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Cooperation and competition as a challenge in and beyond the classroom.

The subject of this ICMI study is beautiful. We all consider that learning and teaching are not only in the classroom, but also beyond the classroom. We all wish mathematics at school and out of the school to be challenging mathematics, meaning interesting and stimulating. And we all know the importance of mathematical competitions, meaning challenges with a mathematical content.

The study will consider this subject from different points of view, and we expect to grasp new ideas and information during this meeting. The point of view I chose for this introductory lecture is cooperation versus competition. I shall begin with very general ideas, then focus on teaching and learning outside and inside the classroom, and introduce some mathematics as kind of a playtime.

Competition and cooperation are key words in our world, and competition comes first. Our societies are based on economic competition inside a free market. It is a fact and it is a rule. The stock exchange is part of the daily news. The European treaties establish free market and free competition as a principle, and they consider scientific research as a basis for economic competitiveness. There is no more competition between different economic systems, but there is a hard competition between firms and groups, between states, between politicians, between individuals and particularly among scientists. There are competitions for entering universities or other institutions of higher education in every country. At the level of individuals, the motto is "struggle for life". At the level of humankind, it looks a necessary condition for our adaptation to a moving environment. Darwin taught us that the evolution of species results from a selection process, the selection of the fittest, and molecular biology confirms the views of Darwin : the evolution of species corresponds to a double process, random mutations and selection of the most efficient. At an intermediate level, the level of human history as we know it, peace and cooperation is just an intermediate step between wars – it was the conception of Nietsche -, and wars are more and more destructive. In the present state of affairs, the improvement at our diaposal seems to replace whenever possible real war by economic war.

Even so, cooperation is unavoidable. It is also a fact of life. Moreover, it is established as a result of the game theory. The game theory originated as a theory of economic or military competition, but the models usually deal with two or a small number of players, looking for the best strategy. A classical paradox is the dilemma of the prisoner : two gangsters are suspected to have taken part in a holdup, and they can be convicted also for a minor crime. They are isolated each from the other and each of them is offered a bargain :

1) if he testifies against the other and the other doesn't, he is free,
2) if he testifies against the other and the other does the same, he gets 2 years,
3) if he doesn't speak and the other does the same, he gets 1 year,
4) if he doesn't speak and the other testifies against him, he gets 3 years,

Now, suppose the prisoners are you and me : if you don't speak, I get either 0 (if I speak) or 1 year (if I don't) ; if you speak, I get either 2 (if I speak) or 3 (if I don't). Whatever you do, it is better for me to speak. The same for you. Therefore we choose to speak (it is an equilibrium in the language of the theory of games). As a consequence each of us gets 2 years, instead or 1 year if both of us had decided not to speak. There are many variations around this example. They show that "chacun pour soi" (every man for himself) can lead to the worst for the community. Cooperation is a need for the welfare of a group.

Actually competition between groups doesn't exclude cooperation inside each group: this will be discussed about mathematical competitions. The real question is: what type of cooperation ? The history of ancient Greece provides us with various examples. Greece was not one country, but a set of independent cities. There were terrible struggles and wars between them, the most awful being the Peloponnesian war, with Athens, Sparta, Corinth and Thebes at the forefront, and all other Greek cities involved. There were moving coalitions and cooperation among them, based on power and empires more than on free choice. The social and political life inside each city was described by one of the fathers of historical studies, Thucydides. You had all political systems, the power belonging either to one ruler, a tyrant or a king, or to a group of rich people, the aristocracy, ot to all citizens considered as equal in right, the democracy. The democratic power, power of the people, excluded the women, the foreigners and the slaves. However, the theory and practise of the democracy in Athens was the source of its power at the time of Pericles, and it is still a model of cooperation: all citizens have the same rights, all of them discuss important issues, there are long and serious dabates before any decision of action, leaders are elected for short periods and for a specific purpose, and that doesn't prevent pre-eminent people to emerge as moral authorities, as it was the case with Pericles. Pericles argued that cooperation between citizens was more efficient on a free basis in a democratic regime than under pressure in another regime. This is a profound idea: the most fruitful cooperations are based on free choice and equality of partners.

