Mathematical Tasks and Teaching Mathematics in MOFET classes

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ABSTRACT

In this article we introduce MOFET association aimed at realization of the students' intellectual potential. Its pedagogical approach is rooted in the ideas of Vigotsky, Freudental, Davydov and Polya. Mathematical investigations are an integral part of the challenges the students meet in their mathematics classes. We present several problems to exemplify some mathematical experiences of MOFET students.

INTRODUCING THE PROGRAM

MOFET is an association established in Israel in 1992, with the purpose to meet the academic needs of gifted and talented students in mathematics and science. Its aim was to enrich learning skills and inspire creativity in Israeli middle school students studying mathematics and other exact sciences. Currently MOFET's programs are meant for students of different levels (see for details: www.mofet.org/en). They allow each student to improve his/her performance while special emphasis is made on the development of the intellectual potential of gifted students. The project is geared to elementary school students as well, thereby providing early intensive exposure to the spirit of MOFET.

Within the MOFET framework there exist both regular school and evening classes where children are taught according to the original programs of MOFET. The program's aim is to let each student achieve a high level of education, regardless of his/her initial level of competence. The primary emphasis is placed on developing creative thought based on the study of mathematics and physics, together with intensive learning of language, and understanding of Jewish heritage and world culture. The program is intended for students with high learning motivation, it addresses learning skills of each individual student.. This project has also served as a basis for a fruitful dialogue between native Israeli and ex-USSR immigrant teachers.

Every educational program is eventually judged by its students' achievements. Yet MOFET views development of learning skills, creativity, and original thinking as superior to high grades in final examinations. The program seeks to develop study skills at an early age and motivate the child throughout his/her education in MOFET. This approach will help pave the way for the future generation to cope with the challenges that face the country and humanity in the 21st century.

THE BACKGROUND

MOFET's pedagogical activities are based on a variety of methodologies rooted in the ideas of Vigotsky, Freudental, Davydov and Polya ([1], [2], [3], and [4]). Following Freudental's ideas of learning mathematics through problem solving and Davydov'd theory of developing education, recently, task-based programs have been developed for the study of mathematics, physics, and biology in the United States, Russia, and Israel. In addition, systems of tasks, games, and other tools for stimulating the thought processes have been developed to shape the basis for successful learning of 4-5 year olds.

Another important MOFET source is the experience of special schools for gifted and talented student established approximately 30 years ago in Russia at the initiative of Kolmorogov and other enthusiasts ([5], [6], [7]). The teaching staff of the schools included highly educated teachers including university level scientists. Many graduates of these schools completed a PhD in mathematics and science, and became well known scientists, including many active members of the academic communities in Israel, Europe and the United States.

LEARNING PROCESS

A MOFET classroom utilizes the natural children's curiosity. The level of studies in each particular MOFET class is adapted to the level of the students, taking into consideration their cognitive styles, abilities, learning skills. Classroom learning combines students' cooperation, their mutual help, excellence aspiration and supportive competition among the students.

MOFET mathematics curricula are based on the following *principles*:

- Learning of each concept relies on the preceding concepts, expands upon those concepts, and connects between them ([8], [9]).
- The implementation of multiple representations of mathematical concepts in problem-solving ([3], [10], [11], [12]);
- The connection and continuity between the subject matter covered at the elementary school level and the curricula in the secondary school ([12], [13]);
- Mathematical investigations, which include conjecturing, testing and proving, are the integral and the central part of the learning activities ([14], [15], [16]).
- Development of mathematical language and thought through mathematical discussions ([3], ([4], [17], [18])

In elementary school, MOFET students study mathematics 5-6 hours per week in keeping with the curricula of the Ministry of Education. The first graders are randomly divided into two smaller groups. Both groups study the same subject matter so that re-division of the class into two new groups at a later date is possible. Children may move between groups according to their wishes and the judgment of the teacher. Studies are organized with the focus on five basic components of mathematical thinking: topological, sequential, quantitative, algebraic, and projective ([19]).

From 7th grade onward, MOFET students' weekly class schedule includes 8 hours of mathematics and 4 hours of physics. In addition to secondary school Israeli curriculum MOFET secondary school program includes extra-curricular topics aimed at developing students' critical and creative thinking and advanced learning skills. Mathematics is viewed as a basic language for the study of scientific subjects, such as physics, chemistry, computers and biology, whereby physics provides practical expression for mathematics. For this reason, students begin the study of physics as a separate subject in the 7^{th} grade. Most students end up with matriculation examinations at advanced level.

