Mathematical Learning: Insights from a Challenging Problem

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Introduction

In this paper we report on the mathematical activity of a group of five high school students (15 – 16 year olds) working on a probability problem, in the context of a longitudinal research involving Rutgers University researchers and students from the Kenilworth schools in New Jersey, USA. The longitudinal study was an after-school program with no time constraints and no grades were assigned. The students worked collaboratively on challenging mathematics problems as a research context for the development of particular mathematical ideas and ways of reasoning in areas of algebra, combinatorics and probability .A more comprehensive description of the longitudinal study is available elsewhere (Maher, 2002; Kiczek, 2000; Maher & Martino, 1996). Our purpose for this paper is to describe the challenge that the problem posed to the students and extract implications for mathematical learning. In the context of the ICMI 16 study, we examine the question regarding what research using challenging problems can tell us about the teaching and learning of mathematics.

Theoretical Framework

This paper contributes insights on to how a challenge helps promote mathematical learning in problem solving. There has been extensive research on how problem solving is related to mathematical thinking (Verschaeffel, 1997; Greer, 1997). However, there has been no "coherent explanatory frame" as to how problem solving and mathematical thinking fit together (Schoenfeld, 1992). This study seeks to address this issue in a case involving a well-defined open-ended challenging problem in probability. Our perspective is that challenges stimulate mathematical reasoning and the latter is a driving force in promoting mathematical learning based on sense making.

Problem solving has been associated with different meanings, which reflect different views about mathematics and mathematical learning. Schoenfeld (1992) reports three uses of problem solving in mathematics classrooms. In one, problem solving is the act of solving problems as a means to facilitate the achievement of other curricular goals, such as teaching mathematics, motivation, recreation, and developing and practicing mathematics skills. In another, problem solving is a goal in itself, out of many, of the instructional process. It is a skill or piece of knowledge that is worth teaching in its own right. Finally, problem solving can be viewed as a form of art, as what mathematics is ultimately all about, when challenging problems are involved. Our problem solving perspective arises from an extensive body of experience that comes from both longitudinal and cross-sectional research on how learners build mathematical meaning. In particular, it recognizes the learners' ability to construct their own personal knowledge in conditions, where they explore patterns, make conjectures, test hypotheses, reflect on extensions and applications of learned concepts, explain and justify their reasoning and work collaboratively (Maher, 1988). This view brings together problem solving and the mathematical process by which thoughtful learning takes place. This paper addresses the question what research that uses challenging problems can tell us about the teaching and learning.

Method

The longitudinal study made possible a large collection of data in the form of videotapes of students' problem-solving activities, transcripts of such videotapes, observation notes, and copies of students' written work. The data for this paper consists of a 2-hour videodata showing five students working on the World Series Problem:

> In a World Series two teams play each other in at least four and at most seven games. The first team to win four games is the winner of the World Series. Assuming that the teams are equally matched, and there are not ties, what is the probability that the World Series will be won: a) in four games b) in five games c) in six games d) in seven games?

The four male and one female student are known in this research as Romina, Mike, Jeff, Brian and Ankur. All of them participated continuously in the Longitudinal Study since its inception in first grade, 1989. They worked on the World Series Problem in the 12^{th} year of the longitudinal study. Data analysis followed an emerging methodology for the analysis of mathematical reasoning using videodata of students working on mathematical tasks (Powell, Francisco & Maher, 2003). In this study, it involved identifying and characterizing three critical or significant episodes of the students' work on the World Series Problem. The episodes became the units of analysis of this paper. They were chosen for the challenge they posed to the students' mathematical activity and the potential insights regarding mathematical learning. The episodes are described in the next section. The following section discusses the implications for learning and teaching from the episodes.

Three Challenging episodes

<u>Episode One</u>: Searching for a strategy

This episode occurred early in the problem-solving session. It refers to the students' collaborative work in searching for a strategy for solving the World Series Problem. We describe three instances of the students' mathematical activity. In the first instance, Romina suggested a solution to the first question in the World Series Problem. She claimed that the probability of the series ending in four games [P(4)] was one-half because of the two ways in which it could be won in four games, AAAA and BBBB, respectively. However, Jeff challenged the idea by arguing that winning the series in four games is [intuitively] "hardest". The students subsequently dropped the idea:

ROMINA:	They can go all seven or they could go all four. So, it would be A, A, A, A and B, B, B, B -Team A and Team B?
ROMINA:	So, in four games, would it be, like, one-half of a chance? Or would we have to write it out with -- using all seven?
JEFF:	See, I think that it's the hardest to win it in four games.
JEFF:	Definitely the hardest.
ROMINA:	Yeah, exactly.
JEFF:	So, it wouldn't be one-half.