If we try to guess something of the time to come, it is likely that cooperation between individuals and people will prove more important than competition. Competition would not disappear, but it would be subordinate to cooperation, just as cooperation is subordinate to competition in the present time. The lesson of Darwin is not struggle for life. Darwin himself observed that the evolution of humankind has some specific features: the weak people are protected (more or less) by the community, and this develops new capacities and new possibilities for the community as a whole. Humankind will not adapt itself to a moving environment by the mechanism of random mutations and selection of the fittest. Its specific tools are curiosity, inventiveness, and transmission of knowledge – this is the theoretical ground for scientific research, arts and technologies, and education. These tools proved efficient in the past, they should be used and developed on a universal basis now, because all problems we face are global problems. Therefore cooperation should involve all people on the earth, and as far as possible all individuals in each group or community. This is an Utopia, in the sense that it never was realized, but I think that it is a good Utopia, as a guide for what we have to promote now. As positive Utopias, cooperation and democracy have the same role, and they are linked together.

It means that we have to pay attention to all germs of extensive cooperation that exist by now. I shall restrict myself to scientific research and to education.

Scientific research is a field of competition. The highest winners are Nobel prize or Fields medal or Abel prize winners. But the competitive aspect of scientific research is overestimated. The real life of a scientist is hard work, communication and cooperation. The ways of communication are changing by now, and collaborations are made easier by e-mail. In mathematics, some laboratories “without walls” are created, involving researchers of different countries linked by e-mail. The best ways to test an idea or to check a result is to communicate with other people in laboratories or in seminars. Ed Barbeau suggested that I say a word on Andrew Wiles and his proof of the so-called great Fermat theorem. The proof was a challenge, Andrew Wiles was absorbed in this challenge for years, alone, and his success is due to a very competitive temper. But it is mainly a success for the community of mathematicans. At the end, Andrew Wiles had to take advantage of the collaboration with other people, who discovered a mistake in the proof and helped him to repair it. But from the beginning of his investigation he took advantage of what mathematicians did in different areas since the time of Fermat. All living mathematicians are in a strong way collaborators of mathematicians of the past. And in another way they collaborate in small group, then in larger and larger structures, in order to build the mathematics of our time.

I shall take another example of a pre-eminent and very pesonal and competitive mathematician, André Weil. André Weil was born on May 6, 1906, and the hundredth anniversary of his birthday was celebrated in Paris a few weeks ago. André Weil had a very artistocratic view of mathematics and mathematicians. Here is what he wrote in 1947.

« En mathématiques plus peut-être qu'en aucune branche du savoir, c'est toute armée que jaillit l'idée du cerveau du créateur ; aussi le talent mathématique a-t-il coutume de se révéler jeune, et les chercheurs de second rang y ont un rôle plus mince qu'ailleurs, le rôle d'une caisse de résonance pour un son qu'ils n'ont pas contribué à former ».

(In mathematics, more maybe than in any other branch of knowlege, the idea comes fully equipped to the mind of the creator. Therefore mathematical talent is usually revealed in youth, and second order researchers have a lower role than elsewhere, the role of echoing and amplifying a sound that they have no part in producing).

Actually André Weil is the main founder of the branch of mathematics called algrebraic geometry, and he made essential contributions to number theory, algebra and analysis. But he is also, together with Henri Cartan (who will be 102 on July 8), the initiator of the most famous cooperative enterprise of mathematicians in the 20^{th} century, the "Eléments de mathématiques" (elements of mathematics) of Nicolas Bourbaki. This collective work is not appreciated now as it used to be 50 years ago, and as it will be in times to come. I never was a Bourbakist but I consider the Elements of Bourbaki as a great achievement of the 20^{th} century, just as Euclid was in his time. Moreover it was a model of cooperation: hard work of each individual, several proposals for each chapter, hard debates before the final choice and the final decision about the way the chapter should be written, final writing and hard control, with a mixture of friendness and harshness in the personal relations, and a remarkable unselfishness. Clearly the extension of the Bourbakist way of exposition to the whole of mathematics education was a failure, but it is remarkable that about all members of the Bourbaki group stayed apart of the reform of the so-called modern mathematics. Anyhow it is not necessary to be a follower of Bourbaki in order to admire the enterprise and the achievements.

I insisted on hard work in scientific research, and it is true that a competitive atmosphere can be an incentive for a hard work. But it can be an incentive also for severe blockages and deviations, such as secrecy maintained on a domain of research, or in the oppositive way publicity given to unestablished results. It is also a way to exploit the working force of young scientists without leaving them the time of free investigation and personal choice. My theme is that what is basic in scientific research is not competition: it is curiosity, freedholm, communication and cooperation.

