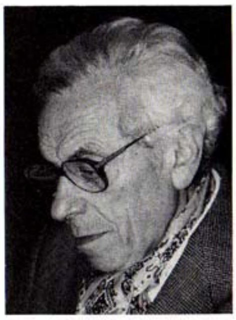
CHILD PRODIGIES [1]

Paul Erdös



Paul Erdös, born in 1913 in Hungary, is an indefatigable traveller, disseminating mathematics challenges to researchers in many countries of the world. He has authored or co-authored 1500 articles or books and so has collaborated with more mathematicians than anyone in history.

I will talk about the child prodigies which I have known. Perhaps I should start with myself since I was something of a child prodigy when I was three years old. I knew how to calculate very well with fairly large numbers. I suppose two of my biggest discoveries were the following ones. When I was four years old I informed my mother that if you take 250 away from 150 you get a number 100 below zero. Also, once when I was five years old I computed the distance to the sun when I knew how many years it takes a train to travel there. Incidently, my parents were both mathematicians. And it is supposed that I learned to compute in the following way:– my father was a prisoner of war in Siberia during the first world war, my mother taught high school and I was left with a German governess. So I was naturally interested in when my mother would be at home and as a result I played with a calendar and knew how many months ahead there would be a holiday. It is possible that this is the way I learned to count. Actually, I started to do mathematical research fairly early. My first paper was written when I was eighteen years old when I gave a new proof for the theorem of Tchebychev that there is always a prime between N and $2N$. But enough about myself.

[1] The author presented a keynote address with the title *Child Prodigies I have Known* at the WFNMC Congress in Pravetz, Bulgaria, July 1994. This paper consists of the paper of his which appeared in the proceedings on *Number Theory*, Washington State University, Washington, 1971, 1–13, with some endnotes written in 1994.

Not all mathematicians were child prodigies but some of them were. For example, Guass was a child prodigy while Hardy certainly was not, von Neumann and Norbart Wiener were both child prodigies. But now I will talk only about the child prodigies which I had personal contact with.

The first two whom I will mention I did not have much contact with so I will be brief.

Peter Lax, who was recognised as a child prodigy in Hungary when he was about twelve, came to the United States when he was sixteen and had a letter of introduction to me so I met him almost on his arrival. We met a great deal in New York and Princeton. I still remember when Halmos said at the Institute for Advanced Study that there was a baby here from Hungary and that you could talk mathematics with him. Peter Lax wrote his first paper in fact on a problem of mine on polynomials but shortly after he moved off to a far away field about which I know nothing, differential equations, and I did not have much mathematical contact with him after that.

The second child prodigy was Peter Ungar who was discovered in Hungary when he was very young, about fourteen or fifteen years old. By the time he was fifteen Turán quoted some of his results in a paper on the Riemann ζ - function. One of his problems in the *American Mathematical Monthly* he composed when he was only fourteen years old. I did not have much contact with him and there is not much more I can say. Both Peter Ungar and Peter Lax are now professors at New York University.

I will now talk about Pósa who is now 22 years old and the author of about eight papers. I met him before he was twelve years old. When I returned from the United States in the summer of 1959 I was told that here was a little boy whose mother is a mathematician and who knows all there is to know in high school. I was very interested and the next day I had lunch with him and Róza Péter, a Hungarian mathematician who worked with him. While we had lunch and Pósa was eating his soup I gave him the following problem: Prove that if you have $n+1$ integers less than or equal to $2n$ then there are always two of them which are relatively prime. It is quite easy to prove that the theorem is not true for n integers since, if you take the multiples of two, no two of them are relatively prime. Actually I discovered this simple result some years ago, but it took me about ten minutes to to find the simple proof. Pósa ate his soup and then said 'If you have $n+1$ integers less than or equal to $2n$ then two of them are consecutive and hence relatively prime.' It is needless to say that I was very much impressed, and I think this is on the same level as Guass when he was seven years old and found the sum of the integers up to 100. Incidentally, the following problem is still unsolved: Denote by $f_k(n)$ the largest value of r so that there is a sequence $1 \leq n_1 < \cdots < a_r \leq n$ so that one cannot select $k+1$ a's which are pairwise relatively prime. I believe than $f_k(n)$ equals the number of multiples not exceeding n of the first k primes. After that meeting I worked systematically with Pósa and I wrote him many letters with problems during my travels. Before he was about twelve years old he proved the following theorem which I had given him. If you have a graph of $2n$ vertices and n^2+1 edges it always contains a triangle. This is a special case of a well known theorem of Turán. Also, I gave him the following problem: Take an infinite series whose n^{th} term is defined as follows: the numerator is 1 and the denominator is the least common multiple of the integers from 1 to n. Prove that the sum of the series is irrational. This is not very difficult to prove but is certainly surprising that a twelve year old could do it. When he was a little over thirteen, I explained to him Ramsay's Theorem for the case $k=2$. The theorem states as follows: Suppose you have an infinite graph, then the graph either contains an infinite complete graph or infinite independent set. In other words, there is an infinite set so that any two of those vertices are joined or no two of the vertices are joined. It took about fifteen minutes until Pósa understood it and then he went home, thought about it all evening and before going to bed he had the proof. By the time Pósa was fourteen you could talk to him as a grown up mathematician. I called him on the phone and asked him about problems and if the problem was about elementary mathematics it was very likely he had some relevant and intelligent comments. It is perhaps interesting to remark that he had some difficulty with calculus and while he understood it and could use it, he was never as much at home as he was with combinatorial analysis and elementary number theory. He never liked geometry. I tried to give him some problems in elementary geometry but he never liked them. He always liked to do only what he was really interested in, and at that he was extremely good. Our first joint paper was written when he was fourteen and a half years old, and I will tell you of his contribution. Pósa wrote many significant papers also by himself. Some of these still have a great deal of importance. His best known and most quoted paper is on Hamiltonian lines and he wrote it when he was fifteen. Unfortunately, for about 4 or 5 years he has not proved or conjectured much and I often comment sadly that he is dead and but I very much hope that he will come back to life soon. I first got worried about him when at age sixteen he would rather be Dostojewsky than Einstein.

