GETTING AWARE:
Secondary School Teachers Learn about 'Mathematics Challenges in Education'

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ABSTRACT

This paper addresses one of the questions raised by the discussion documents of the ICMI STUDY 16: 'How can teachers be made aware of the existence of different types of challenges in school mathematics?' The paper attends to the complexity of the teachers' knowledge structure and suggests several principles for the design of the courses for mathematics teachers. The paper suggests that a Big Idea for the course should be chosen and describes several modes of work with secondary school mathematics teachers. A description of one particular course focusing symmetry exemplifies the principles as applied to one of the modes.

RATIONAL

As mentioned in the Discussion Document of ICMI Study 16 (Barbeau & Taylor, 2005) word 'challenge' has different meanings. In Cambridge Advanced Learner's Dictionary[1] we find that 'challenge' may be defined as 'difficult job', namely, something needing great mental (or physical) effort in order to be done successfully and which therefore tests a person's ability. Thus 'to challenge' may mean 'to test someone's ability or determination'. Correspondingly teachers' challenging role in Mathematics classroom is to stimulate students' mathematical reasoning, to set up their participation in mathematical explorations, to design situations in which students are required to prove mathematical statements, to let students participate in competitions.

In order to develop pupil's mathematical understanding, and allow their successful problem-solving performance a teacher must create situations that demand from the students great mental effort. "In this endeavor, the role of the teacher is critical" (Barbeau & Taylor, 2005; p. 2). Teachers' choices of mathematical tasks for their classes and the ways in which these tasks are introduced determine the quality of mathematics in the classroom (Stigler & Hiebert, 1999). However many teachers choose "stereotypical tasks" for their lessons and guide students towards "stereotypical solutions" (Barbeau & Taylor, 2005; Leikin, 2003). The word 'stereotypical' denotes kind of tasks and solutions, which are included in a particular textbook or prescribed by educational authorities.

One of the ways that can help teachers apply challenging mathematics in their classes is changing the textbooks so that series of challenging tasks will be available for the teachers (Barbeau & Talor, 2005). However, simply providing teachers with ready-to-use challenging math activities is not sufficient for their implementation (Leikin et al., in press). The following conditions appear to be important in employing challenging mathematics in school: Teachers should be aware and convinced of the importance of such implementation; Teachers should 'feel safe' when dealing with such kind of mathematics (mathematically and pedagogically); Teachers should have autonomy (Krainer, 2001) in employing this kind of mathematics in their classes. Additionally the teachers themselves have to be able to choose mathematical tasks from "outside the textbook", to create those tasks, to change stereotypical tasks so that they will become challenging and stimulating and finally – which is most relevant for this paper – solve stereotypical problems in non-stereotypical ways.

TEACHERS' KNOWLEDGE AND BELEIFS

Teachers' knowledge (Shulman, 1986), and beliefs (Cooney, 2001) determine their decision making when planning, performing, and reflecting. Teachers' knowledge and beliefs are interrelated and both have very complex structure. Leikin (in press) suggested using 3D model of teacher's knowledge, which describes this complexity (e.g., Scheffler, 1965; Shulman, 1986). In the context of the ICMI Study -16, I present this model in relation to challenging mathematics (see Figure 1).

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[1] http://dictionary.cambridge.org/define.asp?key=12394&dict=CALD

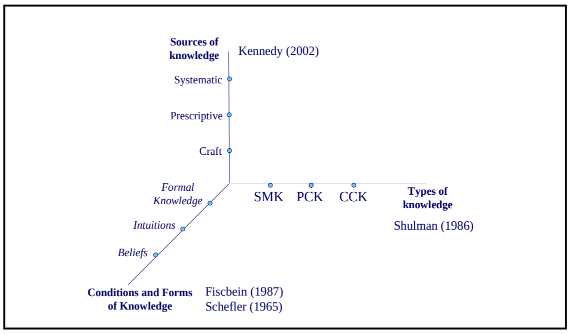

Figure 1: Three dimensions of teachers' knowledge (from Leikin, in press)

Dimension 1 'KINDS OF TEACHERS' KNOWLEDGE' is based on Shulman's (1986) components of knowledge: *Teachers' subject-matter knowledge* comprises their own knowledge of challenging mathematics. *Teachers' pedagogical content knowledge* includes knowledge of how students cope with challenging mathematics, as well as knowledge of appropriate learning setting. *Teachers' curricular content knowledge* includes knowledge of different types of curricula and understanding different approaches to teaching challenging mathematics.

