MATHS À MODELER: RESEARCH-SITUATIONS FOR TEACHING MATHEMATICS

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INTRODUCTION

Several reports from the EU and from the French Ministry of Education (Real Jantzen, 2001; Blandin, 2002; Hamelin, 2003) detect a gap between the teaching of sciences and scientific research: they recommend the involvement of professional researchers in the classroom. But all these reports recognize that the procedures for enacting such a cooperation are still to be constructed. Thus, collective efforts are required to work out ways of connecting teaching and learning on one hand and professional research on the other: didacticians and mathematicians are of course both concerned by this problematic. Our aim is actually to shorten the distance between classroom teaching and the constructs of professional mathematical research.

The context of research and the solving of open questions form the breeding ground of scientific knowledge. We shall assume that an epistemological and didactical study of what we call ‘real mathematical research-situations’ (meaning: situations which are crucial to the core of ongoing mathematical research, still partially unsolved i.e. open situations) are promising and have an innovative teaching and learning potential.

Since 1991, we have devised and studied several ‘real mathematical research-situations’ from a theoretical and experimental point of view. They have proved to be original in many ways: in the type of situations and task to be achieved, as well as in the knowledge and skills summoned and in the teacher’s position.

RESEARCH-SITUATIONS FOR THE CLASSROOM: A DEFINITION

Our “**R**esearch-**S**ituations for the **C**lassroom”, coded RSC, can be considered as a transposition of the mathematical researcher’s activity, for the class-room at different levels, giving access to the study of mathematical growth … We would like to point out that pupils and students work in group (size: 3-5 per group) on RSC (for cognitive and didactical reasons).

From our viewpoint, a RSC must fulfil the following criteria (Godot-Grenier, 2004).

1. *A RSC is akin to professional research strategy*. It must be related in some way to unsolved questions for the following reason: the student will be confronted with tough questions, putting him/her in a real research situation. Both teacher and student are exactly in the researcher’s position.
2. *The initial question should be easily accessible*. In particular, the question can be easily understood by pupils, the problem does not demand heavily formalized mathematics.
3. *Possible initial strategies are in view*, can be considered without requiring specific prerequisites.

4. *Several research strategies and several developments are possible*, from the point of view of mathematical activity (construction, proof, calculation) as well as from the point of view of the mathematical concepts involved.
5. *A solved question can possibly lead to other fresh questions*.

The main incentive in studying and implementing such situations in the classroom lies in the fact that these RSC offer an opportunity to grasp specific transversal knowledge and skills (i.e. skills and knowledge which straddle mathematics, used in the whole variety of mathematical contexts), such as the following: proving, conjecturing, refuting, creating, modelling, defining, extending but also transforming a questioning process, being able to mobilize a non-linear reasoning, experimenting, decomposing-recomposing, having a scientific responsibility. All these points may have a place in French curricula under the heading key word "scientific activity". Of course this has only partially to do with Problem Solving, but the way in which we consider RSC leads us to go beyond the frame of Problem Solving and Heuristics as defined by Schoenfeld for instance (Schoenfeld, 1985).

THE CONTEXT OF OUR RESEARCH: "MATHS À MODELER" PROJECT

The RSC we implement in classroom have been used for a long time in various workshops, from elementary school to university, and have been studied from a theoretical point of view by a group of researchers from various departments and by teachers. Since 2003, a project has been built ("Maths à Modeler"): the current team is composed of researchers both in discrete mathematics and mathematics education.

The mathematical discrete problems in the ongoing research concern mainly graph theory (colouring, extremal graph theory...) and discrete geometry (packing and covering, error-correcting codes...). The specific study of such problems brings us ideas for designing RSC (an illustration will be presented below).

