

**The International Commission on Mathematical Instruction (ICMI)
The Sixteenth ICMI Study**

Challenging Mathematical Problems in the Classroom

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Abstract

In the new Chinese Mathematics Curriculum Standards, there is time scheduled for students to engage in investigative activities. This poses great challenges to mathematics teachers to design challenging tasks so as to help students develop creative thinking and problem solving abilities. However, how this is to be achieved is not explicated in the Standards. Through lesson study of one challenging mathematics problem in authentic classroom context, this study examines what research into challenging problems tells us about the teaching and learning of mathematics. This study illustrates how slight modification of existing problems and a shift of focus to different levels of mathematization can help tailor the mathematical investigations to the different mathematical competence levels of the students.

Keywords: mathematization, mathematical challenge, mathematics problem solving

A. One problem without much mathematical challenge

Problem 1: Given a square cardboard paper, construct a box without a lid that contains things as much as possible.

This problem is an example illustrated in the new Chinese Mathematics Curriculum Standards for use in the secondary classrooms (Cheung, 2004). In Macao, senior secondary students would attempt a solution using standard algorithms taught by their teachers. Very often, teachers find that if students cannot formulate $V = f(x)$, they are unlikely to proceed further. For example, when the paper is of size $12\text{cm} \times 12\text{cm}$, students can easily tackle the problem by differentiation of the volume equation $V = f(x)$ ($V =$ volume of box formed, $x =$ length of the side of the corners cut).

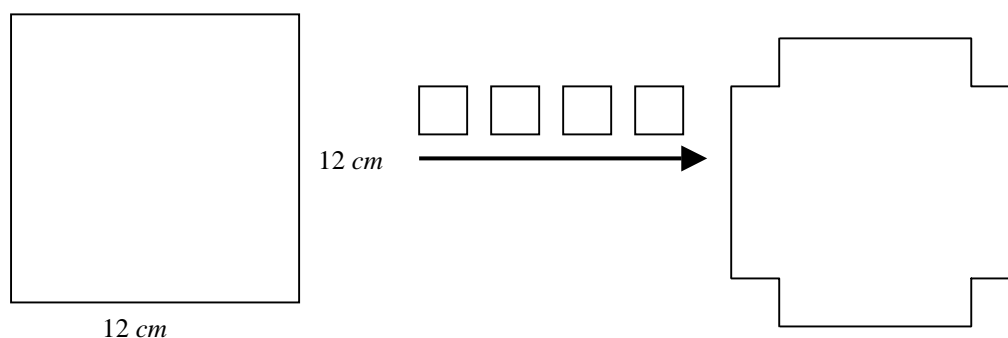


Figure 1: Construction of a box without a lid

Below is the required proof using differentiation:

$$V = (12 - 2x)(12 - 2x)x$$

Setting $V' = 0$ gives $x = 2$ or 6 . V is maximum when $x = 2$ and the value is 128 .

Students may check the correctness of this result using the *Arithmetic Mean/Geometric Mean Inequality*.

Let the length of the side of each of the four square cut corners be x cm, then volume of the open box is:

$$\begin{aligned} V &= (12 - 2x)(12 - 2x)x \\ &= \frac{1}{4}(12 - 2x)(12 - 2x)4x \\ &\leq \frac{1}{4}\left(\frac{12 - 2x + 12 - 2x + 4x}{3}\right)^3 \quad (\text{Using the Arithmetic Mean/Geometric Mean Inequality}) \\ &= 128 \text{ cm}^3 \end{aligned}$$

Equality holds if and only if $12 - 2x = 4x$, i.e. $x = 2$ cm.

As shown above, this is a problem solved without much mathematical challenges. It is envisaged that slight modification of the existing problem or a shift of focus to the lower or higher levels of mathematization may pose challenges to students of different mathematical competences.

B. Slight modification of an existing problem to more challenging ones

Problem 2: Given a $12 \text{ cm} \times 12 \text{ cm}$ square cardboard paper, construct a box without a lid that contains things as much as possible. You are not allowed to waste any of the cardboard paper. Any paper cut out may be pasted back to the net to be folded into the box without a lid.

