Challenging mathematics: Challenging to whom? And to what end?

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Abstract: This article focuses on two particular aspects of the general theme: Who is it that should find our mathematical activities “challenging”? And what are we trying to achieve when we devise “challenging mathematics in and beyond the classroom”? We concentrate on the first of these two questions – with reference to two examples from the UK. However, the form these examples take is determined in part by the second question, so we begin by indicating why the second question seems important, even if we can only hint at a tentative answer.

What are we trying to achieve?

This ICMI Study has to take as given what constitutes “school mathematics” in a particular country, and then consider how, in different contexts, *supplementary provision* of various kinds can contribute to students’ (and teachers’) experience of mathematics. The effort, imagination and devotion that go into providing such supplementary challenges is truly remarkable. But in implicitly recommending these examples for the consideration of others, it is important that we should consider how one might assess the effectiveness of the challenges we recommend.

We are clearly convinced of the benefits of the challenges we struggle to provide. But we operate on the margins of the educational system. We can hardly be held responsible for wider social trends, yet we should perhaps recognise that the impact of our efforts may be more complicated, and less satisfactory, than we often admit. To illustrate this we consider just one example – though one which has been replicated elsewhere.

In the UK, the national pyramid of popular mathematics competitions began to expand from 1988. As in Australia and France, the growth has been phenomenal:

	1987	2005
Numbers taking part in the secondary “mathematical challenges” in the UK	**7 000**	**600 000**

Yet over the same period the number of students choosing to study mathematics at age 16-19 has collapsed:

	1989	2005
Number of 16 year olds choosing to continue studying mathematics in the UK	**85 000**	**53 000**

One is therefore obliged to ask what this growth in mathematics competitions (which are called “Mathematical Challenges” rather than “competitions”) has actually achieved?

Many of those who contribute (mostly voluntarily) to providing such additional challenges view such a question as unwelcome. This is entirely understandable: most contributors do not feel responsible for national statistics, and do not see themselves as being in a position to influence the relevant parameters. They love mathematics, and enjoy passing on that appreciation to others through whatever challenges they contribute to. They might reasonably argue that, if the number of students who are willing to translate their adolescent enjoyment into continued study of mathematics as young adults is in decline, how much more important must it be to provide suitable challenges for those who *are* willing to make this transition!

However, the above statistics are too striking to be shrugged off in this way: we have clearly failed to generate the kind of response many of us assumed would follow automatically from such “mathematical challenges in and beyond the classroom”.

The goal of this paper is to address some of the complications which need to be faced, to hint at the need to revise some of our basic assumptions, and to give two examples that respond in different ways to this challenge. The first example (a student problem solving journal) illustrates what such a revision implies when working with that minority of students who are, in some sense, potentially committed; the second example (a projected supplementary extension programme) highlights the level to which one has to "sink" in order to bridge the gulf which currently prevents most students ever learning what elementary mathematics is about.

Many people in many countries are doing valiant work to devise "challenging mathematics in and beyond the classroom". However, there is often confusion about the kind of "challenge" that is most appropriate. The most widely acclaimed examples of "challenging mathematics" involve problems or activities that are deliberately *hard*, and which presume a level of commitment on the part of the students and their teachers which in many countries is so rare as to make the model highly *exclusive*. If the associated event or publication survives, the level achieved by the most successful participants may be so impressive as to lead the casual observer to judge the event a "success". But what if (as can be true with national olympiads) the outcome is that large numbers of genuinely talented students despair of reaching the required "elitist" level, and conclude wrongly that they are not cut out to study mathematics? In assessing the educational effectiveness of any "challenge" we are obliged to take into account possible negative effects, not just the visible "successes".

As a tentative answer to the question "What ought we be trying to achieve?" we therefore suggest that, at the very least, we should look for activities which improve mathematical culture at secondary level in such a way as to increase the number of students who continue with the study of mathematics through to age 19 *and beyond*.

Who is it that should find these mathematical activities "challenging"?

The meaning of the expression "challenging mathematics" depends on the context within which one is working. "Mathematics" has a distinctive universal character, and this has to be respected by any meaningful "challenge". But to be effective in a particular context we have to find an approach that resonates with the target audience, and with specific local circumstances that may include an impoverished official curriculum, teachers under considerable pressure and with a narrow view of elementary mathematics, and regular superficial high-stakes assessment. Thus, while we are obliged to respect the character of elementary mathematics, we cannot presume that there is some universal interpretation of what constitutes "challenging mathematics"; indeed, an effective local strategy may run counter to the assumptions of one or more interested parties (whether academic mathematicians, or olympiad enthusiasts, or mathematics educators, or those who draft official curricula and national assessments).