What about education? Basically, education is transmission of knowledge and this creates an unequal situation, either in the family between parents and children, or in the class between teachers and pupils. Moreover, education extends now on a lager and larger period of life, with the development of higher education and continued education. In all countries it is considered as a very important social and political issue.

In our competitive societies, there is more and more competition between universities, between schools, between teachers and between students or pupils. There are all sorts of competitions, and I shall not try to make an extensive list of them. The most popular of course are competitions in sport, football, tennis, etc... Tennis is a competition between individuals, football a competition between teams. Mathematics plays an important role in competition exams for entering universities or other institutions, and there is a large variety of mathematical competitions that are similar to competitions in sport. I made a report on mathematics competitions for the Seventh Southeast Asian Conference on Mathematics Education (S.E.A.C.M.E -7) held in 1996 in Hanoï, with a number of examples, and here was the conclusion.

"The aim of this paper was to give matter for future discussions. Mathematics competitions are developing all around the world and that is excellent. They can be more and more in contact with living mathematics. Each of them needs strict rules and perfect observance of the rules. The rules are diverse and more and more formulas should be experienced. Competitions in mathematics are flexible, more than in sports, much more than in economics matters. They may contribuate to make mathematics more human and more popular. They already constitute an incentive for the teaching and learning of mathematics. Is it not time to emphasize their importance among our colleagues as well as in the general public?"

Time has come. An enormous amount of information can be found in the journal of the World Federation of National Mathematics Competitions (W.F.N.M.C.), called “Mathematics competitions”. There are regular reports ont the International Mathematical Olympiades (I.M.O). An important section of the Discussion Document that gives the frame of the present study is devoted to competitions.

What I wish now is to discuss the relation between competition and cooperation in mathematics at a school level.

First, there is always a kind of cooperation hidden behind any competition There is no competition without a cooperation between the organizers. Actually, all mathematical rallies I know need an enormous amount of work from mathematics teachers and a strong cooperation between them.

Competitions are not always in incentive for cooperation among students. For example, competitive exams are very often the fields of a “struggle for life”, meaning that other students are fighters to face. However, it is easy to see that isolated competitors are disadvantaged with respect to competitors that belong to a good group or a good class. This is not only due to the teacher. The conversations and discussions between young people play a decisive role in their capacity building.

Some kinds of mathematics competitions need a very elaborate cooperation inside the class. I am thinking of the competitions between classes, an excellent form of rallies of which I know several versions in France, involving sometimes neighbouring countries. Typically, when the subject is given to the class, it is too long and difficult to be treated by one student, even very good, during the given time. The class should be organized in order the subject to be read, cut into a number of parts, dispatched to a corresponding number of teams, discussed inside each room and if possible solved with the solution written down. The partial solutions are collected and form the contribution of the class to the competition. Preparing the competition may be an apportunity to create a cooperative atmosphere inside the class.

However, in all these examples, competition comes first, and cooperation is at the second place.

Is it possible to put cooperation in the first place? I think so and I shall develop this idea on one only model, the so-called mathematics laboratories.

Mathematics laboratories in secondary schools is an old idea. Hundred years ago, Emile Borel, who had an important role in the reformation of currricula in French high school around 1900, gave a lecture for math. teachers. He urged them to pay attention at all pratical aspects of mathematics, computations and figures in relation with physics and with graphic design. And he suggested to have a mathematics laboratory in every high school, with wood to play with and a carpenter to help, in order that everyone, teachers, pupils and parents get a clear idea of the experimental aspect of mathematics.

This idea never took form in France until a few years ago. It was rediscovered by the association of French teachers (A.P.M.E.P) and formulated again in the proposals of the commission de réflexion sur l'enseignement des mathematiques (C.R.E.M.) in 2000. It was agreed by the minister of education in 2002, but the first realizations are quite recent. They depend entirely on the initiative of teachers and local possibilities. Here is the general frame. First, a room, equipped with some material, as it is the case for laboratories in natural sciences ; the material includes computers, books, and all kinds of objects that can be used for mathematical experiments or constructions. Then, a given time in the timetable of the pupils and in the service due by the teachers. Finally, but that is the first thing to think about, a good set of open activities to propose to the children.

The main feature of math. laboratories is that they are places for experiments. Experiments in mathematics need time and freedholm. The pupils should be provided with subjects to explore, they should not have a task to stick to. They should feel free, not under pressure.

