Mathematical "Vorstellungen"

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The paper will reflect some details related to the process of learning and understanding mathematics when working on challenging problems. Basically the development of powerful mental representations is most important (Vorstellungen in German). There are two types of Vorstellungen, but the traditional kind of teaching mathematics gives strong emphasis only to one of them, to a reflective, logical and analytical thinking. To face challenging mathematics we also need many spontaneous and intuitive Vorstellungen. Some examples will be given.

1. Cognitive Aspects

Studying the Discussion Document for the ICMI study 16 (ICMI 2005), I appreciate very much the broad, deep and balanced discussion of the topic "Challenging Mathematics in and beyond the Classroom". Challenges are not just difficult problems. A difficult problem becomes a challenge only with respect to an individual or a group (ICMI, p. 2). Thus "to reflect the state of the art in providing mathematics challenges in and beyond the classroom" (ICMI, p. 1) also should include in some detail the mental and cognitive aspects the problem solver(s) are confronted with.

Eventually the document could stress a bit more in general the psychological aspects related to the teaching and learning of mathematics. What do we know about advanced mathematical thinking (TALL 1991)? Does challenging mathematics further the development of procepts (GRAY-TALL 1994) or enhance processes of encapsulation (DUBINSKI 2000)? Being confronted with challenging mathematics, which is the role of reification (SFARD 1994)?

To "suggest directions for future developments in research and practice" (ICMI 2005, p. 1) also should include investigations on the interaction processes between the external representations of a problem, which we call "***Darstellungen***[1]" of the problem and the internal mental images or cognitive structure(s) of the problem solver(s), which we call "***Vorstellungen***[2]" (MEISSNER 2002).

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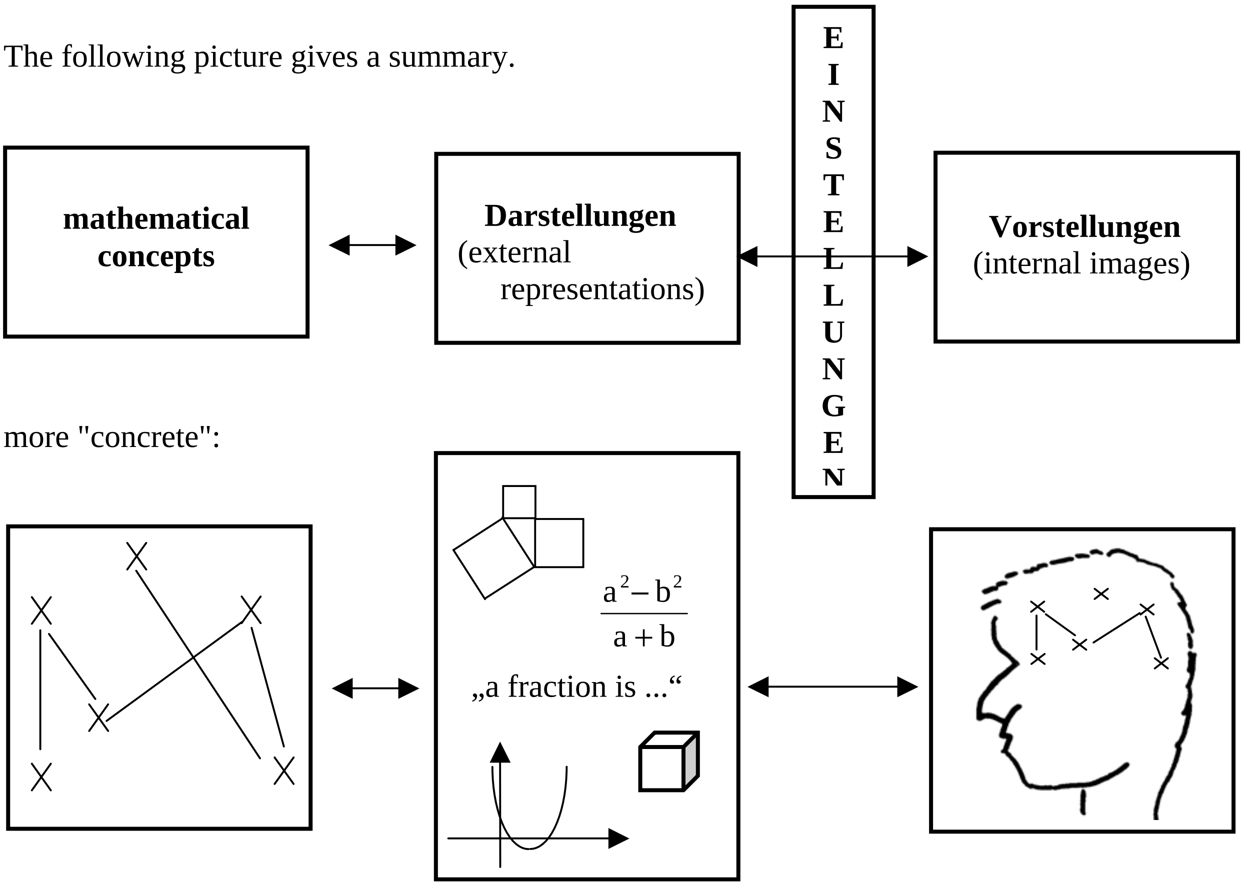
[1] We call external representations of mathematical ideas ***Darstellungen***. Darstellungen we can read, or see, or hear, or feel, or manipulate, ... These *external representations* or *Darstellungen* can be objects, manipulatives, activities, pictures, graphs, figures, symbols, tags, words, written or spoken language, gestures, ... In such a *Darstellung* the mathematical idea or example or concept or structure is hidden or encoded. There is no one-to-one-correspondence between a mathematical idea, concept, etc. and a *Darstellung*.