Provided this is understood, we could all benefit from a communal attempt to identify a shared interpretation of what constitutes the "distinctive character" of mathematics *at school level*. Successful mathematics teaching requires care in shaping and sequencing individual topics so as to allow pupils to accumulate a bedrock of reliable routines that optimises their *long-term* technical development and growth. But if students are to appreciate what mathematics is about, they also need to experience elementary mathematics in a way that nurtures the twin perceptions

- that everything has to "make sense" (within the traditional objective framework of mathematico-linguistic reasoning), and
- that mathematics proper *begins* only when one is expected to cultivate a flexible willingness to select and to coordinate simple routines to tackle what at first sight seem elusive problems, that this process cannot be made deterministic, and that it is at first bound to be unsettling.

An abstract interpretation of the word “challenge” is to some extent captured by the second bullet-point; but the kind of practical projects or activities that are needed, or that might be understood and welcomed, is highly dependent on local circumstances!

Ideally, the official training provided for teachers and the consequent respect for their professionalism would suffice to ensure that “challenge” constitutes a regular part of pupils’ everyday classroom experience. But in many countries the reality at the chalkface is very different. In the UK the real challenge of initial and in-service training has been neglected, and a severe shortage of suitably qualified mathematics teachers has been allowed to accumulate. Instead of addressing these complex and demanding issues, politicians and civil servants have tried to “drive up standards” by imposing central controls – with the consequence that traditional “mathematics teaching” has been largely replaced by an anti-mathematical game of “test preparation”. Responsibility for addressing the long-term professional challenge of developing technique rooted in understanding has been replaced by desperate attempts to satisfy political demands for *short-term* “success”. In such a climate emphasis on objective “sense-making” and on a flexible willingness to grapple with multi-step problems have no place.

PSJ: a problem solving journal for secondary students

The dilemmas facing those who try to use “challenging mathematics” for the public good come clearly into focus when one considers problem solving journals for students. In countries with a sufficiently rich mathematical culture at secondary level one still finds wonderful examples – such as *Kvant* (in Russia) and *Kömal* (in Hungary) – though several similar publications that flourished in Eastern Europe up to 1989 have found it impossible to survive without state support. Thus *Kvant* and *Kömal* cannot simply be copied: in particular, attempts to replicate their tradition of “serious problem solving” in western countries have faced difficulties – the most notable example being *Quantum* in the United States.

These two examples seem to be widely known and respected because *their mathematical level is sufficiently high* to impress those who judge them *in isolation*, knowing nothing of local circumstances. In contrast, relatively little is heard about examples, which may have survived for an impressive length of time but which are on a more modest mathematical level. For example, *Mathematical Digest* in South Africa has survived for nearly 40 years by combining very short articles on mathematical topics with a very different mix of mathematical problems.

PSJ could be described as seeking to combine the mathematical and social philosophy of *Kvant* with the honest realism of *Mathematical Digest*. Part of the material used should be accessible to the top 5-10% of secondary students in a country where the official mathematics curriculum and assessment have been extensively “de-mathematicised”, where teachers no longer see themselves as having the freedom to supplement the official diet *in the classroom*, but where some still recognise the need to provide supplementary challenges for some of their students. In such a context, where most students in the target population have no experience of “traditional” problems, there is no need to aim for “originality”: the problems used only need to be unfamiliar to those who choose to tackle them! Thus we try to take from *Kvant* and *Kömal* their “aesthetic sense” of what constitutes a good problem (interpreted on a lower level) without making the assumption that readers have a sufficiently rich experience of the mathematical literature to render traditional “chestnuts” unusable. We also embrace the goal of offering young mathematicians the opportunity to belong to a “virtual community” of adolescent problem solvers – by naming successful solvers and by making an effort to attribute especially elegant solutions to the students responsible. In this spirit each issue contains an extended discussion of problems from the previous issue, so that the journal could constitute an extended “problem solving course” for those who continue to contribute over a number of years.

