Can high school students think mathematically like Archimedes?
A design research experiment in challenging mathematics.

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A class of academically oriented high school students is offered a problem, and a set of resources they could use to solve the problem. For the use of each resource, however, there is a point-cost which reduces the outcome of the team. Up to 100 points will be awarded for a clear and accurate solution. The resources are:

A tube container just large enough for the three tennis balls it holds: 60 points
A tennis ball: 30 points
A cloth tape measure: 20 points
A metrestick: 10 points
A 1-metre string: 5 points.

Students have ten minutes to decide on the resources they will use, and will then have a further ten minutes to answer the question. The question is about the tube container with three tennis balls. By what percent does its height exceed the distance around the tube (or vice versa)?

In a series of design experiments (Cobb et al, 2003; Lobato, 2003), we have been using this question to introduce a unit of mathematics that engages students in looking more deeply into the formulas for the circumference and area of circles. The students who have participated in the design experiment have learned in previous years that $C = 2 \pi r$, $C = \pi d$, and $A = \pi r^2$, but it has been a repeated source of disappointment for us how few students actually consider using the relationships summarized by the formulas to answer the above question. Comparing circumference with three diameters should be possible without any numbers—thus, without 'buying' any of the resources. (The height of the can is three diameters, and its circumference is π diameters, and so the ratio of $\pi / 3$ provides an answer that is worth 100 points.) It appears that the students' conception of formulas is as recipes for the creation of a desired numerical answer from given numerical values. On one level, the design experiment's research purpose has been to explore the nature of students' understandings of mathematical relationships and formulas as it relates to their sense of the nature of the mathematics they learn, while its instructional purpose has been to develop a unit of challenging mathematics that reorients students to actively pursuing richer understandings of mathematics.

However, we perceive a deeper level of pedagogic problematic in the situation described above. We are asking, both generally and within the framing of the design experiment research being described here, *why* the capable students we teach have *chosen* to develop a purely procedural understanding of the mathematical relationships they are studying. We believe that the students do not perceive the potential benefits, including the intrinsic satisfaction, available to them if they accept the challenge of developing richer understandings. It isn't just about diameters and circumferences or the purpose of formulas. It's also about the nature of school math, and their participation within it. In the instructional element of this research, we are offering students a greater level of challenge than they are used to in mathematics: the challenge of understanding and expressing in multiple ways the *sense* of the relationships among those formulas, including, for example, *why* the constant value in the circumference and area formulas is the same remarkable value, π. Our challenge as educators is not just to make challenging mathematics available in school. It is to enable, invite, and scaffold students to accept and exploit the challenge such mathematical understandings can offer them (Mason & Janzen Roth, 2004).

Design experiment research (Cobb et al., 2003; Shavelson et al., 2003) is a focused form of action research. Its structure involves the development of instructional resources and strategies for potential implementation beyond the scope of the research itself. However, its purpose as a form of educational research is to explore the qualities of student understandings and their development of further understanding as the resource development progresses through its cycles of design, implementation, and evaluation. In this case, the instruction deliberately and aggressively attempts to reorient students' approaches to learning mathematics toward the challenge of developing conceptual understandings rather than the stockpiling of additional procedures, and it uses challenging mathematical activities drawn from an academic study of the history of mathematics to accomplish its educational goals. The design process has generated instructional resources ready for broader use. The research process has addressed the feasibility of reorienting students toward richer forms of learning mathematics.

The history of mathematics as a source of instructional inquiries

The curriculum design for this teaching experiment began with a study of the history of mathematicians' inquiries into circle relationships, especially Archimedes (Cuomo, 2000; Eves, 1960) and early Chinese mathematicians (Liu, 2003). There are a variety of reasons for the use of the history of mathematics as a source of mathematical activity for classroom use. One, it is our responsibility to offer students an opportunity to understand mathematics as more than a static body of knowledge. History of mathematics provides a context for presenting a narrative of mathematics as an ongoing process developing through thoughtful effort our communal mathematical understandings (Arons, 1991; Mason, 1999). The history of mathematics legitimates mathematics as an aspect of our culture, an enriching aspect of what humankind is accomplishing. Two, historical framings of mathematics provide a way to put a human face on the mathematics that students encounter, offering context and story-lines to enliven the content and role/process models for students to view as examples (positive and negative) for their own mathematical efforts (Stinner & Williams, 1998; Mason, 2003b). Three, although individuals' understandings do *not* necessarily follow the historical order in which mathematics developed, the historical structure of the discipline offers a framework within which educators can think about the educational sequence and structure of topics (Rudge & Howe, 2004; Mason, 2001). No matter what the theoretical reasons, however, it is now necessary for those of us who are promoting the sincere use of the history of the discipline to demonstrate its actual value in educational terms (Lederman, 2003; Mason, 2003a). This work contributes to that goal, in part by demonstrating the possibilities of offering more challenging, more rewarding mathematics to students in their school mathematics.

