An Ordinary Problem, But a Difficult One...

Alexander Karp,
Teachers College, Columbia University
Box 210, 525 West 120th Street, New York, NY, 10027
apk16@columbia.edu

Abstract. The present article contains an analysis of the Russian experience of working with difficult problems in the classroom. Certain examples of such problems are examined, as well as the purpose of using them and the ways in which they can be used. Also discussed are issues related to preparing teachers who are capable of organizing the solving of such problems in the classroom. The article concludes by looking at the experience of employing such problems in teacher education outside of Russia.

This article has three aims. First, it represents an attempt to describe and analyze the Russian tradition of solving difficult problems in the classroom. The problems in question are not those of mathematics Olympiads (although the distinction is sometimes a difficult one to make), but plain, ordinary problems—problems that can be solved in ordinary classes and appear on ordinary tests. The origin and character of such problems, as well as the development of methods for solving them and techniques for working with them in the classroom, are of considerable interest. The second aim of this article is to analyze what kind of preparation a teacher requires in order to be able to organize the solving of such problems in the classroom. Finally, the third aim is an attempt to look at the traditions that have evolved in Russia over the course of several decades from a different point of view, and to consider which of their components might be beneficially carried over to countries with different traditions, and how such a transfer might be accomplished (the author restricts himself in this respect to describing his own experience of working with American graduate students in mathematics education).

What is a difficult classroom problem? Any classification of problems is bound to be somewhat nominal. Consider the following two problems:

1. Find all values of a for which the equation $1-|x|=|x-a|$ has exactly two solutions.
2. The numbers 1, 2, 3, … 100 are written out on the blackboard. One is allowed to erase any two numbers a and b and to write down the number a+b instead. If this operation is repeated over and over again, only one number will remain on the blackboard in the end. Can this number be 5000?

Seeing these problems, Russian teachers would likely say that the first one is a "classroom problem," while the second one is an "Olympiad" problem. This by no means implies that the first problem is an easy one. In order to solve it, for example, students must examine the graphs of the functions $y=1-|x|$ (Fig. 1a) and $y=|x-a|$ (Fig.1b shows the case when a=2), and by "shifting" the second graph, to see that the graphs will have exactly two points of intersection when $-1<a<1$ (Fig. 1c).

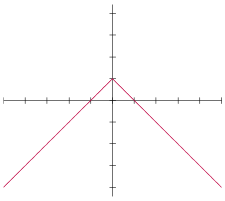
Fig. 1a

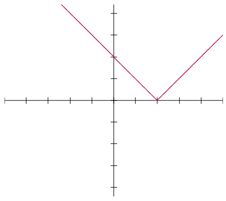
Fig. 1b

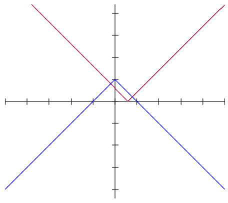
Fig. 1c

The solution to the second problem is shorter: students must merely take note of the fact that the sum of all the numbers written on the blackboard does not change. Therefore, in the end it must be equal to 1+2+…+100=5050, which is not difficult to obtain using the formula for the sum of an arithmetic progression.

The second problem is clearly different from the first. From the teacher's point of view, the difference consists in the fact that problems similar to the first one are usually solved in advanced classes, whereas problems resembling the second one are typically solved only in after-school "mathematics circles." The difficult problems discussed below also differ from "Olympiad problems" thematically. Contemporary "Olympiad problems" can, in principle, be much less closely connected to the school curriculum than difficult "classroom problems."

Finally, the most significant difference between these two kinds of problems consists in the motivation for assigning difficult "classroom problems" in the first place. The aim in this instance is not to identify gifted students or to stage a competition, but to teach students freely to apply the concepts and algorithms covered in the school curriculum. This is not something that can be achieved merely by solving what Polya (2004) described as "routine problems," problems that can be solved "either by substituting special data into a formerly solved general problem, or by following step by step, without any trace of originality, some well-worn conspicuous example" (p. 171). Students must be confronted with situations in which they themselves are required to determine which of the concepts that they have studied are relevant to the given problem, and to decide which of the algorithms that they have studied must be applied to solve it. Teaching students to solve such problems consists in helping them to develop a specific kind of pattern recognition—the situation presented by the problem must not be seen as entirely unfamiliar; and the students must construct the solution from blocks that are already known to them, by trying out and selecting familiar algorithms (and this is yet another difference between the "classroom problem" and the ideal "Olympiad problem," in which the creation of something genuinely new must play a greater role).

