MAKING a heavy task LIGHTer

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Abstract We clarify the significance of the notion of *challenge* in mathematics by presenting its historical, epistemological and pedagogical aspects, together with some practical examples to be presented in class.

Introduction – Why challenges?

From an etymological point of view, *solving* a problem is the same as *unravelling* a twisted rope. In the figurative sense, this means finding out the way that leads from one end to the other: at the beginning, you know that the connection *is there*, but you just do not *see* it. You stay in front of a confused picture, which has to be cleared up. You can accomplish this task step by step, by winding up the thread and patiently undoing each knot you come across. This is a sort of routine work, such as the algorithmic procedure that takes you out of maze. In both cases there is a rule you can follow, there is a prescribed sequence of transformations or choices you have to perform in order to obtain a certain outcome. In games and riddles, the author has hidden this result, and the reader has to find it out through a work of *deciphering* or *(re)construction*. Sometimes the solving path is given explicitly, e.g., in the puzzles where you are asked to create a picture by
(1a) colouring only the squares of a grid that are indicated by a dot, or
(1b) connecting points according to the numbering.
Sometimes it is only described implicitly in form of a list of constraints and includes at each stage a finite number of possibilities, e.g.,
(2a) in word-seeking puzzles, where one must recognize words in a rectangular arrangement of letters, or
(2b) in crossword puzzles, where words must be inserted in the rows and columns of a given grid.
In all the above examples the solution of the problem consists in arranging a finite set of pieces that is made available right from the start. A great part of standard exercises of school mathematics (Euclidean division, solving linear equations, determining areas and volumes by formulas) are of this kind. They contain no challenging questions, but only require basic technical skills. Hence they are little motivating (because too tedious) for pupils with learning difficulties and little appealing (because too trivial) for the others. Thus it seems that the only way to comply with both needs is to propose problems that can be solved by a *simple* method, which, however, *does not lie at hand*. This approach, first of all, can encourage team-work, with a clear distinction of roles, which is expected to produce two positive effects: on the one hand it is likely to increase the self-confidence of the "weaker" students, on the other hand, it should induce the "stronger" students to share their ideas, making them more willing to interact. Both "parties" can profit from a situation that forces them to communicate, to put mathematics in words: trying to expose a procedure or an idea in a neat and orderly manner is perhaps, at any level, the most efficient comprehension test. Furthermore, splitting a problem into *reasoning* and *elaboration* demonstrates to the whole class that mathematics is a twofold activity: it is composed of *intuition* and *formalization* (see its historical development) of *theory* and *application* (see its contributions to science and technology), of *speculation* and *verification* (see its research achievements). The second aspect is not merely a logical consequence of the first one, on the contrary, the two aspects influence each other in a highly dynamical interplay: this is the reason why *all* pupils, regardless of their knowledge and abilities, can take advantage from exchanging views with their class-mates.

1. Too much or too little

The examples presented in the introductory section, all published in popular puzzle magazines, belong to two different classes: in (1a) and (2a) the solution is found by eliminating redundant elements, in (1b) and (2b) by adding the missing parts. These two categories also apply to (challenging) mathematical problems, but in a more abstract context: since the solving procedure is supposed not to be pre-assigned, the first crucial step is to search for the right clue. This should be a fundamental *structure*, which we do not see immediately, but which we can *make visible* by doing some appropriate work. Either

(a) we reduce an object to an essential model it contains (*underlying structure*), or

(b) we extend it to a general scheme it belongs to (*surrounding structure*).

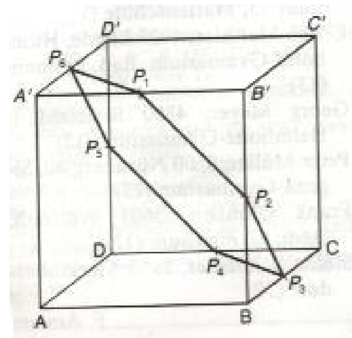
In case (a) we gain something which is simpler, and easier to handle, in case (b) we get something more complicated, but carrying more information. The teacher can pose problems of both kinds, telling the pupils each time which of these two options they should choose, and why: sometimes the reason is evident, and one only needs to point at it, as, e.g., in the following two elementary examples, both taken from European journals of school mathematics.

