Challenging problems in first lessons of geometry

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Abstract

This paper deals with teaching of elementary geometry at school. We mostly concentrate on the first stage of geometric courses, when students start their systematic study of geometry. How to awake the children's interest to the subject and, on the other hand, to follow the rigorous theory based on the axiomatic approach? We discuss possible ways to involve students in solving challenging geometrical problems in the early stage, when the theoretical background of students is little. We share our experience in teaching geometry for beginners and consider some examples of problems on various topics of geometry.

1. Geometry for beginners: the rigor or the interest ?

One of the most difficult questions for teachers and authors of the school courses of geometry is how to start the course. How can we build the first year of teaching geometry in order to realize at least two conditions:

1) to awake the interest of the students to the subject;

2) to provide the students with a rigorous theory based on axioms and on proofs of all statements.

Sometimes these conditions seem to be incompatible. On the one hand, the provability principle is on the essences of a course of geometry. This is the only discipline at school, including even mathematical courses, entirely based on the principle of the consecutively deriving of all statements. Therefore, any course of geometry should be irreproachable in formal logic. On the other hand, if we start teaching geometry with memorizing axioms and with proving obvious things, we make this subject boring and may frighten even mathematically gifted students. The main problem is beginners know too little to solve interesting problems. Providing them with more substantive examples is prohibited till they do a big preliminary job proving all basic properties of geometric objects. For instance, formally we are not able to use the notion of a square, until we justify its existence. For this we need first to prove that the sum of angles of any quadrangle is 360^0, which takes us back to the properties of alternate interior angles, and, finally, to the Euclidean axiom about parallel lines. Without this auxiliary job the use of a square is formally illegal. In practice this means

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at least half a year. So, if we follow a mathematically rigorous course, we have to manage for a long time without squares or square lattice. This is strange for students, because all of them have lattices in their exercise books. The same takes place with many other notions of geometry: area, a cone, a cube. Many objects familiar for us since the very childhood are inaccessible on the first lessons of geometry.

The question arises how to begin a school course of geometry in order to combine the logic rigor of the theory with the interest to the subject? How to teach students the formal logic and the axiomatic method and, on the other hand, to involve them in solving challenging and meaningful problems?

In this paper we try to discuss possible solutions. Our main point is that from the first geometrical lessons students should be involved in solving challenging problems. Such problems can keep children's interest to the subject, and, on the other hand, helps absorbing the main ideas and methods of geometry. We see several sources of challenging problems for beginners. The first is the own experience and knowledge of children about the world around. We discuss such problems in the next section. The others are the use of algebra in geometric problems (section 4) or of simple stereometric problems (section 5). In section 3 we consider two types of challenging problems in geometry: creative problems and thematic problems. Each section is provided with several examples of challenging problems, in which we share our own experience in teaching geometry for beginners and in writing textbooks and books of problems for Russian school. Some principles and methods introduced below were formulated by the author (jointly with I.Sharygin) in [1]. Most of examples of geometric problems is taken from [2] and [3].

2. How to start ?

What sort of challenging problems can be given for beginners, whose theoretical background is too narrow so far? A student comes to the first lessons of geometry with his own knowledge about the World and the Nature around him. Everybody knows what a cube is, what a ball is. We should not ignore this knowledge, but exploit and develop it. We should introduce the main geometric objects, including three-dimensional ones, in the first lessons and use them in the problems.

Example 1 *There are several figures in the picture. It is needed to assemble a disk with them. One of the figures is redundant. Which one?*

According to our experience, beginners solve this problem as well as high school students. This problem has also many variations: making two discs, other figures, there are two redundant details etc. Sometimes the solution is not unique. The next two problems develop the spatial imagination:

Example 2 *There is an object in the picture, say, column, that consists of a cylinder and a ball on it. Draw pictures of these objects seen from the positions of given points A, B, C.*

Example 3 *Draw the shape of a shadow of a given cube to a given plane. What sort of figure will you obtain? (The cube and the plane are in the picture, the plain is orthogonal to the diagonal of the cube.)*

This problem is a "joke", because this shadow is exactly as the picture of a cube, which is given. So, the answer is already available. However, it is rather difficult for beginners to notice this. The next problem can be put to a whole-class discussion as a small competition between student.