The choice of the tasks for MOFET classes has empirical and theoretical background. On the one hand, it is determined by positive pedagogical experience of many mathematics educators including our own experience. On other hand, this choice is grounded in Vygodsky's notion of ZPD (Zone of Proximal Development, [1]). Vygodsky [1, 20] has shown the following: a child's mental development as a whole and the level of his /her mathematical thinking in particular, is determined not only by what the child has already learned to do, but also what he is capable of learning. Differentiating between the tasks done independently and those solved with the help of adults Vygotsky came to the following conclusion about children's ZPD: with the help of adults a child can solve only such tasks that lay in the scope of his/her own intellectual abilities. Krutetsky [21] demonstrated that these ideas are fruitful in the case of children gifted in mathematics. Davydov [3] suggested that education could be developing only when students cope with tasks that belong to their ZPD. These conclusions led us to working out tasks and programs which are rich in investigation activities and allow addressing each student’s ZPD.

As mentioned above, mathematical investigations and solving open problems are central for the learning process both at the elementary and the secondary level. In the following section of the paper we present several examples of typical mathematical problems used in MOFET classes.

EXAMPLES

Task 1

Task 1 presented bellow was given to the fifth grade students in many classes. The task is a non-routine one which is not common for standard mathematics lessons. The students usually started with trial and error strategy, which at first glance seemed natural for them. When realized that required much time, they tried to find a different problem-solving strategy. One of the ways in which a teachers promoted this process was formulation of a new problem equivalent to the given one but easier to cope with. The new problem allowed students to conjecture and discuss different hypotheses. We consider the task within ZPD of MOFET 5^{th} graders, since the students usually could not solve it individually but with a teacher's assistance did it well. We describe a scenario that took place in one of the MOFET classes.

> In a supermarket there are different goods with weights from 1 kg to 40 kg (in whole numbers only). The boss liked math problems and puzzles very much. He decided to buy only 4 different weights for weighing all goods (1-40). Which weights did he buy?

First stage [after trial and error the students employed]**:** The teacher asked his class to reformulate the question considering smaller, simpler questions. One of the students suggested finding which goods can be weighed by 2 weights only. They found that using 1kg and 3 kg weights they could weigh the goods of 1 kg, 2 kg, 3 kg, 4 kg.

Second stage: As the situation stands, no other suggestions could be made so the teacher supplied the 3rd weight that should be added. She asked: "What is the maximum number of goods from 1 kg that can be weighted by 3 weights only?" The students answers were: "1 kg, 3 kg, 9 kg and we can weigh all goods from 1kg to 13 kg."

Following this discussion the students designed the table:

Number of weights	Weight	Weights of the goods
1	1	1
2	1,3	1,2,3,4
3	1,3,9	1-13
4	**1,3,9,27**	**1-40**

Third stage: After restructuring the data and completing the table, the students came to a conclusion. Since in 5th grade the students did not have yet the necessary tools to prove the complete justification for their answer, they examined the assumption and answered the question.

Task 2

Solve the equation in natural numbers: $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=1$

$$x\in N, y\in N, z\in N$$

This Task was given to 8th-grade MOFET students. Some of the students easily found a partial set of solutions: (2,4,4), (4,2,4), (4,4,2), (3,3,3) while several students in the class found all the solutions: (2,3,6), (2,6,3), (3,2,6), (3,6,2), (6,2,3), (6,3,2), (2,4,4), (4,2,4), (4,4,2), (3,3,3). None of the students solved the task systematically. Nobody could prove the task does not have other solutions. Like in the case of Task 1, teacher's guidance was necessary to complete the solution. The teacher has to manage guiding discussion, ask prompting questions and help students to formulate their ideas and assumptions. Students under the teacher's guidance ended up with reasonable explanations. Thus once again we claim the task was within the students' ZPD. Bellow (*in italic*) we present prompting questions the teacher in one of our classes asked the students when managing the task solution. Teachers' hints are followed by the conclusions the students received eventually.

Can one of the variables be equal to 1?

$x, y, z \neq 1$, because $\frac{1}{1}+\frac{1}{y}+\frac{1}{z}>1$

Can all unknowns have a value higher than 3?

No. Because in this case we’ll have: $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}<\frac{1}{3}+\frac{1}{3}+\frac{1}{3}=1$.

Now we can conclude that one of the unknowns must be 2 or 3!

Let’s assume that x=2; what do we get?