Some may find these admirable goals, yet still find the structure adopted in ***PSJ*** surprising. This structure assumes (on the basis of having worked with hundreds of groups of “able” students and dozens of groups of teachers, and having marked tens of thousands of olympiad scripts over the last 15 years) that most “able” students in the UK have very little experience of the simplest traditional multi-step problems, that teachers may recognise that their best students need something more demanding yet have a very hazy idea of what elementary mathematics is really about, and that most teachers see “failure” as entirely negative, so are understandably protective towards their students.

PSJ problems are therefore provided on two levels (“E-type” and “H-type”) and for three age groups (“Junior” age 11-13, “Intermediate” age 14-16, and “Senior” age 17-18).

Roughly speaking, “E-type” problems are those which should ideally be part of everyday classroom culture, but which in the UK in 2006 have become almost totally invisible. They are included both to provide a resource which can be used by a relatively large number of subscribers, and as a necessary stepping stone for those potentially able students who might be enticed to enter more seriously into these “easier” mathematical waters, and later move on to the harder problems. E-type problems use a very simple format to convey to both students and teachers the crucial difference between one-step routines and multi-step problems. For E-type problems the editors ask for “Answers only”, on the grounds that this is likely to prove less daunting to those who are totally unfamiliar with the art of writing mathematical solutions. The problems may look trivial to the initiated, but they are certainly not seen in this way by those in the target audience. The ingenuity being cultivated is sufficiently tested by the challenge to find the right answer – which many potentially able youngsters can manage even where they would have no idea how to present a written solution. E-type problems use very simple material to engage large numbers of students in a kind of thinking which is quintessentially mathematical, and which is (sadly) at a level way beyond what is expected of them day-to-day.

Those who need the additional challenge of presenting full written solutions are encouraged to concentrate their efforts on the second type of problems – the “H-type” problems – and to write up and send in their solutions. H-type problems require a greater knowledge base, a more sophisticated ability to calculate, or a willingness to engage in conceptual analysis. The following sample material combines problems from Issue 007 and solutions to these problems given in Issue 008.

Issue 007

Junior E

1. My Christmas present cost £1 plus half its price. What did it cost?

2. In how many ways can the two missing digits be filled in to make this equation true?

“□ 3” ÷ □ = “6 remainder 5”

3. What number is halfway between $^1/_5$ and $^1/_7$?

4. If everyone including me gets five apples, there will be six left over. If everyone except me gets six apples, there will be five left over. How many people are there?

5. Granny Smith drove at 30mph for 30 minutes, then at 40mph for 40 minutes and finally at 50mph for 50 minutes. What was her average speed for the whole journey?

6. Find a pair of prime numbers which, between them, use each of the four digits 1, 2, 3, 4 exactly once. How many different pairs are there?

Senior E

1. Solve these simultaneous equations in your head:

$$6753x + 3247y = 26\,753, \qquad 3247x + 6753y = 23\,247.$$

2. Find all x such that $\sin 20^\circ + \sin 40^\circ = \cos x^\circ$.

3. ΔABC has $AB=5$, $BC=7$, $CA=8$. The point K on AC is such that $CK=3$. Calculate $<CKB$.

4. In sending 6 Christmas cards to 6 people, I first address the 6 envelopes; then I write the 6 cards; finally I insert the cards in the envelopes. In how many different ways can the cards be inserted in the envelopes so that each card goes in the *wrong* envelope?

> The E-type problems and H-type problems are always separated by an *Interlude* – a short extract from a popular mathematics book. This is chosen to try to make readers aware of the

existence of "popular" mathematical literature, and to tempt some of them to get hold of the books from which the extract is taken and to read more widely.

Junior H

1. Find all solutions to this long multiplication.
Explain clearly how you can be sure that you have found all solutions.

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      2 * *
    ×   9 *
    -------
    * * * 7
  * * 4 9 *
  ---------
  * * * * *
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2. I am thinking of a positive integer. When divided by 6, the remainder is 5; when divided by 7, the remainder is 4; when divided by 8, the remainder is 3. How many such integers are there ≤ 1000?

3. A, B, C, D, E, F are six different towns in order along a winding road. The total distance from A to F is 101 miles, the distance from B to E is 78 miles, from A to D is 83 miles, and from C to F is 42 miles. How many possible values are there for:
(a) the distance from C to D? (b) the distance from B to C?

