in this case, the modified method is at least 8.7% more efficient than the unmodified method ($100(1 - \frac{105}{115}) = 8.7\%$). For larger values of n, the modified method is approximately 10% more efficient than the unmodified method.

3. A Sequential Stopping-Rule Procedure with the One-Step Dependent Allocation of Questions

It can be seen that when using a fixed number of question-pairs, an examinee is sometimes asked questions even though the examinees performance on these questions will not effect, or is highly likely not to effect, the final grade (A, B or C). The efficiency of the assessment procedure can be enhanced by not asking such 'unnecessary questions'. In this way the number of question-pairs, n, becomes the maximum number of questions given to any examinee, and the expected number of question-pairs asked (less than n) contributes to the improved efficiency.

The stopping-rule considered was as follows. The examinees overall performance is observed after each pair of questions is attempted [1], and an examinee whose performance is such that:

1. the number of H questions correct minus the number of H questions incorrect is greater than or equal to m after $n-3m$, $n-3m+1$ or $n-3m+2$ questions-pair, is awarded an A, provided the examinee has greater than or equal to 50% of all questions correct (m=[$n/3$],[$n/3$]$-$1,[$n/3$]$-$2,...,3,2,1 and $m = 0$ for the case of exactly n question-pairs), or

2. the number of H questions incorrect minus the number of H questions correct is greater than or equal to m. After $n-3m$, $n-3m+1$ or $n-3m+2$ question-pairs, is awarded a B, provided the examinee has greater than or equal to 50% of all questions correct (m=[$n/3$],[$n/3$]$-$1,[$n/3$]$-$2,...,3,2,1), or

3. the number of E questions incorrect minus the number E questions correct is greater than or equal to m after $n-3m$, $n-3m+1$ or $n-3m+2$ question-pairs, is awarded a C, provided the examinee has less than 50% of all questions correct (m = [$n/3$],[$n/3$] $-$ 1,[$n/3$] $-$ 2,...,3,2,1), or

4. the number of E questions correct minus the number of E questions incorrect is greater than or equal to m after $n-3m$, $n-3m+1$ or $n-3m+2$ question-pairs, is awarded a B, provided the examinee has greater than or equal to 50% of all questions correct ($m = [n/3], [n/3]-1, [n/3]-2, \ldots, 3, 2, 1$), or

[1] It is noted that the pair of questions can be considered in a preferential order with the stopping-rule applying after just one question, rather than after the pair. This was not considered, although it is noted that such an approach would lead to a small additional increase in efficiency.

5. the number of E questions incorrect minus the number of E questions correct is greater than zero after n question-pairs, is awarded a C, provided the examinee has less than 50% of all questions correct, or

6. it is not covered by (1) to (5) above, is awarded a B.

Table 4 shows the total number of misclassifications (similar to Tables 1 and 2) for the same (P_H, P_E) values as earlier. It also tables the average number of question-pairs for various values of n, the maximum number of question-pairs, and the mean of these averages, Av1. The average Av1 can be seen to be considerably less than n for these values of (P_H, P_E). In Table 5 we repeat these average calculations for examinees on the A/B and B/C boundaries ((P_H, P_E)) equals (0.5, 0.9) and (0.1, 0.5) respectively), and note that these averages are somewhat larger than the corresponding ones in Table 4, as expected.

In Table 6 we compare this sequential stopping-rule procedure (with the earlier proposed 1-step dependent allocation rule) to the earlier proposal with a fixed number of question-pairs which was described in Section 1. The n_2 in Table 6 is the largest odd value such that the total number of misclassifications for the fixed case is greater than or equal to the total number of misclassifications for the sequential stopping-rule case in the same row of Table 6.

It can be seen from Table 6 that, for (P_H, P_E) values of (.1, .4), (.1, .6), (.4, .9) and (.6, .9), the sequential stopping-rule procedure is approximately 32% more efficient than the fixed number of questions procedure when $n_1 = 11$ (100(1−0.684)=31.6%). For values of (P_H, P_E) close to (.1, .5) and (.5, .9), the sequential stopping-rule procedure is approximately 29% more efficient when n=11 (100(1−0.713)=28.7%). Thus, it can be seen that the increase in efficiency by using the sequential stopping-rule, is not insubstantial. Indeed, for large values of n, the increase in efficiency, for (P_H, P_E) values of such as (.1, .4), (.1, .6), (.4, .9) and (.6, .9), is approximately 43% (100(1−0.567)=43.3%).

