"Here is a Situation …!"
Team Challenges with "Pictorial Problems"

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In this paper some unusual open-ended problems are presented, which have been "tried and tested" in mathematics team competitions for lower secondary school students in Germany. Our goal was to foster team work competences in small groups of students, and the main focus is not on calculation but rather on all the steps necessary before the calculations can begin. "Here is a situation. Think about it!" (Henry Pollak)
These rather unusual problems turned out to be very interesting and challenging to all the students involved. Exercises like these are indispensable to the introduction of skills inherent in mathematical modelling where the emphasis is not on algorithmic procedures but rather on the higher order skills of translation, interpretation, and evaluation of the real life problem in terms of the mathematical model and its solution(s).

1 Here is a situation – think about it!

In the minds of the masses, "Doing Math means calculating". True, but this is certainly not the whole story. There is far more to mathematics than "mere" calculation! In this paper, some unusual open-ended problems will be presented which we successfully used in mathematics team competitions for lower secondary school students in Germany. All students – from grade 5 to grade 10 – had to solve the same problems within one week, and then send in their solutions, including their presentation of the solving process.

In these tasks, calculating is not at the forefront, but rather all the thinking and planning skills necessary before the calculations can begin. "Here is a situation. Think about it!" (Henry Pollak)

All the students' teams participating in our competitions picked up these unusual tasks with matter of course. Moreover, they all found out their own individual way of thinking about the pictorial problems. The solutions we obtained were in some sense journals of the students solving process. It was interesting to read in their answers how they learned step by step to understand the problem, to go in search of helpful information, to activate useful mathematical knowledge and to bring together all their solving ideas to write down their answer.

Unusual problems need unusual solutions. All students worked under this maxim. Thus the results were very different, and it was not an easy job to compare the solutions and fit out the best ones. From grade 5 to 10, the creativity of our students impressed us. Even students who do not prefer usual mathematics problems are inspired by such an unusual pictorial problem. This is proven again by the extraordinary resonance of the second com-

Wie eine Werbeaktion für die Fußball-WM in Deutschland wirkt die Kuppel auf dem Dach der Brockenherberge. Unter der mehrere Meter großen runden Kuppel befindet sich Technik zur Überwachung des nord- und mitteldeutschen Luftraumes. Die Deutsche Flugsicherung hatte die Anlage wegen der besonderen geographischen Lage auf dem Brocken 1999 in Betrieb genommen. *Foto: dpa*

Figure 1

- **What size should be a goalmouth fitting to this giant "soccer ball"?**
- **How many usual soccer balls would fit into this giant "soccer ball"?**

petition of this type in 2006.

In the first exercise presented here (Figure 1), a newspaper article depicting a giant “football” or “soccer ball” is used as a starting point: What size is this giant “soccer ball”? What size should be a goalmouth that fits to this giant “soccer ball”? How many usual soccer balls would fit into this giant “soccer ball”? Everyone seems to find a task like this rather unusual, and it is always intriguing to read the different ways of solving the problem.

2 Many different ways of solving the problem

The standard approach is to use an object in the picture as an estimator or yardstick, e.g., the windows, the height of the floors. It is quite easy to measure these things, both in the picture and in reality. A few simple calculations suffice to give us the real diameter of the giant “soccer ball”.

Furthermore, a suitable search in the internet could help to find some information – about the size of usual soccer balls, about the size of usual soccer goalmouths, and, maybe, about this building – this could well turn out to be an interesting research project!

But now, a problem arises, and we are faced with a real challenge: We still have to pack the usual soccer balls into this giant one. How to cope with this challenge? How to fit the “small” soccer balls into this giant “soccer ball”? Now, some mathematical ideas are requested. We could look to develop some formula – that’s not too easy. Or we could make experiments with some “giants’ balls” and “dwarves’ balls” and collect data by measuring, or … You can imagine many different ways of solving this problem, especially, because all students got this very same problem – from grade 5 to grade 10!

How reliable, however, are the different approaches to the problem? How accurate are in fact the various measurements and estimates? In the end, a critical comparison of each method might well reveal a slight difference but we still haven't come up with “the right solution”!

Very precise … and very rough!