> Problem 1a *Bluebeard's daisies* (inspired by *Le Kangourou des Mathématiques*, France).
> In a sunny clearing, Kangaroo meets Bluebeard, who was resting in a daisy field. "Where do you come from, sir?" – "I am back from a trip in Provence, where I made a successful business. I wished my last wife could profit from this treasure, but I am sure that in my absence she disobeyed, so that she will have to die like the others! But I have an idea. On this graceful day, I propose you to play with me: let us pluck the 13 petals off this daisy. We agree to pick each time one or two petals in turn. The one who will pick the last petal loses, and if it is me, my wife will be saved. Since my name is Bluebeard, I will start.
> How must Kangaroo play in order to save the life of Bluebeard's wife?

The game apparently involves a number of possibilities that is too large to be managed by human beings in a reasonable time. On the other hand, the way in which the question is formulated already suggests that the desired strategy should be based on a general rule, which is independent of the duration of the game, but probably exploits the fact that Kangaroo is always the second to play. How should he counter each move of Bluebeard?

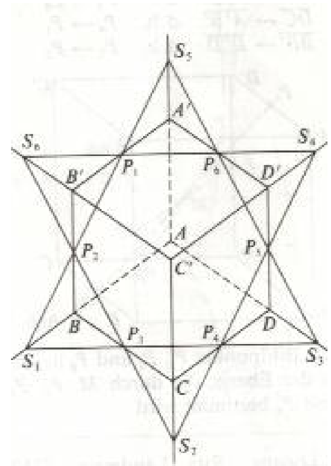
Solution The answer is that he should always leave an odd number of petals to his opponent. Since the initial number of petals is 13, he should always reply to 1 with 1 and to 2 with 2. Here the *underlying structure* is binary arithmetic, as in all *Nim*-like games [3].

Pedagogical considerations The teacher should first propose the problem in a simplified form, with a small number of petals (e.g. five) and ask the pupils to reconstruct the sequence of possible moves in the reverse order, starting from the winning situation for Kangaroo. The first question to be posed is: how many petals were left after the previous move of Bluebeard? As soon as the pupils have found the two possible cases (two or three), they should consider Kangaroo's reaction in each of these cases, and then proceed backwards, up to the start of the game. Examining Bluebeard's move and Kangaroo's countermove at each step should allow the pupils to recognize the general rule. At this point, the teacher can verify whether they are able to apply it to the case of 13 petals.

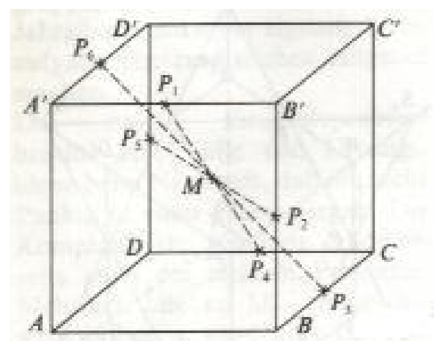


Problem 1b *The Hexagon in the Cube* (from *mathe-plus*, 1985, Germany).
The middle points of six edges of a cube are connected as shown in the picture. Prove by elementary geometric tools that the resulting figure is a plane regular hexagon.

The concise presentation of this problem calls for completion. Two different solutions proposed by readers go in that direction: they both rest on the symmetry properties of the cube, one in a static way, in terms of congruent figures, the other in a dynamical way, in terms of rigid transformations. The cube is placed in its natural *surrounding structure*, the Euclidean space.