4. The Modified Question-Allocation Procedure and the Sequential Stopping-Rule Combined

The effect of combining the efficiency-enhancing methods of Sections 2 and 3 was considered, and Table 7 gives the relevant results for this combination, for the same parameter values considered earlier.

By comparing Tables 4 and 7 it can be seen that, for all values of n from 15 to 145, the combination of the two efficiency-enhancing methods produces fewer misclassifications with smaller average numbers of question-pairs, indicating improved efficiency for all these values of n. For example, it can be seen that the number of misclassifications in Table 7 when $n = 125$ (namely 802) is less than the number of misclassifications in Table 4 when $n = 135$ (namely 854). By comparing the corresponding averages, 62.85 and 70.27, it can be seen that, for large values of n, the increase in efficiency by adding the modified question-allocation procedure to the sequential stopping-rule method is of the order of 10%$(100(1 - \frac{62.85}{70.27}) = 10.6\%)$.

5. A Comparison with the Static Assessment Method

We now compare the method of Section 4 with the static case considered in the paper referred to earlier. Table 8 gives the total number of misclassifications for the static case, and can be derived from Tables 1 to 4 in the earlier paper. The case when $n = 15$ in Table 7 has fewer misclassifications (namely 83240) than the case when $n = 15$ in Table 8 (namely 84891). Thus, when $n = 15$, the increase in efficiency by using the method of Section 4 is at least 53% $(100(1 - \frac{7.03}{15}) = 53.1\%)$. Also, comparing the case when $n = 35$ in Table 7 with the case when $n = 45$ in Table 8, the increase in efficiency is at least 61% $(100(1 - \frac{17.50}{45}) = 61.1\%)$. And comparing the case when $n = 95$ in Table 7 with the case when $n = 135$ in Table 8, the increase in efficiency is at least 64% $(100(1 - \frac{48.05}{135}) = 64.4\%)$. For even larger values of n, increases in efficiency of about 67% are achievable.

6. Conclusions

The efficiency of assessment procedures can be very substantially increased by using dynamic methods. A particular question-allocation rule and a sequential stopping-rule have each been shown to contribute very substantially to the improved efficiency.

Table 1

The Number of Misclassifications in the Unmodified (1-step dependent) Method

(P_H, P_E) Misclassified as	(1, 4) B	(1, 6) C	(4, 9) A	(6, 9) B	Total
$n = 5$	37112	20580	32318	26277	116287
$n = 15$	20220	12811	17876	15193	66100
$n = 25$	12275	8080	11064	9528	40947
$n = 35$	7979	5292	7085	6077	26433
$n = 45$	5211	3595	4694	4127	17627
$n = 105$	518	364	466	428	1776
$n = 115$	378	262	328	296	1264
$n = 135$	161	115	149	129	554
$n = 145$	118	90	120	104	432

Table 2

The Number of Misclassifications with the Modified Method of Allocating Questions

(PH,PE) Misclassified as	(1, 4) B	(1, 6) C	(4, 9) A	(6, 9) B	Total
$n = 5$	37196	21260	31273	26186	115915
$n = 15$	19917	12055	16376	14604	62952
$n = 25$	11877	7212	9476	8900	37465
$n = 35$	7372	4477	5783	5544	23176
$n = 45$	4764	2908	3668	3631	14971
$n = 105$	374	210	258	310	1152
$n = 115$	266	156	185	213	820
$n = 135$	106	59	68	90	323
$n = 145$	81	49	55	68	253

Table 3

The Difference in the Number of Misclassifications : Unmodified Minus Modified Method of Allocating Questions

	Difference (same parameters)
$n = 5$	372
$n = 7$	1080
$n = 9$	1926
$n = 11$	2527
$n = 13$	2755
$n = 15$	3148
$n = 17$	3256
$n = 19$	3301
$n = 21$	3295
$n = 23$	3526
$n = 25$	3482
$n = 35$	3257
$n = 45$	2656
$n = 105$	624
$n = 115$	444
$n = 135$	231
$n = 145$	179