It is important to outline two premises of our work with the history of mathematics which may differ from readers' anticipation of such work. First, we have not accepted the historical record of mathematics as a linear sequence of formal results. We recognize and understand the utility of mathematicians presenting their mathematics only in final form, with fully developed deductive reasoning as support (Hoffman, 1988; Mason, 2004) but without a record of the processes by which the mathematical inquiry itself progressed. However, we also recognize that mathematics documents, by their very nature as precise and fully grounded works of mathematics, obscure the mathematical cognition that preceded them, and we must be diligent in our search for the mathematical processes that *led* to the ideas in such documents. There are, of course, incomplete records of such processes. The discovery in the early 20th century of a partial copy of Archimedes' *The Method* provides an example of documents that offer glimpses into the thought processes that precede the publication of mathematicians' formal results. Other aspects of those thought processes need to be inferred from considerations of the historical record of the lives and societies in which the mathematicians did their work (Joseph, 2000).

Second, we have not looked to the historical development of mathematics for processes for students to replicate. On a practical level, it's not easy to sort out the apocryphal accounts of mathematicians and scientists from the plausible or likely accounts. In fact, our desire for a good quick story may have favored the apocryphal anecdote over more detailed and accurate accounts of mathematicians' cognition (Fara, 2002; Mason, 2003b). Two examples are the Eureka story of Archimedes and the bathtub, and Newton being struck (literally) by gravity when an apple fell on his head. We cannot afford to view or represent mathematical cognition as bright-flash happenstance moments if we want students to accept the challenge of doggedly but creatively constructing richer understandings of the objects of their attention.

There is a much more important reason, however, for moving beyond replication. Although much can be gained from replicating the actual processes by which mathematical and scientific premises were conceived and tested (Heering, 2005; Mason & Stinner, 2001), it is neither practical or purposeful within the broader educational endeavor to have students using only the tools available to Archimedes (papyrus or sheepskin and ink, sand-tables, clay tablets with a stylus, for instance). Not only do we want students developing capability with the tools of today for pragmatic reasons (National Council of Teachers of Mathematics (NCTM), 2000). It is also in the spirit of the historical record that mathematicians' tools of choice were the tools available to them in the time and place, the culture, where they lived. But most important, we must recognize that our purpose in a classroom is not to relive history, in this case the brilliant development by a single mind of a signal idea. Our purpose is educational, the development by as many students as possible of as rich and as interwoven a conceptual understanding as possible. We must avail ourselves of the most potent supports for student learning as possible, and that means moving beyond historical replication. History of mathematics can provide only the raw material for effective curriculum development; pedagogy—our understandings of cognition and how to cause it—must be foremost in the historically grounded instructional design of activities using today's learning tools (Mason & Janzen Roth, 2005).

In brief, we use a rigorous analysis of the history of mathematics to develop a plausible conception of the cognition that led to the mathematics that our students learn. We then adapt the processes of inquiry that led to the development of that mathematics into curriculum that enables students to inquire into the same mathematics. The challenge for the students is in part the mathematics content—to develop a rich conceptual understanding of the mathematical relationships behind the formulas (e.g, the concurrent precision and irrationality of the constant in the formulas for the circumference and area of circles). The challenge is also to appreciate and develop one's skills in mathematics as inquiry, including mathematical thinking (Huckstep,2000;

J. Mason, Burton, & Stacey, 1985), ~~mathematical problem posing (Walter, 2003), and~~ mathematical problem solving (NCTM, 2000; Reid, 2002). The history of mathematics gives us the mathematical version of the inquiry processes behind the content. Design experiment research may give us the mechanism, over time, to develop instructional versions of those processes which preserve the spirit, challenge, and intrinsic rewards of the mathematician's original inquiries.

The instructional processes

Details of the instructional processes in our design experiment continue to be redeveloped as the experiment continues. However, the unit in its current design consists of about ten hours of instruction. It consists of three general strands. Mathematical inquiries into the relationships among quantities of the circle are the primary strand. Attached to that strand are processes of interaction, oral and written, about the relationships and the processes within the inquiries. Third is a sequence of anecdotes with associated student activities that bring Archimedes into the students' imagination.

