# The Challenge of Discovering Mathematical Structures: Some research based pedagogical recommendations for the secondary classroom

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*Abstract*
This contribution addresses the use of number theoretic and combinatorial problems in facilitating the discovery of deeper mathematical structures. In particular, the paper reports on the outcome of teaching experiments in secondary classrooms with the use of problems involving Steiner triple systems, the Dirichlet principle and Diophantine n-tuples. Data collected from the teaching experiments are analyzed and used to construct research based pedagogical recommendations for the secondary classroom, particularly reflections on the types of mathematical challenges that are accessible to secondary students.

*Introduction*
The ICMI 16 Study on challenging mathematics in and outside the classroom comes at an opportune time, given the growing discrepancy between what constitutes "appropriate" K-12 mathematics among professional mathematicians and mathematics educators in many countries. A particularly troubling manifestation of this is evidenced in the "math wars" in the United States and Canada with each "side" prescribing mathematics content and pedagogy for schools. The continual clash of political and ideological forces in many countries on what is mathematics has resulted in schools being unable to implement any mathematics curriculum for a sufficiently long period to time, nor to train their teachers on effectively implementing the particular curriculum in use (traditional, reform or otherwise). The student is ultimately the one whose learning experiences are compromised. Given this broad preamble, this ICMI focus on the role of mathematical challenges with a broad ensemble of mathematicians and educators in participation offers an opportunity to tackle core issues which when answered offer teachers and students the possibility of experiencing afresh the joy of mathematics, its teaching and learning. As a former public school teacher, research mathematician and an active mathematics educator, I find the first four questions particularly appealing. More importantly, I have conducted numerous studies in high school classrooms, which offer a start at understanding how challenges contribute to the learning process, how challenges can be used in the classroom, and opportunities to challenge that enhance the teaching and learning experience in the regular classroom. In the remainder of this paper, I present same research based pedagogical recommendations for the secondary classroom, relevant for this particular ICMI study based on the results of various teaching experiments conducted in high school classrooms.

*Mathematical Challenges in the secondary classroom*
The author conducted several teaching experiments at a rural mid-western high school with ninth grade students enrolled in various sections of a beginning algebra course, consisting of a heterogeneous group of learners. The author was also the teacher of each of these algebra courses. The goal of these teaching experiments were to offer mathematical challenges not provided by the regular curriculum and to study how students abstract and generalize

advanced mathematical concepts, with implications for new ways of differentiating the curriculum for all learners. Each teaching experiment typically lasted one semester, involved a different group of students and was designed as follows. The students in each class had to maintain a mathematics journal, in which they solved one non-routine problem every other week. Students were asked to restate the problem in their own words, devise a strategy to solve the problem, and then write a summary of their solution. They were also encouraged to include all their scratch work in their journals. Full credit was granted to all students who followed the cues and included all their work, irrespective of the correctness of their solutions. The problems for each of these experiments were carefully chosen to facilitate the discovery and formulation of mathematical generalizations. The problems were chosen from combinatorics and number theory, which were not covered by the $9^{th}$ grade curriculum. The author chose these topics in particular because (a) counting and arrangement problems were easy to understand, and (b) previous mathematics education research indicated that students brought considerable intuition to such problem situations.

*Experiment 1*

In the first experiment four problems that resulted in the construction of Steiner Triple Systems (STS) were assigned over the course of three months (Sriraman, 2004a). A STS is an arrangement of 'n' objects in triplets such that every pair of objects appears in a triplet exactly once. Formally speaking, a STS consists of a set *X* of *n* points, and a collection *B* of subsets of *X* called *blocks* or *triples*, such that each block contains exactly 3 points, and any two points lie together in exactly one block. One of the oldest results in Block Design Theory is Kirkman's existence theorem for STS, which states that a STS of order *n* exists if and only if *n* = 0 or n is congruent to 1 or 3 mod 6. The sequence of possible values for n are 3, 7, 9, 13, etc. There exists ½ n (n-1) such pairs and the number of required triplets is one-third the number of pairs It can be mathematically proven that there exist ½ n(n-1) such pairs and that the number of required triplets is one-third the number of pairs. It is possible to construct a Steiner Triple system only when each object is in ½ (n-1) triples, with the restriction that these numbers are integers. The goal was to present students problem solving situations which would facilitate student discovery of this beautiful mathematical structure and the formulation of the statement of this theorem.

*Experiment 2*

In the second experiment, five problems whose solutions were characterized by the pigeonhole principle were assigned over the course of three months (Sriraman, 2003a, 2004b). The pigeonhole principle states that if 'm' pigeons are put in 'n' pigeon holes and if m>n, then at least one pigeonhole will have more than one pigeon. The beauty of this obvious principle is the wide range of mathematical situations to which it can be applied.