As far as Problem 1 is concerned, the box formed is a cuboid ($8 \times 8 \text{ cm}^2$ base area; height 2 cm) and the resulting net resembles a cross (see Figure 1). Because Problem 2 necessitates no wastage of cardboard paper, the four corner squares are pasted together to form a square ($4 \text{ cm} \times 4 \text{ cm}$) or a rectangle ($2 \text{ cm} \times 8 \text{ cm}$). These can be cut into four strips of equal width (0.5 cm) so that they can be adhered to the perimeter ($4 \times 8 \text{ cm} = 32 \text{ cm}$) of the opening edges of the box (labeled as Method 1; see Figure 2 for detail). The size of this box is then 160 cm^3 ($8 \text{ cm} \times 8 \text{ cm} \times 2.5 \text{ cm}$). Increase in volume of this box is $160 - 128 = 32 \text{ cm}^3$. Students should continue to explore whether this method of enlarging the size of a box towards maximum size is necessarily producing a box of maximum size. They would be surprised to find that this is not necessarily the case.

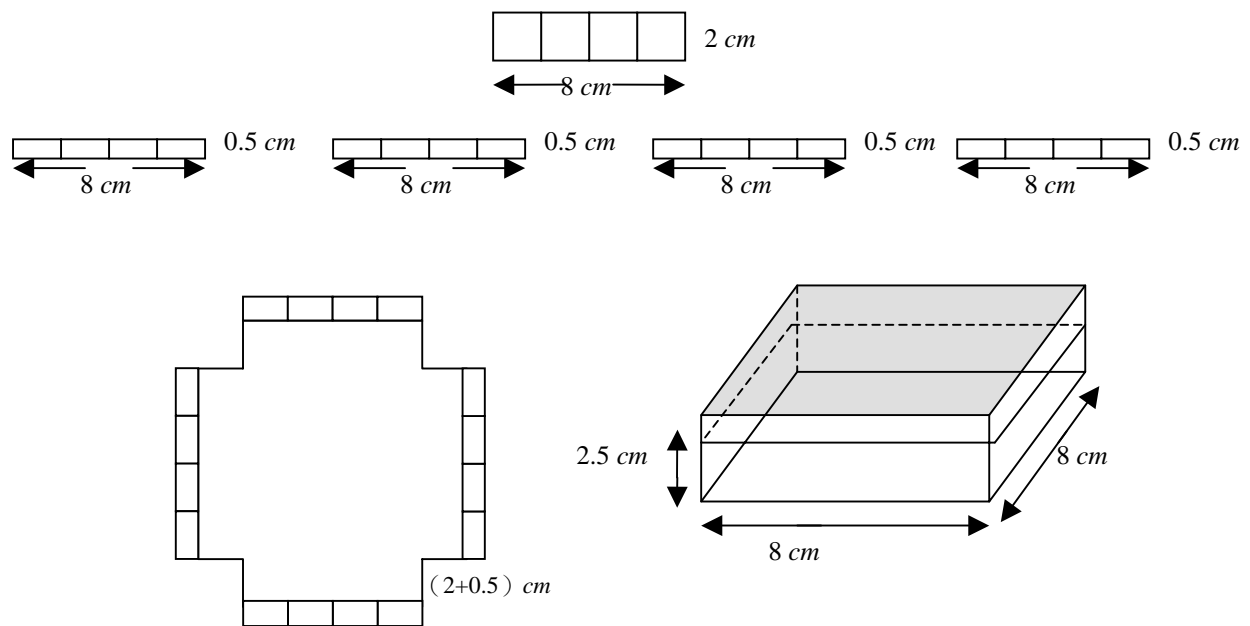


Figure 2: Increasing the volume of open box using Method 1

Another way of producing a box of bigger size is to cut two $3 \times 3 \text{ cm}^2$ from two adjacent corners and pasted them back at the middle of the side opposite to that the two corners are cut. The net formed in this way can be folded into a box ($6 \text{ cm} \times 9 \text{ cm} \times 3 \text{ cm}$) without a lid. The size of this container is found to be 162 cm^3 (labeled as Method 2; see Figure 3 for details).