4. A power company offers three packages, each with guaranteed "savings": one saves 30% of current bills, the second saves 45% of current bills, while the third saves 25% of current bills. If all three packages could be commbined, what percentage could you save on your current bills?

5. You can use matches to make regular polygons as long as you use the same number of matches for each edge. What can you say about the number of matches in a matchbox if, for each pair of polygons in the following list, it is possible to make both polygons at the same time using up *all* the matches in the box each time?
an equilateral triangle, a square, a regular pentagon, a regular hexagon

Senior H

1. Factorise $(a-b)^3 + (b-c)^3 + (c-a)^3$.

2. (a) Seven matchboxes are arranged in a row: the first contains 19 matches, the second 9, the third 26, then 8, 18, 11, and 14. Matches may be moved from any box to an adjacent box. The goal is to finish with equally many matches in each box. What is the least number of matches one must move?
(a) Suppose the matchboxes are arranged in the same order, but in a circle, with the box containing 14 matches next to the box with 19. How many matches must now be moved?

3. You are given n glasses – all initially "bottom down" position. A "move" involves choosing some k glasses and turning them all over. Decide (with proof) for which values of n, k it is possible to find a sequence of such moves which results in all n glasses being inverted.

4. Two congruent rectangles $ABCD$ and $WXYZ$, with $AB < BC$ and $WX < XY$, are arranged with $ABCD$ on top of $WXYZ$ so that A lies somewhere on the short side WX, B lies on the long side WZ, and BC passes through the point Y. Let $\angle WAB = \theta$.
Find, and solve (as simply as possible) a polynomial equation that determines the angle θ when the two rectangles are A4 sheets of paper – that is, when the ratio $AB:BC = 1:\sqrt{2}$.

5. $ABCD$ is a cyclic quadrilateral whose circumcircle has diameter 50. If $AB = 15$, $BC = 15$, $CD = 50$, find AD.

SOLUTIONS TO H-TYPE PROBLEMS IN ISSUE 007

Senior-H

Sergey Goryunov (Grange S) solved all six problems – and did so in fine style. Ian Fraser (Torquay Boys GS) and Nimrod Gileadi (Cherwell S) solved five problems, while Anna Hall (Hills Road SFC), David Weston (Cherwell S), and Lee Zhao (Nottingham HS) solved four and a half. James Folliard (Cheltenham C) and Jiexi Zhao (Cherwell S) solved four problems while Tom Anthony (Hampton S) and Kathryn Atwell (Emmanuel C) solved three and a half.

1. *Factorise* $(a-b)^3 + (b-c)^3 + (c-a)^3$.
[Though a pleasing number produced the result, it was not always clear how it was found. In general, you must learn to resist the urge to "multiply out", and look for ways of factorising *from the outset*.]
We use the standard factorisation: $x^3 + y^3 = (x+y)(x^2 - xy + y^2)$ with $x = a-b$, $y = b-c$.
$[(a-b)^3 + (b-c)^3] + (c-a)^3 = (a-c)[(a-b)^2 - (a-b)(b-c) + (b-c)^2] + (c-a)^3$

$= (a - c).[(a - b)^2 - (a - b)(b - c) + \{(b - c)^2 - (c - a)^2\}]$
$= (a - c)(b - a)[(b - a) + (b - c) + (b - 2c + a)]$
$= 3(a - b)(b - c)(c - a).$