The didactical problematics[1] consist in studying transversal knowledge and skills on the one hand and discrete concepts formation on the other hand. We assume that the field of discrete mathematics is actually a good environment for learning, training and popularizing. The core of the problematics of the researchers in didactic of mathematics concerns also the nature of the relation to the knowledge and the management of RSC. We should like to point out that a lot of RSC have been conducted in the classroom both by teachers and researchers, from elementary school to university level, and also for teacher training and sessions aimed at the general public.

For the conduct of our epistemological and didactical research, we enlisted several partnerships in different context (primary and secondary schools, university, scientific museum):

- teachers
    - at the IUFM: French university for teacher training,
    - at the university: center for the formation of PhD students teaching at the university;
- specific publics
    - underachieved pupils and students

[1] We use the French word "problématique" in the following sense: the problematic area of a subject.

- persons trying to reintegrate the school system
- students experiencing difficulties
- psycho-pathologic children (in this specific context, clinician psychologists and medical staff joined our team),

- and the 'general public' (we hold different events aimed at the public at large: sociologists explore the common mathematical culture and mathematical representations in every day life in cooperation with our team).

THEORETICAL BACKGROUND

We have to determine a specific theoretical background for the design and the study of such novel situations. This background is now constituted by the theory of didactical situations (Brousseau, 1997), itself inspired by the mathematical 'combinatorial game theory'.

Different theoretical points of view are represented in our research team. For instance, a theory of models, based upon Tarski's work, is developed in order to clarify the "validation" stage of the theory of didactical situations (Durand-Guerrier, 2005). For some specific RSC such as situations of definitions construction (called SDC), a more specific theoretical framework is elaborated in order to grasp the process of 'defining' (Ouvrier-Buffet 2003, 2004).

All these theoretical efforts are directed at the same object i.e.: to characterise effectively research-situations in order to implement them in the classroom (Grenier-Payan 2003, Godot-Grenier 2004, Duchet 2003), to determine all knowledge and skills, to refine the analysis of the mathematical processes involved in such situations: modelisation (Rolland, 1999), implication (Deloustal, 2004), refutation, definitions construction (Ouvrier-Buffet, 2003, 2004 & 2005).

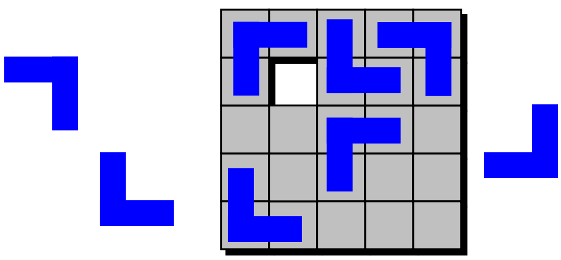
The designing of a full theoretical framework is actually at stake, for the design, the devolution and the management of RSC as a whole: such a framework takes on board both the mathematical processes involved in RSC and the teacher's position (it is actually noteworthy that the teacher has to take a specific position, similar to the posture of a researcher faced with a mathematical open question: his interventions has to be subtly characterised). Our aim is to characterize RSC, including their mathematical, didactical and pedagogical aspects: such features allow dissemination of RSC to the users (teachers for the class and wide public for popularization).

RESEARCH-SITUATIONS: SOME EXAMPLES

We will give to the reader some examples of RSC and detail some of them from a mathematical and didactical point of view.

Tiling with the triminos L

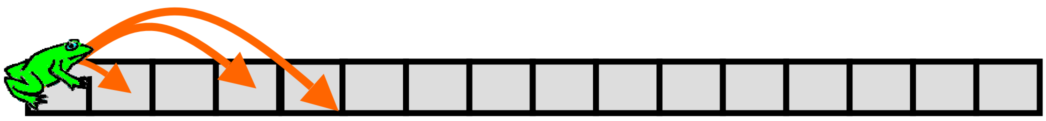
The question is to tile rectangles with one square removed, with the triminos L.