[2] Human beings are able to "associate" with these objects, activities, pictures, graphs, or symbols a meaning. That means each *Darstellung* evokes a personal internal image, a ***Vorstellung*** (cf. concept image, TALL/VINNER1981). Thus *Vorstellung* is a personal internal representation, which can be modified. Or the learner develops a new *Vorstellung*. A *Vorstellung* in this sense is similar to a cognitive net, a frame, a script or a micro world. That means the same *Darstellung* may be associated with many individual different internal representations, images. Each learner has his/her own *Vorstellung*. And again here, there is no one-to-one-correspondence between a *Darstellung* and a *Vorstellung.*

The process of building up a *Vorstellung* also depends very much on the basic mentality of the learner, i.e. on his or her ***Einstellung***. The *Einstellung* includes affective components like attitudes, emotions, values, etc. A central term is *beliefs* (Leder e.a. 2002). The *Einstellung* affects attitudes towards learning in general, towards mathematics in general, towards problem solving, or towards the specific learning "environment". The *Einstellung* is a product of social interactions (with parents, teachers, peers, etc.), of genetic factors, of cultural or historical impacts, etc. Positive *Einstellungen* in the class room are necessary for a successful teaching-learning process. A learner with a negative *Einstellung* probably will not be very successful. *Einstellungen* work like a filter or a catalytic converter in the transformation processes *Darstellungen* ↔ *Vorstellungen*.

The following picture gives a summary.



In this paper we will reflect *Vorstellungen*. Which *Vorstellungen* are necessary when the problem solver gets confronted with a challenge? Already KIENEL (1977) distinguishes in his dissertation five categories of mathematical "problems". Problems of Types I – III can be solved by applying a rule or an algorithm or a procedure. In problems of Type I the rule or algorithm or procedure is mentioned explicitly, in problems of Type II the rule or algorithm or procedure is known to the problem solver, but it is not mentioned explicitly. The rules or algorithms or procedures necessary to solve a problem of Type III first must be constructed by the problem solver via combining known rules or algorithms or procedures. Problems of Type IV are given verbally as "a real world" problem, and the mathematical content first must be analyzed and transformed into a mathematical problem to get then a problem of Type I-III.

Type V puts together all those "real" word problems, where neither the knowledge of rules or algorithms or procedures nor a specific knowledge of facts, of data, of relations, of properties, of structures, etc. is sufficient to get a solution. Especially "challenges" are problems of Type

V. To solve these problems we need a new idea or a "cognitive jump", "a divergent or creative thinking is necessary" (KIENEL, p. 122).