2. *(a) Seven boxes in a row: the first with 19 matches, the second 9, then 26, 8, 18, 11, and 14. Matches may be moved from any box to an adjacent box. What is the least number of matches that must be moved to finish with equally many matches in each box.*
19 + 9 + 26 + 8 + 18 + 11 + 14 = 105 = 7×15, so each box must finish with 15 matches.
There is only one way to fill up (or to empty) the end boxes, so we must move 4 from the 19 to the 9; then move 2 from the 26 to the 9+4; and so on, moving 4 + 2 + 9 + 2 + 5 + 1 = **23** matches altogether.
(b) Suppose the boxes are in a circle, with 14 next to 19. How many matches must now be moved?
It now matters where we start and how many matches we move.
Observe that we can manage by moving just 19 matches: 19 gives 2 to 9, and 2 to 14 – who passes on 1 to 11; 26 gives 4 to 9 and 7 to 8; 18 gives 3 to 11 – a total moved of 2 + 2 + 1 + 4 + 7 + 3 = 19.
Is this least? At first, the boxes differ from their final states by 4, 6, 11, 7, 3, 4, and 1. So we need 4+6+11+7+3+4+1 = 36 changes.
Each move changes the state of **two** boxes, so we have to move at least $^{36}/_2$ = 18 matches.
If moving *exactly* 18 matches is possible, *no match moves more than once*; i.e. all corrections can be done by moving matches from the three "over-full" boxes to their *immediate* neighbours: so 19 gives just 1 to 14, and hence 3 to 9; so 26 must give just 3 to 9+3, and 8 to 8 – which is then over-full!
Hence 18 moves is impossible, so 19 is the smallest possible number of matches moved.
[An alternative approach was suggested by Angela Li and Dominic Yeo. Suppose that 19 passes x matches to 9; then 9+x passes x – 6 matches to 26; then 26 + (x – 6) passes x + 5 matches to 8; then 8 + (x + 5) passes x – 2 matches to 18; then 18 + (x – 2) passes x + 1 matches to 11; then 11 + (x + 1) passes (x – 3) matches to 14; finally 14 + (x – 3) passes x – 4 matches to 19.
"The total number of matches moved" = $f(x) = |x| + |x - 6| + |x + 5| + |x - 2| + |x + 1| + |x - 3| + |x - 4|$.
We need to find the minimum of this function (compare this with problem 3). The function is "concave" and the minimum occurs when x is an integer, so we can use trial to find $x = 2$, $f(x) = 19$. The method is developed in *Functions and graphs* by I.M. Gelfand, E.G. Glagoleva, and E.E. Shnol (Dover).]

3. *n glasses are initially all "bottom down". A "move" involves choosing k glasses and turning them over. For which values of n, k is there a sequence of moves which results in all n glasses being inverted.*
We follow Sergey Goryunov. **If *k* = *n***, all can be inverted in one move. So **assume *k* < *n***.
If *k* is odd, arrange the glasses in a circle: $G_1, G_2, G_3, \ldots, G_n$, and turn over $G_1, G_2, G_3, \ldots, G_k$, then $G_2, G_3, G_4, \ldots, G_{k+1}$, then $G_3, G_4, G_5, \ldots, G_{k+2}$, ... and finally $G_n, G_1, G_2, \ldots, G_{k-1}$. In this sequence, each glass is turned over k times, so finishes bottom up".
If *k* is even, consider first the case where ***n* is even**. (The same argument works for k odd and n odd, but this has just been dealt with.) Since $k < n$, we know that $k \le n - 2$. First invert $G_1, G_3, G_4, \ldots, G_{k+1}$, then invert $G_2, G_3, G_4, \ldots, G_{k+1}$. These two moves invert G_1, G_2, and leave the remaining $n - 2$ glasses "bottom down". So we have reduced the original problem with n, k to the same problem but with $n - 2$, k. If we repeat such moves $(n - k)/2$ times, we eventually reach the case where "$n = k$" and finish in one more move – using $(n - k) + 1$ moves altogether.
If *k* is even and *n* is odd, then the glasses cannot be inverted. The proof should by now be familiar.
Each glass must be inverted an odd number of times, so the total number of "single glass inversions" must be odd.
However, k (even!) glasses are inverted each time, so the total number of "single glass inversions" must be even – a contradiction!
So there exists a sequence of moves to invert all the glasses if and only if n is even or k is odd. **QED**

4. *Two congruent rectangles ABCD and WXYZ, with AB < BC and WX < XY, are arranged with ABCD on top of WXYZ so that A lies somewhere on the short side WX, B lies on the long side WZ, and BC passes through the point Y. Let <WAB = θ. Find, and solve (as simply as possible) a polynomial equation that determines the angle θ when the two rectangles are A4 sheets of paper: that is, when the ratio AB:BC = 1:√2.*
[Several solvers produced a quartic and an approximate solution.]
Lee Zhao went one better than last time and produced a beautiful quadratic with an exact solution.
Let $AB = WX = 1$. Then $AX = 1 - \cos\theta$,. $\therefore$ $AY^2 = AX^2 + XY^2 = (1 - \cos\theta)^2 + 2$.
$BZ = (\sqrt{2} - \sin\theta)$, so $BY^2 = 1 + (\sqrt{2} - \sin\theta)^2$
$\therefore$ $AY^2 = AB^2 + BY^2 = 1 + [1 + (\sqrt{2} - \sin\theta)^2] = (1 - \cos\theta)^2 + 2$
$\therefore$ $(\sqrt{2} - \sin\theta)^2 = (1 - \cos\theta)^2$
$\therefore$ $(\sqrt{2} - \sin\theta) = (1 - \cos\theta)$ (since $0° < \theta < 90°$, and $0 < \sin\theta, \cos\theta < 1$)