Although there are more in the actual unit, three mathematical inquiries may portray the quality and range of these curriculum elements. In one of the inquiries, students inscribe and circumscribe a series of regular polygons within circles, measuring the edges and apothems and calculating lower and upper bounds for the circumference of the circle. The activity stops short of Archimedes' method of exhaustion for determining circumference to any desired degree of accuracy, but the activity gives students the opportunity to (first) construct and think about circles and polygons without using measured lengths and angles and (then) use measurements to confirm the progression in the relationships among the lengths. In another inquiry, students build "radius squares" to match a set of progressively larger circles, and graph the quadratic relationships of the edge of each square (the radius) with the areas of both the square and the circle. They then graph the relationship between the area of the squares and the area of the circles, and discover it to be, counter to their intuition, to be linear. In another activity, the students cut paper circles into equal sectors and rearrange the sectors within a rectangle whose base is the circumference and height is the radius. They repeat the process, with twice as many sectors each of half the size; they repeat it a third time, now with sectors much smaller than the sections of an orange-slice. In the manner of a thought experiment (Winchester, 1991), and approximating closely the thinking of Archimedes that foreshadowed the calculus of Newton, the students have spontaneously extended the process, imaging sectors that are infinitesimal in width; the students can readily see that the sectors would exactly half-fill the rectangle. They can 'see' (and express in words and symbols) the relationship between the linear construct, circumference, and the two-dimensional construct, area; and they see the presence of the same constant, π, in both. (The activity deserves to be shared as a process, rather than summarized in the few sentences used here.) Each activity contributes to the experiences, the imagery, and the communicative competence of the students. It is not the presumption of the instructional design, however, that the challenge to understand a specific formula in a specific way is represented by a specific mathematical inquiry. Rather, it is the generalization and

abstraction of experiences from the group of inquiries overall that seems to be leading students to accept the challenge and understand the mathematics of circles as Archimedes did.

The second element of the design is the communicative element. Communicative processes include the use of *groups of four* (Johnson & Johnson, 2004) for collaboration when the whole class does the same inquiry. With other inquiries, half of each group does a different inquiry, followed by pairs teaching pairs the processes and outcomes they experienced. The students' written work for each inquiry includes a piece of interactive writing (Mason & McFeetors, 2002), giving the student the chance to think about and express their own mathematical thinking or values. As well, the teacher's personal responses help the student to recognize the teachers' purposes and concern for their achievement. Closure interviews with each person used narrative inquiry methods (Clandinin & Connelly, 2000; McFeetors & Mason, 2005) to foster students' retrospective thinking about the unit when it was completed. In part, communicative elements provide the teacher-researchers with the data that the research process requires. However, the communication processes are essential aspects of the students upgrading their intellectual engagement with the mathematics beyond minimal computational competence.

The third element brings the historical record into the students' view. It consists of short verbatim anecdotes, taken from ancient (e.g., Cicero, in Newman, 1956) and modern (e.g., Barrow, 1992) versions of Archimedes' life. Each anecdote comes with a demonstration or an activity that enables the students to be intellectually active with the anecdote. For example, after the Sand Reckoner story, students are led to use exponential representations to calculate the grains of poppy-seeds that would fit in a cubic metre, after estimating the grains that would fit in a centimetre cube. To date, we have made progress with keeping the anecdotes and activities from expanding beyond reason the time the unit takes. Our goal is for the historical elements of the unit to be recognized as inherent to the overall experience, not as interludes in the flow of mainstream activity.

Research results

The research has now completed three full cycles of design, implementation, and redesign. The first cycle of curriculum development incorporated the responses of mathematicians and educators to instructional activities attempting to represent the cognition of Archimedes that is summarized by the π –based circle formulas all students learn to use. In the second cycle, academic grade nine students interacted with a prototype unit of instruction, including a sequence of activities built around historical vignettes. In this year's cycle, the unit was adapted to affect the understandings of the nature of mathematics held by a group of academic grade twelve students.

Each cycle has provided opportunities to better understand how to engage students in the challenge of deep understanding of algebraic formulas. We have found first and foremost that students hold a wide range of beliefs and values about the nature of academic mathematics and its rightful purpose in school. When probed, students have portrayed their beliefs as deeply embedded in their lived histories as learners of mathematics, products of their personal experiences and their social environments. Second, we have found that students' initial understandings of the functions and relationships that the circle formulas summarize (for mathematicians) are disappointingly shallow. Yet, the shallowness is clearly remediable, through engaging students in challenging mathematical inquiries related to those relationships. Third, we have been happily surprised by students' openness to perceiving mathematics as our historically grounded curriculum presents it, as a complex human enterprise available to their abilities.