His first theorem which he conjectured and proved by himself and which was original was as follows: Every graph of n vertices and $2n-3$ edges contains a circuit having a diagonal. The result is the best possible: There is a graph of n vertices and $2n-4$ edges which does not contain a circuit with a diagonal.

Denote by $f(n,k)$ the smallest integer so that every graph of n vertices and $f(n,k)$ edges contains k disjoint circuits. In our first joint paper we determined $f(n,k)$ for $n > 24k$. I previously had proved that $f(n,2) = 3n-5$ for $n \geq 6$ but my proof was complicated and did not generalise for $k > 2$. Later I found out that Dirac also had proved $f(n,2) = 3n-5$. I told Pósa the problem of determining $f(n,2)$ and in a few days he found a very simple proof of $f(n,2) = 3n-5$, and I could very easily extend his proof for general k. I will now give the outline of his proof. We use induction for n. It is easy to see that $f(n;2) = 3n-5$ holds for $n = 6$. Assume that it holds for $6 \leq m < n$ and we prove it for n. Our graph must have a valency ≤ 5 (since

$$\frac{1}{2}\sum_{i-1}^{n} v(X_i) = 3n-5,$$

where $v(X_i)$is the valency of the vertex X_i.) If our graph C_j has a vertex of valency ≤ 3, we omit it and by the induction hypothesis the remaining graph has two vertex disjoint circuits. Thus we can assume that our graph has a vertex x_1 of valency 4 or 5. Assume first that $v(X_1) = 5$ and let $X_2, \ldots, X_6$ be the vertices joined to X_1. Since our theorem holds for $n = 6$ the subgraph spanned by the vertices $X_1, \ldots, X_6$ has at most 12 edges–hence without any loss of generality we can assume that the edges (X_2, X_3) and (X_3, X_4) are missing from our graph. Omit now from C_j the vertex X_1 and all edges incident to it but add (X_2, X_3)

and (X_3, X_4). The new graph has $n-1$ vertices and $3n-8$ edges, thus by the induction hypothesis it contains two vertex disjoint circuits, only one of these can contain the new edges (X_2, X_3) or (X_3, X_4). Let us assume that one of these circuits contains (X_2, X_3). We then omit (X_2, X_3) and replace it by (X_2, X_1), (X_21, X_3). Thus our original graph has two vertex distinct circuits, as stated. Assume next that X_1 has valency four and is joined to x_2, x_3, X_4 and X_5. All the edges $(X_i, X_j), 1 \leq i < j \leq 5$ must be in C_j, for if say (X_2, X_3) would not be in C_j, we would add (X_2, X_3), omit X_1 and the previous proof would apply.

Now count the number of edges of C_j incident to one of the vertices X_1, X_2; $X_3, X_i, (i > 5)$ cannot be joined to X_1 and if it is joined to both X_2 and X_3 the two triangles (X_2, X_3, X_i) and (X_1, X_4, X_5) are vertex disjoint. Thus we can assume that each $X_i, i > 5$ is joined to only one of X_1, X_2, X_3. This gives at most $n-4$ edges, there are 9 more edges spanned by $X_1, \ldots, X_5$ incident to X_1, X_2 or X_3. Thus there are at most $n+4$ edges of C_j incident to X_1, X_2 and X_3. Omit now the vertices X_1, X_2, X_3 and all the edges incident to them. The remaining graph has $n-3$ vertices and at most $3n-5-(n+4) = 2n-9$ edges. Since $2n-9 \geq n-3 \quad (n \geq 6)$, this graph contains a circuit which is vertex disjoint from (X_1, X_2, X_3). This completes the proof.
You will agree that it is a remarkable proof for a child of 14.