Math lessons are typically characterised by precision. For example, if three sides of a rectangular box are 3 cm, 5 cm, and 7 cm respectively (and precisely, of course!), then find the volume of the box. But this obsession becomes an exercise in futility the moment mathematics becomes involved with “the rest of the world”. There, most of the numbers which crop up are only approximately correct. This is inevitable and unavoidable! Likewise, the results are only rough estimates.

In mathematics education, therefore, one of our tasks, indeed obligations, should be to bridge the gap between these two different worlds: the world of accuracy so typical of mathematics, and that of lack of precision in the rest of the world (cf. Herget/Torres-Skoumal, 2006). This is imperative because both worlds are important and both are indispensable. How can we possibly learn the true value of the precision and certainty of mathematics if we have not yet learnt that, in the “rest of the world”, this precision and reliability is something which is very difficult to achieve? On the other hand, one can only learn to cope well with this inaccuracy and blatant lack of precision if one has learned to exploit the many possibilities offered by the very precise field of mathematics.

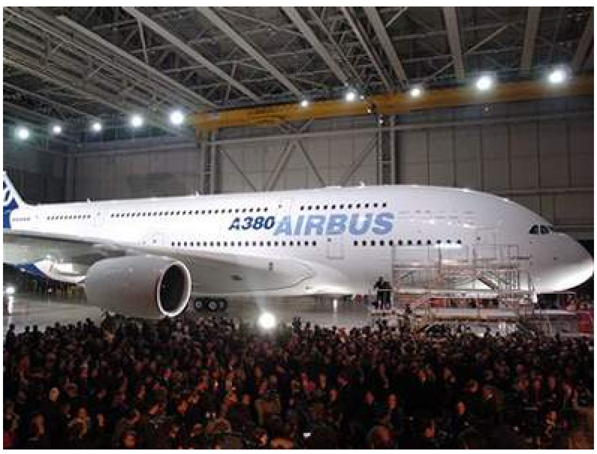

Figure 2

- **How many tennis balls would fit into an Airbus A380?**

3 A picture tells a story of (well over) 1,000 words!

How is it then possible to bridge the gap between mathematics and the “rest of the world”? How do we carefully and sensitively introduce the young students to the uncertain world of mathematical modelling? At this point we propose a very special method: setting tasks mostly based on rather unusual newspaper cuttings that we are apt to call “Pictorial Problems” or “Picture Mathematics” (cf. Herget/Torres-Skoumal, 2006). Many tasks based on real-life situations are often far too cluttered with text to be truly effective for the young mathematics student. This is where a picture, supplemented by the students' general knowledge and imagination, comes in handy: “A picture can indeed say far more than a thousand words!”

Figure 3

Since yesterday, Arminius is a real football fan

The Arminus statue, 26 meters high, reminds of the German hero Arminus, who, two thousand years ago, defeated the roman soldiers.
Since yesterday, Arminus is a real football fan – kitted out with a tricot of the local soccer team, probably the world’s biggest football fan tricot.

- **How many square meters of cloth were used for this stately football fan tricot?**
- **In winter time, of course, sometimes it will get cold.**
  How many meters of wool would be needed to knit some gloves, suited to the Arminus statue?

4 Different ways but common ideas

Doing math means calculating. Yes, but doing math means as well setting up mathematical models for better understanding parts of the real world. But the complexity of such modelling tasks makes it difficult to itemize it in the classroom. At this point the proposal of the pictorial problems comes into the game.

Pictorial problems can help the students to develop techniques setting up models - learning step by step thinking about an interesting, but easy to grasp situation. Of course, often the situation might look a little bit strange, but on the other hand it is simple enough to discuss it successfully on a way the students can manage.

> *What is the problem given by the picture? What aspects of the situation are mathematically relevant? How to obtain the information necessary to solve the problem? How can we cope with putting our ideas together to find the answer? How can we prove our answer to be correct (or, at least, not too bad)? ...*

When thinking about the pictorial problems the students answered these questions by challenging their own strategies. The ways to solve the respective problem were different but the important ideas were often the same. So it is possible to itemize some basic steps inherent in the process of seeking solutions to these examples (cf. Herget/Torres-Skoumal, 2006).

