CHALENGING MATHEMATICS IN AND BEYOND THE CLASSROOM:
How far could we go? How to lead students to succeed with excitement?

Valeria Pandelieva, Ottawa, Canada,
641 Kirkwood Avenue, Ottawa, K1Z5X5
valeria.pandelieva@sympatico.ca

Iliana Tsvetkova, Sofia, Bulgaria,
D. Nestorov blvd. N 24, app. 16, 1612 Sofia
iliana_tzvetkova@yahoo.com

ABSTRACT

The authors share examples of their experiences to lead students to an appreciation and enjoyment of mathematical challenges at two schools with very different profiles. Special attention is drawn on the role topics from history of mathematics and interdisciplinary connections and projects play in classrooms to inspire students' interest and excitement about Math. Based on their observations as teachers of students between age of 10 to 13, and analysing their high achievements in math competitions and challenges, the authors present their opinion that students are capable of reaching significant goals and challenges if the timing is right and the environment and approach are appropriate. Many of the issues in the focus of the ICMI Study 16 are touched by providing well supported personal views and opinion on :

- The significance of exposing younger students to math competitions and various forms of challenging Math activities
- The benefits of early differentiation
- The importance of teaching Math to grades from 5 up by Math specialist teachers
- The place of technology in Math education

INTRODUCTION

Trusting our observations throughout the years, we have the feeling that a large percentage of people in general consider Math as boring and difficult matter. Many of them could not wait to finish their compulsory Math studies at school, hoping that they would not have to deal with it till the end of their lives. Very often, the first question asked in class after announcing the title of a new Math lesson or unit is: "Well, when in my life will I use this and why do I need to study it?" All of us teachers had to answer similar questions many times and, honestly, it is sometimes not easy. We have to admit that not many people use functions, quadratic equations, or even the Pythagorean Theorem on a daily basis. Unfortunately, few children and adults realize that the main purpose of studying Math is to develop one's thinking skills to the extent to be able to make logical connections in familiar and non-familiar situations, to see multiple solutions and answers, to be able to model and put in a structure processes so that they could be forecast and explained. To look for a reason and explanation of everything that happens around us is such an important life-long skill. Building such abilities and extending them to become a general approach to life is supposed to be the main goal of all school Math programs.

In fact, Math is attractive, interesting and exciting. However, as opposed to sports, arts, shows, or literature, its beauty is not evident until one gets up to a certain level of awareness and proficiency. It is unfortunate that many students give up before achieving that level.

Below, we are sharing examples of our experience in the process of leading our students to the state of being able to appreciate and enjoy Math challenges. Our vision regarding some of the questions posted in the Discussion Document, together with the problems we identified and ideas that suggest some solutions, are integrated in our comments and are summarised in the end of this paper.

I. WHERE AND WHEN?

Most of our examples and observations presented below are from the two schools where the authors have been teaching in the past fifteen years.

- ***Sofia Mathematics High School, Sofia, Bulgaria*** (SMG) is one of the 28 Bulgarian schools specialized in offering a full-day enriched and extended Math program to students from grade 5 to 12 (approximate age 10 to 18). Students are selected after a comprehensive entrance exam. The academic program consists of all compulsory courses required by the Ministry and three levels of Math classes, where students follow a rigorous Curriculum in Math and get additional training for Math contests and Olympiads.

- **Academie de la Capitale, Ottawa, Canada** is an independent bilingual school for students from JK to Grade 12, with a significantly different educational environment. The Academic program follows the Curriculum of the Ontario Ministry of Education, which is delivered through a student – centered inquiry-based approach. In addition, the school offers advanced art classes, sports activities and Math Enrichment classes. The school was certified as an IB PYP school in 2003. The students differ in their learning abilities and styles. In the same classroom, along with gifted children, there are students with learning disabilities. One of the major strengths of the inquiry-based approach is the flexibility to accommodate all types of learners. The tasks and projects address students' varied intelligence levels, so that they can find the most effective form to acquire knowledge and skills and to reveal themselves to the best of their potential.

II. WHAT AND HOW?

1. The history of Mathematics as a tool for providing challenges in class.

Most of the Math concepts and theories exposed in School Math originate from ancient places and ages. The "youngest" ones are about 300-400 years old. This is an excellent reason to introduce many topics from their historical perspective, describing the process of their development through the years. The historical approach to teaching math allows each new concept to be placed in a "larger puzzle picture" showing its connections to other previously studied concepts and forecasting new concepts that are to be studied in future.