Example 4 *There are three round stubs on a glade. Three foxes are sitting at given points A, B and C on the glade (the corresponding picture of the view from above is provided). Where can a rabbit hide so that the foxes would not see him behind the stubs? Draw the set of safe points for the rabbit.*

Solving this problem children face an unfamiliar notion of visible or invisible objects. The first challenge for the students is to formalize this notion. As soon as they realize that "invisible" means that the line from the rabbit to a fox intersects the stub (circle), the problem immediately becomes simple. In addition this problem helps students to understand the notion of a tangent to a circle.

This problem also possesses many variants. For instance, one can vary sizes, locations and number of the stubs. One can also consider an opposite problem: arrange three foxes so that a rabbit could not hide.

3. Two types of of challenging problems for beginners: creative problems and thematic problems

The problems in Examples 1–4 can be called *creative*. Such problems develop geometric skills of the students, but do not teach them the rigorous theory. The other kind of problems, which can be called *thematic*, is devoted to the axiomatic method and develop the skills of rigorous proof. In contrast to creative ones, thematic problems are more restrictive, they require formal reasoning and deriving of all the statements. Most of challenging problems are certainly creative. However, we believe that thematic problems can also be made challenging, although this is, certainly, more difficult task, especially in the beginning of the course. Challenging thematic problems can put creativity in those stages of teaching geometry that are usually considered to be boring.

Example 5 *[Theme: properties of a straight line]. There are six points A, B, C, D, E, F in succession on a line. Consider three segments AD, BE, CF. Find those points of the line belonging to*

a) exactly one segment; b) at least one segment; c) all three these segments; d) at most two segments.

Example 6 *[Theme: properties of a straight line]. There are four segments on a straight line, all of them contain a given point A. Prove that one can always choose two segments that cover the remaining two ones.*

The solution for Example 6 is rather short: it suffices to choose the first point from the left among the ends of the given segments and the first point from the right. Then two segments that have ends at these points give the answer. If they are the ends of one segment, then this segment covers the three remaining ones. In this case each of the remaining segments can be chosen as the second one. For beginners this problem is rather difficult, because they have to give a rigorous proof based on the axiom of a strait line (each point split a line into two half-lines with certain properties). On the other hand, it demonstrates the power of axiomatic method for children. In fact, instead of considering many possible cases of mutual location of the segments, we find a formal proof valid for all the cases.

Example 7 *[Theme: measuring of lengths]. There are three points* A, B, C *in succession on a line,* $AB = 3, BC = 6$*. How many ways do we have to put a point* D *on this line so that there would be precisely two equal segments among the four segments* AB, BD, BC *and* CD *?*

The answer to this problem: there are 3 points. The first one is from the left from A such that $DA = 3$, the two others are from the right from C such that either $CD = 3$ or $CD = 6$. Note that the midpoint of BC is not feasible, since in this case we have three equal segments (this is the most common mistake of students). This problem helps beginners to understand deeper the geometry of a straight line, and to develop their skills in calculus of lengths. A better result can be achieved if we make a small competition among the students, how many points they have hound. Let us give two more examples of challenging problems on this theme.

Example 8 *[Theme: measuring of lengths]. Two segments of lengths* 3 *and* 6 *are on a straight line. The distance between their midpoints is* 4*. Do they intersect?*

Example 9 *[Theme: measuring of lengths]. Draw a set of points on the sides of a given triangle, for which the distance to the closest vertex is smaller than 3. The lengths of the sides are* $5, 6$ *and* 8*.*

The next problem concerns the geometric properties of a plane. Its solution involves basic notions and properties of half-planes.