First Solution The lines AB and P_1P_2 meet in one point S_1, since they are coplanar and not parallel. Since the triangles P_1P_2B' and P_2S_1B are congruent, we have that $|B\,S_1| = |B'P_1| = a/2$, where a is the edge length of the cube. Similarly one shows that the lines AB and P_3P_4 meet in one point S_1' and $|B\,S_1'| = a/2$. It follows that $S_1 = S_1'$. The lines P_1P_2 and P_3P_4 thus meet in S_1, so that they lie in a plane. This holds for every set of four consecutive points P_i. Since the triangles $S_1S_3S_5$ and $S_2S_4S_6$ have six points in common, and these are not collinear, the two triangles are coplanar. In particular this holds for their intersection points $P_1,\ldots, P_6$. The regularity of the hexagon easily follows from the symmetry of the cube.



Second Solution We only prove coplanarity. Let M be the centre of the cube. It can be easily shown that the lines MP_2 and P_3P_4 are parallel. Hence the points P_2, P_3, P_4, and M are coplanar. Due to the symmetry of the cube, the reflection about point M maps P_2, P_3, P_4 to P_5, P_6, P_1 respectively. Hence P_5, P_6, P_1 lie in the plane determined by P_2, P_3, P_4, and M.

Pedagogical considerations This is apparently a typical challenging question, reserved to the "best" pupils. The approach to the problem, however, is accessible to all levels of problem solvers (regardless of their ability to complete it and to achieve the solution). All pupils can be asked, e.g., to investigate the symmetries of the cube, and can be expected to give, at least, a partial answer, by indicating some pairs of sides of the hexagon which must have equal length, and some of its angles must have equal width. This little success can have an encouraging effect, especially if the teacher has initially announced the problem as a difficult one.

Both problems 1a and 1b require a certain amount of creativeness if the student can rely only on his own intuition. When working in a group, however, even the so-called "bad solvers" can contribute. It suffices that the teacher reveals the heuristic recipe to be applied.

(a) For the first kind of problems, the underlying structure is a *pattern* that is common to all games where the opponent has no chance to win; it can be detected by examining special cases, mainly those where the number of remaining petals is small and/or there are only few moves left before the end of the game.

(b) For the second kind of problems, the surrounding structure is a *system of relations* between the vertices of the hexagon; it can be figured out by recovering what is known about the geometric figures presented.

All pupils can help *collecting/verifying examples* in case (a), and *collecting/verifying properties* in case (b). Both tasks resemble standard activities (computing, learning) with ordinary syllabus material, and, therefore, they turn out to be especially useful to pupils who have gaps to fill. Furthermore, unlike in the usual lessons, the motivation here does not come from the threat of a bad mark or a failure to be ashamed of, but from the prospect of a success, which can enhance one's reputation within a group.

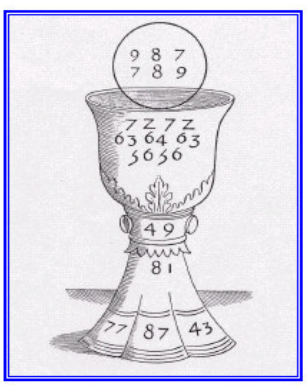
2. Mathematics as an experience at hand

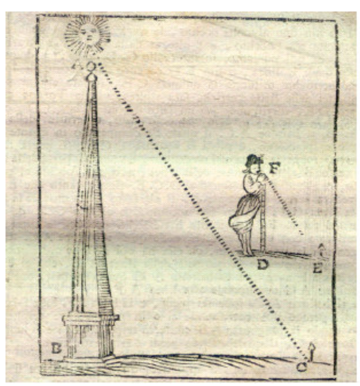
A challenge is not a challenge if it does not provide a strong motivation and an achievement to aim at. The challenges we are interested in must appeal to positive attitudes and feelings: they ought to arouse curiosity, show the fun of investigation and the joy of discovery. This, after all, is the way mathematics was created in the course of the centuries, originating from mental endeavours of passionate intellectuals, or from the practical ingenuity of skilful craftsmen. They all had a part in establishing

(a) *algorithms*, where an *underlying structure* turns into a *procedure*;

(b) *theorems*, where a *surrounding structure* turns into a *rule*.