In the second instance Brian suggested multiplying the odds [sic probability] of a team winning a game. More specifically, he suggested multiply 1/2 [the probability of either of the equally matched teams winning a game] four, five, six and seven times to obtain P(4), P(5), P(6), and P(7), respectively. However, as the students used it to compute the probabilities, Romina pointed out that the probabilities were getting smaller. The students immediately abandoned the strategy:

BRIAN:	So, it's like, half times a half – no, wait – remember the odds get harder to win two in a row, like a coin flip?
ROMINA:	Yeah, that's how you do it: a half times a half times a half times a half.
ROMINA:	Four – hold on – four times –
BRIAN:	That's sixteen.
ROMINA:	Oh, never mind, I get it. Now, would you have, for five games, like, would it be like that [1/2*1/2*1/2*1/2*1/2]?
ANKUR:	Hopefully, the odds of winning are –
JEFF:	We're never going to get – it's never gonna equal up to one, though.
BRIAN:	Does it have to?
ROMINA:	[Looks at her paper] Wouldn't you have easier odds of winning in six games than in four?
JEFF:	Yeah.
ROMINA:	Doesn't it get less, though?
JEFF:	That's why it's wrong.

In the third and final instance Mike intervened and explained his reasoning. However, when Ankur and Jeff pointed out that it suggested that P(4)=1 or 100%, the students no longer pursued the idea:

MIKE:	Okay, that's good enough. There's gotta be a different way of, um, of looking at it, then, 'cause if you just say, multiply the probability in four games and seven games, it'd be –
ROMINA:	It's gonna be too – it's too small.
MIKE:	It's gonna be harder in seven, but actually, it's really not
BRIAN:	'Cause you've got more chances.
ANKUR:	Yeah, 'cause you could win one, lose one.
ANKUR:	You could win three and lose three.
MIKE:	Then you go like this. You have – with the four games, you have a maximum of four and you have to win four, so it's like you have to be a hundred percent a winner. And, with seven games, you have seven possible, all you have to do is win in four. So you got a four out of seven chance of–
JEFF:	But then, so then –
ANKUR:	But what's the overall probability?
JEFF:	But then, what's the probability of winning four games, if that's the case?
MIKE:	I don't know.
JEFF:	You can't say a hundred percent.
MIKE:	Yeah.

The students eventually decided to solve the problem by using the notion of probability as a ratio. They listed by "brute force" the series winning game combinations for the value of the numerator in the probability ratio. However, computing the denominator posed another challenge, which is described below. The key aspect of this episode is how the students continuously proposed and critically evaluated each other's ideas.

***Episode Two*: Determining the sample space**

This episode focuses on the students' discussion on determining the sample space for the World Series Problem in the context of probability as a ratio. Early in the problem-solving session, Romina suggested that the denominator for P(4) would be 7, which is the maximum number of games before a series ends. Ankur and Jeff disagreed:

JEFF:	All right. So after the four, for winning in four games –
ROMINA:	Should it be over seven, though?
MIKE:	It should be over – over seven, 'cause it's four out of seven games.
ANKUR:	But this one wouldn't be over seven.
JEFF:	It wouldn't be.
ANKUR:	It wouldn't. None of this would be over seven.

In particular Ankur explained that they could compute the denominator of P(4) as a power of two, or $2^4 = 16$. The students decided to use the idea. However, as they tried to compute P(5), Romina asked if the sample space would be 2^5 or 2^7:

ROMINA:	Wouldn't it be two to the fifth? Or would it be two to the seventh?
JEFF:	Two to the seventh is a hundred twenty-eight. And, but, like B, B, B, B, B – like B seven times wouldn't count. But on the other hand, the sum –
ROMINA:	But, I'm saying for this one, it's eight over – would it be eight over two to the seventh or two to the fifth? Maybe to the fifth? Well, because –

Ankur repeated his idea computing the denominator as a power of two. The students used it to obtain the remaining probabilities. However, the issue about the sample space still came up, although in different context. The students found it difficult and time consuming to list all winning game combinations for P(7). Jeff suggested a "short cut." The students would obtain the number of winning game combinations for P(7) by subtracting from 128 [Sample space computed as 2^7] the numbers 2, 8 and 20, the number of winning game combinations for P(4), P(5), and P(6), respectively. However, the students abandoned the idea, after Jeff could not find the total out of which to consider the result:

JEFF:	You have twenty-eight here, so you'd subtract two because of the four games. There's two that would cancel out, like –
ANKUR:	What do you mean?
JEFF:	And then –
ANKUR:	You'd subtract eight.
JEFF:	Eight. Ten – no, then eight, twenty –
ANKUR:	Twenty.
BRIAN:	Those aren't the best [inaudible].
JEFF:	Thirteen.
BRIAN:	They can't be factors in the seventh game.
ANKUR:	That's what I'm thinking.
JEFF:	Well that's why they're not – that's why you subtract them.
ANKUR:	That's why you subtract them. Yeah.
JEFF:	But then you get like a number like ninety-six.
BRIAN:	Out of what? One twenty-eight?
ANKUR:	Out of what?
JEFF:	I'm not sure.

The key aspect of this episode is that the students' work to determine the sample space permeated the entire session, as they attempted to make sense of competing answers for the sample space.