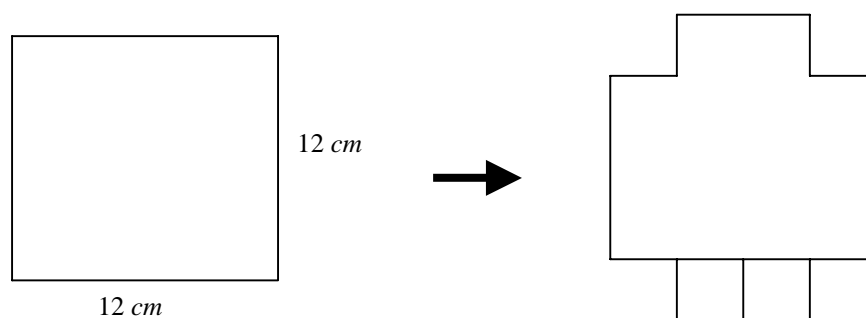


Figure 3: Increasing the volume of open box using Method 2

Comparing the two boxes using Method 1 and Method 2, the main difference is that the former produces a box with a square base and that the latter needs not be the case (i.e. a rectangular base of dimensions $6 \text{ cm} \times 9 \text{ cm}$). Using experimentation alone cannot produce clear-cut maximization results. Therefore, the ultimate solution is rested on the application of the Lagrange Multiplier Method. The constraint is that the area of the net equals 144 cm^2 because no cardboard paper is to be wasted. After derivation, V can be shown approximately equal to 166.28 cm^3 . Details of the calculations are not presented here.

C. Making a shift of focus to lower levels of mathematization

A shift of focus to the lower or higher levels of mathematization may pose challenges to students of different mathematical competences. For example, Problem 1 may be assigned to primary six students instead of secondary students in order to let them experience how to convert a practical problem into a mathematical problem for its solution. For the purpose of the present study, the researcher chose 12 cm x 12 cm square cardboard papers so as to allow manageable manipulation and experimentation by the students. At the same time, mathematical investigation allows the researcher to study the phenomenon of mathematization – an idea stemming from Freudenthal’s notion of Realistic Mathematics Education (RME) (Freudenthal, 1973 & 1991; Gravemeijer, 1994 & 1999). RME provides an analytic framework for teachers to examine students’ level of mathematical competence in terms of levels of vertical mathematization and extent of horizontal mathematization (Cheung, 2005a). Using Problem 1 as an illustrative example, four levels of mathematization are envisaged when students are engaged in the problem solving activities and these are summarized concisely below:

(1) **Situation Level** – Form the open box as suggested. Measure the size of the box formed so as to calculate its capacity/volume. May conduct experiments systematically to find out whether boxes of bigger sizes may be formed or not.

(2) **Referential Level** – After forming one box, students become familiarize with the construction process and they need not make more boxes. Instead, based on the first box, they can proceed to tabulate relationships of length of side of corner cut (i.e. x cm) with volume of box formed (i.e. V cm³). Graph of V against x may be plotted to reveal the non-linear relationship. This graph is then a model of the given problem and maxima can be located accordingly.

(3) **General Level** – No box needs to be made. Students make rough sketches of the plan for making the open box and later come up with a non-linear equation relating volume of box formed (i.e. V) with length of side of corner cut (i.e. x). Using the non-linear equation, students find the maximum value of V without recourse to the original problem situation. $V = f(x) = x(a - 2x)^2$, a being the length of the side of a piece of square paper, is therefore a model for relating variables bearing non-linear relationship of the third degree.

(4) **Formal Level** – Using $V = f(x)$, students solve the maximization problem using formal mathematical methods within the mathematics discipline, e.g. differentiation.

The primary six students in Macao are expected to demonstrate their competences at the situation level and perhaps to a slight extent at the referential level of mathematization. The following scenario was envisaged by the mathematics teacher when students succeed at the situation level of mathematization – students understand that four corners, each of which is a square of the same size, are needed to be cut out from the cardboard paper to form a net (see Figure 1). This net, which looks like a cross, is then folded and adhered together to form a box without a lid. The volume of the box should be as large as possible if it is required to contain things (e.g. rice, sand, jelly) as much as possible. By doing experiments students should be able to find that the four corner squares should be 2 cm x 2 cm each and that the largest volume attained is 128 cm³. Of course, due to their limited mathematical background knowledge there is no way for the students to demonstrate that maximum volume is 128 cm³.