2. Two types of Vorstellungen

"Vorstellungen" are like "Subjective Domains of Experiences"[3] (BAUERSFELD 1983), they are personal and individual. The goal of mathematics education is to develop mathematical "Vorstellungen" which are extensive and effective, which are rich and flexible. We distinguish two kinds of "Vorstellungen", which we call ***spontaneous*** or ***common-sense Vorstellungen*** and ***reflective Vorstellungen***. Thus we refer to a polarity in thinking which already was discussed before by many other authors[4].

"Reflective Vorstellungen" may be regarded as an internal mental copy of a net of knowledge, abilities, and skills, a net of facts, relations, properties, etc. where we have a conscious access to. Reflective Vorstellungen mainly are the result of a teaching. The development of reflective Vorstellungen certainly is in the center of mathematics education. Here a formal, logical, deterministic, and analytical thinking is the goal. To reflect and to make conscious are the important activities. We more or less do not realize or even ignore or suppress intuitive or spontaneous ideas. A traditional mathematics education does not emphasize unconsciously produced feelings or reactions. In mathematics education there is no space for informal pre-reflections, for an only general or global or overall view, or for uncontrolled spontaneous activities. Guess and test or trial and error are not considered to be a valuable mathematical behavior. But all these components are necessary to develop spontaneous or common-sense Vorstellungen. These Vorstellungen mainly develop unconsciously or intuitively.

Both types of Vorstellungen together form individual "Subjective Domains of Experiences" (SDE). For a well developed and powerful SDE both is essential, a sound and mainly intuitive "common-sense" and a conscious knowledge of rules and facts. Both aspects belong together like the two sides of a coin. And whenever necessary the individual must be able, often unconsciously, to switch from the one side to the other. But the way of looking at things is different. Spontaneous Vorstellungen and reflective Vorstellungen often interfere, positively or negatively. The view on facts, relations, or properties "suddenly" changes.

There also is a competition between different SDEs to become dominant when a new problem is presented. Intuitively, the then chosen SDE often remains dominant even when conflicts arise. The individual rather prefers to ignore the conflict than to modify the SDE or to adopt another SDE. And in mathematics education it is quite natural that an analytical-logical behavior remains dominant and that conflicting common-sense experiences or spontaneous ideas get ignored[5]. The chosen SDE even then remains dominant when the reflective Vorstel-

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[3] "Subjektive Erfahrungsbereiche" in German

[4] e.g. VYGOTZKI talks about spontaneous and scientific concepts, GINSBURG compares informal work and written work, or STRAUSS discusses a common sense knowledge vs. a cultural knowledge. STRAUSS (1982) especially has pointed out that these two types of knowledge are quite different by nature, that they develop quite differently, and that sometimes they interfere and conflict ("U-shaped" behavior).

[5] There are interviews where children give different results to the „same" problem. For example adding "50+25" in a "number world" may have a different result than adding the same numbers in a "money world". Those children often do not see a conflict, because unconsciously they just react in two different SDEs and "in mathematics it is different".

lungen obviously are not sufficient to solve the problem. Then the related rules and procedures just get reduced or simplified or get replaced by easier mechanisms.

Summarizing, we distinguish two types of internalizing our experiences from interacting with "Darstellungen". On the one hand we develop conscious reflective Vorstellungen and on the other hand we (mainly intuitively) create spontaneous Vorstellungen. Related to the momentarily situation both types together create or modify an individual Subjective Domain of Experiences in which this situation is imbedded then. To develop more balanced both types of Vorstellungen we must further individual and social abilities, we need challenging problems, and the nature of activities in the class room must integrate more spontaneous ideas and more (unconscious and intuitive) common-sense knowledge.

3. How to further "Challenging Mathematics"?

The emphasis of the traditional mathematics teaching obviously lies on the development of powerful and conscious reflective Vorstellungen. Overemphasizing the algorithmic and procedural approach we must face the danger that our children get trained in skills but not in getting enough insight. "The importance of the ability to serve as a poor imitation of a $4.95 calculator is rapidly declining" (KAPUT).