Why are difficult classroom problems important? We will focus here on only two aspects of this complex issue. The first of these concerns students' impressions of mathematics and the beliefs that are inculcated in them and developed by them in the process of problem solving, beliefs that subsequently exert a considerable influence on their behavior (Schoenfeld, 1985). As Polya (2004, p. 172) wrote, "Teaching the mechanical performance of routine mathematical operations and nothing else is well under the level of the cookbook." Within such a framework, it is impossible to develop an interest in mathematics, an understanding of its significance, the ability to apply it in even slightly nontrivial situations, and above all, even the ability to conceive of any such application.

But there is a second consideration which is no less important. The memorization of basic techniques, which is precisely the object of solving myriads of nearly identical exercises at the cost of avoiding more substantive assignments, is in fact also not achieved most effectively through routine problem solving. The studies of memory conducted by Luria (2004) support the conclusion that "the more difficult an intellectual activity is, the more conducive it is to the memorization of the materials to which it is devoted" (p. 220). Moreover, even in the process of deliberately learning by rote, "organizing elements into self-contained meaningful structures… makes the traces of memory incomparably more durable" (p. 216). The crucial fact regarding problem solving is that the operations which they require the students to perform can be made sense of—can be seen as "meaningful structures"—only when they are contextualized within a broader framework; in other words, only through solving more difficult problems.

Working teachers are familiar with situations in which, on the fourth or fifth exercise out of, say, the fifteen given on a worksheet, students at last seemingly begin to perform the operations that are demanded of them correctly. On the following day, however, everything turns out to have been forgotten again. This is hardly surprising: there has been no interpretation, no "making sense of," no intellectual work—not to mention the absence of any emotional reaction, the importance of which in the process of memorization was also emphasized by Luria.

Where do difficult classroom problems come from? The corpus of difficult classroom problems has been assembled over decades, if not centuries. Many of the problems used today go back to problem books from the 19th-early 20th centuries, often published in languages other than Russian (Aubert and Papelier, 1941). A fair number of problems have been contributed by the Olympiads, which in their early stages in Russia were thematically much closer to the school curriculum than they are now (at least V. Krechmar (1972), the author of one of the first and most influential Soviet problem books devoted to difficult problems in mathematics, noted that his book was born in 1937 during his work on putting together problems for a mathematics Olympiad). A substantial role has likewise been played by high school final exams, which are taken by all students in Russian schools; as well as by college entrance exams, which are conducted by every Russian college and which offer dozens of new interesting problems every year.

It must be noted, however, that among the difficult problems that have appeared and continue to appear on exams, many are quite artificial in character. Such problems attain their level of difficulty not by virtue of any substantive conceptual thought, but through a rather mechanical combination of various separate exercises whose connection to one another is often quite tenuous. This attribute is most conspicuous in the "sewn-together" problems that were popular on final exams before the 1917 Revolution, in which students were asked, for example, to determine the age of a father and a son, using the fact that their sum was equal to the solution of a given exponential equation, while their difference was equal to the coefficient of a particular term in the expanded form of a given expression (Karp, 1998). Such problems gradually went out of use, but the poor taste that was manifested in piling up complexities only for the sake of complexity was never fully eradicated. Therefore, both today and in the future, teachers who are thinking not only about preparing their students for exams, but also about their genuine development, must evaluate new problems with a sufficiently critical eye.

In any event, when a difficult problem is introduced into a lesson in the classroom, it undergoes substantial transformations, since its purpose changes—from an instrument for testing it turns into an instrument for teaching. In this way, the problem enters into a fundamentally new system, both in mathematical terms—since it frequently becomes contextualized within a system of related assignments—and in terms of organization and pedagogy, since it is now posed and solved in a new setting. A large number of problems that have become something like classics in the field were originally composed in schools. Schools with advanced teaching of mathematics have played a particularly important role in this respect (Vogeli, 1997). These schools, which were created on the initiative of the Soviet

Union's top research mathematicians, were originally aimed at nurturing mathematical creativity. Therefore, even while going over the elementary mathematics curriculum in the classroom, teachers sought out opportunities for deeper study, that is, opportunities for formulating substantive and difficult problems.