However, mathematical investigation is not only posing challenges to our primary six students but also to the teacher (Ms. T) who is involved in the series of lesson studies. Figure 4 detailed a basic model of teacher action education so as to frame the conduct of the three lesson studies of student mathematical investigations (Cheung, 2005b)

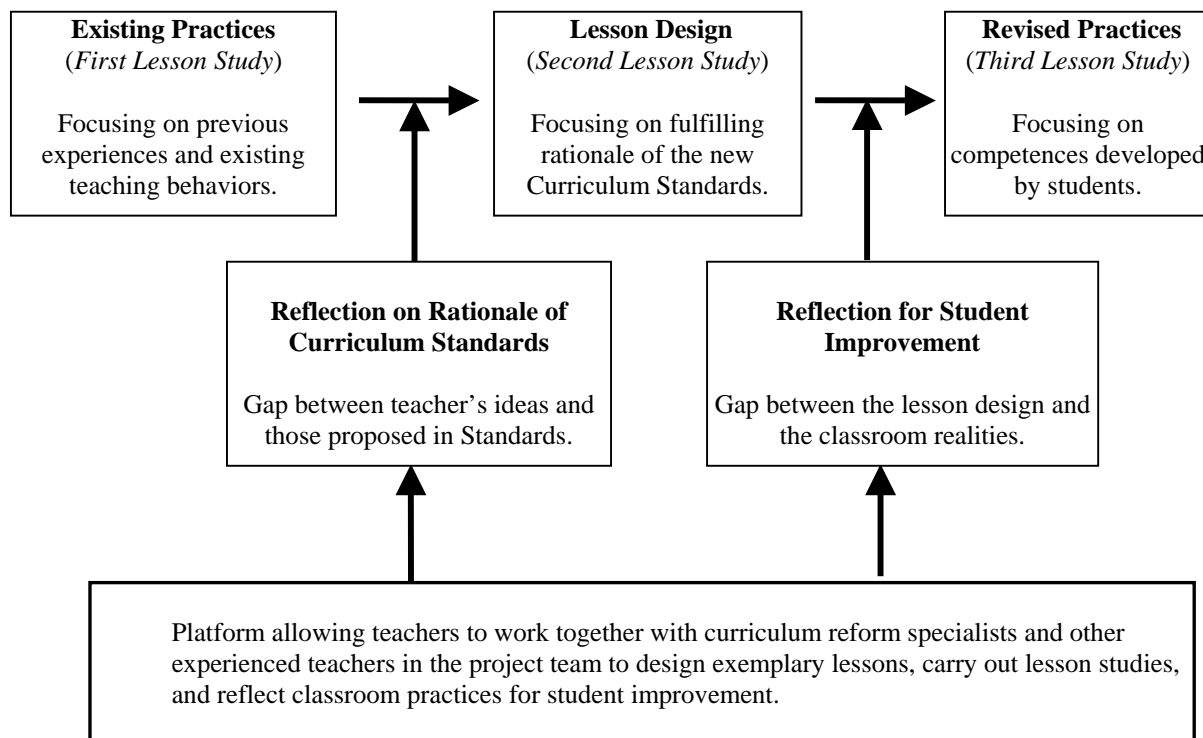


Figure 4: Basic Model of Teacher Action Education

D. Results of the first lesson study

In the first lesson study, a class of 33 primary six students studying in a private school in Macao was engaged in solving the “open box” problem (Problem1). These students have learned how to conceptualize and calculate the volume of a cuboid in their earlier grade levels. They have not received instruction on how to form a net to construct an open box without a lid, although they have experiences of making such boxes during the art and craft lesson.

In order not to induce students to use the cuboid volume formula right away, Problem 1 was deliberately phrased in colloquial language, i.e. *Given a sheet of cardboard paper of size 12 cm x 12 cm, cut away the four corners and fold the resulting net to form an open box without a lid. Investigate how to cut the corners so that the box can contain things as much as possible.* Appendix A details the lesson plan with explanatory notes of the first lesson study.

Points worthy of attention when planning the group practical work for the proposed mathematical investigation are summarized below:

1. **Grouping of students for student work** – the first criterion is that the group members should not be in conflict amongst themselves. The second criterion is that there should be at least one member in the group who is good at mathematics and can understand and think about what the teacher would like them to know or find out.
2. **Use of shopping context to set the context of the problem** – the teacher would like to convey the message to the students in everyday terminology that “the larger the space inside the box the more the things it can contain”. She expected that the students would have no difficulty in visualizing boxes of different sizes in the shopping context. However, she was amazed to find that one student could not meet this expectation, i.e. the one box that looked the smallest was perceived by him can contain the most of things inside.
3. **Methods of comparison of size of boxes formed** – three methods have been planned by the teacher in advance, i.e. using ruler and calculator, using water, and using candies. Apart from the three methods planned, there was one method anticipated by the teacher but somehow has not been observed during the brainstorming session of the lesson, i.e. through logical analysis students should be able to rule out the smallest box by putting this box into another larger box so as to come up with a direct comparison.
4. **Conflicting results using the experimental approach** – the teacher did make some attempts to explain the conflicting experimental results to the students. Given the tight time constraint the teacher has to give up attempting to explain to the class the concepts of experimental errors.
5. **Experimental and investigative skills of the students** – the teacher noticed that the students were not measuring properly, e.g. faulty measurements using a ruler, faulty use of calculators, the colored water has not filled up the boxes fully before pouring into the measuring cylinder.
6. **Colloquial use of instructional language** – the problem was phrased in everyday language i.e. cut away the four “corners” of a piece of cardboard paper so as to fold and form a box without a lid. As it was desirable that the students should do some autonomous mathematical investigations, the teacher was trying hard not to tell the students directly how to cut the corners. As a result of this, different groups have different ways of cutting the corners, and some even do not cut at all.
7. **Misconception of cutting the corners of the square paper** – students possess an interesting misconception that the less the size of the corners cut the larger the capacity of the box formed. Therefore, some groups are found reluctant to cut away the four corners as required by the problem. When asked to do so, they simply cut away tiny bits from the four corners in order to fulfill the problem solving requirements.
8. **Different patterns of cooperative exchanges amongst groups** – the teacher found that different groups were cooperating in different ways and noticed that the students had not engaged in genuine cooperative exchanges as anticipated. Most students were in fact doing things on their own most of the time. It was observed that before starting the mathematical investigation the groups have not communicated well amongst themselves in order to arrive at a plan. Without much thinking most students started immediately and plunged into making boxes of various sizes by cutting and folding the given cardboard papers.

After analyzing existing practices and before commencing the second lesson study, the researcher convened a meeting with Ms. T and the other five teachers participating in the series of three lesson studies to review how the first lesson study may be improved and taught to a second class of primary six students (see Figure 4 for the design of three lesson

studies). Comments worthy of special attention were remarked to Ms. T and these are summarized below:

1. **The original investigative problem is phrased in colloquial language.** The researcher deliberately uses the words “cut away the corners” to hint to the students that this is not a paper-folding activity. The students need to visualize how to form the net of the open box first and then subsequently fold it up into a hollow cuboid. There is a need to draw a connection between the space inside the hollow open box and its volume. This is why the word “volume” is deliberately not mentioned in the problem statement. Ms. T needs to decide on how to increase students’ understanding of the relationship between a net as a two-dimensional mathematical object and a solid occupying space in the three-dimensional everyday world. In this regard, Ms. T may need to cut some edges of a hollow open box and show the students how the net looks like. In this way, students should recognize what is meant by “cut away the four corners” in the demonstration.
2. **The teacher’s own mathematical knowledge is limited.** Ms. T knows that given a $12\text{ cm} \times 12\text{ cm}$ paper the maximum size of the open box formed is 128 cm^3 . This is done by cutting four identical squares from the corners of the cardboard paper so as to form the required net. The Powerpoint shown to the class at the end of the first lesson study was a case in point. During the self-reflection session after the first lesson study, Ms. T confessed that she did not know why cutting corners in this way can attain maximal result. She was worried that she could not explain correctly to students why cutting 2 cm corners would produce the correct answer. In the second lesson study, Ms. T needs to decide whether the wording of the investigative problem should be changed from “contains things as much as possible” to “capacity is as large as possible”, but not “volume is greatest”.
3. **During the self-reflection session after the first lesson study, Ms. T confessed that she felt busy during the lesson.** This is because every student appeared to be moving hurriedly coming up with an open box to calculate its capacity, and then use it to contain the Swiss candies. There was no group planning observed, and cooperative exchanges were only superficial. Ms. T needs to redesign the investigative problem as a group work activity so as to allow students to engage genuinely in cooperative exchanges. Moreover, she needs to take note of the suggestions on student learning as stipulated in the New Chinese Mathematics Standards. Ms. T was reminded that the main purpose of the investigative problem is not to experiment on how to make an open box (i.e. like those done in the first lesson study), but to understand the relationship between volume and the height of the open box. Therefore, what is most important is to make clear to the class that the height is actually same as the length of the squares of the four identical cut corners. Without making this clear to students, it is not possible that mathematization can be elevated to the referential level.