- "Real world" mathematics remains the focal point for the duration of the activity until a solution is reached – the problems do not exist merely as a desperate attempt to superimpose a real world problem on analytical techniques previously learned.
- The facts are analyzed and the mathematically relevant details are filtered out while the perhaps interesting, but irrelevant information (for the solution's sake) is laid aside.
- Often, an appropriate object is chosen to serve as a yardstick for the necessary measurements which have to be made in the solution process.
- Necessary simplifications are performed.
- The interesting measurements are taken from the picture; through the measurement process one is constantly conscious of the unavoidable element of uncertainty furnished by the approximations.
- Common knowledge is activated, e.g., how tall is an average person, an average floor, an average window, which size is a tennis ball, a soccer ball, a goalmouth, an airplane, a tricot, etc. If necessary, information from other sources will be obtained.
- The relationship between the chosen yardstick and the measurements obtained will be mathematically defined and refined.
- A suitable mathematical model and methodology for the solution of the problem will emerge from this process, as opposed to students being handed pre-conceived ones. The students may choose the model they feel is most suitable, i.e., they must choose the model themselves.
- Often, by the way, technology allows for solutions previously denied students until much later in their mathematical development, or not at all.
- This entire process is guided and enlightened by fundamental mathematical considerations, strategies and concepts, which make a solution possible.
- Throughout this process other questions or ideas emerge, mathematical or otherwise, which can then be expanded upon, time permitting.
- At the end, the emphasis should be on good mathematical writing, thoughtful reflection, and convincing presentation of the results.

A discussion of these examples highlights the essential aspects of the process of mathematical modelling. In mathematics education knowledge and skills are necessary pre-requisites which assist us in various stages of the process, but to accomplish the entire task at hand, certain central ideas or concepts are necessary, namely the concepts of measurement, approximation, and linearization (Herget/Klika, 2003).

All of the above is in accordance with Hans Freudenthal's view of mathematics (Freudenthal, 1968) – 'mathematizing' as the activity of looking for problems and solving them, by organizing all the information you have about this problem situation and then choosing and using suitable mathematical tools.

When working with these "pictorial mathematics" exercises, the role of the accompanying teacher – if there is one – changes from being mainly the disseminator of information to becoming a moderator or facilitator of knowledge. The teacher must carefully consider the various methodologies chosen by the students, and gently guide and direct their efforts in their quest for a solution. (For example, the most common error for wide discrepancies of approximations was incorrect handling of units, e.g., incorrectly changing cm^3 into m^3, or m^3 into liters, etc.) Furthermore, the accompanying teacher should encourage discussion and reflection on the various strategies employed, and point out the central concepts and ideas contained in the various solutions. Now more than ever expertise is needed in handling information explosion, performing thorough research, discerning the important from the unimportant and the correct from the questionable. Good communication skills are required in order to make the procedures and processes accessible to others. Lastly, all of the incorporated and integrated information should lead to a higher level of knowledge.

