STRANDS OF CHALLENGING MATHEMATICAL PROBLEMS AND THE CONSTRUCTION OF MATHEMATICAL PROBLEM-SOLVING SCHEMA

Authors: Keith Weber Carolyn Maher Arthur Powell
Contact: khweber@rci.rutgers.edu, Graduate School of Education, Rutgers University, New Brunswick, NJ, 08901

Abstract. In this paper, we illustrate the processes that students in a longitudinal study on problem solving used to construct and apply mathematical problem solving schema while solving challenging problems.

1. Introduction

1. 1. Importance of schema in mathematical problem solving

The goal of this paper is to illustrate how engaging students in solving strands of challenging problems can lead them to construct powerful problem-solving schemata. One characteristic that distinguishes expert mathematical problem solvers from less successful problem solvers is that experts have and use schemata—or abstract knowledge about the underlying similar mathematical structure of common classes or problems—to form solutions to problems. A summary of the research on the role of schema in mathematical problem solving (taken from Schoenfeld, 1992) is given below:

- Experts can categorize problems into types based on their underlying mathematical structure, sometimes after reading only the first few words of the problem (e.g., Hinsley, Hayes, & Simon, 1977; Schoenfeld & Hermann, 1982).
- Schemata suggest to experts what aspects of the problem are likely to be important. This allows experts to focus on important aspects of the problem while they are reading it, and to form sub-goals of what quantities need to be found during the problem solving process (e.g., Hinsley, et. al., 1977; Chi et. al., 1981).
- Schemata are often equipped with techniques (e.g., procedures, equations) that are useful for formulating solutions to classes of problems (e.g., Weber, 2001).

To illustrate, consider the following problem: Two men start at the same spot. The first man walks 10 miles north and 4 miles east. The second man walks 4 miles west and 4 miles north. How far apart are the two men? According to Hayes (1989), when experienced mathematical problem solvers read this statement, it will evoke a "right triangle schema" (problems in which individuals walk in parallel or orthogonal directions to one another can often be solved by constructing an appropriate right triangle and finding the lengths of all of its sides). The keys to solving such problems are framing the problems in terms of finding the missing length of a right triangle, setting as a sub-goal to find the lengths of two of the sides of the triangle, and using the Pythagorean theorem to deduce the length of the unknown side.

1. 2. The use of challenging problems to promote schema construction

In this paper, we define a *mathematical task* as a situation in which an individual is given an initial situation and to accomplish this task, the student needs to apply a sequence of mathematical actions**.** A mathematical task is an *exercise* to an individual if, due either to the individual's experience or the situation in which the problem is presented, it is obvious to the individual what actions should be applied. A task is a *problem* if it is not obvious which actions should be applied, either because the individual does not immediately recall appropriate actions or because there are several plausible actions to choose from (cf., Schoenfeld, 1992; Weber, 2005). We call a problem *challenging* if the individual is not aware of a subset of the mathematical actions that are critical for solving the problem and will have to invent or discover these actions to be successful. For instance, most proofs in high school geometry are problems, and sometimes difficult ones, since the prover needs to decide which theorems and rules of inferences to apply from many alternatives (e.g., Weber, 2001). However, proofs that require the prover to create new mathematical concepts or derive novel theorems would make these proofs challenging problems.

Because "problem solving expertise is dependent upon the acquisition of domain-specific schemata" (Owen & Sweller, 1985, p. 274), many researchers argue that an important goal of the mathematics curricula should be to provide students with the opportunities to construct problem-solving schemas (e.g., DeCorte et. al., 1996; Reed, 1999; Nunokawa, 2005). What is less clear is how this goal should be obtained; Marshall (1996) argues that the issues of how students construct problem solving schemas and what types of environments or instruction techniques might foster these constructions are open questions in need of research.

Some psychologists and mathematics educators have suggested that students construct schema by transferring the solution of one problem to another superficially different but structurally analogous problem (e.g., Owen & Sweller, 1985; Novick & Holyoak, 1991). Unfortunately, students often have difficulty seeing the deep structure of problems and transferring the solution of one problem situation to another (e.g., Novick & Holyoak, 1991; Lobato & Siebert, 2002). Accordingly, it is suggested that schema construction can be facilitated by providing students with basic problems to which that schema applies, both to increase the likelihood of successful transfer and to minimize the cognitive load that students use to solve these problems, thus leaving more resources available for learning (Owen & Sweller, 1985). Our research on the long-term development of students' mathematical reasoning suggests the opposite is true. We have found evidence that students often develop a rich understanding of essential ideas in the context of solving complex, challenging problems (Francisco & Maher, 2005). In this paper, we will illustrate how students developed a powerful combinatorial schema while solving strands of problems that were challenging (in the sense that was described earlier in this paper).