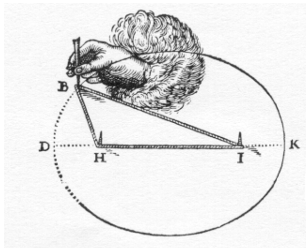
Algorithms and theorems made their first appearance in solving problems. A sequence of digits representing an integer rests upon its decomposition into increasing powers of ten, which gives rise to the method of casting out nines, or to the addition in columns: these techniques belong to the tradition of late medieval arithmetic, which includes many other curious ways to perform operations by manipulating regular arrangements of digits.





The proportionality between the sides of similar triangles can be used to derive unknown lengths from known ones: it was exploited since the antiquity to measure lengths lying beyond human reach, from the height of pyramids to the distance between the Sun and the Earth.
The development of mathematics is a story of discoveries and inventions that give rise to *tools*: this reveals its similarity to other sciences, such as physics or medicine. In this respect, it is not only something you can talk and think about in class, but also something you can see and touch in every-day life. A tool is made to be seized and moved, and it must designed so as to match the object that it must create or modify. In this sense it may be regarded as a *model* of a work project:

(a) a straightedge and a T-ruler resemble, in their contours, lines and right angles;



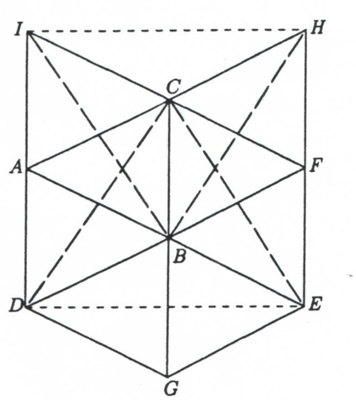
(b) a compass forces one of its points to move at a constant distance from the other one; in the so-called “gardener’s device”, the sum of the distances of the pencil from two pins remains the same at any moment.

The instruments in (a) *explicitly* reproduce the visible shape of figures (*underlying structure*), those in (b) *implicitly* include the metric constraints defining figures (*surrounding structure*). They all can be constructed and employed by pupils for exploring geometric objects, but they also show a possible approach to mathematical problems: first construct something that is accessible and is similar to the desired result, then try to modify it accordingly. After all, it does not matter if a straightedge has a bounded length: it can be shifted and give rise to any line segment. If the ellipse constructed by gardener’s device is too large, you can shorten the lace, or reduce the distance between the pins. The point to focus on is no longer the problem setting, but lies close to the desired result. A question should be viewed as a flexible matter, which can be turned around, approached indirectly, and if necessary, the final object can be

(a) first sketched or guessed, then successively adjusted;

(b) first transformed in something more familiar, then transformed back.

Examples of (a) are the method of false position, already used by the Egyptians, and Euclid’s division algorithm; (b) generated a large part of modern mathematics, which studies properties remaining invariant when others are changed, e.g., Affine Geometry, or Relativity. Approaches (a) and (b) also apply to problems that do not request construction of objects, but deduction of properties: tools are not only used for shaping and building, but also for measuring and comparing. Rough quantities can be determined by operating directly on the object (with a calibre, a protractor, and so on), finer ones are frequently derived indirectly, analysing a physical phenomenon (compression, refraction) produced by the object, or observing a transformed image of the object, obtained by casting shades, mirror reflection, optical magnification, and so on. In any case, a tool is invented for enhancing human capabilities. Challenges may sound frustrating if one feels helpless, therefore the teacher should emphasize that in mathematics, like in the real world, difficulties can often be overcome by resorting to suitable instruments.
When looking for school problems to be attacked by the two above approaches, one would perhaps instinctively think of numerical approximation (going straight to the goal!) for (a) and of Euclidean geometry (moving all around the place!) for (b). The proposals we are going to present intend to *challenge* this commonplace: in the first one, we cross the plane hunting for points, in the second one we just consider a simple shift in numbers.



