A SOLUTION TO 1988 IMO QUESTION 6
(The Most Difficult Question Ever Set at an IMO)

Theorem

If a, b are integers ≥ 0 such that

$$q = \frac{(a^2 + b^2)}{(ab + 1)}$$

is integral, then

$$q = (\mathrm{GCD}(a,b))^2$$

Proof

If $ab = 0$ (i.e. if $a = 0$ or $b = 0$) the result is plain. This suggests using induction on ab.

If $ab > 0$, we may suppose (from the symmetry of the problem) that $a \leq b$, and the result proven for smaller values of ab.

The next step is to find an integer c satisfying

$$q = \frac{(a^2 + c^2)}{(ac + 1)} \qquad \textbf{(1)}$$

and $$0 \leq c < b \qquad \textbf{(2)}$$

It will then follow by induction (since $ac < ab$) that

$$q = (\mathrm{GCD}(a,c))^2 \qquad \textbf{(3)}$$

To obtain c, we solve

$$\frac{(a^2 + b^2)}{(ab + 1)} = \frac{(a^2 + c^2)}{(ac + 1)} = q$$

Because these ratios are equal, we may subtract numerators and denominators to give

$$\frac{(a^2 - b^2)}{(ab - ac)} = q,$$

i.e.

$$\frac{(b + c)}{a} = q \qquad \text{(since we want } c \neq b\text{)}.$$

so that

$$c = aq - b.$$

Notice that c is an integer, and

$$\mathrm{GCD}(a,c) = \mathrm{GCD}(a,b).$$

Therefore the proof will be finished if we can prove (2).

To prove (2) we note, on the one hand, that

$$q = \frac{(a^2 + b^2)}{(ab + 1)}$$

$$< \frac{(a^2 + b^2)}{ab} = \frac{a}{b} + \frac{b}{a}$$

giving

$$aq < \frac{a^2}{b} + b \leq \frac{b^2}{b} + b \text{ (since } a \leq b\text{)}$$
$$= 2b$$

Thus

$$aq - b < b,$$

i.e.

$$c < b.$$

On the other hand

$$q = \frac{(a^2 + c^2)}{(ac + 1)}$$

implies

$$ac + 1 > 0$$

implies

$$c > \frac{-1}{a}$$

implies

$$c \geq 0 \qquad \text{(since } c \text{ is integral!)}.$$

This completes the proof.

Some further remarks

Let $q = k^2$ (k a positive integer) be given. A careful study of the above proof shows that all solutions

$$(a,b) \qquad (a,b \text{ integers} \geq 0)$$

to

$$\frac{(a^2 + b^2)}{(ab + 1)} = q \tag{4}$$

are obtainable by applying a sequence of the operators (transformations)

$$S: \qquad (a,b) \rightarrow (b,a)$$

$$T: \qquad (a,b) \rightarrow (a,aq - b)$$

to the solution $(k,0)$. Because S^2 (S applied twice) and T^2 leave (a,b) unchanged, every solution to (4) has the form

$$W(k,0)$$

where W denotes a (possibly empty) sequence of S's and T's in which *adjacent operators are never the same.* Conversely, if W is such a sequence then $W(k,0)$ is a solution to (4) because the operators S and T leave the function

$$\frac{(a^2 + b^2)}{(ab + 1)}$$

unchanged.

In order to identify those solutions

$$(a,b) = W(k,0)$$

with $a,b \geq 0$, we display the first few applications of S and T to $(k,0)$ as a tree:

$$\begin{tikzcd}
 & (k,0) \arrow[dl, "S"'] \arrow[dr, "T"] & \\
(0,k) \arrow[d, "T"] & & (k,k^3) \arrow[d, "S"] \\
(0,-k) \arrow[d, "S"] & & (k^3,k) \arrow[d, "T"] \\
(-k,0) \arrow[d, "T"] & & (k^3,k^5 - k) \arrow[d, "S"] \\
(-k,-k^3) & & (k^5 - k,k^3)
\end{tikzcd}$$

Evidently, the solutions (a,b) to (4) with $a,b \geq 0$ are

$$(k,0),\ (0,k);\quad \text{and}\quad W(k,0)$$

where the sequence W ends in T (so that T is applied *first* to $(k,0)$). One final remark (which we leave the reader to verify):

If $q = 1$ there are exactly 3 such solutions;
If $q > 1$ there are infinitely many such solutions.

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