2. Research context

This research takes place within the context of a longitudinal study, now in its 18th year, tracing the mathematical development of students while they solve open-ended but well-defined mathematical problems (cf., Maher, 2005). Many of these problems are challenging in the sense that students often initially are not aware of procedural or algorithmic tools to solve the problems but are asked to develop them in the problem-solving context. In this environment, collaboration and justification are encouraged, and teachers and researchers do not provide explicit guidance on how problems should be solved. One aspect of this study was that students worked on *strands* of challenging tasks—or sequences of tasks that may differ superficially but pertain to the same mathematical ideas. The use of strands of related challenging tasks allows researchers to trace the development of students' reasoning about a particular mathematical idea over long periods of time (e.g., Maher & Martino, 1996).

Most studies examining schema construction or transfer take place over a short period of time in conceptual domains in which students have limited experience (Lobato & Siebert, 2002). However, meaningful mathematical schemata are likely constructed over significant stretches of time after students become accustomed with the domain being studied. Hence, Anderson, Reder, and Simon (1996) argue such studies seek evidence of schemata usage and transfer in places where one is least likely to find it. We are not aware of longitudinal studies in mathematics education that address schema acquisition. Hence, the longitudinal and empirical nature of the study that we will report has the potential to offer unique research findings in an important area.

One strand of task used in this study were variants of the following questions:

- Suppose that you have three yellow Unifix® cubes and two red Unifix® cubes. How many different five-tall towers could you make with these Unifix® cubes? Justify your answer.
- Suppose that you are ordering a pizza where you have five toppings to choose from. How many three topping pizzas can you order? Justify your answer.
- Imagine that you have a coordinate-grid map of your city with the origin of the grid being your taxi stand and further that you are dispatched to pick up a passenger located 2 blocks south and 3 blocks east. Is there a shortest route to this passenger? How do you know it is the shortest? Is there more than one shortest route to each point? If not, why not? If so, how many? Justify your answer.

These problems all have the same underlying structure and can be understood in terms of a “Pascal’s triangle schema”. The answer to each question is $\binom{5}{3}$, or the third entry in the fifth row of Pascal’s triangle. In the longitudinal study, students were given extended opportunities to work with each of these questions. As the study progressed, they were able to link the first two problems to Pascal’s triangle. Later they used Pascal’s triangle and the other two problems to solve the Taxicab geometry problem. We will present a fine-grained analysis of how students were able to make these constructions. Each meeting with students was videotaped. Videotapes were analyzed using the methodology of Powell, Francisco, and Maher (2003). The results of these analyses were narratives that described how students reasoned about and learned mathematical ideas.

3. Results

In this section, we examine how a group of five students (Ankur, Jeff, Brian, Michael, and Romina) solved three problems in 10^{th} and 12^{th} grade. The three problems were: 1) How many different pizzas can you make with four available toppings? 2) How many different five-tall towers can you make with three red and two yellow cubes? and 3) The taxicab geometry problem given in the last bulleted point of the methodology section.

3. 1. How many pizzas are there with four different toppings?

In a 10^{th} grade session, Ankur, Jeff, Brian, and Romina used case-based reasoning and various counting strategies to obtain the correct answer—fifteen pizzas with toppings plus one pizza with only cheese. Michael developed a binary representation to create each of the pizzas. Each of the pizzas was represented using a four digit binary number, where each topping was associated with a place in that number, where a one signified that the topping was present on the pizza and a 0 signified that the topping was absent. For instance, with the four toppings- pepperoni, sausage, onion, and mushroom- the binary number 0010 would refer to a pizza with only onions. Michael was able to use this notation to explain why 16 pizzas could be formed when there were four toppings available and convince his group that there would be 32 pizzas if there were five toppings available (the other group members believed that there would be 31, not 32 pizzas).

At the end of the session, the researcher asked the group if this problem reminded them of any other problems, Brian responded “towers”—referring to the problem of forming four-tall towers from red and yellow cubes. However, Ankur noted the problems were “similar, but not exactly the same”, since more than one yellow could appear in an acceptable tower, but you couldn’t list mushroom more than once on the toppings of the pizza. All of the students at this time accepted Ankur’s explanation. The following week, Michael represented the towers problem using binary notation—the n^{th} digit in the notation refers to the n^{th} cube in the tower, with a 0 signifying a yellow cube and a 1 the red cube. For example, 0010 would represent a four-tall tower in which the third block was red but the other three were yellow. Hence, via this binary notation, Michael was able to show his group a correspondence between the towers and the pizzas.