This RS has similarities with dominos tiling of troncated rectangles (see Grenier-Payan, 1998). These tiling problems mobilize the following concepts: counter-examples, proof by induction, divisibility, implication...
Tiling problems with polyminos (pieces of grid) are very hard from a mathematical and computational point of view and many conjectures are still opened; for instance, characterising polyminos which can be tiled by dominoes is not completely solved. We can find a partial result in Thurston (1990).

Frog game

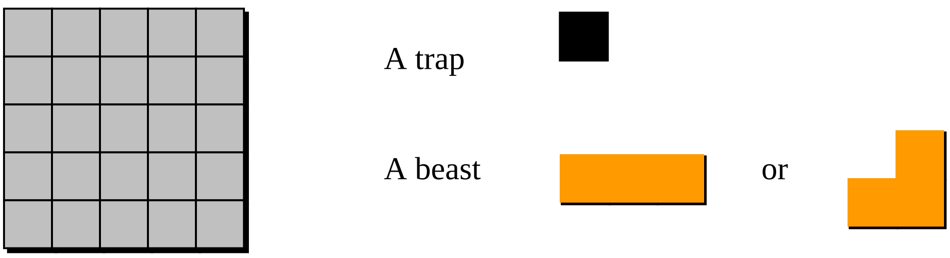
It is a two players game, which can be considered as a generalization of 'course à 20' (Brousseau, 1997). Let a frog on a band with a fixed length. The players play alternatively by moving the frog always on the same direction. The lengths of the jump of the frog are given (it can be 1 or 2[2], or 1, 3 or 4...). The first player, who pushes the frog outside, wins.



As in most of two players games, the main difficulty lies in the understanding of what is a *winning strategy*. The other concepts involved in this situation are induction and periodicity, the latter amounting to a generalization of divisibility.
Moreover, we would like to point out that pupils or students can modify the input of this problem by modifying the length of the jump. This allows the study of a new game, which can be mathematically rather different. The free modification of the problem constitutes one of the most important features of scientific research.
We can extend such a game by playing with several frogs and/or bands. The formalization of this kind of games can be found in Berlekamp, Conway and Guy (2000).

Hunting the beast!

Let us consider a given territory (a rectangle on the grid in the present case). A beast is a given polymino constituted by few squares (see below).



We have to position traps (here a single square) on the territory in such a way that no beast can be placed.
The aim is to position the smallest number of traps.

[2] for instance, if the lenght is 1 or 2, it is 'course à 20'

This problem is emblematic of optimization problems. Indeed, in order to prove the optimal value, it is necessary to produce a solution with this value (sufficient condition) on the one hand, and to prove that we cannot do better (necessary condition) on the other hand. The problem, which has given us[3] food for thought this RSC, is due to Golomb (1994).

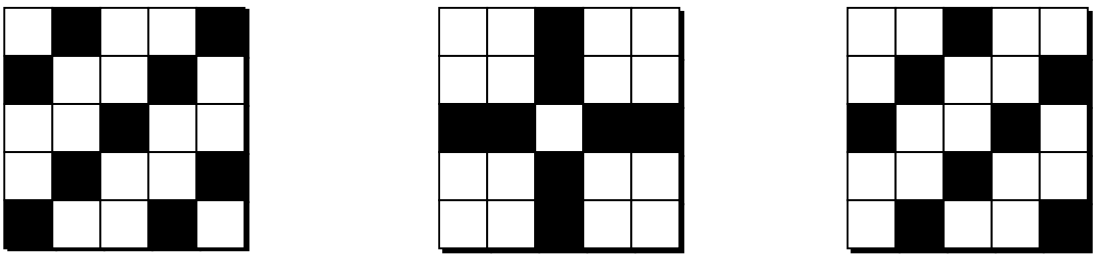
We shall now describe the chronology of an experiment consisting of five sessions (one hour each). ‘Hunting the beast’ was experimented and observed by didacticians and mathematicians in several contexts (psycho-pathologic pupils and at secondary level).
The progress of this RSC was as follows:

- **devolution of the situation and of the problem of optimalization**: each group of pupils **chooses** a beast (consisting of at most 5 squares) to hunt on an eight by eight square. We notice that a mathematician does not necessary know the optimal value of each case.