Benefits:

- ✓ Provides a sense of continuity and the idea of the holistic nature of the Math science.
- ✓ Connects scattered pieces of knowledge into a larger structure, thus making it easier for students to remember and understand them.
- ✓ Feeds students' natural curiosity and affinity for a good story or joke.
- ✓ Conveys a message that the existential purpose of math concepts is different from stuffing the textbooks and making children's lives complicated. Students see how these concepts have arisen by the natural evolution of scientific ideas as a response to practical needs of mankind.
- ✓ Shows that in fact not all mathematicians were weird and boring people and inspires students' interest in further inquiries about their personalities and lives.

Examples from practice:

- ❑ Young students (grade 3-4-5) learn with enthusiasm how to calculate sums of terms of arithmetic sequences, without knowing anything about the theory of sequences and series, through the well-known **story of how a 7-years-old boy named Carl Gauss** calculated the sum total of the numbers from 1 to 100 in a few minutes, and thus disappointed his teacher, who believed that this task would keep his students busy for hours. They get really excited about it, and they always remember later to look for appropriate application of this idea, naming it after Gauss and remembering the story, as if it were about an old friend of theirs.

- In connection to divisibility of numbers, fractions, geometry, Pythagorean theorem, students do a research project about **Pythagoras and his secret society** and make a presentation about the Pythagorean philosophy.
- A long-term project on the **history of numbers** is appropriate for all grades from 1 to 8. Depending on the age and the abilities of the students, it may include different parts such as inquiry, research, presentation, comparison between the studied numerical systems and the Base 10 numerical system, analysing advantages and disadvantages of the systems, etc.
- **Research projects and presentations about famous mathematicians** are an excellent opportunity to learn interesting facts and stories about people who usually appear as abstract names in the text books. While working on the task, students touch upon math terminology and concepts that they have never heard of before and, naturally, they start asking questions and looking for answers. As a result, they get a "larger picture" where the familiar concepts belong, and thus they understand them better and apply them better. In the discussions on their projects, one group of grade 7-8 students came up with a lot of open-ended questions (e.g. What else do the numbers in the Pascal's triangle represent, and where are they applied?; What is Euclidean Geometry, is there a non-Euclidean Geometry and how could it be different?; What are the conic sections?). Answers could not be provided at that point, however just the fact that another door for further inquiries has been open testifies for the benefits from such projects.
- Studying different **calendars throughout history** is an excellent topic for younger students to learn how concepts of keeping time developed, and understand the calendar in use nowadays. A lot of interesting and challenging logical and pattern-related problems refer to this matter.
- **Mathematical Paradoxes and Sophisms** have their place in teaching some math concepts. For example, a reference to the Zenon's paradox while teaching infinite series is a good illustration of the fact that sometimes sums of infinite number of terms have a finite answer.
- Students are really excited when given opportunity to learn how **math constants (π, e)** or math concepts (such as "**infinity**") were introduced and what the process or their evolution was. Again, there are facts around that boost their curiosity and motivate them to ask questions and look for answers.

2. Connecting Math to other subjects and areas of study.

Most education systems emphasise making links between different subjects and working on inter-disciplinary projects. This idea also appears naturally in the IB MYP educational philosophy, and is one of its most significant components. As far as the impact on studying Math is concerned, students are more interested, involved and motivated when they are given the opportunity to apply what they have studied in an attractive or useful project. Often, interdisciplinary projects challenge students' creativity and let them apply and express themselves in a very personal way, thus they are a great opportunity for enhancing Math knowledge and skills in the regular program.

*For **Examples of Interdisciplinary Projects** click here*

or send an e-mail to valeria.pandelieva@sympatico.ca

3. Exposing students to competitions and challenges as often as possible.

Young people are natural competitors. They like challenges and contests. Participation in any form of competing brings a lot of

Benefits:

- ✓ Provides an opportunity to compare with others and to elevate standards.
- ✓ Builds a character and life-long skills such as perseverance, reasoning, communication, independence.

- ✓ Contests are the only force that can make children sit for hours and work on math problems, because they set achievable goals and reach real benefits from making the effort.
- ✓ Participating in some competitions, summer Math camps and schools provides interesting opportunities for socialising – students travel, meet new friends, experience exciting moments, build a network for further contacts.
- ✓ Students willingly push themselves towards improving academically and achieving great results. The process of training for a contest is often more beneficial than the contest itself, in the long run. A healthy portion of stress is a powerful positive force in this process.
- ✓ Students learn to manage stress, they learn how to cope with negative emotions in case they do not win and how to benefit and learn from their mistakes.
- ✓ It is a rewarding activity and brings the joy of success, pride of work well-done, recognition by society; It builds self-esteem and motivates for further efforts.