It was not difficult to extend this proof for $k > 2$ by induction with respect to k. Pósa also found a very beautiful proof that every graph of n vertices and $n+4$ edges contains two edge disjoint circuits.

There was another child progedy in the university town of Szeged called Máté. He was a year older than Pósa and in fact the two children met in my mother's apartment. As many of you know, I call children "ϵ's" and Máté when he was phoning me once (I was abroad), introduced himself to my mother: 'I am the ϵ from Szeged'. The two ϵ's those days were really enthusiastic and always wanted new problems. I remember once I had a bad headache and wanted to go home to rest and had difficulty getting away from them.

Máté got to university two years early by winning the Kurshak competition, Pósa could have gone to university two years ahead but he liked high school very much and refused—I will tell you more about his high school later.

Máté works mainly on set theory now, has written several papers and is writing a book with the well known set theorist G. Podor. I proved some time ago the following theorem: To every real X there corresponds a nowhere dense set of reals $A(X)$. Two points X and Y are independent if $X \notin A(X)$ and $Y \notin A(X)$. A set is independent if every two of its elements are independent. I proved that there is always an uncountable independent set. This question is still open. Máté proved the following result: To every ordinal $\alpha < \omega_1$ there is an independent set of type α. He found this proof by himself and the proof uses category arguments very cleverly.

In Hungary the children go to elementary school for 8 years and there are 4 years of high school. A few years ago they started a special high school for children gifted in mathematics. The school had just opened when Pósa was due to go to high school and he liked this high school very much. It was not long before he told me that there were boys in his class who were better at elementary mathematics than he was. I would like to say a few words about two of them, Lovász and Pelikán, who are both now 22 years old. Lovász is perhaps the most successful of the prodigies up to now, his career showed no breaks like Pósa's. He started scientific work a little later at the ripe old age of nearly 17, but has done outstandingly well mainly in combinatorial mathematics. He was the first to give a construction for a graph of arbitrarily large chromatic number and arbitrarily large girth. He did this while he was still in high school and the construction is very ingenious and difficult.

Pelikán also worked mainly in graph theory and is the author of several papers. Those who were at the graph theory meeting in Tihany in 1966 will remember him, he was just out of high school and gave a talk in excellent English about his recent results. It was a masterfully clear lecture and nobody could have believed that this was his first lecture. I had much less contact with Lovász and Pelikán than with Pósa.

Before I continue I would like to make a few conjectures about the reason there are so many child prodigies in Hungary. First of all there was for at least about 80 years a mathematical periodical for high school students, also they have many mathematical competitions—the Eötvös-

Kurshak competition goes back 75 years (see the Hungarian problem book). After the first world war a new competition was started for high school students who were just finishing high school and after the second world war several new competitions were started, the most interesting being the Schweitzer competition. In this competition there are about ten problems given and the competitors have more than a week to send in the solutions. Collaberation is not allowed but any books can be used. Recently, a book was published containing these problems. In the last few years I have heard many reports about child prodigies in this country as well. I have met some of them, eg Grost from Michigan State, but I do not know enough about them. I would like to mention only one, namely Turansky. I met him at the University of Pennsylvania when he was 17. He was extremely talented, but, due to unknown causes he never gained a Ph D and was killed in a traffic accident when he was 35.

A few years ago a new kind of competition was started in Hungary for high school students which was held on television. The competitors are given questions which they have to answer in 2 or 3 minutes.The questions are usually very ingenious and the competitors are judged by a panel of leading mathematicians, eg Alexits, Hajos and Turán. It seems that many people watch these competitions with great interest even if they do not understand the problems.

Graph theory and combinatorial analysis is a very suitable field for young mathematicians to do original work—there are still many unsolved problems whose solutions require only ingenuity and not much knowledge. There is also some danger in this since there is a temptation for young geniuses just to prove and conjecture and not learn other branches of mathematics.

Finally I want to talk about a child prodigy who is still in high school, I. Ruzsa. Pósa introduced him to me two and a half years ago when he was not fifteen at the time. His special interest was number theory and he was especially good at raising new and interesting problems. He has several papers which will soon be published. Ruzsa, Sárközi and I have a long forthcoming paper on additive number theoretic functions. Let me mention some of his problems and results. Let $f(n)$ be an integer valued function which satisfies $f(a) \equiv f(b) \pmod{m}$ if $a \equiv b \pmod{m}$ for every a, b and m. Polynomials satisfy this condition. Ruzsa proved that if $f(n)$ satisfies this condition and is not a polynomial then

$$(1) \qquad \lim_{n\to\infty} \frac{|f(n)|}{n^k} = \infty$$

for every k and he also proved that (1) is the best possible result. He further shows that for infinitely many n

$$|f(n)| > (e-1)^{n(1+o(1))}$$

and he conjectures that $f(n) > e^{n(1+o(1))}$ infinitely often.