We'll get: $\frac{1}{2}+\frac{1}{y}+\frac{1}{z}=1 \quad \Leftrightarrow \quad \frac{1}{y}+\frac{1}{z}=\frac{1}{2}$

Can one of the variables (y or z) be equal 2?

It is clear that $y,z \neq 2$, because $\frac{1}{2}+\frac{1}{z}>\frac{1}{2}$

Can't both unknowns (y and z) have a value higher than 4?

No. Because in this case we'll get: $\frac{1}{y}+\frac{1}{z}<\frac{1}{4}+\frac{1}{4}=\frac{1}{2}$.

Now the student concluded that one of the unknowns **must be** 3 or 4!

Further students solved the task individually as follows:

> Let's assume that y=3. So z=6. We have gotten 6 different solutions: (2,3,6), (2,6,3), (3,2,6), (3,6,2), (6,2,3), (6,3,2).
>
> Let's assume that y=4. So z=4. We have gotten 3 different solutions: (2,4,4), (4,2,4), (4,4,2).
>
> Now, let's assume that x=3. In the same way we'll get that the last solution is (3,3,3).

Answer: (2,3,6), (2,6,3), (3,2,6), (3,6,2), (6,2,3), (6,3,2), (2,4,4), (4,2,4), (4,4,2), (3,3,3).

Task 3:

Task 3, presented bellow, exemplifies cases in which standard textbook problems are used for mathematical investigations. Teachers' aim in such cases is to guide students' investigation so that students are involved in a real mathematics experimenting, hypothesizing, examining the hypotheses and proving them.

The Original textbook task was: "Prove for any natural n the next statement: $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+...+\frac{1}{n\cdot(n+1)}=\frac{n}{n+1}$". We have reformulated this task as follows:

> For any natural *n* calculate the sum
>
> $$S(n)=\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+...+\frac{1}{n\cdot(n+1)}$$

This task was introduces to 10^{th} graders by several teachers. In one of the classes the lesson developed as follows:

***The first stage:* experimenting**

At the first stage, the teacher asked the students to analyze and study the values of sums *S* as the functions of number *n*: S=S(n). At this stage, she didn't establish the number of tests and didn't restrict pupils in doing those tests. When experimenting the students used table as suggested by the teachers. They completed the following table:

n	1	2	3	4	5	6	7	8	9	10	11
S(n)	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{3}{4}$	$\frac{4}{5}$	$\frac{5}{6}$	$\frac{6}{7}$	$\frac{7}{8}$	$\frac{8}{9}$	$\frac{9}{10}$	$\frac{10}{11}$	$\frac{11}{12}$

***The second stage*: raising a hypothesis**

At this particular stage the teacher had to enable the pupils to establish their own independence and scientific activity. After the performance of a set of experiments, raising a hypothesis was the crucial stage in solving the task. After completing the table she asked students to formulate their hypotheses. Many students in the class found that one rule was rather obvious:

$$S(n) = \frac{n}{n+1}$$

***The third stage*: examining the hypothesis**

The teachers' purpose at this stage was to allow the students to rely on the logic of their solutions. Our pupils quickly acquired the ability to show that in order to demonstrate the invalidity of any hypothesis it is enough to show the invalidity in one particular case. The validity of the hypothesis must be proved by rigorous deduction.

***The fourth stage*: proving hypothesis**

At this stage some students proved the equality $S(n) = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + ... + \frac{1}{n\cdot(n+1)} = \frac{n}{n+1}$ using inductions (as required by the original task). Other students proved it differently:

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + ... + \frac{1}{n\cdot(n+1)} = \frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + ... + \frac{1}{n} - \frac{1}{n+1} = 1 - \frac{1}{n+1} = \frac{n}{n+1}$$

We claim that approaching the task as an investigation problem opened students minds and allowed them prove the equality in different ways.

CONCLUSION

Tasks, such as described above, exemplify systematic implementation of mathematics investigations at different levels. These tasks involve students in research and discovery activities that can contribute to student's development as independent researchers. These activities enhance motivation, provide multiple opportunities for participation in discussions, offer the basis for conclusions and new questions, and allow the students to analyze the outcome of the mathematical operations. Implementation of such activities can contribute to changes in students' views of mathematics and, moreover, encourage them to search for connections and generalities, as they continue to develop their mathematical knowledge. In this way students develop their mathematical thinking, their intellectual curiosity. The 12-year study program promotes students' beliefs in their own abilities, which will allow them to have a better approach to complex situations, and difficult assignments in future studies and, perhaps, in life.

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