Let $s = \sin\theta$.
$\therefore\ \sqrt{2} - s = 1 - \sqrt{(1 - s^2)}$
$\therefore\ s^2 + s(1 - \sqrt{2}) + 1 - \sqrt{2} = 0$
$\therefore\ s = [\sqrt{2} - 1 + \sqrt{(3 - 2\sqrt{2} - 4 + 4\sqrt{2})}]/2$ (since $s > 0$)
$= [\sqrt{2} - 1 + \sqrt{(\sqrt{8} - 1)}]/2$. **QED**

Nimrod Gileadi came up with a more direct solution:
Let *AD* cut *XY* at *M*, and the line through *M* parallel to *AB* meet *BC* at *K*. Pythagoras in ΔAXM gives:
$2(1 - \sqrt{2}.\mathrm{cosec}\theta) + XA^2 = -2\mathrm{cosec}\theta.\cot\theta + (\cot\theta)^2$
$\therefore\ 2(\sin\theta - \sqrt{2}) + XA^2.\sin\theta = -2\cot\theta + (\cot\theta)^2\sin\theta$
But $\cot\theta = BZ = WZ - WB = \sqrt{2} - \sin\theta$
$\therefore\ 2(\sin\theta - \sqrt{2}) + XA^2.\sin\theta = 2(\sin\theta - \sqrt{2}) + (\sin\theta - \sqrt{2})^2.\sin\theta$
$\therefore\ XA = \sqrt{2} - \sin\theta$ (since $XA > 0$)
But $XA = 1 - \cos\theta$
$\therefore\ \sin\theta - \cos\theta = \sqrt{2} - 1$
$\therefore\ \sqrt{2}.[\sin\theta.\cos 45° - \cos\theta.\sin 45°] = \sqrt{2} - 1$
$\therefore \sin(\theta - 45°) = (\sqrt{2} - 1)/\sqrt{2}$. **QED**

5. *ABCD is cyclic with circumcircle of diameter 50. If AB = 15, BC = 15, CD = 50, find AD.*
We follow Kathryn Atwell: ΔCBD is right angled, so $BD = 5\sqrt{91}$, $\cos\angle BDC = {}^{\sqrt{91}}/_{10}$, $\sin\angle BDC = {}^{3}/_{10}$.
$AD = 50.\cos(\angle ADC) = 50.\cos(2\times\angle BDC)$
$\therefore\ AD = 50.(\cos^2\angle BDC - \sin^2\angle BDC) = 50({}^{91}/_{100} - {}^{9}/_{100}) = {}^{50.82}/_{100} =$ **<u>41</u>**.

***Extension mathematics*: bridging the gulf?**

Those who are committed to providing "challenging mathematics in and beyond the classroom" may feel that the sort of problems posed in ***PSJ*** could and should be enjoyed by large numbers of students. However, the reality in the UK is that the "spirit of adventure", which should underlie elementary mathematics – where basic routines are mastered *in order to combine them in slightly unexpected ways to solve simple problems* – has been almost completely lost. Instead, political and financial pressures on schools have been exerted in such a way as to remove the simplest challenge to think flexibly, to make sense of mildly unfamiliar formulations, to identify intermediate stepping stones that might lead to a complete solution, or to give mathematical reasons: tests which are obliged to "deliver year on year improvement" dare not include questions that make such demands, so they are no longer taught! There is therefore a gulf between what it is that gives elementary mathematics its "power" and "human appeal" and the way most teachers and students now perceive school mathematics.

Traditionally it may have been enough to provide challenging materials and expect interested students to take advantage of them (with the help of their teachers). But the gulf has been allowed to grow to such an extent, and the professional freedom of teachers has been so curtailed, that such a strategy may no longer be viable. One therefore has no choice but to try to devise and make available resources that struggle to link these two antithetical worlds – the rich universe of elementary mathematics and the largely de-mathematicised classrooms dominated by one-step routines and test preparation. This is what *Extension mathematics* (Oxford University Press 2007) seeks to do. Each of the three books straddles two Grades (ages 10-12, ages 11-13, and ages 12-14 – with hopes for a fourth book covering ages 13-15).