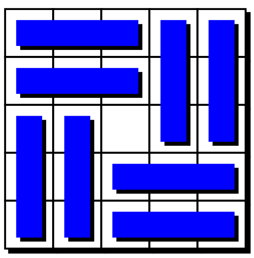
Groups are quick to propose solutions. The Manager-Observer (coded MO: it can be the teacher and/or the researcher) removes one trap from pupils’ solution in order to reveal a beast: it allows the devolution of the optimization problem. This kind of interaction may be repeated. It leads pupils to a critical question: they think that they cannot do better. At this stage, the fact that they are not able to do better convinced them that they rushed to the optimum, except if another group produces a better solution.
In order to go further, the MO proposes simpler cases.

- **research on a particular case**: the next session concerns the hunt for this beast  on a five by five square.

After solutions have been found for 11, 10 or 9 traps, pupils produce solutions using only 8 traps.



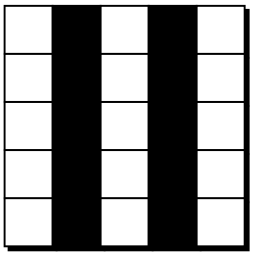
Pupils are persuaded to have the optimum with 8 because they tried and failed several times with 7 traps. A crucial point appears then: the necessity of a *proof*.
The position of the MO becomes important to engage pupils in a rational proof problematic. He can refer to simplest cases (to prove that it is impossible with 1, 2 or 3 traps...).
A ‘tiling’ argument may appear during the proving process: if one can put 5 disjointed beasts on the territory, then at least 5 traps are required. In our case, one can prove that 8 traps are necessary.



[3] Gravier-Payan, 2001.

- **research on a particular case**: the next session concerns the hunt for this beast  on a five by five square.

The organisation of this session is similar to the previous one: the pupils reinvest their own proof process. Nevertheless, the tiling argument for this beast is not enough because one can put at most 8 beasts on a five by five square, but an optimal solution need ten traps.



Therefore we get here only bounds on the optimal value (it is between 8 and 10). Such results are common in mathematical research. However, we can refine the tiling proof in order to prove that 10 traps are necessary. We have obtained this proof in some groups of pupils. The following figure shows a sketch of the proof.

- **reinvestiment of pupils' strategies** (ideas and/or methods) on larger territories (7 by 7 squares for instance).

- **realisation of a poster and an oral presentation** by pupils about their own research processes: ideas, results, methods and also conjectures. Pupils present that in a seminar called "Maths à Modeler Junior" in our laboratory.

The above presentation about 'Hunting the beast!' is conducted through mathematical and didactical problematics. We would like to point out that the position of the MO has to be more characterised in order to devoluate RSC to a teacher (the teacher has to take the responsibility of the management of research-situations).

CONCLUSION

The proposed problems, inspired from current mathematical research, are designed to engage participants with concepts and reasoning in a manner better aligned with that experienced by active research mathematicians. They give an opportunity to work on scientific processes which are constituted by students' experiments with different cognitive attitudes: doubting, conjecturing, refuting, generating new counter-examples, testing etc. A gate is now open on the investigation of the potential educational outcomes of this type of experience. We shall actually collaboratively develop criteria for the design and evaluation of such experiences, geared towards students at various levels of schooling.

SCIENTIFIC PRODUCTIONS (since 2003 – in discrete mathematics and mathematics education)

- SCIENTIFIC
  - PhD thesis (4)
  - International Conferences (13)
  - International Journals (26)
  - Books (2)

- DISSEMINATION
  - CD-Rom ("les 7 énigmes de K'stêt", edited by Generation5, Chambéry, France – in French)
  - Website www-leibniz.imag.fr/LAVALISE

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