There are two things worth noting about these problem-solving episodes. First, when students were initially comparing the pizza and towers problems to one another, they did not seem to see the deep structure between the problems. In fact, Ankur argued the problems differed significantly. The connections between the problems were not immediately perceived but were only constructed by Michael after reflection. Second, the notational system that Michael developed while working on the pizza problem was critical for the construction of his correspondence.

3. 2. Linking the pizza problem, the towers problem, and Pascal’s triangle

One month later, students were invited to further explore the relationship between pizza problems and tower problems. They were asked to determine how many five-tall towers could be formed with three yellow blocks and two red blocks. Using Michael’s binary representation, they translated this problem to determining how many five-digit binary numbers with three 0’s and two 1’s could be formed. By controlling for where the first one in this sequence occurred, the students were able to deduce that 10 such towers could be formed. Note that the methods Michael developed to cope with the previous pizza problems were now a scheme that the students used to make sense of a new pizza problem (see Uptegrove, 2005). After obtaining their solution, a researcher introduced students to Pascal’s triangle, explained how the n^{th} row of Pascal’s triangle were the coefficients of the expression $(a + b)^n$, and that the terms in Pascal’s triangle are often represented using combinatorial notation. For instance, the fourth row—1, 4, 6, 4, 1—can be written as $\binom{4}{0}\ \binom{4}{1}\ \binom{4}{2}\ \binom{4}{3}\ \binom{4}{4}$ She then asked the students to try to understand what these coefficients might mean in terms of what they’ve just done. After thinking about these problems, the students were able to make these links. They noticed the 10 that appears in the fifth row in Pascal’s triangle corresponding to the expression $\binom{5}{2}$ also corresponded to five-tall towers with two red blocks (and three yellow blocks). Further investigations led these students to describe the relationship between Pascal’s triangle and the pizza problem—namely, that the $\binom{n}{i}$ entry in Pascal's triangle corresponds to the number of pizzas that could be formed with i toppings if there were n to choose from. These students could also explain why $\binom{n}{i} = \frac{n!}{i!(n-i)!}$ and $\binom{n}{i} + \binom{n}{i+1} = \binom{n+1}{i+1}$ (Pascal's Identity) were true by using the towers problem and the pizza problem.

3. 3. Solving the taxicab problem

Two years later, Michael, Romina, Jeff, and Brian (now in 12th grade) were given a version of the taxicab problem discussed in the methods section. In essence, they were asked how many ways that a taxi could take a shortest route along a grid to go four blocks down, one block right; three blocks down, four blocks right; and five blocks down, five blocks right. This qualifies as a challenging problem for the students. The solution to this problem more or less requires the application and use of combinatorial techniques, yet the students solving this problem had not used such techniques before to solve novel problems. The initial stages of the students' activity were exploratory in nature. They worked to make sense of the problem, posed some initial conjectures that turned out to be incorrect (e.g., the distance from the starting point to the endpoint would tell you the number of shortest routes), and tried to answer the question by explicitly drawing and counting the routes.

Romina asks if it would be possible to "do towers" to the problem. Michael and Romina note that the distance to one of the points is 10 and wonder if the total number of shortest routes to that point is 2^{10}. Later, the students attempted to solve the problem by finding the number of shortest routes to corners close to the point of origin (e.g., there are two shortest routes to go one down, one right; three shortest routes to go two down, one right). This yields a table like the following:

1	1	1	1	1
1	2	3	4	5
1	3	6	10	12
1	4	10	15	
1				

(where the m^{th} by n^{th} cell in the table represents the number of shortest paths to go m units to the east, n units south).

Romina notices that the fourth diagonal of this table is the sequence 1 4 6 4 1 and declares, "It's Pascal's triangle", where the diagonals in the table correspond to the rows of Pascal's triangle. Jeff notes that the 12 and the 15 in the next diagonal would not be correct if this was the case and ask Brian to re-evaluate the number of routes it takes to go four over and two down. When Brian announces that he found 15 routes, Michael comments, "it means that it is the triangle". A little later, Romina writes a 20 in the box for three right, three down while Brian worked on re-computing this value. At this point, Michael asked his colleagues how they knew it was 20. Jeff responded that if they can show the triangle works, they don't need to verify that it's 20.