The benefits of competitions become even more significant when children start competing while in elementary grades. It is great that the number of contests suitable for younger students increases each year.

It is very important to choose the contest, its form and level of complexity in relevance to students' abilities and training. Otherwise, children might easily get discouraged. There are suitable challenging choices for students from all achievement levels. Even students with limited abilities can benefit from being exposed to a contest-like activity. They will be involved in the preparation process beforehand, and they will improve their test-taking and stress-managing skills, which is, no doubt, something that they have to face anyway.

It is also important to set achievable goals for each student. The principle of the Olympic Games is often the most reasonable in Math Olympics - let everybody participate in the game and do their best and let the best ones (as of this moment) win.

*For **Examples of contests and competitive activities** click here*

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4. How far can we go? Why should we go as far?

A simple back-in-time walk through the content of school math textbooks, both in Bulgaria and in Canada, shows that in the past several decades the level of difficulty and the depth of the concepts studied has gradually decreased. Trying to understand the cause for such a tendency, one can assume that children nowadays are less intelligent, and their learning abilities are rapidly going down, which is definitely not true. In fact, **today's students** are children of the Technology and Computer Era and, **as far as learning and challenges are concerned they are more capable and more equipped than ever.**

It has been probably assumed that decreasing the volume and the depth of math studies at school will take the pressure away from math classes and will make most students succeed. Actually, **the lower the expectations, the less students push themselves toward reaching them**, and the ratio between succeeding and struggling students almost does not change. One of the unfortunate results of this process of "making school Math easy" is that **more and more students who are naturally talented in Math become bored** by a program where many repetitions and hands-on manipulations are provided in order to give everybody a chance to understand the basics. Despite the fact that all teachers are aware of the necessity of providing differentiation in their classroom, this is the real situation in a lot of schools. **Without implementing a real differentiation in studying Math, most of the excellent students simply give up waiting for their challenging time**.

Our experience proves that **students are capable of dealing with a lot of challenging concepts, if they are exposed to them at the right moment and in the proper way**. The best time to start teaching topics that require higher level of abstract thinking and reasoning is around the age of

ten. At this age, students already have a significant knowledge on basic arithmetic and geometry concepts required for their level. They are mature enough to learn to support all statements with thorough explanations and to investigate problems making multi-step logical connections. Moreover, they already intuitively experience a need for stronger, structured methods and schemes in solving harder problems. If they are not shown the shortcut, they would find their way intuitively, however what is the point in putting them in a situation of "reinventing the wheel"?

Many concepts and types of problems from the senior school math curriculum have a simple arithmetical or logical solution based on the knowledge their young children already have. Once they are introduced to these concepts in an appropriate manner, students do not have trouble understanding and applying them independently. **It is astonishing**, even for teachers who did not specialize in Enrichment programs for younger grades, **how far students can go in their achievements and learning**. It is so exciting to see grade 5-6 students prove with confidence that the three medians in a triangle intersect at the same point, using only their knowledge about the area of a triangle, and the simple statement about the median that it splits the triangle into two triangles of equal areas!

In fact, the ratio of segments and areas of figures, together with the properties and concepts of divisibility of natural numbers, are the best fertile soil for raising excellent thinkers and building foundations for future mathematical growth. They make it possible, for the first time, to expand students' awareness about what Math actually is, far beyond the simple manipulations with numbers and recreational puzzles, and to introduce them to the general method of deductive proof as early as in grades 5-6. This is when talented students really get interested and attracted by Mathematics, because now there is a lot of investigations and discoveries beyond the evident in the problems. Math now sounds serious, and at the same time it is like a spy-game. **Once students get used to the importance of proving every statement, they have already the right critical approach in place and are well prepared for further advanced studies.**

*For **List of sample challenging topics, per grade of suitability** click here*

or send an e-mail to valeria.pandelieva@sympatico

III. WHO?

The success of any project related to *Challenging Mathematics in and beyond the classroom* significantly depends on the united effort of several groups of active participants.