<u>Problem 2a</u> *The chromatic plane* (from *Crux Mathematicorum,* 1991, Canada)
Each point in the plane ($\mathbf{R}^2$) is coloured by one of the two colours *A* and *B*. Show that there exists an equilateral triangle with monochromatic vertices.

<u>Solution</u> Let the two colours be black and white and let *ABC* be an equilateral triangle. We may assume, without loss of generality, that the vertices *A* and *B* are black. Suppose *C* is white and consider the figure, where *ADGEFC* is a regular hexagon (with *B* the centre). Now proceed as follows:

- *ADB* is equilateral. *A*, *B* are black; so suppose *D* is white.
- *CDE* is equilateral. *C*, *D* are white; so suppose *E* is black.
- *BEF* is equilateral. *B*, *E* are black; so suppose *F* is white.
- *CFH* is equilateral. *C*, *F* are white; so suppose *H* is black.
- *BHI* is equilateral. *B*, *H* are black; so suppose *I* is white.

Now *IDF* is equilateral and *I*, *D*, *F* are white, completing the proof.

In this exercise, approach (a) consists in picking one object at random, and then going on searching systematically, supposing that the outcome is always bad. After a finite number of steps, the conclusion is necessarily positive.

<u>Pedagogical considerations</u> The solution presented above can be discovered by the pupil in a kind of game played against the teacher. In the picture, points *A* and *B* are initially black. At each step, the pupil is asked to create an equilateral triangle with monochromatic vertices by colouring one of the points depicted. But each time the teacher counters his move by changing the colour of this point: he pretends that he/she will always be able to prevent the pupil from creating an equilateral triangle with monochromatic vertices. The surprise comes at the end: the challenge has an "embarassing" outcome for the teacher, since the triangle sought by the pupil results from three white points inserted by the teacher himself/herself.

<u>Problem 2b</u> *Calculus with integers* (from *mathe-plus*, 1984)
The polynomial

$$p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$$

has integer coefficients, and takes the value 5 for five distinct integers. Show that there is no integer *k* with $p(k) = 11$.

<u>Solution</u> Let $z_1, ..., z_5$ be distinct integers such that $p(z_i) = 5$ for all $i = 1, ..., 5$. Consider the polynomial $q(x) = p(x) - 5$, which also has integer coefficients, and has $z_1, ..., z_5$ as distinct integer roots. Hence it can be decomposed as follows:

$$q(x) = (x - z_1) \cdots (x - z_5) r(x),$$

for some polynomial $r(x)$ with integer coefficients. Suppose that $p(k) = 11$ for some integer *k*. Then

$$6 = p(k) - 5 = q(k) = (k - z_1) \cdots (k - z_5) r(k) \ ,$$

so that 6 would be decomposable in the product of six divisors, five of which are pairwise distinct, and this cannot be true. Hence no integer *k* with the above property can exist.

Approach (b) is used here to replace the given polynomial with another polynomial, which is easier to be handled, since it has five known roots, and is closely related to the original one, from which it only differs by a constant summand.