Table 4

The Total Number of Misclassifications and the Average Number of Question-Pairs Attempted for the Sequential Stopping-Rule Procedure with the One-step Dependent Allocation of Questions

n	Total Misclassifications	Average Number of Question-Pairs (1, 4)	(1, 6)	(4, 9)	(6, 9)	Average Av1
5	134061	2.41	2.42	2.42	2.40	2.41
9	111649	3.84	4.04	4.03	3.84	3.94
11	102001	4.62	4.96	4.95	4.62	4.79
15	83922	6.89	7.23	7.22	6.89	7.06
25	54415	11.92	12.72	12.72	11.92	12.32
35	35760	17.26	18.42	18.44	17.23	17.84
45	23709	22.51	24.10	24.09	22.48	23.30
55	16126	27.63	29.69	29.68	27.63	28.66
65	10878	32.72	35.24	35.20	32.73	33.97
75	7481	37.76	40.72	40.70	37.79	39.24
85	5164	42.77	46.16	46.17	42.75	44.46
95	3701	47.73	51.60	51.58	47.72	49.66
105	2504	52.66	57.02	56.97	52.69	54.84
115	1716	57.56	62.40	62.36	57.63	59.99
125	1196	62.50	67.78	67.76	62.52	65.14
135	854	67.41	73.16	73.12	67.39	70.27
145	607	72.29	78.51	78.50	72.30	75.40

Table 5

The Average Number of Question-Pairs Attempted for Examinees on the B/C and A/B Boundaries

n	Average Number of Question-Pairs		Average Av2
	(1, 5)	(5, 9)	
5	2.44	2.44	2.44
9	4.06	4.06	4.06
11	4.99	4.99	4.99
15	7.50	7.50	7.50
25	13.80	13.79	13.79
35	20.88	20.90	20.89
45	28.40	28.38	28.39
55	36.12	36.13	36.12
65	44.05	44.08	44.06
75	52.20	52.20	52.20
85	60.33	60.39	60.36
95	68.74	68.75	68.74
105	77.17	77.14	77.15
115	85.70	85.60	85.65
125	94.21	94.23	94.22
135	102.85	102.87	102.86
145	111.58	111.59	111.58

Table 6

A Comparison of the Sequential Stopping Procedure with the Fixed Number of Question-Pairs Procedure

Sequential Stopping -Rule Case				Relevant Fixed Case Comparison			
n_1	Total misclassification	Av1	Av2	n_2	Total misclassifications	Av1/n_2	Av2/n_2
5	134061	2.41	2.44	3	134482	0.803	0.813
9	111649	3.94	4.06	5	116287	0.788	0.812
11	102001	4.79	4.99	7	102374	0.684	0.713
15	83922	7.06	7.50	9	91179	0.784	0.833
25	54415	12.32	13.79	17	59624	0.725	0.811
35	35760	17.84	20.89	27	37635	0.661	0.774
45	23709	23.30	28.39	37	24430	0.630	0.767
55	16126	28.66	36.12	47	16243	0.610	0.769
65	10878	33.97	44.06	57	11006	0.596	0.773
75	7481	39.24	52.20	65	8139	0.604	0.803
85	5164	44.46	60.36	77	5201	0.577	0.784
95	3701	49.66	68.74	83	4002	0.598	0.828
105	2504	54.84	77.15	95	2600	0.577	0.812
115	1716	59.99	85.65	107	1722	0.561	0.800
125	1196	65.14	94.22	115	1264	0.566	0.819
135	854	70.27	102.86	125	854	0.562	0.823
145	607	75.40	111.58	133	616	0.567	0.839

Table 7

The Results for the Modified Question-Allocation Procedure Combined with the Sequential Stopping-Rule

n	Total Misclassifications	Average Av1
5	132648	2.46
15	83240	7.03
25	52775	12.17
35	33688	17.50
45	21781	22.76
55	14258	27.93
65	9368	33.03
75	6171	38.08
85	4210	43.07
95	2838	48.05
105	1904	52.99
115	1287	57.92
125	802	62.85
135	571	67.75
145	400	72.65

Table 8

The Total Number of Misclassifications for the Static Case

n	Total Misclassifications
5	126362
15	84891
25	61443
35	45630
45	34592
85	12399
135	3778

References

[1] Pollard, G and Noble, K, (2001), The Increased Reliability and Efficiency of Dynamic Assessment Methods, *Mathematics Competitions*, Vol 14, No 2

Graham Pollard
Division of Management and Technology
University of Canberra ACT 2601
AUSTRALIA
email: ghp@ise.canberra.edu.au

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