Dimension 2 'SOURCES OF TEACHERS' KNOWLEDGE' (based on Kennedy's; 2002): Teachers' *craft knowledge* is largely developed through classroom experiences with challenging mathematics. *Systematic knowledge* is acquired mainly through systematic studies of mathematics and pedagogy in colleges and universities, and through reading research articles, journals, and professional books. *Prescriptive knowledge* is acquired through institutional policies, which are transparent in tests, accountability systems, and texts of diverse nature.

Dimension 3 'FORMS AND CONDITIONS OF KNOWLEDGE' differentiates between teachers' *intuitive knowledge* as determining teachers' actions that cannot be premeditated (e.g. in Atkinson & Claxton, 2000), their *formal knowledge,* which is mostly connected to planned teachers' actions, and *teachers' beliefs,* which are expressed in teachers' conceptions of teaching (Scheffler, 1965, p. 76).

Courses that are aimed at the development of teachers' awareness of the existence of different types of challenges in school mathematics have to take into account complexity of teachers' knowledge. Such courses should take into account teachers' intuition as the basis for the development of their formal knowledge. On the other hand, they should consider teachers' beliefs and be aimed at developing teachers' enthusiasm about introduction of challenging mathematics in school. In the next section I present some principles for designing courses for mathematics teachers that may develop their awareness of the importance of challenging school mathematics.

COURSES FOR TEACHERS

Underlying principles

Challenging teachers with 'powerful tasks' is fundamental for teacher development (Krainer, 1993). In order to motivate teachers' participation in the courses and concentrate their attention on challenging mathematics, mathematical tasks for the teachers have to be closely related to the secondary school curriculum as well as to create authentic mathematics-learning situation for the teachers. In addition, professional development activities should involve problem-solving situations combining mathematical and pedagogical issues (Cooney & Krainer, 1996). To stimulate teachers' pedagogical reasoning the courses should be based on the same principles as teaching mathematics to school students. The teachers at the courses play the roles of students according to the following principles, which they are expected to learn through their participation in the course:

Meeting appropriate difficulties: Learners have to face difficulties when coping with mathematical tasks. Principles of 'developing education' (Davydov, 1996), which integrate Vygotsky's (1978) notion of ZPD (Zone of Proximal Development), and Leontiev's (1983) theory of activity claim that to develop students mathematical reasoning the tasks should not be too easy or too difficult. According to Polya (1973), Schoenfeld (1985), and Charles & Lester (1982) mathematics tasks should meet the following conditions: "First, the person who performs the task has to be motivated to find a solution; second, the person has to have no readily available procedures for finding a solution; third, the person has to make an attempt and persist to reach a solution; fourth, the task or situation have several solving approaches. Obviously, these criteria are relative and subjective with respect to a person's problem-solving expertise in a particular field, i.e., a task that is cognitively demanding for one person may be trivial (or, vice versa) for another" (Leikin, 2004b, p. 209).

Mathematical connections: Learners have to develop mathematical connections of different types: between representations of the mathematical concepts, between different mathematical tools and concepts from the same field, between different mathematical topics (e.g., NCTM, 2000). In this perspective Russian educators (Sharygin, 1989; Davydov, 1996) advised *divergence principle* of learning simultaneously several topics while NCTM (2000) standards recommend solving problems in different ways.

Active participation in learning process: Learning has to be active (NCTM 2000), meaning that learners construct their individual knowledge through *mathematical explorations* with emphasis on conjecturing that leads to mathematical discussions, proofs and refutations. With respect to the principle of 'active participation' Sharygin's (1989) recommended '*changing priorities*': priority for ideas when learning a new topic and solving non-routine (heuristic) problems vs. priority for complete answers when working with known ideas and solving standard problems.

Individual and cooperative: Learners have to study systematically in individual mode in classroom and at home in order to develop their problem-solving competence (Sharygin, 1989). They have to be involved in cooperative learning activities in order to advance in problem solving by supporting each other's ZPD (NCTM, 2000; Davydov, 1996). The balance between individual and cooperative may provide learners with better opportunities for realization of their potential.