- **Dedicated Math Specialists, in grades 5 and up.**

Teachers, who are to provide challenges in class, have to like them and be confident in the matter. In Bulgaria, after grade 5 (the first grade after the Elementary school) only specialist teachers, who have at least Undergraduate degree in Math, are eligible to teach Math, in all schools. These regulations guarantee that the concepts will be taught properly and systematically, without gaps and misinterpretation of basics; the opposite is disastrous in studying Math, especially if it appears at the early stages.

In other countries however, students in Junior grades (4-6) and Intermediate grades (7-8) are usually taught by teachers-generalists. Some of them not only are not trained to teach challenging Math but also, in their time as students, struggled and did not like their Math classes. They cannot inspire interest and appreciation to a matter they do not appreciate. Often students from such classes build wrong approach and vision, even for the basic concepts. By the time they get an experienced Math Specialist teacher in High school, it is too late to correct or to catch up.

- **Students, with an active attitude willing to try and perceive in their Math studies.**

Students who were born "math wizards" enjoy Math Enrichment activities and any challenges they face in Math classes, because they have the opportunity to develop their natural talent and interest. However Math challenges also target students who claim they are bad at it. By showing them that Math is not necessarily hard and that it has its own beauty once you can open your eyes

to see it, hopefully educators will succeed to make more students realise that Math makes sense if only you would put your mind to it.

- **Parents, to support the teachers and encourage the students.**

The role of the parents in the process of increasing the challenges is of great importance. They have to accept the fact that challenges usually go together with higher degree of involvement, and more effort. Students who feel the parents' approval, positive attitude and dedication go through difficult moments and stages with responsibility and confidence

- **Universities and Colleges, Unions and professional organisations of mathematicians and math educators.**

These are the institutions that must provide the academic expertise; the specialised teachers' training; the workshops, schools, and training for highly motivated students; the networking and contacts at the National and International level. It is our pleasure to mention that, both in Bulgaria and Canada, professors from the Math Departments of Colleges and Universities and the two Math societies, UMB and CMS respectively, play excellently their active part in planning, organising and providing resources for all activities and events from the Math Calendar. Thanks to the inspiration, enthusiasm, professionalism, spirit and dedication of many people from these institutions, the new generation of young talents has the opportunity to develop and work for excellence.

IV. REFLECTION, QUESTIONS, PROBLEMS, SUGGESTIONS

As a reflection on the aspects considered so far, several issues seem to have an impact on all of them and leave extra room for open discussions.

- **The necessity for real differentiation no later than grade 5, in most schools**.

The classes have to consist of students with uniform abilities – this is the optimal environment for the benefit of all of them. A class with various levels of ability is a sacrifice of either the strong or the weak students for the others. As a result, none of them get the best they need.

The forms of accommodation of differentiated classes in the school's structure and schedules are to be discussed and chosen by administrators of schools. There are many working solutions to this problem in various countries.

Definitely, there must be an opportunity for every student to transfer from one level to another, within the same school, by establishing a flexible process of regular (annual) evaluation.

The form of early selection and advanced Math education for talented students that is a practice at the specialised Math schools in Bulgaria yields excellent results for these students. It is worth providing such an opportunity for students who have great and promising potential.

In fact, the early specialisation and differentiation is a controversial issue for some psychologists and educators. They insist that it is damaging for children's harmonic personal development since the emphasis on one type of intelligence prevents other ones from developing. This is probably true in some cases, but not for children who take advanced Math programs. On the contrary, it seems that developing higher level of competence in Math supports and fosters the students' excellence in all areas of study and personal development.

- **Hands-on approach must be used wisely**

It is no doubt recommended and invaluable in introducing and explaining concepts, especially as far as elementary students are concerned. However, if used for longer than necessary it prevents students from moving ahead. Besides, for some concepts, attempts to present them by hands-on manipulations are more confusing than a simple abstract explanation. Moreover, there are concepts that can never be "touched" or modelled by sticks and coins. Students who were not

required to go beyond the hands-on observations and investigations early in their Math studies, are helpless when a need for abstract connections arises.

- **To what extent is it appropriate to use Guess-and-Check or estimations in problem solving?**

It is common to ask young children: "Can you guess what the answer is?" If they are too young or do not have the knowledge, this is the only way to challenge their natural number sense or intuition.