According to KIENEL we must further the abilities to solve problems of Type V. In other words, we will need more gifted and/or creative children. But how to further giftedness and creativity? We first will try to give some kind of definition.

According to KAEPNICK (1998) being ***mathematically-gifted*** is the potential for special abilities in the mathematical area. Characteristics[6] of mathematically-gifted children are:

- The ability to remember
- The ability to structure
- The ability to switch levels in representation
- Being reversibility capable
- The ability to convert
- Having spatial thinking
- Being mathematically sensitive
- Being original and having fantasy

But what does ***creativity*** mean in this context? Many experts from different disciplines give various descriptions, but there is no standardized answer. Studying the diverse descriptions we can detect some common aspects. We need more than reflective Vorstellungen (not only "simply repeating other's old tricks"). Flexible thinking is demanded (KIESSWETTER 1983), especially in two complementary modes (BISHOP 1981, KRAUSE 1999). And now we should add, we need flexible thinking with regard to reflective Vorstellungen **and** with regard to spontaneous common-sense Vorstellungen **and** we need a flexible thinking to combine or interweave or expand these two types of Vorstellungen[7].

[6] investigations with children in the third and forth grade of primary schools (age 8 - 10).

[7] in other words, according to TALL, GRAY and others, we claim to further the development of "procepts".

Flexible thinking includes independence. We must not rely on a few dominant SDEs, we need the chance to experience, to construct, and to reflect many divergent SDEs. We then can detect and discuss analogies and differences and multiple classifications. Reflecting these activities we gain more insight and reduce the complexity. A social communication is the vehicle to widen consciously the horizon.

To further creative thinking in mathematics education we need more than powerful reflective Vorstellungen. Intuitive and spontaneous components are necessary. Each SDE is a mixture which allows different ways of looking at things. A balance between a reflective arguing and a common-sense thinking must become the goal in the class room: Let us start with a typical situation and try to collect and to discuss then all the individual SDEs coming up. Let us try not to separate artificially our daily life knowledge and experiences from the development of the "scientific concepts" in mathematics education. A creative mathematics teaching needs additional aspects:

- We need more spontaneous ideas and more (unconscious and intuitive) common-sense knowledge,
- we must further individual and social abilities (exchange of experiences and ideas; via communication intuitive and spontaneous ideas become conscious),
- we need ***deep*** problems (solutions on various levels are possible),
- we need ***broad*** problems (various types of Darstellungen to describe solutions are possible).

When we put together these two lists for describing giftedness and creativity we may get hints how to emphasize more "Challenging Mathematics" or in other words how to present mathematics to the students more challenging.