While above principles are general ideas for learning challenging mathematics discussion of the learning processes, in which the teachers are involved, leads to the development of teachers' pedagogical knowledge. For example, when learning challenging mathematics in cooperative learning settings the teachers both developed their knowledge of learning method and develop their sensitivity to students (in the intuitive form) (Leikin, 2004a). Teachers' awareness of ZPD and its role in problem solving is an integral part of their pedagogical knowledge. The next principle is specific for the courses for mathematics teachers. It allows focusing teachers' attention on an issue which is the most interesting for them and working on the development of different kinds of teachers' knowledge simultaneously.

Focusing Big Ideas: Choosing Big Ideas ('Habits of mind' in Goldenberg, 1996) as unifying topics for the courses is one of the main principles for the courses design. A Big Idea has to be in the focus of the course long enough in order to preserve continuity in the teachers' learning, to allow connectedness of knowledge and meaningfulness of reflection. Big Ideas may be of different types; among those that I used in my courses were the following: Mathematical Concepts, e.g., symmetry (Leikin, 2003), equivalence; Meta-mathematical concepts, e.g., mathematical definitions (Leikin & Winicky-Landman, 2001); Didactical ideas, e.g., multiple-solution connecting tasks (Levav-Waynberg & Leikin, in press), and cooperative learning (Leikin, 2004a).

Types of courses

As big ideas incorporated in the courses change and the forms of work with the teachers vary, different modes of professional development may be considered within the framework of the three dimensions of teachers' knowledge. I suggest considering four main modes of development: Mathematical, pedagogical, research and implementation. These different modes vary with respect to the goals stated explicitly, the role of the challenging content, and thus processes and mechanisms of teachers' knowledge development.

Mathematical mode: Teachers' main purpose is learning mathematics. A course is focusing on challenging (mathematical) tasks, types of the tasks and types of mathematical challenges. The teachers are coping with challenging mathematics as learners. Development of SMK is the explicit purpose of the course whereas PCK and CCK are developed implicitly through facing different learning experiences (e.g., Leikin & Winicky-Landman, 2001)

Pedagogical mode: Teachers' main purpose is to lean pedagogical approaches/methods (didactics). A course is focusing on the ways of implementation and stimulation of mathematical challenges in classes while mathematical challenge is an integral component of all the activities. Development of PCK is the explicit purpose of the course whereas SMK and CCK are developed implicitly through facing different pedagogical experiences (e.g., Leikin, 2004a).

Research mode: Teachers main purpose is learning about students' reasoning. In such a course, for example, reasoning of gifted and talented students may be compared with the reasoning of regular students of the teachers' themselves. Mathematical challenge is the content for research on students' mathematical reasoning. The courses may include reading studies by famous researchers and performing individual or collaborative research with gifted students. The course ought to demonstrate viability of applying challenging mathematics with students. Knowledge of students is developed in a direct mode while subject matter and curricular content knowledge are developed implicitly (e.g., Leikin, 2003)

Implicational mode: Teachers main purpose is teaching. The purpose of a program is learning through teaching. The program may include Teaching Experiment or Teacher Development Experiment. The program aim is development of teachers craft knowledge of different kinds mathematical challenge is the unfamiliar content to be taught in the classroom. Knowledge development depends on the initial state of teachers' knowledge especially for this mode (e.g., Leikin, 2005; in press)

Obviously some courses may combine different modes. Moreover, to develop teachers' awareness of the existence and of different types of challenges in school mathematics, combinations of different modes are necessary, no one of the modes is sufficient.