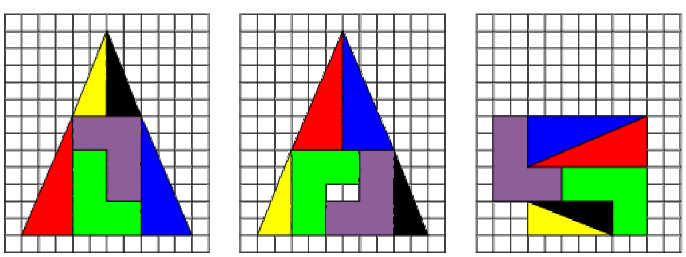
<u>Pedagogical considerations</u> The teacher can present this solution step by step, asking the pupils for the consequences of the single arguments introduced: Consider the polynomial $q(x) = p(x) - 5$: what can be said about its coefficients and its roots? What kind of number is $q(k)$? If we know

that it can be decomposed in the form $q(x)=(x-z_1)\cdots(x-z_5)r(x)$, what can be said about the divisors of the integer $q(k)$?
After the proof has been completed, the teacher may ask the pupils to go back to the initial step, and try to recognize the role of the constant 5: would the above arguments have worked on the polynomial $p(k)$? Or on the polynomial $p(k)-c$, for some other constant c? Why?

In both examples 2a and 2b, the solutions are found by grasping at what lies next: the first equilateral triangle encountered in the two-colour plane, and then its neighbours, or the decomposable polynomial that most resembles the one proposed. One could dare to say that the strategy is determined by the solver's ignorance. A challenge must not be too ambitious, it should not induce us to trespass our natural limits. On the contrary, it should show us how to use them as landmarks in our endeavours. Mathematics must be regarded as a human experience, a matter that concerns an individual as a whole, with his qualities and his drawbacks, as a member of society and part of the world. There are literary works by and on mathematicians that you can borrow from the library or buy in bookstores, museums and exhibitions of mathematical creations and models that you can visit, personally, or virtually on the web. These resources show that a mathematical *structure*, due to its generality and abstraction, can govern arithmetic progressions as well as plant growth (Fibonacci numbers), can reside in geometric constructions as well as in technical devices (parabola), can produce the laws of logical thought as well as the basic techniques of electronic computation (binary number system). The works by Pacioli, Descartes and Leibniz derive the principles of aesthetics, optics and philosophy from numbers and figures. Mathematics also concerns human categories such as personal taste, practical sense, and common reason. These abilities are normally activated when someone faces a challenge in life; this should also happen when the challenge is a mathematical one.

3. Mathematics has two faces

There is at least one life lesson that mathematics can teach us: everything/everyone can be considered from different angles, a superficial glance often is insufficient, if not misleading.



Recreational mathematics is full of examples of dissection paradoxes that warn us from relying upon our visual impression without verifying our conclusions analytically (e.g., by counting squares and comparing right-angled triangles). But the contrast is not only between the outward appearance (*visible shape*) and the deep essence (*verifiable structure*). Mathematical objects are many-sided. This should not be surprising to those pupils who have tried the challenge of Problem 1a: there usual arithmetic blows down to binary arithmetic. Problem 2a successfully explores a continuous geometric object through a finite procedure in two-valued logic. Even more significant examples can be given: the solvability of the 15-puzzle, or of the game "Lights out" (see [6]) are different combinatorial questions that can be completely answered by arithmetical and linear algebraic tools respectively.
All three examples lead to binary arithmetic; it is no wonder, since we have already remarked that structure is "exportable". And a skilful transfer of structure is the key to ingenuity: tools like the sliding rule or the pantograph combine arithmetic and geometry with mechanics.
We have seen cases where the field in which a mathematical problem is posed differs from the one in which it is solved. There are other problems where the solution really rests upon two *equivalent* descriptions: two examples are reported below. The first one is based on the fact that adding the integers 1,…, n in this order is the same as performing the operation with respect to the opposite arrangement. The basic identity is obtained by rewriting the sum according to the commutative

property. In the second example the two sides of the crucial equation are the cardinalities of two sets linked by a bijection.

Problem 3a *Sums of consecutive integers*.

Show that for every natural number n, the following formula holds: $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$.