4. Examples stimulating spontaneous Vorstellungen

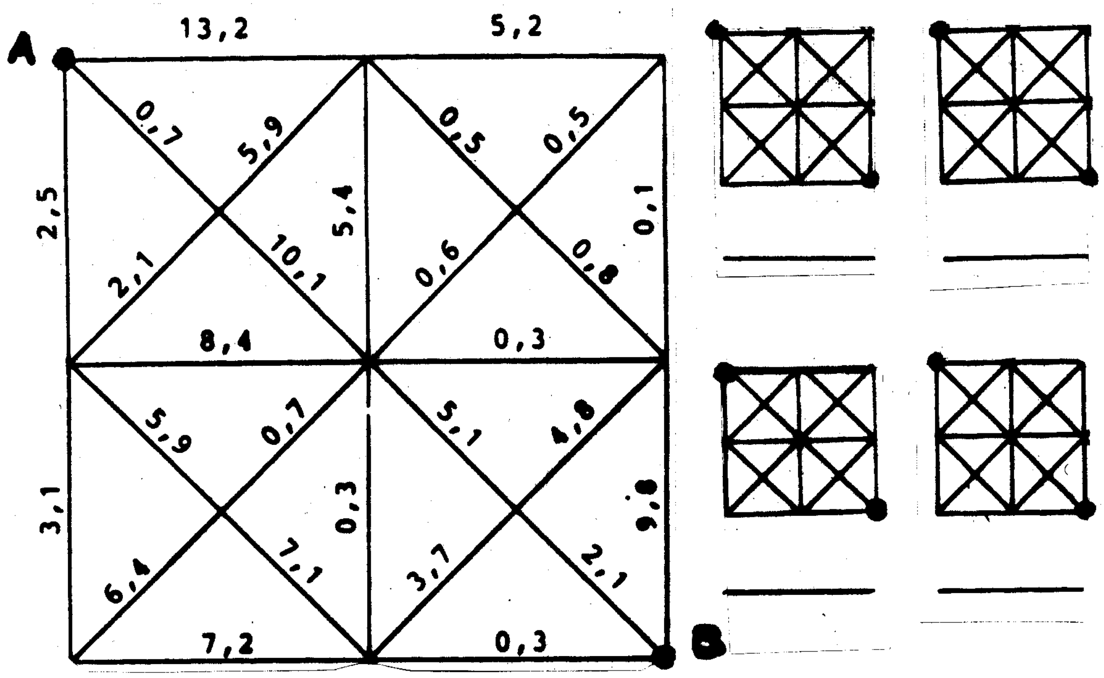
Reflecting the term *challenging mathematics* usually we think on "difficult" problems, "difficult" with respect to an individual or a group. But we should specify. The problems may be "difficult" with respect to the mathematical structure behind and/or to the mainly unconsciously existing problem solving habits, experiences, and practices of the problem solver. These habits and experiences pre-determine an appropriate development of necessary cognitive processes. To further *challenging mathematics* then means, we also must further adequate problem solving habits and experiences. Especially we must concentrate on and train activities which develop spontaneous and common-sense Vorstellungen. And to reach this goal we need problems where the mathematical difficulty is not too demanding.

We will present some examples. Interacting with the problems in 4.1 to 4.4 will stimulate a diversity[8] of spontaneous Vorstellungen which must be realized, verified/falsified or specified. Examples 4.1 and 4.4 are *deep* and *broad* because they are open to various interpretations. In 4.1 the instructions deliberately are left vague and in 4.4 first the reflective descriptions must be transformed into common-sense Vorstellungen. Examples 4.2 and 4.3 concentrate on the development of spontaneous Vorstellungen about functional properties via guess-and-test. This is an intermediate step to develop first intuitive Vorstellungen before we then start to discuss these experiences and try to describe them in algebraic notations (transforming intuitive experiences into reflective Vorstellungen).

[8] We suggest that the reader works on each of the four problems to observe his/her mental processes during that problem solving process. Which Vorstellungen became conscious and changed? How? Why?

4.1. Decimal Grid

Select a path from A to B. Change the direction at each crossing. Multiply (with a calculator) the numbers of each step you go. Find the path with the smallest product. You have 4 trials.



At a first glance, the problem is easy, let's start immediately: Take always the smallest number, write down your result. But then:

- How to find another better path?
- Are there rules to find the best path?

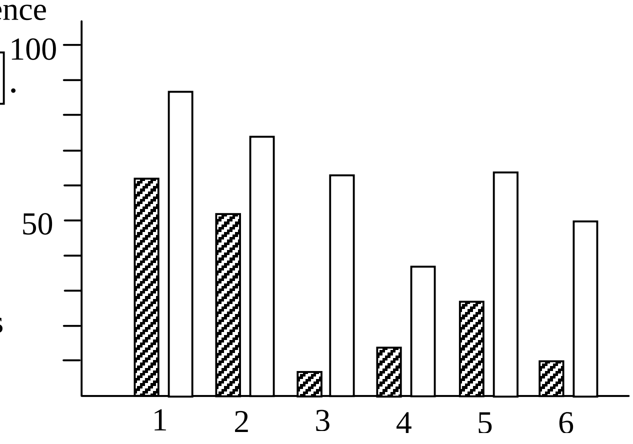
Cognitive jumps (SDEs get changed):

- Multiplication not always makes bigger
- More factors may give a smaller product
- Running in a circle forth and forth (i.e. … 0.3 x 0.8 x 0.6 x …)
- infinite path (→ intuitive concept of limit)

4.2. Teaching Percentages

There are calculators which work syntactically like we speak in our daily life: "635 + 13 % = ..." needs the key stroke sequence

[6] [3] [5] [+] [1] [3] [%] [=].