Some types of difficult classroom problems. Without attempting to put together an exhaustive catalogue, we will give a sketch of some possible themes and categories of difficult problems in algebra (including the study of elementary functions under this heading as well). It should be noted, however, that difficult problems have been developed for virtually every section of the school curriculum. What needs to be identified, therefore, are the particular properties that make these problems important in the teaching process.

In historical terms, the problems that deal with algebraic transformations are probably the oldest. Typically, they revolve around discovering one or another pattern or—as in the case with the popular problems involving the factoring of expressions—the strategic application of one or another technique. Although such techniques may sometimes also be criticized for being excessively artificial, the process of finding and applying them often substantially increases the depth of the students' assimilation of the concepts being studied.

As an example, consider the classic problem of factoring the polynomial x^4+4. Its solution involves altering the polynomial to fit familiar formulas that express the difference between two squares and the square of a sum. By adding and subtracting the "missing" term $4x^2$, the given expression may be transformed as follows: $x^4+4=x^4+4x^2+4-4x^2=(x^2+2)^2-(2x)^2$. All that remains is to notice that the expression thus obtained is equal to $(x^2+2-2x)(x^2+2+2x)=(x^2-2x+2)(x^2+2x+2)$.

The process of solving this problem in class, however, goes far beyond simply mastering a trick. Students who are used to mindlessly carrying out simple operations in order to factor straightforward expressions are now forced to think about what exactly it is that they wish to do. At the beginning of the discussion, such answers as $x\cdot x^3+4$ or $x\cdot\left(x^3+\frac{4}{x}\right)$ are not uncommon. These attempts enable the students to form a more precise grasp of the concept being studied, something for which there was simply no need previously. Discussing whether the desired factorization can have a linear factor also turns out to be highly illuminating. Finally, the very fact of factoring an expression that has no roots turns out to be quite instructive for the students. Naturally, solving subsequent problems of the same type turns out to be somewhat less important, with the students' focus typically shifting to the technical side of things; however, even then the students continue to assimilate the algebraic concepts involved in the solutions more thoroughly and deeply.

Another category of difficult classroom problems involves investigating functions using elementary methods. What is the range of the function $y=x^2+4x$? How can one show that the function $y=x^3+3x$ is an increasing function? How can one prove that the function $y=x^2$ is concave upward? Students armed with graphing calculators can become convinced that all of these assertions are correct by pressing a few buttons, and this represents a valuable use of technology in education. Less welcome, however, is the fact that many students jump to the conclusion that these assertions need no further foundation. Students who have completed a course in calculus will have no difficulty solving, say, the last of the aforementioned problems, but often the solution will be obtained only by mechanically following rules, which requires no understanding of what is being done. Solving these problems using elementary methods (without calculations involving the derivative) often helps to clarify the essence of the concepts involved. What does it mean, for example, to say that “a function increases”? How can a geometric image that the students have formed be translated into a verbal description, and how can this verbal description then be translated into formulaic language? Finally, how can one prove various inequalities that do not come from nowhere, as unmotivated exercises, but arise as steps in the solution of a given problem? Technology, which is unfortunately only starting to penetrate into Russian schools, renders the first steps in solving such problems more accessible, and problems of this type as a whole constitute an excellent school of searching for and applying various forms of representation.

Perhaps the most widespread type of difficult problem has to do with solving equations. Solving equations has evolved into something like a science, which includes many specific techniques (which, again, can sometimes be artificial and cumbersome), and is occasionally reminiscent of 16th century algebra, whose depth and even elegance cannot be denied.

Consider the following example of a difficult classroom equation (with a less archaic, more “functional” ideology): solve the equation $2^x + 5^x = 7^x$. One solution is obvious: x=1. But how can one find other solutions or determine that no other solutions exist? Note that, in general, the graphs of two increasing functions—and both $y = 2^x + 5^x$ and $y = 7^x$ are increasing functions—can intersect more than once. None of the standard methods—such as taking the logarithm of both sides—work here.

However, an algebraic technique greatly simplifies the situation. By dividing both sides of the equation by 7^x, one obtains the equation $\left(\frac{2}{7}\right)^x + \left(\frac{5}{7}\right)^x = 1$. Now it is obvious that the equation has one and only one solution—the graph of the decreasing function can intersect the horizontal straight line $y = 1$ only once (the function $y = \left(\frac{2}{7}\right)^x + \left(\frac{5}{7}\right)^x$ is evidently a decreasing function because it is the sum of two decreasing functions).