Space limitations preclude reporting of students’ mathematical investigations in the second and third lesson studies. In spite of this, what is reported so far hopefully is enough to reveal that teachers generally lack experiences to involve students in mathematical investigations, and that mathematical investigation is not only daunting to students but also to the teachers as well.

E. Conclusion

This paper demonstrated that mathematical challenges can be designed in at least two different ways, one by ingeniously modifying existing problems to non-routine forms, another is by making a shift of focus to lower or higher levels of mathematization. Admittedly, it takes time and energy for both students and teachers to respond well to mathematical challenges. In the present study, lesson study is shown to be a viable approach of teacher professional development within which students' active engagement and success in mathematical investigations can be achieved by outcome.

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Acknowledgement: I would like to thank Dr Huang Rongjin of University of Macau for his help in conducting the Macao-Shanghai ICMI-15 Study. This paper was based on some of the collected data and research findings of this comparative mathematics education study.

Appendix A: Lesson plan of the first lesson study (with explanatory notes)

[This is an unmodified version of the submitted lesson plan of a mathematics in-service teacher teaching senior primary mathematics at Choi Ko Primary School. She is currently studying her final year of BEd(Primary Education) at the Faculty of Education of the University of Macao. The original lesson plan is written in Chinese]

Time: 2:30-3:30 pm, 30 May 2005

Class: Primary 6C

[This is a class of students with varying degree of aspiration for study. Some students are rather willing to learn, whereas some are very passive. Some students are often in conflict with their classmates. Ms. T is not her mathematics teacher. She needs to consult the class mathematics teacher to form study groups of 5-6 students for mathematical investigations.]

Lesson Format: Small group teaching (5-6 students in a study group)

[There are two criteria in the formation of a study group. The first is that the group members should not be in conflict with each other. The second is that there should be at least one group member who is good at mathematics and can understand well teacher's instruction.]

Number of students: 33

Lesson Content: Given a sheet of cardboard paper of size 12 cm x 12 cm, cut away the four corners and fold the resulting net to form an open box without a lid. Investigate how to cut so that the box can contain things as much as possible.

[This is the first lesson study. Ms. T was told to conduct a lesson pertaining to practical work and integrated activity. The problem statement is purposefully phrased in colloquial language. There is no mention to students on how to cut, how to fold, what things are contained inside, and what is meant by "as much as possible". In fact, the word "volume" is purposely not mentioned in order to set the mathematical investigation in realistic everyday context.]

Design Consideration: Through conjectures, group discussion and hands-on manipulation to stimulate students to engage in autonomous investigations.

[Ms. T is aware of recent developments of the new Mathematics Curriculum Standards in China. Therefore, this consideration is in line with what is proposed in the curriculum standards. However, she still lacks experience in practicing mathematics teaching in the said manner.]

Teaching Objectives:

1. Know the interrelationship between "space" and "volume".
2. Given a sheet of cardboard paper of size 12 cm x 12 cm, able to cut away the four corners and fold the resulting net to form an open box without a lid.
3. Able to make an open box having dimensions 8 cm x 8 cm x 2 cm as the solution to the assigned problem – after cutting the 12 cm x 12 cm cardboard's corners and folding of the resulting net this box can contain things as much as possible.
4. Foster group cooperation and develop investigative skills.

[Ms. T does not have any intention to demonstrate to the class how to cut and fold to produce the open box. This is because she thinks that this is an investigation and the findings need to be discovered by the students themselves. During the briefing session, she was informed that the open box having dimensions 8 cm x 8 cm x 2 cm as a result of cutting and folding the 12 cm x 12 cm cardboard paper can contain things as much as possible. However, she does not understand why this is the case.]

Teaching Aids and Materials: four transparent boxes, one big empty cardboard carton, cuboids of dimensions 12 cm x 8 cm x 8 cm, 10 cm x 4 cm x 4 cm, 8 cm x 4 cm x 4 cm, 8 cm x 4 cm x 6 cm, seven calculators, water, measuring cup, 33 sheets of 12 cm x 12 cm cardboard paper, one bucket of Swiss candies, chalks, computer, overhead projector, visualizer, courseware.