'MATHEMATICAL CHALLENGES IN EDUCATION' COURSE: MEETING THE PRINCIPLES

As an example, I present here a course 'Mathematical challenges in education' specifically designed for graduate program for secondary school mathematics teachers. The course included 13 meetings, each lasted 4 hours. Nine of the 13 meetings were focused on symmetry in its broad sense (Leikin, Berman & Zaslavsky, 2001, 1998). The themes of the meetings were the following: (1) Definition of symmetry; types of symmetry, (2-3) Solving geometrical problems using geometric and logic symmetry, (4) Geometric symmetry in space, (5) Role symmetry in algebra and geometry, (6) Functions and symmetry, (7) Solving maxima-minima problems using symmetry, (8) Defining mathematical objects using symmetry, (9) Symmetry and Number sets

Course design incorporated five abovementioned principles. It was focused the 'big idea': "Symmetry as a way of thought in mathematics". The course combined teachers' individual and cooperative learning. Both during the meetings and in home assignment the teachers were asked to work in different modes. The teachers reported that some of the activities stimulated them to use different learning methods implemented in the course in their classes. The following exemplifies the tree other principles as applied in the course.

Note that all the examples bellow are regular textbook tasks that become challenging for the teachers when connected to the concept of symmetry. Incorporation of the "regular" tasks in the course was especially important in order to develop teachers' awareness of the importance and the applicability of mathematical challenges in their classes, taken into account the prescriptive component of teachers' knowledge

Meeting appropriate difficulties: As found in the previous studies, usually teachers' knowledge of symmetry is associated with geometric line symmetry (reflection about a line) whereas other types of symmetry (i.e., rotation, translation, central symmetry (Leikin, et. al, 1998) and especially role symmetry (Polya, 1973): algebraic and logical) are usually unfamiliar to the teachers. Additionally, as this concept is not a part of mathematical curriculum, the teachers do not use symmetry in problem solving. Thus the choice of symmetry as a unifying concept for the course allowed authentic and difficult learning situation for the teachers.

Examples

> *We expect that any symmetry found in the data and condition of the problem will be mirrored by the solution* (Polya, 1981; p. 161)

The following task is considered by the teachers as difficult one since school calculus -- to which the task is usually attributed by the teachers --does not allow solving it. Logical (role) symmetry allows solving this problem with junior high school students:

Problem 1.1: Among all the triangles inscribed in a given circle which one has the maximal area, assuming that such a triangle exists?

***Solution*:** If side b and side c of the triangle are not equal, then the area of the triangle can be increased. Hence $b=c$. **In the same way**, $a=b$. Thus the triangle is equilateral.

Note that the Problem 1.1 was presented to the teachers as an exploration task within DGE. The symmetry based solution was used for proving teachers conjectures.

Application of algebraic role symmetry simplifies meaningfully solution of the following task. However, since this type of symmetry is unfamiliar to teachers the task can be considered as non-trivial for them:

Problem 1.2: How many real solutions can the following system of equations have?

$$\begin{cases} x+y+z=a \\ xy+yz+xz=b \\ xyz=c \end{cases}$$

Solution. The given system consists of three symmetrical equations with three unknowns. Each equation is invariant under any permutation of the unknowns.
Thus, if a triplet (t_1, t_2, t_3) is a solution if the system, then the triplets (t_2, t_3, t_1), (t_3, t_1, t_2), (t_2, t_1, t_3), (t_1, t_3, t_2), (t_3, t_2, t_1), are also solutions of this system.
According to Vieta theorem a triplet (t_1, t_2, t_3) is a solution of the given system if and only if t_1, t_2 and t_3 are the solutions of an equation t^3-at^2+bt-c=0.
If these solutions t_1, t_2, t_3 are all different, then the system has **6 real solutions**.
If $t_1 \neq t_2 = t_3$ it has **3 different solutions** (t_1, t_2, t_2), (t_2, t_1, t_2), (t_2, t_2, t_1)
If t_1=t_2=t_3, then it has only **one real solution** (t_1, t_1, t_1).
The system **does not have a real solution** when the equation t^3-at^2+bt-c=0 has only one real solution.

Mathematical connections: Symmetry was considered in the course as a concept connecting different branches of mathematics: it was used in geometry (problems 1.1, 3.1), in calculus (problem 1.1.) and algebra (problems 1.2, 2.1). Symmetry appeared to be a useful tool in defining, conjecturing and proving. It also allows emphasizing importance of the connections between different representations (problem 2.1).

Examples
Solution of the following problems connects between, function properties, geometric and algebraic symmetry. It is based on the ability to translate algebraic representation of the object into its geometric representation:

Problem 2.1 For which value of the parameter ***a*** the system of equation

$$\begin{cases} (|x|+1)a = y+\cos x \\ \sin^4 x + y^2 = 1 \end{cases}$$

has only one solution?