Finally, no list of difficult problems solved in Russian classrooms can fail to include problems involving parameters. The exercise given at the beginning of this article is an example of such a problem. Problems involving parameters are traditionally given on exams and are likewise traditionally criticized for their arbitrariness and cumbersomeness. At the same time, many of them present wonderful opportunities for developing logical reasoning and the ability to find connections with other areas of mathematics.

How difficult classroom problems are used. The basic distinguishing characteristic of the use of difficult problems in school, which has already been alluded to, consists in the fact that such problems are employed as part of a system, that is, surrounded by and in connection with other problems. The difficulties they present, which in another context might well intimidate the unexceptional student, are ameliorated by the gradual fashion in which they are approached, and by the variety and multiplicity of situations in which they are encountered. Techniques for arranging groups of problems, as well as techniques for organizing classwork, can vary infinitely. The teacher (or the author of the problem book) can begin immediately with a difficult problem, constraining the students to dive headlong into the world of deep concepts and their complex relations, and rely on a group discussion to shed light on the problem and its solution. Such a problem might be followed by others which, without literally repeating what has already been done, will lead the students, collectively or individually, once again to think through the solution.

The opposite approach is also possible, in which work begins with sets of relatively simple assignments and leads up to a difficult problem whose solution must be assembled by the students from previously prepared blocks. The ability to put together such a sequence of exercises, as well as the ability to evaluate the results obtained by studying the given situation, also can and must be taught (Karp, 2002).

The equation examined above, for example, may appear as part of a system of problems that make use of completely different functions, or completely different algebraic techniques for obtaining the solution, or completely different properties of functions rather than whether they are increasing or decreasing. Such sequences of problems may be given at once during a single lesson, or they might be developed gradually and separated by weeks and months. They may be offered in class as homework assignments, projects, or even voluntary “challenges” to think about during summer vacation. The only crucial factor is that each problem must be perceived as part of a complex process with many stages.

How can work with problems be structured and sequenced? What is the psychological framework within which students go through the process of problem solving? What is the role of the mathematical and pedagogical aspects of the teacher’s activity? All of these topics are highly important and insufficiently studied.

From the Russian experience of preparing teachers who teach solving difficult classroom problems. Effective teacher preparation must contain many stages and devote unflagging attention to solving difficult problems at every stage. Briefly, one can say that in order for teachers to be able to teach students how to solve difficult problems, they themselves must solve them. It is sometimes somehow forgotten that the famous questions which Polya (2004) recommended for his second stage of problem solving—“*Have you seen it before? Or have you seen the same problem in a slightly different form?”*—presuppose familiarity with many problems. The teacher must have seen many problems prior to the moment when he or she begins assigning them to students.

Such an acquaintance with problems can take place at the very beginning of the teacher’s college education. For example, the Herzen University in St. Petersburg offers an introductory course that is formally devoted to mathematical language, but in reality involves solving problems in which concepts from the school curriculum are studied more deeply and rigorously (Curcio et. al., 1997; Michailov et. al., 2001). Such a course can provide the student who has just entered college with a new orientation—its purpose is not carrying out computations, not carrying out previously memorized algorithms, but a deeper understanding of the subject matter.

The more traditional courses usually offered to future teachers familiarize them with many difficult problems, acquaintance with which will serve them well in their future work. It is crucial to understand that teachers must become familiarized with problems that are, generally speaking, more difficult than the ones which are usually given in class. It may be that, in working with students, some teachers will never make direct use of the problems that they have solved in teacher preparation classes; nevertheless, this is the only way in which they themselves will be able to attain a sufficient grasp of what is being studied, and thus be in a position to find and pose difficult problems that are accessible to their students.

This kind of work must continue during the stage of in-service training (Karp, 2004), in the course of which teachers can become actively involved in discussions of the pedagogical aspects of problem solving: When can one or another problem can be posed? What are the aims of its application? What kinds of problems ought to precede it, what kinds of problems out to follow it? Writing and formulating difficult problems must also be a part of this activity.

From the experience of using difficult classroom problems in working with American teachers. The central goal that teacher educators must set for themselves in working with future teachers on problem solving is the overcoming of those negative beliefs which many of them have developed in school. Future teachers often genuinely believe that a problem is either solved in one minute or not solved at all; that the most important thing is to give an answer, and that where the answer came from does not much matter; that it is far more important rapidly to climb up the ladder of courses from algebra to, say, differential equations, than truly to get to the bottom of the concepts being studied. Solving difficult problems can be enormously beneficial in this regard, not only by helping to furnish future teachers with useful materials, but above all by showing them the importance of a different view of mathematics.