*Experiment 3*

Finally, in the third experiment, which lasted nearly the whole school year, the Diophantine n-tuple problem was developed starting with the simplest cases, resulting in the search for the general 5-tuple pattern (Sriraman, 2003b). A diophantine n-tuple is a set of 'n' positive integers such that the product of any two is one less than a square integer. It was the authors' hope that a very elementary version of the problem would kindle student interest and eventually result in an attempt to tackle the as yet unsolved 5-tuple problem in integers. Does there exist a diophantine 5-tuple?

*The Pedagogical Set Up*

In all experiments students were given one to two weeks to solve each problem, after which the journals were collected. The author communicated exclusively on a one on one basis with the students through the journals. The author was also available before and after school if students had any questions. In the first experiment the communication between teacher and student was predominantly written. In the second experiment all nine students were extensively interviewed over the course of three months. In the third experiment, the communication was written with one period per week for student presentations of progress made and informal discussions before and after school. There was an extensive debriefing session at the end of each teaching experiment. The results of these three experiments were as follows.

*Results*

<u>*Experiment 1*</u>

In the first teaching experiment, students were presented STS structure problems framed in the context of recreational arrangement problems such as inviting people over for dinner, schoolchildren on a walk (a variation of Kirkman's famous problem), prisoners chained in triplets. Over the course of the first semester, out of the sixteen students in the class, three students (Leah, Farram and Seth) were able to gain a substantial insight into the structure of STS. Out of the remaining thirteen students, five attempted to solve the generalized problem, namely: In an arrangement of "n" objects in triplets, how many ways can each pair of objects appear in a triplet once and only once? How many pairs are possible? How many triplets are required for the pairing to occur?, but were unable to fully discern the underlying pattern detected by Leah, Farram and Seth. Six students were able to solve the dinner and schoolgirl problems and partially solve the prisoner problem. Two students were able to solve the dinner and schoolgirl problems. Even though only half the students were able to make a genuine attempt at the general problem, all students in this class were engaged in constructing mathematical representations reasoning, abstraction and generalization over an extended time period. Thus the goal of getting students to "think mathematically" and leading to a discovery and understanding of STS was realized. At the end of the semester I presented the individual student solutions to the various problems and discussed the history and structure of STS. The students were very pleased to hear that they had achieved partial success at problems normally encountered at the undergraduate level. It was also aesthetically pleasing that the students ' understanding of STS paralleled the historical evolution of ideas that came out of the recreational problems leading to the formalization of STS.

<u>*Experiment 2*</u>

In the second teaching experiment involving problems with the pigeonhole structure, five non-routine combinatorial problems in varying but understandable contexts were assigned as journal problems, at increasing level of complexity. The hope was that some of the students would eventually be able to discover this general principle by discerning similarities in the structure of the problems and their solutions. The data for this study was collected through students' journal writings, clinical interviews and my journal writings/notes. The interviews were open-ended with the purpose of getting students to elicit their thought processes in solving a given problem. In all five rounds of interviews were conducted with the nine students over a three month period. The author documented the evolving strategies of the students and classified them according to Lester's (1985) problem solving model, which consists of four categories. *Orientation* refers to strategic behavior to assess and understand a problem. It includes comprehension strategies, analysis of information, initial and subsequent representation, and assessment of level of difficulty and chance of success. *Organization* refers to identification of goals, global planning, and local planning. The category of *execution* refers to regulation of behavior to conform to plans. It includes performance of local actions, monitoring progress and consistency of local plans, and trade-off decisions (speed vs. accuracy). Finally, *verification* consists of evaluating decisions made and evaluating the outcomes of the executed plans. Data gathered through journal writings, open-ended interviews and classroom observations was analyzed using techniques from grounded theory (Glaser & Strauss, 1977). Lester's (1985) model was operationalized in order to code data and to identify variables related to successful and unsuccessful generalization strategies. Four students were successful in discovering, verbalizing, and in one case successfully applying the generality that characterized the solutions of the five problems, whereas five students were unable to discover the hidden generality. However the important point here was that even the "unsuccessful" students made valiant attempts at solving these problems and were able to experience both the joy and frustration of mathematics. Two new categories emerged as the result of adapting Lester's (1985) model, namely *generalization*, and *reflection*. *Generalization* was characterized as the process via which students identified commonalties in the structure of the problems and their solutions. It included making analogies as well as specializing from a given set of objects to a smaller one. *Reflection* in this study was characterized as the process by which the student abstracted knowledge from actions performed on the problems. In other words reflection consisted of thinking about similarities in the problems and solutions, and abstracting these similarities over an extended time period.