For the teachers also, math. laboratories are a field of experimentation. They can try new subjects, out of any curriculum.

The atmosphere of a laboratory should be an atmosphere of cooperation : cooperation between pupils, cooperation between the teacher and the pupils.

Moreover what we already observe in the existing realizations is an extension of cooperation at many different levels : cooperation between math. teachers, if only because they now have a common room to meet and discuss when they like ; cooperation between math. teachers and teachers of other disciplines, cooperation between school teachers and professional mathematicians, or researchers in other fields, or engineers, because the teachers have something to ask, new subjects, new ideas, and something to offer a room, an equipment, a meeting spot, an audience of young people. The situation may be different in different countries, but in France school teachers have no personal office, and laboratories are at least a way to feel at home somewhere in the school.

As far as the activities in laboratories are concerned, there is already a mine at the disposal of mathematics teachers, with all types of challenging mathematics that are practised in competitions, rallies, clubs, exhibitions, popular lectures etc... Clearly the present study will charge and deepen this mine. But any random walk through the existing literature can provide us with new ways to look at old things, and I shall try to give an example before ending this talk.

Before leaving the laboratories, I should insist on an aspect of education that is not particular to mathematics. Most teachers have more and more difficulties with their classes, they have no time to perform the program as they should like to do, there is a lack of work an motivation among the pupils. This is still more true for scientific matters, in particular mathematics. On the other hand, we know that mathematics offer a large field to free investigation, imagination and creative activities. In order to take advantage of this aspect of mathematics, that is, of “challenging mathematics”, we can observe the practise of our colleagues in the natural sciences: they develop new relations wiht the pupils when they work in small groups, in laboratories, or when they walk together in a geological excursion. Challenging mathematics is a way to establish new relations between teachers and pupils, either beyond the school or in the school. In the school, laboratories may offer possibilities of other relations than in the classroom. On the other hand, never laboratories and free activities can replace the classroom. Curiosity and inventiveness have a natural place in math laboratories. But the transmission of knowledge cannot rely on what can be done in laboratories. Formal definitions, statements, constructions and proofs are essential in mathematics and they need a special attention of the teachers and the learners. My assumption is that what is done in the classroom can be concentrated in a shorter time if mathematics appear also under other forms out of the classroom, and if the pupils are more motivated for its more formal aspects.

Where is it possible to grasp ideas of new mathematical topics to play with? Everywhere, in all the existing literature, including on the Web. At a research level, there is plenty of new material provided by other sciences, physics, cognitive sciences ans so on. I shall look in another direction, ancient history and the views of Plato.

Plato was not a mathematician, but he had a good taste in mathematics and he got advice of very good mathematicians, in particular Thaetetes. He lived at the time of the terrible wars between Greek cities that I mentioned before. Some cities were destroyed completely. When they were not killed the people of these cities enmigrated somewhere and built new cities. Each city had its own constitution, rules and laws, depending of its founder. The book Plato wrote at the end of his life, “Laws”, was a kind of political book in this context. Plato considered all problems of the cities, education, families and civic rules. He had several ways to consider what should be an ideal city, the so-called Platonic utopia. One way is to consider the family, including the slaves, as the unit of the city. What is the ideal size of a city, the ideal number of units? Plato answers: 5040. And he explains that 5040 is divisible by 2, 3, ... all numbers until 10, and that it is very convenient to have so many ways to decompose the city into equal groups. Actually he says that 5040 has 59 divisors. This raises a series of questions.

1) How did Plato, or the mathematicians around Plato, compute the number of divisors of 5040?

For those who don’t know the decomposition into prime factors, it is a difficult question, and I believe that it leads to this decomposition. It would be worth trying with children.

Anyhow,

$$5040 = 7\,! = 2^4 \times 3^2 \times 5 \times 7,$$

the divisors are the numbers

$$2^a \times 3^b \times 5^c \times 7^d \text{ with } a = 0,1,2,3,4\ ;\ b = 0,1,2\ ;\ c = 0,1\ ;\ d = 0,1,$$

therefore the number of divisors is 5 x 3 x 2 x 2 = 60, including 1 and 5040.

Likely Plato excluded 1, and that gives 59.

There is no written trace of a decomposition into prime factors at the time of Plato. The experiment that I suggest with children could support or not the idea that mathematicians of this time knew this method (the beginning of the dialog Theetetes would support this idea anyhow).