To understand why Pascal's triangle would provide the number of shortest routes to any points on the grid. Romina announces that she will try and relate the triangle back to the towers and focuses on the 1 2 1 diagonal. She notes that all of the points on this diagonal are two away from the starting point and this also forms the second row of

Pascal’s triangle. Further, she notes a connection between the middle entry in that column—with towers, the middle entry would refer to a two-tall tower with one yellow and one red block; with taxicabs, this refers to a trip with one across and one down. Likewise, the entry two down, one right, would refer to a tower that was three tall, with two yellow and one red block, or the taxicab location three away, with two down and one across. The students filled in the rest of their grid in accordance with Pascal’s triangle. For instance, when they filled in the cell for five down, two over, they reasoned that the number of routes would correspond to the fifth entry of the seventh row of Pascal’s triangle (not counting the beginning 1) since it would be “five of one thing and two of another thing”. At a researcher’s request, Michael also explains the connection between Pascal’s triangle and the pizza by using his binary number notation. For the taxicab geometry problem, a 0 would indicate going down and a 1 would indicate going across. Hence, using the example of going two down and one across, one would need to find the number of binary strings that have two 0’s and one 1. In their work relating Pascal’s triangle to the pizza problem, the group had already established that this would be the first entry (ignoring the first 1) of the third row of Pascal’s triangle. Finally, the group was able to use these constructions to answer the given questions. For instance, the number of shortest routes to the point that was five right and five down would be Correspond to the fifth entry of the tenth row of Pascal’s triangle.

4. Discussion

In the first two excerpts above, we illustrated how students constructed a powerful problem-solving schema for solving combinatorial problems. We then illustrated how students applied that schema to solve the challenging Taxicab geometry problem. The application of this schema not only allowed them to construct the solution to the problem, but it also provided them with a deep understanding of their solution and enriched the schema that they constructed. In this section, we will discuss four aspects of our problem-solving environment that enabled students to make these constructions.

First, students were asked to work on *challenging* problems. If students were asked to work on problems in which they had already learned techniques for addressing them, they may have attempted to see whether various techniques that they had learned would be applicable to the problem. As the students needed to develop techniques to make progress on these problems, this was not an option for these students. A particularly important precursor toward developing the schema that these students constructed was the development of useful ways of representing the problem. Michael’s binary representation of the towers and the pizza problem, in particular, paved the way for students seeing the deep structure that these problems shared. One general finding from the longitudinal study was that students developed powerful representations in response to addressing challenging problems (Davis & Maher, 1997; Maher, 2005).

Second, students were asked to work on *strands* of challenging tasks—or problems that were superficially different but shared the same mathematical structure. This provided students with the environments in which schema could be constructed. Researchers also fostered this construction by encouraging students to think about how the problems they were solving might be related to problems that they had solved in the past. However, we believe that having students work on strands of challenging tasks is a necessary but not sufficient condition for schema construction and usage. Students also need time to explore the task and benefit from heuristics that guide their explorations in productive directions.

Third, students were given *sufficient time* to explore the problems and were given the opportunities to *revisit* the problems that they explored. The students did not instantly see the connections between the towers and pizza problems, nor did they see how the taxicab problem was related to either of these problems. It is especially noteworthy that students initially believed that the towers and pizza problems were similar but also differed significantly and that Romina's initial suggestion to relate the taxicab problem to the towers was not immediately pursued. Further, as students revisited problems, their representations of the problems became increasingly more sophisticated, enabling them to see links between the problem being solved and previous problems on which they worked. As Uptegrove (2005) illustrates, many of the connections students made could be traced back to problem-solving sessions on which they worked months or years before.

Finally, as Powell (2003) emphasizes, the heuristics that students used in their problem solving enabled them to relate the problem situation to their schema. Among the heuristics used by the students were the following: solve a difficult problem by solving easier ones (before finding the number of shortest routes to a location ten blocks away, find the number of shortest routes to a location two blocks away), generate data and look for patterns, and see if there is an analogy between this problem and a familiar one. Without the use of these heuristics, the links to existing schema may not have been made. However, the disposition to use such heuristics was likely developed during the students' years of solving challenging problems (cf., Powell, 2003; Uptegrove, 2005).

Most research on schema construction has been done using traditional psychological paradigms, investing how and (more often) to what extent individuals can construct and apply schema in a short period of time. Our research differed from this paradigm significantly, looking at how students developed schema over time while solving challenging problems. We believe this change in perspective radically altered the nature of our findings. If our students were given straightforward problems, they would not have had the need to develop the useful representations for these problems that were critical for their schema construction. If they were only given a short period of time to explore these problems, the schema also would likely not have been constructed. In fact, students initially did not see the deep connections between the various problems on which they worked. Looking at the processes that individuals use to form and use schema in relatively short periods of time is looking at only a subset of the processes used in this regard. We believe that studying the way that students solve challenging problems provides a more comprehensive and useful look at how students can construct and use problem-solving schema.

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