*Experiment 3*

The rationale for the third teaching experiment involving Diophantine n-tuples was that students in their study of two years of high school algebra encounter very little experience with proof. In fact most traditional algebra curricula takes the properties of the real number system as a priori truths, focuses on algebraic representations of simple geometric curves on the Cartesian co-ordinate system with endless manipulations of polynomial equations and inequalities. A substantial amount of time is also spent on solving contrived "word problems" under the guise of "applications" in order to somehow justify the topics that have been covered. There is very little room for 'play' or exploration within such an approach. This approach also does not convey to the student that mathematics is a process of ongoing conjecture, proof and refutation (Lakatos, 1976). This begs the question as to how does a teacher go about implementing an approach that allows for the process of conjecture, proof, and refutation. As stated previously, one of the course requirements was that students maintain a mathematics journal. It is the author's belief that writing can be a powerful medium of expression. However, the idea of a math notebook or a journal is not new. In the history of mathematics, many mathematicians like Gauss, Euler, Fermat and Ramanujan wrote their calculations, conjectures and results in notebooks. There is also a substantial body of research in mathematics education that expounds on the value of math journals in mathematics teaching and learning (Borasi & Rose, 1989; Chapman, 1996; Elliot, 1996; Waywood, 1992). Further, journals are useful for diagnosing misconceptions, an avenue for establishing one on one communication with each student, as well as tailoring a portion of instruction to cater to the needs of the individual student. The teacher provided written feedback to each student. Students were allowed to work together after they had attempted the problem on their own and received individual feedback from the teacher. It took two months for all students to get used to writing mathematics and about mathematics. One class period per week was set aside for students to present their solutions to the class, and defend their strategies with the teacher acting as the facilitator in this process. This set the stage for investigating an unsolved diophantine problem.

The Greek mathematician Diophantus (200 A.D) is renowned for his work on solving equations with rational number solutions. Number theorists relish tackling diophantine equations with integer solutions. The beauty of many diophantine equations lies in the fact that they are easy to understand, yet very difficult to solve. Fermat's Last Theorem is a notorious example to illustrate this point. Elementary number theoretic concepts such as prime numbers, and tests for divisibility are introduced in most middle school curricula. However at the high school level, the curriculum offers students very little opportunity to tackle number theoretic problems. For example, many questions in number theory may be posed as diophantine equations -- equations to be solved in integers. Catalan's conjecture, which was only very recently resolved, was whether 8 and 9 were the only consecutive powers? This conjecture asks for the solution to $x^a - y^b=1$ in integers. The Four Squares Theorem states that every natural number is the sum of four integer squares. In other words, it asserts that $x^2 + y^2 + z^2 + w^2 = n$ is solvable for all $n$. But the attempt to solve these equations requires rather powerful tools from elsewhere in mathematics to shed light on the structure of the problem (Oystein, 1988). Students in this course were introduced to elementary diophantine equations under the guise of recreational journal problems. The problem chosen for investigation was the classic n-tuple diophantine problem supposedly posed by Diophantus himself for solutions in the rationals. Simply put, a diophantine n-tuple is a set of n positive integers such that the product of any two is one less than a square integer. It was the authors' hope that a very elementary version of the problem would kindle student interest and eventually result in an attempt to tackle the as yet unsolved 5-tuple problem in integers. Does there exist a diophantine 5-tuple? The author initiated this problem by simply mentioning in class off hand the 3-tuple problem, if one considers the integers 1,3, and 8, then it is always the case that the product of any two is always one less than a perfect square. Indeed $1\times3 = 2^2 - 1$; $1\times8 = 3^2 - 1$; and $3 \times 8 = 5^2 - 1$. This remark led students to wonder if other such 3-tuples existed? This problem was then assigned as a recreational journal problem. The journal response of Heidi is presented below.

Problem: Are there other 3-tuples?

Heidi: I first looked for 2-tuples cause looking for 3 numbers right away was kinda confusing and complicated. I started with 2 as the first number and then changed it to 3 and 4 and 5. This is what I found.

(2,4); (2,12); (2,24); (2,40)….So 2 times what number is one less than a square, and the numbers were 4,12,24,40,60 and it goes on and on. Then I did the same thing for 3, and found (3,1); (3,5); (3,8); (3,16)…so 3 times what number is one less than a square and the numbers were 1,5,8,16,21,33, and on and on.

Now with 3-tuples, it was hard. I started with 2,4 and looked for a third number.