2) Are there numbers smaller than 5040 with as many divisors ?

The answer is no. In order to get the answer, it may be convenient to write all positive integers n in the form

$$n = p_1^{a1} . p_2^{a2} ....p_j^{aj} \text{ x } ...$$

(p_1, p_2, ...p_j,...) being the prime numbers and aj $\in$ **N**, with $a_j = 0$ for j large, and the number of divisors of n in the form

$$d(n) = (a_1 + 1) (a_2 + 1)...(a_j + 1)...$$

Then d(5040) = 60

Now, let m be the smallest number such that $d(m) \geq 60$. Clearly m $\leq$5040.

Let p be the largest prime number dividing m, and let us write

$$m = 2^a . 3^b . ... \ p^x$$

If $p \geq 11$, consider m’ = 8 m / p. We have m'<m and

$$d(m') = ((a + 4)/(a+1))/ ((x+1)/x$$

therefore $d(m') \geq d(m)$ if $a \leq 2$, a contradiction. Therefore $a \geq 3$

In the same way (considering m” = 9 m / p) we have $b \geq 2$, and all prime numbers $\leq p$ divide m , that is,

$$2^3. 3^2. 5 . 7 . 11$$

divide m. But this is larger than 5040. Therefore $p \leq 7$, and

$$m = 2^a . 3^b . 5^c . 7^d$$

$$d(m) = (a + 1)(b + 1)(c + 1)(d+1).$$

When d(m) is given, m is minimum for $a \geq b \geq c \geq d$, therefore this can be assumed.

Since both $2^2 . 3^2 . 5^2 . 7^2$ and $2^2 . 3^2 . 5^2 . 7$

are larger than 5040, we necessarily have d = 1 and c = 1, and the conclusion follows easily.

3) We can say that 5040 is a “highly composite number”, meaning a number m such that

d(m) > d(m’) whenever m’< m. How to find other highly composite numbers, and how to describe their properties ?

This is not an easy question, but it is possible to give partial answers, just by the same method as for 5040. For example, if m is a very large highly composite number and

$$m= 2^a . 3^b . ...,$$

the ratio a / b should be near ln3 / ln2

A way to look at highly composite numbers is to look at the points

$$(n, d(n)), n = 2, 3, 4, ...$$

Let us consider the smallest convex set containing these points. If we denote an extreme point by (m, d(m)) , then m is a highly divisible number. By the way, the figure made by the points (n,d(n)) is quite interesting : the prime numbers correspond to the horizontal line d(n) = 2, the squares of the prime numbers to d(n) = 3, the products of two different prime numbers to d(n) = 4...

As far the asymptotic properties of highly composite numbers are concerned, they depend on the distribution of prime numbers, and it is a real topic of research. There are many ways to play with these numbers, but I stop there.

The term of highly composite number and the first studies on these numbers are due to Ramanujan, and apparently Ramanujan didn't make the relation with Plato. I told you how I discovered 5040 in Plato. How did I discover Ramanujan? Let me just tell the story. I asked Google about "number of divisors", then I saw highly composite numbers with a reference to my colleague from Lyons Jean-Louis Nicolas, and we were just writing a paper together with Jacques Dixmier. I asked him what he knew on the subject, and apparently he is one of the best experts in the world. In particular, he edited and corrected what Ramanujan left on the subject.

As I already told you, challenging mathematics can be found anywhere, from Plato to Internet. There is certainly no competition between the sources, and it is up to us to let them cooperate.

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Highly composite numbers

References.

S.Ramanujan, Proc.London Math.Soc.(2)14 (1915),347-400
Jean-Louis Nicolas and Guy Robin, The Ramanujan Journal 1 (1997),119-153

List of the first, with the exponents in their decomposition into prime numbers

2	1
4	2
6	1,1
12	2,1
24	3,1
36	2,3
48	4,1
60	2,1,1
120	3,1,1
180	2,2,1
240	4,1,1
720	4,2,1
840	3,1,1,
1260	2,2,1,1
1680	4,1,1,1
2520	3,2,1,1
5040	4,2,1,1
7560	3,3,1,1
10080	5,2,1,1
15120	4,3,1,1
20160	6,2,1,1
25200	4,2,2,1
27720	3,2,1,1,1
45360	4,4,1,1
50400	5,2,2,1
55440	4,2,1,1,1
83160	3,3,1,1,1
110880	5,2,1,1,1