We structured our problem solving work with students along two lines. First, we offered two courses devoted to the solving of problems, "Problem Solving" and "Exploring Secondary School Mathematics." These courses made thorough use of the classroom problems described above and others like them. These courses were also useful because they made it possible to establish links between many of the concepts covered by the students in more advanced courses in mathematics, such as Calculus or Abstract Algebra, and ordinary classroom mathematics.

In this line of work, it was extremely important to choose problems that were free of excessive technical burdens, which would have been utterly useless for American students. No less important were detailed discussions of how such problems are solved, as well as how they arise and how they are connected with ordinary classroom materials. Discussing the answers to the questions which students love to pose—"Is this always the case?" and "Can you always do this?"—made it possible substantially to generalize "ordinary" classroom problems, thus arriving at those comparatively more difficult formulations which were being studied. It would seem that the study of the mathematical origins of difficult problems (and not merely their historical origins, which were discussed above)—that is, of the kinds of questions that might be answered by solving them—helps to create the kind of situation that leads future teachers to become more actively involved in solving them.

The second line of our work consisted in lessons in the group writing of lesson plans devoted to problem solving. The problems dealt with in this case were simpler (most of the lessons were devoted to linear and quadratic functions and their corresponding equations). Attention to the "genetic" aspect of solving problems, mentioned above, was automatic—students had to work within the framework of standard classroom materials and to include these materials in their lessons. The composition of such lesson plans included such elements as determining the overall goals of the lessons, selecting assignments, discussing and developing them, and so on. All of this was done by the students with the participation and support of the teacher. Such a way of scaffolding lesson plans constitutes an important technique that enables future teachers to develop their own methods of working with problems in a more favorable environment.

Conclusion. Virtually every one of the issues mentioned above requires further study and detailed discussion. Let us add a final question: How can one judge whether a difficult classroom problem is successful or cumbersome and superfluous? As is clear from what has been said above, the author of this article is of the opinion that a good problem is characterized first and foremost by connections with various concepts studied in the course that come to light during its solution. A good problem is one whose solution makes it possible, on the one hand, to answer questions that have come up earlier, and on the other hand, to continue developing the investigation by generalizing the problem itself or transferring its solution to other objects. Such problems have been collected over decades. It is important not to lose what has been preserved, but to make it as accessible as possible to every mathematics educator.

References

Aubert, P.; Papelier, G. (1941). *Upragnenia po elementarnoy algebre* [Exercises in Elementary Algebra]. M: Uchpedgiz.

Curcio, F.R., Evans, R.C., Plotkin, A. (1997). “Restructuring Mathematics Teacher Education: the Evolution of an Innovative Preservice Program in Russia.” *Teacher Education Quarterly*, Spring, 53-61.

Karp, A. (1998). *Pis’mennye vypusknye eksameny po algebra v Rossii sa 100 let* [Written Final Exams in Algebra in Russia over 100 Years]. St. Petersburg: SPbGUPM.

Karp, A. (2002). “Math Problems in Blocks: How to Write Them and Why,” *PRIMUS*, 12(4), 289-304.

Karp, A (2004) “Conducting Research and Solving Problems: The Russian Experience of Inservice Training.” In T. Watanabe & D. Thompson (Eds.) *The Work of Mathematics Teacher Educators. Exchanging Ideas for Effective Practice,* AMTE, 35-48.

Krechmar, V. (1972) *Zadachnik po algebre.* [Problem Book in Algebra]. Moscow: Nauka.

Luria, A. (2004). *Lekzii po obschey psihologii* [Lectures in General Psychology]. St.Petersburg: Peter.

Michailov, A., Plotkin, A., Riss, E., Yashina, E. (2001). *Matematicheskii yasyk v sadachah.* [Mathematical Language in Problems]. St. Petersburg: RGPU.

Polya, G. (2004). *How to Solve It*. Princeton, NJ: Princeton University Press.

Schoenfeld, A. (1985) *Mathematical Problem Solving*. New York, NY: Academic Press.

Vogeli, B.R. (1997). *Special Secondary Schools for the Mathematically and Scientifically Talented. An International Panorama.* New York: Teachers College, Columbia University.