One of the suggested problem-solving strategy in books for elementary students is "Guess and Check", along with "Make a diagram or a table", "Use logic", "Make an organised list and explore all options", "Work backwards", etc. This is fine, for younger children it is sometimes enough to figure out the number that works. Often they feel the answer, and it is satisfactory just to make sure that the guess is correct. In addition, estimation skills are necessary and very helpful when exact answers are not required or when a rough idea about the range of quantities must be obtained quickly. Estimating also provides an excellent feedback for verifying calculation results and students should be encouraged to use it to discover and correct careless mistakes in their work. Our concerns are about students who keep "guessing" their answers and try to solve the problems by randomly combining the numbers from the given, without bothering to think and elaborate some logically connected steps. Not being able to "guess" the answer, for them is the same as to lose in a Bingo game, so they simply state: "It is too hard, I do not get it". It must be admitted though that this is not completely the students' fault. In some programs, guessing and checking just goes too far. Sometimes it is predetermined to be the most common problem solving strategy by the selection of the practice questions and examples. When they are too simple, the answers are evident. Students simply guess them, check in the answer sheet that they got them right and do not see the point in going further making connections or explanations. From their point of view, once they have found the correct answer, they do not need to do anything else. As a result, such students are not given the chance to learn how to deal with problems where they cannot guess.

- **When and how to introduce specific Math terminology and the problem solving jargon**

Math educators agree that the most difficult skills to teach are the communication ones. Students find it more difficult to write down their solutions and communicate their reasoning than to solve the problems. Recently, many programs emphasise on building and improving students' ability to communicate using the proper math terminology. The assessment of the communication skills has become an important part of the formative and summative assessment strategies and tools. At the same time, professional math notation barely appears in the textbooks, up to grade 8-9. Mathematics no doubt needs, and has developed, its own professional language as all other academic fields do. We all know how convenient it is to write anything related to Math in a professionally-set environment. Asking young students to communicate Math ideas BEFORE teaching them the language is as unfair as to ask them to write a five-page creative essay, using their baby-language only. At each grade level, it is appropriate to introduce some specific terms and signs, on a need-to-know basis, and to teach their proper use. Young students are eager to use "shortcuts" like $\in$ (belongs to), $\Rightarrow$ (therefore), $\exists$ (exists), $\varnothing$ (empty set), QED. Just the idea to use a shortcut is already a winning strategy in the teacher's game, especially when the terminology and notation are introduced in a funny way. At each level, the use of as many word problems with written solutions as possible is the only strategy to build good communication skills. Students should be taught structures, hints and steps in writing clear presentable and easy-to-follow solutions.

- **Projects and application assignments have to be related to concepts studied**.

They have to come after the students achieve considerable understanding on these concepts from the pure mathematical or scientific point of view. It has to be clear how the application task

addresses and uses these concepts. Assigning practical tasks just for the sake of giving a "good-practice-related" view of the content sometimes reflects in the students' sliding on the surface of the matter, without even understanding "what this all has to do with our Math".

- **The use of technology in teaching Math**

Students of today will be facing challenges of the High-Technological era and they have to be prepared for it. No doubt, technology provides opportunities for a rich practical experience and more investigations in Math classrooms. It is important to learn how to use it in order to be efficient and successful in tasks that involve a lot of calculations or managing large amount of data. In our opinion however, the technology experience must follow the understanding of the concepts and has to take place after students achieve a considerable level of proficiency at the required basic skills. They have to realise that the computer or the calculator can help them manage routine work faster and easier, but it cannot replace their thinking and making decisions. Computer programs in Math classes are a great tool for illustrating new properties, in explorations and making hypotheses. On the other hand, the significance of abstract thinking and the deductive approach to proving those properties and hypotheses is doubtless, and they must have the predominant place in the learning process.

The right solution is in the wisely set balance between the two aspects.

LAST WORDS

Teaching challenging Math is a difficult and energy-consuming job, but it is also a rewarding one, not only because of the longer summer holidays.

- You stay young and alert – dealing with young and energetic people does not leave you a choice.
- You never get bored – each lesson is a real-time performance. You can never repeat it twice the same way, and you never know what exactly will happen until "the curtain falls".
- You learn every day from your colleagues, from your own mistakes, and also from your students.
- You never reach the top; there is always a worthwhile goal to aim for. The more the students get, the more you want to give them.
- Your best reward is when you see those special sparkles in your students' eyes – once they understood something that they had struggled with before.
- You have a chance to leave a trace and make a difference in so many lives.
- You are the happiest person in the world every time a student of yours overtakes you.

We both had the rare professional chance to teach at excellent schools, to collaborate with colleagues who are great professionals, to work with wonderful and talented students. Without them, any of the experiences we shared here, would not be possible.

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