Solution. The equation $y = (|x|+1)\cdot a - \cos x$ is the equation of an even function the graph of which is symmetrical with respect to the y-axis. The graph of the second equation is also symmetrical with respect to the y-axis. Hence, if a pair (t_1, t_2) is a solution of the given system, then the pair $(-t_1, t_2)$ is also a solution of this system. Therefore, a necessary condition for the system to have only one solution is $x = 0$.

With x=0, the system becomes $\begin{cases} y = a-1 \\ y^2 = 1 \end{cases}$.

The solutions of this system are $y=-1, (a=0)$ or $y=1, (a=2)$.
For $a=0$ the original system has infinitely many solutions.
For $a=2$ the system has the unique solution: $(0,1)$.

Problem 2.2: Consider the function $f(x) = \dfrac{cx}{2x+3}$, where $x \neq -\dfrac{3}{2}$.
Find all values of c, if any, for which $f(f(x)) = x$

Solution.

1. $\frac{cx}{2x+3}=\frac{c}{2}+\frac{-0.75c}{(x+1.5)}$, Thus the graph of the function $f(x)=\frac{cx}{2x+3}$ can be obtained from the graph of a function $h(x)=-\frac{0,75c}{x}$ by translation of the graph by vector $(-1.5,\frac{c}{2})$.
2. $f(f(x))=x$: the function is inverse to itself

   Therefore the graph of the function $y=f(x)$ is symmetrical with respect to the line $y=x$.

   Hence $\frac{c}{2}=-1.5$; $c=-3$ (Figure 2).

***Active participation in learning process*:** About half of the tasks in the course were formulated as inquiry-based problems and teachers had to make explorations in order to solve them. Most of the problems (which are usually proof problems) were formulated as open-ended tasks (e.g., problems 1.1, 1.2, 2.1, 2.2) which teachers were asked to investigate.

To promote teachers' active participation in the course they were asked to identify tasks in school textbooks which may be solved using symmetry of a particular type according to the topics of the meetings (geometric symmetry, algebraic symmetry, and logical symmetry). The following three problems are *teacher-generated examples* of the similar tasks, which were identified and solved using symmetry by different teachers. Surprisingly these solutions encompass different types of geometric symmetry that teachers learned during the course:
(i) The rule "of all the figures with equal perimeter the most symmetric has maximal area" (Problem 3.a - 1st way),
(ii) Reflection about a line, symmetry of a parabola (Problem 3.a - 2nd way; Problem 3.b),
(iii) "Comparing with a symmetric situation" strategy (Problem 3.c),
(iv) Central symmetry (Problem 3.c).

Problem 3 Length of the segment AB is a. Point E is on the segment AB. At the ends of the segment AB, two segments perpendicular to AB are raised so that $AD=AE$, $BC=BE$ (see Figure 3).
What should be the length of the segment BE such that:
(a) Maximal area;
(b) The triangle DEC will have minimal perimeter;
(c) The quadrilateral $ABCD$ will have minimal perimeter.

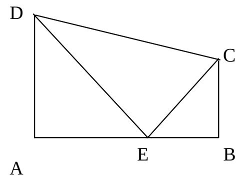

Figure 3

Solution:

(a) Triangle DEC is a right angle triangle (DAE and EBC are right angle and isosceles triangles) thus $\max S(\Delta DEC)=\max(DE\cdot EC)$

1st way:

The area of the triangle DEC equals half the area of rectangle $DECF$ (see Figure 4).
Among all the rectangles with constant perimeter a square (the most symmetric rectangle) has maximal area.
Thus the area of DEC will be maximal when $EC=ED$.