In each of the three design experiment cycles, participants were observed to hold diverse perspectives on the nature of mathematics and what it means to learn mathematics. Some participants held instrumental views of mathematics: it is a set of ideas and formulas to be learned for use in applications and in further study. Others held a conceptual view of mathematics: mathematics is a collection of ideas with internal and interconnected coherence. Some saw mathematics as a field of present-tense inquiry in which they could participate through problem-solving and inquiry in the kinds of thinking that are part of our culture and history; others saw mathematics as a collection of artifacts from past inquiries to be apprehended and remembered. However they saw mathematics, all participants demonstrated their eagerness to perceive mathematics as something other than the credential that school used to differentiate among young people. In the last cycle of research reported here, not only did these different orientations to the subject come into view; it was also clear that the unit of instruction successfully invited people of all orientations to respect the possibilities of mathematics as a challenging yet accessible opportunity to learn.

The data used here to illustrate these claims comes from interactive writing by academically-oriented grade twelve students on the last day of the seven-day unit. Students wrote about their understandings of the relationships summarized by the circle formulas and π, and they wrote about their experience. The particular answers quoted below are part of their responses to a question about the value of the historical aspects of the unit, and the value of the unit overall in a course defined primarily by a provincial examination. The first answer quoted here illustrates the views of Jason, a person who was quite comfortable with mathematics as a culturally vital field of inquiry.

> In the background history, we can see why and how we are doing certain things. Personally I have gained respect for math and those who use and have created it to enrich our lives. We do math and wonder why we don't just use calculators and computers. But Archimedes did not have these available to him, thus he was able to see the other methods and related strategies to further his understanding of math.
>
> The last seven classes have been priceless to me. I fully understand where π comes from and am now more comfortable using the value. I like knowing the reason I use formulas and constants. Being more comfortable in math is key to success in math and will help me in further math for the rest of my life. Learning like this works for me—practicle [sic], theoretical, and historical is the perfect combination for learning, in math and in life.

It is clear that Jason found the unit to fit with his personal conceptions of mathematics and learning. He found synchrony between his desire to understand, his desire to be able to apply what he learns, and the opportunity to learn mathematics in a historically informative way.

Daniel developed similar views, but for him it was the communicative aspects of the unit that provided his greatest opportunity to grow. His comments suggest that unit's historical elements invited the students to develop a critical perspective inherent to that discipline (Stahl &

Shanahan, 2004). However, he also points to the challenge he found and embraced to explain what he was coming to understand, and to explain as a way to come to understand abstract ideas.

> I think that math history gives you an insight on why we are learning the things we are. By learning it, we can better understand the concepts we are learning. It also gives you a better appreciation for the greatness of the mathematicians before us. (Maybe the stuff we are doing IS so hard after all.) But one thing to remember is that history is not always accurate. As we saw from the Archimedes story. We heard two very different endings to the way he died. Which one is true? Maybe neither. The bottom line is that history can be skewed based on what people want to hear not on actual facts. You have to be wary on what sources you use and how much you choose to believe.
>
> These classes have not only showed me how π was derived, but they have also taught me to think more abstractly. The biggest gain from these classes has been the need to explain my thinking when I'm not completely confident on why the answer is the answer I got.

Although Daniel didn't think he was good at explaining, he recognized that his attempts helped him to understand, and helped him to "gain" in his abilities to do so. For Daniel, a historical approach to mathematics was intriguing, but the conceptual aspects of the mathematics offered him his best opportunities to grow.

In contrast, Ernst liked physics more than history. He viewed mathematics instrumentally, as a set of formulas and manipulative processes that enabled him to use those formulas in the subjects he liked. Yet Ernst as well acknowledges the value even from an instrumental perspective of understanding the formulas of mathematics.

> I think that when we used Archimedes' method to find how he narrowed π down to a very accurate approximation of the actual value of π, I though that that was very helpful in understanding what π actually is. The other historical stories I found to be interesting, but did not aid in my learning.
>
> I found that the area of a circle formula now truly makes sense to me, instead of just being a formula I could type in my calculator.

Overall, because we were looking to invite and encourage students to adopt a richer view of mathematics, it was the willingness of instrumentally oriented students like Ernst to engage in the conceptually and historically oriented unit that we value most, from this last cycle of the design experiment. André, who wanted only to be able to remember a formula for the sake of plugging numbers into it, expressed his newly developed view that understanding why a formula works matters to him.