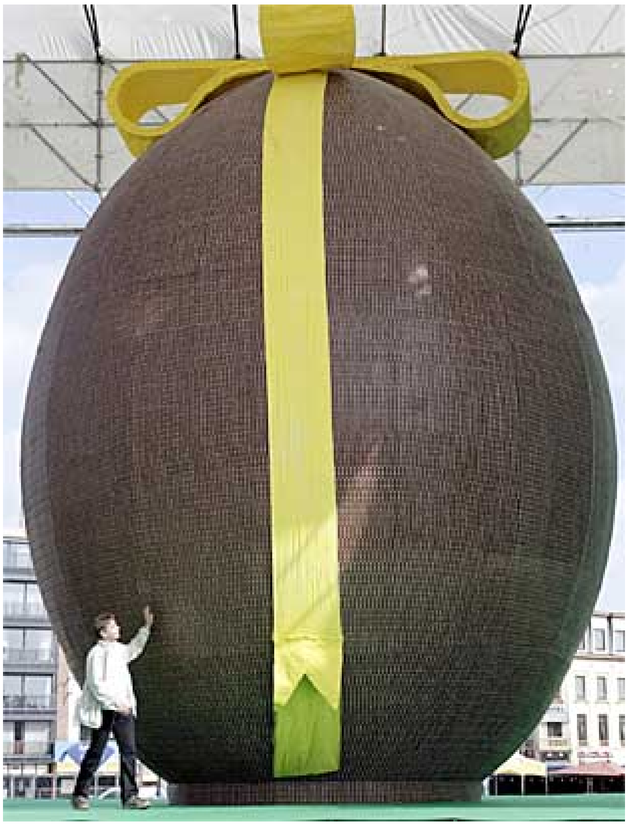

Figure 4

Biggest Chocolate Egg

SINT-NIKLAAS. The biggest chocolate egg of the world, 8.32 metres high and 6.39 metres wide, was built by 26 Belgium chocolatiers. It took 525 hours and nearly 2000 kg chocolate to create this attraction. But, unfortunately, the egg is not edible, for it had to be preserved.

- **How thick was the chocolate shell of this world record egg?**

5 Mathematical modelling as a central theme

Nowadays, tasks requiring pure technical calculation can be solved with the help of a calculator or computer software. Consequently, more demanding activities are gaining importance, e.g., the analysis of problems affecting the "real world", i.e., mathematical modelling. It is to the child's benefit to discover the "discomforts" of uncertainty in mathematics as early as possible. For students (as well as for teachers) the shift from everyday problem solving (with one correct answer) to mathematical modelling (multi-solution paths leading to approximate answers each with possible limitations or potential for extension) requires a new set of teaching and learning skills – symbolizing data, cleverly translating the task into the language of mathematics, i.e., into a mathematical model, the internal treatment of this problem in the field of mathematics right up to its solution(s) possibly with the aide of technology, and finally, a deliberate interpretation and critical examination of the results obtained. "Has our original question really been answered? How accurate and how reliable is the result? How applicable are our results to other (new) situations?"

Thus using carefully selected examples, the typical process of mathematical modelling may become one of the central themes also in the mathematics classroom, including both modelling methods and the accuracy of the mathematics involved, of course without losing sight of the discrepancy between the mathematical model and reality.

6 Assessment of modelling tasks

We all know that in good classroom practice assessment is a natural by-product of the classroom experience and should be a celebration of achievement. In using such activities for assessment purposes we recommend, therefore, a criteria-based assessment that is defined from the sub-tasks themselves. Since in these open-ended tasks "many roads lead to Rome", the focus of the assessment should be uppermost on the selection of the path and the markers that the students have set up along the way to ensure a secure journey toward the goal (Herget/Torres-Skoumal, 2006). Considerations as criteria should therefore be – **Communication**: To what degree have the students used tables, diagrams, graphs, etc. as aids in defining the problem? Have they used correct mathematical notation and terminology throughout the activity? (*Yes*, this is still very important, perhaps now more than ever, the need for correct and clear communication.) **The Model**: What have the students used as a yardstick, and is it well justified? Have they taken into account all vital variables to the problem? Does the model suit the problem – how *well* does it fit the problem? **Mathematical Content**: Are the calculations correct and justified? Were formulas correctly applied? **Evaluation**: To what extent have the students evaluated the meaningfulness of the approximations obtained from the model in light of the real life problem? Have they considered limitations of the model, or possible extensions and applications of it?

The assessment of such activities can provide students with opportunities and rewards for carrying out mathematics without short time limitations of tests and exams and their accompanying stresses. Furthermore it can provide those students who have difficulty performing well on traditional exams a sense of success and achievement in this subject. Hence, the emphasis in this kind of assessment should be on good mathematical writing and thoughtful reflection.

7 Conclusions

Mathematics modelling is a useful tool for encouraging students in learning mathematics. Interesting unusual situations can help them to make by themselves the first steps in setting up mathematical models. Thinking about an inspiring pictorial situation they challenge the most important steps solving a modelling problem – step by step. Here is a situation – think about it! This is really a challenge. The students work and answer with sensitivity and creativity. We, as teachers, have the extraordinary opportunity to learn together with our students: It is an interesting and powerful way to bring life to mathematics teaching and learning.

Additional information and many further examples are available under Herget (2002), Herget/Jahnke/Kroll (2001), Herget/Klika (2003), Herget/Skoumal (2006), Büchter et al. (2006). But, one will surely find up-to-date pictures, perhaps even in the respective local newspaper, featuring events which are of interest to the pupils, and are closely related to *their* world – and *to mathematics*, of course.

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