In this case $E=E^*$ is the middle of AB. ($BE=\frac{a}{2}$)

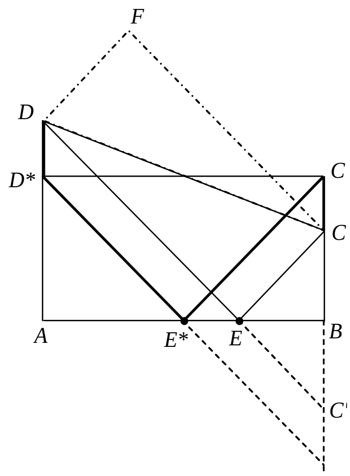

Figure 4

2nd way:

Let's reflect point C about AB (Figure 4).
$EC=EC'\Rightarrow DE+EC=DE+EC'=DC'$,
$AD||BC$ thus DC' is constant and $DE+EC$ is constant.
Thus, $\max(DE\cdot EC)$ is achieved in the symmetric situation, i.e., when $DE=EC$ ($BE=\frac{a}{2}$)

A possible explanation: $DE + EC = b\ (const)$; $DE \cdot EC = x(b-x)$, $f(x) = x(b-x)$ is a quadratic function. $f(x) = 0$ when $x = 0$ and $x = b$. By symmetry $f(x)$ riches maximal value in $x = \frac{b}{2}$

(b) $\min P(\Delta DEC) = \min(\underbrace{DE + EC}_{DC' \text{ constant}} + CD) = \min(CD)$

Thus, DC is minimal when $DC \parallel AB \quad (DC = D^*C^*)$,
Hence the triangle is of minimal perimeter when $E = E^*$ is the midpoint of AB.

(c) *Special cases of symmetric objects are often prime candidates for examination.*
(Schoenfeld (1985), p 81)

Let's consider a symmetrical situation:
$E = E^*$ is the midpoint of AB. In this case ABC^*D^* is a rectangle such that: $AD^* = AE^* = E^*B = BC^* = \frac{a}{2}$ (Figure 4).

Conjecture: In this symmetric situation (when $ABCD = ABC^*D^*$) of minimal perimeter.
Proof: Let's break the symmetry: let's consider $E \neq E^*$ on AB and compare $P_{ABC^*D^*}$ and P_{ABCD}

$P_{ABC^*D^*} = AB + BC^* + AD^* + D^*C^*$

$P_{ABCD} \quad = AB + BC \quad + AD \ + DC$

$BC^* + AD^* = BC \quad + AD$, since $D^*D = C^*C$ (as symmetrical about the intercept of DC and D^*C^*)

D^*C^* is perpendicular to $AD^* \quad and \quad BC^*$, hence $D^*C^* < DC$

That's why for $E = E^* \quad (BE^* = \frac{a}{2})$ perimeter of the quadrilateral ABCD is minimal.

Note that in order to promote teachers' active participation in the course, four last meetings of the course were devoted to presentation of the teachers' individual projects that were assigned at the beginning of the course. Teachers in pairs or in groups of three had to choose, investigate, and connect "Big ideas" to the school curriculum at different grades and difficulty levels. Among others they chose: Groups, Fibonacci series, Equivalence, Continuity, and Conic sections.

CONCLUSIONS: CHALLENGING MATHEMATICS AND TEACHERS' KNOWLEDGE

When teachers take part in different types of professional development activities we aim at development of their systematic knowledge. Teachers develop their understanding of challenging mathematics presented to them, become aware of the notion of challenging mathematics and develop their intuitions regarding students learning of this mathematics. However, teachers' own experience with challenging mathematics appears often to be insufficient for developing their pedagogical knowledge and beliefs. Teachers' mathematical, pedagogical and curricular knowledge is interrelated. Very often teachers consider mathematical tasks through the lenses of their profession. Sometimes they decide that particular tasks may be good for them but not good for their students. When experiencing difficulties with particular tasks they tend to think the students will not be able to cope with them. Additionally, teachers' prescriptive knowledge appears to be a pitfall for their willingness to apply what they have learned in their classes. Teachers often ask the question: "Will they accept this solution in a matriculation examination?", and are skeptical that problems which are not included in the textbooks may be applicable in their classes.

Then again, when implementing challenging mathematics in their classes, teachers develop craft knowledge in the field and begin to believe that this implementation is important for the development of students' mathematical reasoning. One of the important questions in this context is "How can teachers be convinced that employing challenging tasks is possible?" A potential answer to this question is "teachers' micro-experiences with such implementations" which are not described in this paper, e.g., interviewing individual students or pairs of students and analyzing their problem-solving performance;

videotaping successful teaching experiences and discussing them with teachers, and involving teachers in study groups whose focus is on challenging mathematics.

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