[Ms. T went to a department store to buy a set of four transparent boxes in the form of cuboids that can contain different sorts of stationary. She planned to use these boxes for size comparison purposes. The water is colored in blue to increase its visibility. The courseware is actually Powerpoint presentations made by Ms. T for demonstration purposes.]

Teaching Process:

A. Lesson Induction

- Ms. T shows the four transparent boxes to the class, asks the students the following two questions: (a) In your opinions, which of the following four boxes (i.e. the four transparent cuboids) has space most inside? (b) How can you arrange the boxes in order of amount of space inside and rank the capacity of boxes from the largest to the smallest?
- Small Group Discussion: What methods can be used to check which of the four transparent boxes has space most inside? Students report and discuss the findings together.

[This process pertains to teaching objective 1 as detailed above. It was meant to be a lesson induction which takes no more than 10 minutes. In the first lesson study, Ms. T actually took more than 30 minutes to complete this process. The activity of ranking the boxes was delivered in a shopping context. Ms. T would make use of the opportunity to familiarize students with the idea that empty space inside boxes can contain different kinds of objects and that there was a need to compare the sizes of these spaces. She has anticipated and planned four different approaches to rank the boxes, i.e. using candies, using colored water and measuring cup, using ruler and the volume formula, and a combination of direct comparison and logical analysis.]

B. Development

- Ms. T shows in front of the class a cuboid of dimensions $12\text{ cm} \times 8\text{ cm} \times 8\text{ cm}$, and then asks the students, "What is it?"

[Ms. T attempts to make the classroom atmosphere alive by performing a "magic show". She used some small unit cubes to make an open, hollow pencil case in the form of a cuboid. When this case is shown in front of the class, she purposely hides the opening of the pencil case so that the class can only visualize the pencil case as a cuboid. She expects "cuboid" from the students as the answer to her question.]

- Ms. T puts this cuboid into a big empty cardboard carton, and then asks the students to make a guess what will happen to the cuboid (After guessing, Ms. T takes out a pencil case which is in the form of a cuboid from the carton).

[This "magic show" intends to illustrate that one can create empty space in a cuboid to contain objects in realistic everyday life.]

- Ms. T shows three blocks (each in the form of a cuboid) composed of unit cubes. The volumes of the three blocks are all different. Ms. T invites the students to guess which of the three blocks can fit into exactly the hollow empty space inside the pencil case? Students are required to give reasons to their answers.

[The purpose of this activity is to show that empty "space" in the hollow pencil case and "volume" of the block that can be fitted into this space are identical concepts. Since both the blocks and the pencil case are both composed of unit cubes, the students can easily calculate the volume of this space by counting the unit cubes, or apply the volume formula which is related to the three dimensions of a cuboid: length, breadth and height.]

- Using a Powerpoint, Ms. T poses the investigative problem in front of the class -- *Given a sheet of cardboard paper of size $12\text{ cm} \times 12\text{ cm}$, cut away the four corners and fold the resulting net to form an open box without a lid. Investigate how to cut so that the box can contain things as much as possible.*

[This investigative problem is meant to be carried out in small groups of 5-6 students in the form of competition. Each group is required to form the box as stipulated. The one box that can contain things most is the winner.]

- Ms. T guides the students to summarize the points needed attention in order to make the open box. These main points are summarized on the blackboard.

[Ms. T has prepared a courseware in advance before the lesson illustrating different ways of cutting and folding the cardboard paper to form the open box. Students raised some unanticipated answers during the questioning period and this has put Ms. T in a difficult position. Given that the lesson was seriously running behind schedule, Ms. T did not know how to react to the unexpected situation.]

- Ms. T distributes each student in the group with one sheet of $12\text{ cm} \times 12\text{ cm}$ cardboard paper. Each group is provided with a small bag of Swiss candies to fill up the open boxes they make.

[The Swiss candies are meant to be the "things" that may be contained in the boxes formed by the groups. The group that can make a box containing the greatest number of the Swiss candies wins the competition.]

- Each of the six groups displays their investigative results in front of the class. Then the whole class works together to judge which group wins the competition.

[It is noteworthy that the students interpret this activity as a game. Therefore, most group members hurriedly cut and fold all kinds of boxes in their own ways and then try to squeeze in as many Swiss candies as possible.]

C. Ms. T guides the students to arrive at a summary.