***Episode Three*: Justification and reflection**

The students eventually came up with the following answers for the problem: P(4)= $\frac{2}{16}$, P(5)= $\frac{8}{32}$, P(5)= $\frac{20}{64}$, and P(7)= $\frac{40}{128}$. They listed the winning game combinations [for the numerator of the probability ratio] and computed the denominators as powers of two. However, when they explained the solution to the researchers, they acknowledged that they lacked "mathematical backing" for their answers.

JEFF: That was the first one, so for the first one, it's two sixteenths. All right, for the next one, we're going to do the same situation, but this will be two to the fifth, so it's gonna be out of thirty-two, and thirty-two is the bottom number. And then for – I think, for these we were just kind of – we went through 'em. We were – that's why there's strings of A's and B's on everyone's paper. And in order to get these, we went through all the possibilities where there was only – there was five, five different, five places, and A or B was in four of them. And we went through all of them and that's how we got that. And then we ended up with, um, eight of thirty-two for that. Now, that's not too convincing 'cause we just went through 'em, but we went through all the ones that were out of five with four A's and so we got that. I don't think we have a really concrete mathematical backing for that. .

This situation opened up a new challenge for the students. They had to come up with a convincing argument that they had listed all possible game winning combinations for P(7). They brought in their experience with Pascal's Triangle. Mike explained that he had noticed that the number of favourable outcomes that they had listed for the probabilities of the series ending in four, five, six and seven were the same as the numbers in the Pascal's Triangle as indicated below:

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                 1
              1     1
           1     2     1
        1     3     3    (1)
     1     4     6    (4)    1
  1     5    10   (10)    5     1
1     6    15   (20)   15     6     1
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MIKE: All right, um, I just found, like, if you take the fourth number in each one [circles these entries]-- that way, if you double each number, 'cause you have two teams, you can get the possibilities of four games. Four games, um, equals two, right? You got eight, twenty and forty, like they said.

However, Mike was not sure how to explain the connection:

MIKE: Three of them that have two and one of them will have three. Um, now when you go to the next step, those, uh, that last – those last – those three games that they won – The first three games, if they win that, that'll be like, those three possibilities without – would be – if they win the next game or those three – if they win – Uh, I don't know how to explain it. Uh, on the third game – I don't know. I – I have trouble explaining things. I don't even know what I'm trying to do now.

The students discussed the issue for a relatively long period of time. Eventually, Ankur provided the closest argument for a justification of the students' answers using Pascal's Triangle:

ANKUR: Actually, I was going to say, like, that one represents the, like, winning three games in a row, or like three A's.

MIKE: That's the probability.

ANKUR: And then, if you go to the right, that's like getting another A, and there's only one way to get four A's. If you go to the left that's like getting a B, and that's like three A's and a B, and there's four different ways you can write that.

The key aspects of this episode are the students' acknowledgment that they had no justification for their answers, the connections between the problems and the students' collaborative work to come up with come up with a justification for their answers.

Insights on mathematical learning

One insight is that *the nature of students' collaborative work can be a source of mathematical challenge and ways to overcome it*. In the World Series, the students constantly suggested, listened to, and critically evaluated each other's ideas. New challenges and suggested solutions or strategies emerged from the students' collaborative work, which helped them make progress in solving the problem. In Episode One, it helped them generate a number of ideas from which they chose the strategy that they used to solve the problem. It also helped them control for variables when listing winning game combinations. In Episode Two, it led to a refinement of the students' strategy for finding the denominator of the probability ratio as powers of two. In Episode Three, it allowed the group to eventually provide an explanation of Mike's insights on the relationship between The Word Series Problem and Pascal's Triangle.

Another insight is that *complex mathematical relationships or tasks are not the only sources of challenges when it comes to promoting mathematical reasoning. Basic or fundamental concepts, such as probability and, more specifically, sample space, can also be cognitively challenging and engaging to the students*. This follows the claim in Episode Two that the students' attempts to determine the value of the denominator in the probability ratio permeated the entire session on the World Series Problem. An analysis of the students' attempts suggests that the challenge for the students was to determine the type of elements of the sample space, not so much how many such elements were in the sample. This suggests the insight that *challenge in probability is not only about how to count the elements of a sample space. It is also about what to count.*

Episode Three gives another insight on the relation between challenge and learning. The students acknowledged that they did not have a "mathematical backing" for their answers to the World Series Problem and eventually examined the connection between the problem and Pascal's

Triangle. *This suggests that challenge can motivate students to pursue other potentially complex challenges, which can result in the building of more powerful knowledge.*

Conclusion

This paper examines the relation between challenge and mathematical learning in problem solving. The main argument is that challenge stimulates mathematical reasoning in students, which is essential in promoting thoughtful mathematical learning. As the students overcome challenging obstacles, they build lasting and more complex forms of understanding. However, the paper study also suggests that mathematical tasks are not the only source of challenge. Students' constant evaluations of each other's ideas and the need to justify answers also provide forms of challenge. They suggest a dynamic view of challenge, which evolves in the context of the students' mathematical activity and enhances the development of independent learners.

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