> The history of math is an excellent tool to use when explaining theories. Knowing who it was invented by and how it was invented gives us the true feeling of the subject and therefore it will stick in our head and make it easier to remember the formulas without even checking our formula sheets. It will also help us plug in the formulas correctly if asked a question. Because we know why we are putting certain numbers in certain spots.
>
> The value is that when I'm asked a question involving π, not only will I know why the formula works but I'll know if I'm wrong or not.

Is it overly optimistic to think that André is moving toward a conceptualist appreciation of mathematics? Or is his belief that conceptual understanding of formulas props up his instrumental use of those formulas as far as he is going to get? We do not know. However, we feel good about the progress André made in adapting his orientation to the subject matter to incorporate the conceptual and historical elements of this unit.

Another student, Sandie, offered appreciation for having been 'forced' to study some mathematics conceptually and historically, and suggests that she had developed a new appreciation of discovering mathematical relationships.

> There isn't enough history of math being taught in class. Students have the formulas and relationships between numbers thrown at them, but many don't understand who discovered them, or why these formulas and relationships exist. If we had a greater understanding of why the formulas we learn work, we would be able to use them more effectively, and discover more relationships between formulas.
>
> I found these 7 classes valuable because I found out why stuff works the way it does. The thing I found most interesting was the relationship between the area of a rectangle and the area of a circle. I believe that if students were forced to learn this material in class, many would find this time as valuable as I did.

Sandie sees herself and other students receiving mathematics in school, with the nature of that mathematics determined for herself and others by the teachers. Yet, within this one unit, Sandie sees possibilities for receiving mathematics that enables her to understand formulas and discover relationships, and find out why 'stuff' works. She suggests that such a challenge would be valuable to her, and that she could succeed in the face of that challenge. It is almost as if she challenges us to make the mathematics that students receive from us, indeed are 'forced to learn' by us, more meaningful, conceptual, and enriching.

The challenge of mathematics

How can we make sense of the pedagogic challenge of getting students to appreciate challenge in mathematics? Perhaps the giants of 20th-century pedagogy offer valid starting points. Piaget offers us a way to think about students changing orientations to their subject matter. He suggests that as learners we are all more comfortable adding on to our current knowledge and understandings, a process he calls assimilation. Yet it is not by assimilation that students come to change their conceptual orientations: students are reluctant to change how they perceive themselves in relation to their environment (accommodation), and will do so only in the face of relatively persistent discomfort with the fit between their orientation and some aspect of their environment. It isn't hard for teachers to offer discomfort. The challenge lies in having students accept the discomfort as an invitation to change or grow, rather than rejecting the discomfort as a source of frustration. Here, it is Vygotsky's idea of zone of proximal development that helps us to organize our pedagogy: when we offer students opportunities to grow beyond what they are already capable of, we must fit what we offer to students into that region beyond what they can already do independently, and what they can do with the scaffolding we can offer them in their relationship with us and the classroom in which they are learning. (By scaffolding, I mean the interpersonal and strategic supports that teachers and classmates can offer to learners to enable them to do learning that they could not do on their own.) This suggests that we can offer significant invitations to grow or change (dissonance) only when we also offer significant scaffolding that enables students to see their engagement beyond their comfort zone as likely to generate success for them. The instruction described in this manuscript was organized according to these pedagogic principles.

To be true to the spirit of this work, I will offer the last word to one of the students. I hope it offers encouragement to mathematicians and mathematics educators to believe that making high school mathematics more challenging and making mathematics more inclusive of more students are not oppositional goals. The authors of this document believe that greater challenge and greater inclusion are not only compatible, but complementary—the achievement of either goal will require us to achieve both. But it is not us, we theorists and educators, in our curriculum design and pedagogy that will ultimately determine the truth or this claim. It is students. Manuel, whose voice will close this document, sees mathematics now as something that not only enriches his sense of the world, but enriches him through his study of its ideas.

> The history of math is vital, in my opinion, to understanding the math concepts we as students are exposed to in daily life. In learning about Archimedes' methods, we gain an understanding of his thought process, in whatever degree. Understanding the thought process of a great mind in math will help students to approach mathematical scenarios with the discipline of mind that will help them to see it as a whole and to understand a more complete spectrum of mathematics.
>
> The last 7 classes have helped me to fully understand π for what it is, as opposed to a mere value used to calculate various dimensions of a circle. These classes have also given me an appreciation of Archimedes' contributions to science and mathematics. With my understanding of Archimedes' thought processes and methods furthered, I how have a more disciplined mathematical mind.

In his statement, Manuel reminds us of why we want to make mathematics more challenging, and bring more people to success with the challenge of mathematics.

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