So 2 times 4 is 8, 1 less than a square. Now 2 times a different number and 4 times a different number have to both like work at the same time. I started trying any numbers at first, then I went in sequence from 1, and I found 12.

So, (2,4,12) would be another 3-tuple. With 3 and 1, you gave us 8 that worked, so (1,3,8) is another one which we already had.

Other students in the class found different 3-tuples, which led to the following natural questions: What is the pattern for 3-tuples? Are there 4-tuples? These simple questions eventually led to an investigation of the unsolved 5-tuple problem over the course of the school year, whose progression was documented and the mathematics created by the students was reported in Sriraman (2003b). Over the course of the semester the students attempted to solve the open 5-tuple diophantine problem being immersed in the process of conjecture, proof and refutation. Once a week, the teacher utilized one class period to allow students to present their work to the other students. The teacher acted as a moderator in this process to facilitate the discussion. Many of the students were surprised at the difficulty of solving these seemingly easy problems. The mathematical experiences of several students also led them into writing algorithms and computer programs to check for integer solutions. Usually when other students in the class became convinced that multiple integer solutions to a particular problem existed, they formed conjectures on what the underlying pattern was and tried to express them symbolically. This in turn allowed the teacher to use student conjectures as a starting point to initiate the process of mathematically proving or disproving a conjecture. It is important that the reader note that the progress of the problem depended completely on the "will" of the students. The mathematical notation was created as a class only after every student had expressed the particular idea in their own words. There were some weeks when students did not attempt the problem, especially before or after school breaks and during examination weeks. When this occurred, the author continued with the regular algebra curriculum. It was crucial that students initiated this process of conjecture, proof and refutation out of their need to resolve the unimagined difficulties that arose from a seemingly easy problem. It was noteworthy that all thirteen students in this class willingly engaged in trying to solve one of the unresolved conjectures of our time over a seven month time period through the process of conjecture, proof and refutation. The author hopes to have conveyed to the reader the value of the use of conjecture-proof-refutation in the mathematics classroom in order to create meaningful learning experiences. The mathematics created by the students in trying to solve the classic 5-tuple diophantine problem clearly indicates that students are capable of original thought that goes beyond the mimicking and application of procedures taught in the classroom. Although the proof created by the students did not resolve the 5-tuple problem by any means the reader might appreciate the fact that fourteen- year old students worked over an extended time period to solve a difficult number theoretic problem. The use of journals to nurture the process of conjecture-proof-refutation is invaluable to the teacher. It allows for constant communication between the individual student and the teacher, and allows room for students to express themselves, something they might be reluctant to do so in front of the entire class. Journals also allow the teacher to gain an insight into the affective drives of the students, as well as their capacity for originality and creativity. Journal problems also allow for extended investigations that are student driven and convey that mathematics is an evolving process of discovery leading to generalities that are either proved or disproved.

*Conclusions*

The data gathered through all the teaching experiments was analyzed using the constant comparative method from grounded theory (Glaser & Strauss, 1977). Student generalization strategies were qualitatively analyzed for similarities and differences. The generalization strategies constructed by the students evolved with the complexity of the problems and with time. These students were able to reflectively abstract (Dubinsky, 1991; Piaget, 1971) the similarities that characterized their solutions, and discern the underlying structure of the given class of problems. They were also able to conceptually link the problems (Hiebert, 1986; Minsky, 1985; Skemp, 1986), and ignore superficial similarities that did not pertain to the structure of the problems. Students of varying mathematical abilities were also consistently able to devise strategies that began with the simplest cases (Polya, 1954), examine plausible examples, and control the variability of the problem situation. Sometimes students were distracted by the superficial similarities in one or more problems, and were more concerned with constructing just one particular solution to a problem as opposed to pursuing a general solution.

As the results indicate, since students are capable of higher order mathematical processes such as abstraction and generalization, it is vital that high school teachers create learning opportunities that allow these students to utilize these talents. In the author's experience journal problems of varying levels of difficulty, which are characterized by an overarching mathematical generality, is a novel and non-intrusive way of differentiating the curriculum for all students, and not simply for the able students. This approach also allows the teacher the flexibility of picking problems from a diverse range of mathematical topics such as combinatorics and number theory, that are normally not covered by the secondary school curriculum (especially in the U.S), thereby allowing for enrichment. In addition, journal writing provides the teacher and the student a means of one on one communication, which is impossible in the general class setting.

It is hoped that the mathematical challenges discussed in this contribution complement the descriptions of participants from other countries. Another desired outcome is to discuss with other participants in this study, practical recommendations and research avenues for schoolteachers to use in the daily classroom setting.

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