On the positive side, there is some recognition that the existing curriculum does not serve the needs of the top 25% or so of each cohort; there is even a degree of political will to address the problem. However, there is no debate about what this might require, and the policies currently proposed seem likely to make matters worse. So the ultimate success of this particular project may well be limited, but it is important to address the challenge of finding some way to help teachers and educrats rediscover the essential character of elementary mathematics.

These are not textbooks – but rather sequences of problem-sets, on a level which many will see as rather modest, but which trialling has shown is entirely unfamiliar to most students in the target population. Each item

- addresses a topic from the official curriculum, which positively cries out to be used to cultivate more mathematical thinking;
- begins with a short class activity (Problem **0**), in which a *difficult* example is used to bring out the underlying ideas of the problem-set without the teacher imparting a "rule".

A couple of edited examples from the first book (age 10-12) cannot be representative, but may convey something of the intended spirit of this project.

Example 1: When using pencil and paper for multiplication, long multiplication is the most efficient and reliable method. So you must learn to do long multiplication quickly and accurately. But when calculating mentally, keep an eye open for the possibility of "short cuts". For example:
(a) When multiplying by "1 less" or by "1 more" than an easy multiplier:
for example $30 \times 39 = 30 \times (40 - 1) = 30 \times 40 - 30 \times 1 = 1200 - 30 =$ ………
(b) Sometimes it is possible to "trade" factors to make an awkward-looking calculation easy:
for example $15 \times 26 = 15 \times (2 \times 13) = (15 \times 2) \times 13 = 30 \times 13 =$ ………
or to combine 2s and 5s to make 10 or 100:
for example $8 \times 75 = (2 \times 4) \times (25 \times 3) = 2 \times (4 \times 25) \times 3 =$ …………

0. $15 \times 75 \times 40 =$ ___________________

1. $25 \times 12 =$ ____________ **2.** $19 \times 13 =$ ____________
3. $18 \times 15 =$ ____________ **5.** $28 \times 75 =$ ____________
6. $19 \times 15 =$ ____________ **7.** $99 \times 83 =$ ______________
8. $35 \times 26 =$ _____________ **9.** $49 \times 22 =$ ______________
10. $35 \times 202 =$ ______________ **11.** $222 \times 15 =$ ______________
12. $125 \times 35 \times 52 \times 40 =$ ___________________

Example 2: If you know your tables, and think carefully what each of the equations in this section is saying, you should be able to work out the numbers that go in the boxes to make these equations correct. The first one is not too hard.

0. (a) [……] ÷ 4 = "5 remainder 1".

The next is a wee bit harder ("[…] 3" stands for a two-digit number with units digit "3").

0. (b) "[…] 3" ÷ 7 = "[…] remainder 1".

"When you divide 7 into the (unknown) two-digit number "[…] 3", you get "remainder 1".
You have to find the unknown tens digit, and how many times 7 goes into "[…] 3".

Now have a go at these.
1. [……] ÷ 5 = "3 remainder 1". **2.** [……] ÷ 6 = "9 remainder 4".
4. 56 ÷ […] = "6 remainder 2". **6.** "[…] 3" ÷ 7 = "[…] remainder 1".
7. "5 […]" ÷ 13 = "[…] remainder 7". **8.** "[…] 3" ÷ […] = "12 remainder 5".
9. How many different solutions are there to this one? "[…] 3" ÷ […] = "8 remainder 5".

Conclusions

"Challenging mathematics in and beyond the classroom" cannot simply ignore the trend in many countries for fewer and fewer students to continue studying mathematics at a higher level. Elementary mathematics has an eternal human appeal provided two conditions are met: first, basic instruction in the classroom must emphasise *connections* and must cultivate the notion that mathematics *begins* when simple routines have to be selected and combined to solve harder problems; second, the elementary mathematics we exploit to provide supplementary challenges must be seen to be "inclusive", by being presented in a form that is recognisable and appealing to the target audience. Where mathematical culture is impoverished, we must work to improve matters in the long-term; but in the meantime we have to find ways of interpreting "the spirit of mathematics" within the local context, so as to create bridges that might allow more students to progress from the existing classroom reality to an eventual higher appreciation of mathematics.