Example 10 *[Theme: properties of a plane]. There are 3 points on a line. Three rays start at these points and lie in the same half-plane. To how many parts can these rays split the half-plane? Find and illustrate all possibilities.*

The answer is from 4 to 7. We get 4 parts when all rays are parallel; 5 parts, when two ones are parallel, and the third one that goes through the intermediate point is not parallel to them, 6 if all the three rays concur at one point, 7 if they intersect each other in different points. The students are required to illustrate all the cases and to prove that the other numbers are impossible. This problems is also good if given as a competition between students.

4. The use of algebraic problems in geometry

It is a tradition of the Russian school that it is in the 7th grade, when the systematic study of geometry starts. Children of this age have already learned many themes of algebra: four arithmetic operations, solving of linear and fractional-linear equations, linear inequalities, and so on. Therefore, one source of challenging problems for beginners is to involve algebra. The following problem is reduced to a simple linear equation. The main difficulty is to choose the smallest segment LM as a variable:

Example 11 *[Theme: measuring of lengths]. There are 4 points K, L, M, N on a line. The segment LM three times as smaller than KL and four times as smaller than MN. What is the ratio between KN and KN ?*

To solve the next problem a student have to transfer a geometric relation to algebraic inequality:

Example 12 *[Theme: measuring of lengths]. Points A and B are given on a line, the distance between them is 2. Find all points M on this line such that $2AM - BM \geq 1$.*

5. Stereometric problems in the first lessons

Another source of challenging problems in the first lessons of geometry is solving of stereometric problems. Usually stereometry is studied in high school, for one or two last years. We believe that this is reasonable to include some elements of solid geometry in geometric lessons for beginners of 6-7 grades. Children of this age already know many spatial objects and their basic properties. Usually they have pretty good spatial imagination, which is sometimes better than older students have. Some elements of stereometry can be included in planimetric lessons starting from the very beginning of the course. Of course, we should not give them stereometric axioms and theorems. We mean problems about spatial objects (a cube, a ball, a pyramid) that possess "planimetric" solutions, i.e., can be reduced to simple planimetric problems. In most cases they are really challenging for beginners, and, moreover, interesting, because they are less abstract and bring children back to the world of real objects. Let us start with problems for the first lessons.

Example 13 *[Theme: geometric objects]. Find a way how to measure the width of a football.*

The advantage of giving this problems to beginners is that they have no skills in calculus (the length of a circle etc.) and have to look for geometrical solutions. One of many possible approaches is to lean the ball against a wall, put a piece of plane on it, say, a big book, and to mark the corresponding level on the wall. The height of the level equals to the diameter of the ball.

Example 14 *[Theme: geometric objects]. Find a way how to measure the diagonal of a brick with a ruler.*

Again the advantage is that children of 7th grade are not aware of the Pythagoras theorem. One of possible solutions is to assemble a wall with 4 bricks and remove one of them, then measure the distance between two suitable vertices of the remaining bricks. Usually children find various geometrical solutions, including quite tricky ones.

Since such problems do not assume special knowledge of children, they give a good challenge also for backward students. Often backward students demonstrate excellent spatial imagination and good skills in solving such problems. The next examples are thematic problems.

Example 15 *[Theme: equalities of triangles]. The edges AB and BC of a tetrahedron $ABCD$ are equal, the flat angles ABD and CBD are also equal. Prove that the angles DQC and DMA are equal, where Q and M are midpoints of the segments AB and BC.*

The solution consists of proving the following chain of consecutive equalities: 1) $\triangle DBM = \triangle DBQ$, 2) $\triangle DBC = \triangle DBA$, 3) $\triangle ABM = \triangle CBQ$, 4) $\triangle CQD = \triangle DMA$. The aspect, which makes this problem challenging, is that a student have to find this unique way of solution. In addition, a student should see equal triangles in different planes. Let us give another problem of this sort.