Let $f(n)$ be a multiplicative function whose value domain is an abelian group of order N. It is true for every element g of our group the integers n for which $f(n) = g$ have a density $1/N$. If the abelian group has only the elements ± 1 then this is an old conjecture of mine proved first by Wirsing.

Some time ago I asked the following question: Let $1 \le a_1 < \cdots < a_k \le X$ be a sequence of integers so that for every n the number of solutions of $n = pa_i$ for p prime is at most 2. Is it true that $\max k = o(X)$? It is easy to see that if we require that the products pa_i are all distinct, ie the number of solutions of $n = pa_i$ is at most 1, that $k = o(X)$, this easily follows from the fact that the numbers $\dfrac{a_i}{P(a_i)}$ must all be distinct where $P(a_i)$ is the greatest prime factor of a_i.

I gave this problem to Ruzsa who very soon showed that $k > o(X)$ is indeed possible. He argued as follows: Let $b_1 < b_2 < \cdots$ be a sequence of those integers which do not have two prime factors p and q satisfying $p < q < 2p$. It is not too difficult to show that the density of the b's exists and is positive. Thus if this density is α then there are $(1+o(1))\,\alpha X/2$ b's in the interval $\left(\dfrac{X}{2}, X\right)$. A simple argument shows that for these b's the number of solutions of pb_i is indeed < 3. Thus $\max k \ge \left(\dfrac{\alpha}{2} + o(1)\right) X$ which answers my question.

You will agree with me that Ruzsa shows ability of the highest degree and one can hope and expect that he will become a great mathematician. Ruzsa incidentally did outstandingly well this year at the mathematical olympiad.

Before I finish my talk let me tell you a few anecdotes about the child prodigies. Lovász and Pósa when they still went to high school asked me why there are so few girl mathematicians. I told them 'Suppose the slave children (boys) would be brought up with the idea that if they are very clever the bosses (girls) would not like them—would there be then many boys who do mathematics?' Both said 'perhaps not so many'.

In Hungary, many mathematicians drink strong coffee, in fact Rényi once said 'a mathematician is a machine which turns coffee into theorems.' At the mathematical institute they make particularly good coffee and when Pósa was not quite 14 I offered him a little strong coffee which he drank with an infinite amount of sugar. My mother was very angry that I gave the little boy strong coffee. I answered that Pósa could have said 'madam, I do a mathematician's work and drink a mathematician's drink.' I saw a movie many years ago where a lady sees a boy of 16 drink whiskey with the older man and is shocked. The boy says 'madam, I do a man's work and drink a man's drink.'

When Lovász was still an ϵ in the first of high school, he and a friend, a fellow mathematician, courted the same boss-child, also a mathematician and not a bad one as bosses go. The two slave-children asked her to choose. She chose Lovász and, in fact, they were married last year. Milner improved on this story by changing her answer as follows: 'I will choose the one who proves the Riemann hypothesis.' If so, I hope Lovász will become a great mathematician whose name will be remembered though the story will perhaps survive in this form.

Endnotes

I think it is of some interest to state what happened to the child prodigies. I wrote the paper above in 1970, 24 years ago.

Lovász became one of the leading mathematicians, his work in combinatorics and computer science will survive him by centuries, his work on the probability method (local lemma) is very important. His wife Kati Lovász did nice work on geometry and we have two triple papers in geometry.

Pósa perhaps did not do as much research as promised by his talent. However, he became a very successful teacher of clever children. Since 1971 he has done very important work on Hamiltonian cycles of random graphs. We have a significant triple paper with Hajnal and he also has many interesting unpublished results in geometry, set theory and combinatorics.

Ruzsa did wonderful work in number theory and settled many of my conjectures. His work on essential components is very nice but if I were to discuss all his results I would have to write another paper.

Máté is now professor at the City University of New York. He has done important work in set theory and on orthogonal polynomials.

Bollobás was also one of my child prodigies and we have many joint papers. He is one of the leading experts in extremal graph theory and the probability method. Besides his many significant results, he has written many excellent books and has done good work in analysis as well. He has also had many excellent students.

Peter Ungár has done some very beautiful work in combinatorial geometry.

Peter Lax is one of the leading experts in partial differential equations, a subject of which I am shamefully ignorant.

In the U.S., Donald Newman was one of my very successful child progidies. His work on approximation by rational functions is epoch making.

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