Solution If we rewrite the sum after reversing the order of the summands, we obtain the identity

$$\sum_{i=1}^{n} i = \sum_{i=1}^{n} (n+1-i).$$

Expanding the right-hand side we further get

$$\sum_{i=1}^{n} i = n(n+1) - \sum_{i=1}^{n} i,$$

which immediately implies the claim.

Here, once again, like in Problems 2a and 2b, the proof is a quickie that only contains elementary, straightforward arguments.

Pedagogical considerations The crucial idea in the solution of this well-known problem is reversing the order of addition. It thus offers the opportunity of recalling the commutativity of the sum of integers, of practising the conversion of a procedure described in words into a symbolic formula, and of learning how to split an implicit expression with a running index into its components. This example shows the relevance of literal calculus, which is usually frowned upon by the pupils as a useless formal exercise, but here turns out to be a helpful and flexible tool. The teacher should guide the pupils through the *rewriting* process, by requesting them to get a prescribed final form: first they can be asked to obtain an equality between forward and backward addition, then, an equality where the same summation appears on both sides of the equality.

Problem 3b *Meeting people* (from *Crux Mathematicorum*, 1991)
There are n participants in a conference. Suppose (i) every 2 participants who know each other have no common acquaintances, and (ii) every 2 participants who do not know each other have exactly 2 common acquaintances. Show that every participant is acquainted with the same number of people in the conference.

Solution Let α be one of the participants in the conference, and let $A = \{\alpha_1, ..., \alpha_r\}$ be the set of his acquaintances. If $n = 2$, then $r = 1$ and both participants each know one person. If $n > 2$, then $n \geq 4$ and it is easy to check that $r \geq 2$.
Since the elements of A have α as common acquaintance, they pairwise do not know each other; hence, in particular, α_1 and α_2 have exactly one further common acquaintance β_1. Call B_1 the set of acquaintances of β_1. Since α and β_1 have α_1 and α_2 as common acquaintances and by assumption they cannot have more, we have $A \cap B_1 = \{\alpha_1, \alpha_2\}$.
Now suppose that $\beta_2 \neq \beta_1$, with $\beta_2 \notin A$, and call B_2 the set of acquaintances of β_2. Then $A \cap B_2 = \{\alpha_i, \alpha_j\}$ for some distinct $i, j \in \{1, ..., r\}$. If $i = 1$ and $j = 2$ (or vice versa), then α_1

and α_2 would have three common acquaintances, namely α, β_1, β_2, and this contradicts the hypothesis. Therefore $A \cap B_1 \neq A \cap B_2$.

This actually shows that there is a bijective correspondence between the unordered pairs of elements of A and the $n-r-1$ participants α does not know.

Hence $r(r-1)/2 = \binom{r}{2} = n-r-1$, and so

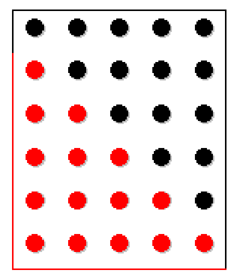
$$r^2 + r + 2 - 2n = 0.$$

Solving this quadratic equation we get the solutions

$$r_1 = -\frac{1}{2} + \frac{1}{2}\sqrt{8n-7}, \quad r_2 = -\frac{1}{2} - \frac{1}{2}\sqrt{8n-7}.$$

Since r_2 is negative, we discard it. So the only solution is r_1, which is independent of the choice of α. Of course, this only makes sense if r_1 is an integer. The claim is proven.

Here a combinatorial constraint has been turned into a numerical relation: once again we have an example of a problem that crosses the borders between different branches of mathematics.



Pedagogical considerations The above proof is one more example which can be presented by the teacher step by step. The suggestion is to guide the pupils through the argumentation by singling out the parts in which it is naturally subdivided:

- consider the situation in special cases, for small values of n;
- apply the general (*global*) rule given in the claim to specific (*local*) situations, i.e., to single participants $\alpha, \alpha_1, \alpha_2$.
- argue by contradiction;
- recognize a one-to-one correspondence between two finite sets;
- apply a combinatorial formula;
- solve a quadratic equation depending on a parameter;
- discuss the possible numerical values taken by a formula with respect to integrity and sign.