We taught percentages with the percent key, without using formulae or reverse functions or algebraic transformations of formulae. If necessary the missing values had to be guessed and verified by pressing always the same key stroke sequence from above. The students became excellent in guessing each value and they developed an astonishing *%-feeling*. We administered the same test with 6 problems in our experimental group (white bars, N ≈ 250) and in a control group with the traditional "reflective" approach (N ≈ 500, dark bars).

4.3. One-Way-Principle

Our method to use always the same key stroke sequence, eventually by guessing and testing the missing variables, we have called *One-Way-Principle* (OWP). The OWP is an intermediate step between simple examples with easy numbers and the algebraic generalization with formulae or functions and reverse functions and algebraic transformations. The OWP is a method to develop intuitive and spontaneous Vorstellungen about the relations between and about the order of magnitude of the many variables. Using guess and test develops an intuitive concept of *percentages*. The students spontaneously can guess quite exactly the result already before they start their computation.

A similar approach is possible to teach the topics *interest, compound interest, growth and decay*, and others. We urge our students to write protocols from their guess and test work because these protocols are an excellent picture of their (mainly intuitive) *Vorstellungen.* Discussions then can bring the shift from an unconscious feeling to a conscious insight. We also used the OWP to teach the concept of *functions* (MUELLER-PHILIPP 1994). The traditional school curriculum has had not much success in developing a deeper understanding between the gestalt of a graph and the related algebraic term. Using guess and test with computers we developed that missing link for linear and quadratic functions: Our students very easily could sketch the gestalt for a given term and write down a term for a given graph.

4.4. Imagine Solids

Given is a solid where the base and the upper face are parallel and congruent. Describe the solid. Base and upper face are regular n-polygons, describe again. Assume the solid is not a prism, but base and upper face still are regular n-polygons, describe. All side faces are regular triangles, describe now. Which is the name of the solid when also the base is a regular triangle? – Many SDEs and the relations between them will be in the center of the discussion.

5. Assessment and Teacher Training

Teachers should be trained to identify their students' Vorstellungen. But observing *Vorstellungen* is a contradiction in itself, we only can see *Darstellungen* which we can interpret. A lot of cases are necessary where the *Darstellungen* shown by a learner allow some interpretations. Only with skilled interviews we might back or modify our first interpretations.

Thus in teacher education we go into schools and observe with a small group. In addition an experienced mathematics teacher conducts the video camera to interesting events. After the lesson (of about 40 minutes) we need at least two or three hours to analyze our observations. First each teacher student reports on his/her own observations and interpretations. Then we come back to specific situations documented on the video clip. We discuss and interpret, sometimes we repeat a single video situation for more then ten times. And suddenly we get an explanation for a certain behavior and we can start to discuss the *why* and *how*.

But there also is the possibility to ***measure Vorstellungen*** up to some degree by multiple choice tests. The most demanding part to develop these tests is the development of reasonable answer possibilities. Each of the given multiple choice answers should be a *Darstellung* of a different (correct or incorrect) *Vorstellung*. Then not "always one of the three given answers" is correct, but the following rules apply:

- For each question as many "reasonable" answers as possible are allowed
- For each question the subject does not know how many answer possibilities are correct and how many are incorrect, especially
- for each question all answer possibilities might be correct, but also

- for each question all answer possibilities might be incorrect.
- Correct answers will bring positive scores, incorrect answers will bring negative scores.
- The scores are weighed, summing up all scores for correct answers will bring ***+100%*** and summing up all scores for incorrect answers will bring ***–100%.***
- The only hint the subjects get is the total of correct answering possibilities and the total of incorrect answering possibilities.

To develop such a test is a very demanding task, it needs experienced specialists. Clinical interviews with students during their problem solving activities are very helpful. We have practiced these tests now since more than twenty years, examples may be given at the conference. (Examples will not be published, otherwise these examples cannot be used any longer for a next test.)

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