Example 16 *[Theme: equalities of triangles]. The edges AB and BC of a tetrahedron $ABCD$ are equal, the edges AD and CD also are. Prove that the distance between midpoints of the edges AD and BC is the same as the distance between midpoints of AB and CD.*

The following simple problem turns out to be surprisingly challenging not only for beginners but also for high school students.

Example 17 *[Theme: constructions with compasses and ruler]. The lengths of all edges of a tetrahedron $ABCD$ are given. Construct a segment equal to the distance between the midpoints of the edges AB and CD.*

The construction itself is rather simple and standard. If we denote by M and N the midpoints of the edges AB and CD respectively, then first we construct triangles equal to $\triangle DBC$ and $\triangle DAC$ on a piece of plane, then draw their medians BN and AN. Finally we construct a triangle equal to $\triangle BNA$ by the lengths of its sides. The median MN of this triangle gives the answer.

Thus the solution involves only standard constructions of triangles by three sides and of medians for given triangles. However, the challenge for students is the process of construction on solid objects. The first difficulty students have to overcome is how to construct on faces of a tetrahedron. They are puzzled with several technical questions. For instance, is that possible to draw a circle on a face with the center at the vertex? Where to put the compass leg? After some discussion they come to the crucial conclusion: there is no need to construct on the surface of a tetrahedron, we can transfer all the constructions to a separate piece of plane. The second question is how to draw a segment MN "through" the tetrahedron? Usually this causes long discussion between the students. First they attempt to find some obvious solutions. For example, to cut the tetrahedron along the plane CMD, or to drill a hole from M to N (which is prohibited, of course, since we are given with a ruler and compasses, not with a knife or drill). Finally they again come to the conclusion that the segment equal to MN can be constructed on a separate plane. This problem helps developing the spatial imagination, also it teaches students the method of auxiliary constructions on separate planes.

Example 18 *[Theme: triangle inequality]. What is the shortest possible way over the surface of a cube between two its opposite vertices?*

The solution is standard: by making a diagram with two faces of the cube and then by using the triangle inequality. However, the idea of diagram is not given for students in advance. The challenge for them is to come to this idea on their own. This is better to give this problem to a whole-class discussion. At the first stage children try to make a guess about the shortest possible way: to go along three edges of a cube, to go over two faces through the midpoint of an edge, etc. However, they are not able to prove that the ways found are really the shortest ones. Eventually they find the crucial idea: to enroll two faces with common edge onto a plane and to draw a straight line between two points.

After considering several problems that exploit the same idea of diagram, we give the next challenging problem. It also uses diagrams, but in a rather unusual way:

Example 19 *[Theme: triangle inequality]. An ant needs to go from the vertex A to the vertex C of a square $ABCD$. The square is divided by a triangular fence BKD that is equal to the triangle BAD and orthogonal to the plane of the square. What is the shortest possible way for the ant?*

In this problem we have to double the triangle BKD into two copies (two sides of the surface of the fence) and make a diagram by those copies with common edge DK. We obtain two squares $ABKD$ and $DKB'C$ with common side DK. By the triangle inequality the shortest way on the diagram is the segment AC. Therefore, the shortest way for the ant is the broken line ADC, or ABC. In the both cases the ant walks around the fence and do not claim it. Then we can treat similar examples showing to the students that this strategy for the ant is not always optimal. For instance, if the fence BKT is an obtuse-angled isosceles triangle, then the shortest way has a part on the surface of the fence. Usually beginners are not worse in solving this problem than high school students. Sometimes during the discussion they get unusual ideas about the solution. Once a student who obtain a correct solution, decided to consider two possible cases: "when the and can fly, and when it cannot" (it turned out that there is a sort of flying ants).

6. Conclusions

We have discussed possible ways how to provide beginners in studying geometry with challenging problems. We have considered two types of challenging problems: creative problems and thematic ones, and tread several examples of both kinds of problems. This work can also be considered as an invitation to a wide discussion on this theme.

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