A complex task (such as the comprehension of a longer proof) can be achieved by concentrating on one passage at a time.

Finite quantities can be evaluated in more than one way: this achievement from elementary arithmetic and college algebra is a powerful argument for deriving combinatorial identities. The formula for the sum of consecutive integers actually arises from a mere formal transformation. A visual proof can be given by comparing areas. Note that passing from a sum to the reversed sum or realizing the area of a rectangle as the sum of two triangles has little to do with the famous *pattern of two loci* introduced by Polya [5, Ch. 1]. In our examples there are not two sets, with different definitions, to be intersected, but the same number expressed in two distinct forms, which can either reflect

(a) the way in which the number is defined as the result of an algorithm or a counting process (*underlying structure*), or

(b) the way in which the number can be determined by relating it to other ones (*surrounding structure*).

The pedagogical relevance of structure in algebra was emphasized by [4], whereas its role in popularization was discussed in [1].

4. Conclusion - A lesson from algebra

We have seen that, under certain circumstances, a polynomial $p(x)$ may deserve to be thought of, more awkwardly, as $(p(x)-5)+5$. By now we have become aware that the most concise form is not necessarily the one that leads to the solution, since it may be too implicit, it may conceal useful information. Solving a linear equation means bringing it to its shortest possible form. This is no longer true for algebraic equations of higher degree, or for transcendental equations. Often an expression, like an engine under examination, needs to be blown up or disassembled in order to inspect it inside and find its zeros. The form

$$(x-1)(x+2)(x-3)=0$$

is more complicated, but certainly more convenient than

$$x^3-2x^2-5x + 6=0 .$$

Similarly, the equation

$$4^x+2^{x+1}+1=0$$

can be solved only after the equivalent transformation

$$\left(2^x\right)^2+2\cdot 2^x+1=0 ,$$

where the well-known rules for powers

$$\left(a^u\right)^v=a^{uv}$$

$$a^u a^v=a^{u+v}$$

are read from right to left, unlike in ordinary arithmetic. In this way, the structure dominating the left-hand expression becomes apparent: $\_^2+2\_+1$, allowing us to treat the above equation as a special kind of quadratic. The practice of algebra requires seeing through the formulas, taking a closer look, going beyond exteriority, and examining the inner features. There are large areas of algebra dealing with various types of processes called *decompositions* and *resolutions*, where objects are "opened" and "unfolded". In other contexts, the objects are "compressed" and "minimized", and only studied in terms of their role in a network of maps or relations. Sometimes these *underlying/surrounding structures* are not contained in known theorems, but must be invented *ad hoc*. This confirms the deep analogy between mathematical challenges and the arts: both need to *look* at things under a new light, finding brand-new descriptions that disclose unexplored paths. And both cannot be disjoint from creativeness, which means to *use* things in an unconventional way, detaching them from the framework in which they were born to make them flourish on a foreign ground. This, however, should not be considered as a purely mental activity, reserved to a few talented people. Ideas can be induced by a systematic analysis of connections and similarities; inventions can be produced while one tries to combine the existing ingredients according to innovating schemes. This (more or less blind) search can be assisted by technology [2]: patterns may be hard to recognize only by experience and memory. On the other hand (more or less focused) discussions widen the range of possible findings: a single viewpoint certainly cannot overlook the whole situation. All pupils, even the "weaker" ones, have a chance to succeed, provided they know that the initial step must always be to *seek help*. One should keep in mind that the meaning of “help” is not “replacement” (another person doing the job for you), but is “support” (a resource that backs up and/or amplifies your abilities). Standard school mathematics essentially teaches us *how to use* given instruments. Challenges should train us in tackling the more intriguing question of *what to use* and *where to find it*.

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