Understanding mathematics through resolution of paradoxes

Margo Kondratieva
Department of Mathematics and Statistics, Memorial University of Newfoundland,
St. John's, NL, Canada A1C 5S7
e-mail: mkondra@math.mun.ca

"*Everything I tell you is a lie*"

Abstract. Brain-challenging puzzles have attracted people for a very long time. Paradoxes constitute a special type of puzzle aimed to reveal and emphasize an inconsistency or contradiction resulting from some mental experiments in mathematics. Their resolution teaches us to stay alert and be aware of possible flaws of various kinds. Many paradoxes, such as those of Zeno and Russell, greatly influenced the shape of mathematics as we know it today. That suggests a possibility to incorporate the study of paradoxes in standard mathematical courses. But how productive may it be? At which stage of their study will students benefit from being exposed to paradoxes? How one can practically do it in the classroom? This paper is an attempt to address some aspects of these important questions. We discuss the nature and role of paradoxes in the process of understanding, along with potential problems and advantages of their use in study. We give several examples of mathematical paradoxes in both the historical and the classroom context. A short survey results outline an idea of the audience reaction and suggests further directions for research. We conclude that the pedagogical payoff of the use of paradoxes in the classroom is currently underestimated and a consistent study of the impact of paradoxes on learners will allow us to develop a teaching portfolio which takes a comprehensive advantage of the natural curiosity of the mind towards puzzles.

1. The role of paradoxes in the process of understanding and learning.

" In his discussion of metaphor in the *Rhetoric*/ Aristotle says there are 3 kinds of words./ "Strange, ordinary and metaphorical./ Strange words simply puzzle us;/ ordinary words convey what we know already;/ it is from metaphor that we get hold of something new & fresh"/ (*Rhetoric*, 1410b10-13)./ In what does the freshness of metaphor consist?/ Aristotle says that metaphor causes the mind to experience itself/ in the act of making a mistake./ He pictures the mind moving along a plane surface/ of ordinary language/ when suddenly/ that surface breaks or complicates./ Unexpectedness emerges./ At first it looks odd, contradictory or wrong./ Then it make sense./ And at this moment, according to Aristotle,/ the mind turns to itself and says:/ "How true, and yet I mistook it!"/ From the true mistake of metaphor a lesson can be learned./ Not only that things are other than they seem,/ and so we mistake them,/ but that such mistakenness is valuable./ Hold into it, Aristotle says,/ there is much to be seen and felt here./ Metaphors teach the mind/ to enjoy error/ and to learn/ from the juxtaposition of *what is* and *what is not* the case." [1]

A *paradox* in the broad sense is a suddenly emerged unexpectedness, a statement that looks wrong and contradictory. As we see, its presence facilitates the process of understanding things in an attempt to fix an error and to make sense. The metaphor, as described by Aristotle, presents the paradox at the initial moment of establishing non-obvious relations. Later, the paradox becomes resolved by confirming or accepting the identification. In a multi-layer metaphor, another kind of paradox may occur, when an unacceptable conclusion emerges from parts that make sense as they are, but the integrated whole does not. That forces one to review the entire way of reasoning and give up one of the parts.

If we accept the view [6] that mathematics has a metaphorical nature, we have to admit, following Aristotle, that in the whole history of mathematics, in the course of its development, the mind has been permanently and constantly exercised in the resolution of paradoxes of various kinds and levels.

Over the centuries, this resulted in the mathematical theory with its definitions, rules and statements, the justification and meaning of which can be fully understood only via personal experience of the resolution of the paradoxes for yourself. In a way, the event of each discovery repeats at the level of an individual learner who is trying to understand the essence of a statement. Having said that, we do not suggest a rediscovery by a learner of the entire theory from scratch, but rather a rediscovery of the meaning of the existing theory.

As an example of learning from a paradox, let us consider the following algebraic derivation.

P1 *Assume that* $a = b$*; Multiply both sides by* a *to get* $a^2 = ab$*; Substruct* b^2 *from both sides:* $a^2-b^2 = ab-b^2$*; factor both sides* $(a + b)(a - b) = b(a - b)$*; divide both sides by* $(a - b)$*:* $a + b = b$*; recall that* $a = b$*, so* $b + b = b$*; therefore* $2 = 1$*.*

The arrival at the contradictory statement makes one analyze the derivation line by line and eventually realize why *division by zero* is prohibited in algebraic manipulations, as opposed to just blindly accepting the rule from a book or teacher.

Knowing rules is very important for doing mathematics. But mathematical rules cannot substitute for understanding. First of all, according to Gödel's discovery made less than a century ago, a complete system of formal rules is simply impossible. This discovery ruined Hilbert's agenda for *complete axiomatization* of the theory with further deductions using a perfect artificial language of reasoning as it was seemingly suggested by the ancient Greeks' geometry and philosophy of Euclid and Plato. On a positive note, the discovery by Gödel tells us that the door is open for insight, creativity, informal approaches and imagination in mathematical thinking ([8], p.72). To survive and be transmitted from one generation to another, mathematics needs an interpretive, active mind, not just a structured storage of its facts. The understanding of mathematics seems to appear through this interpretation of the *real imperfect world* in pure abstract terms, through a desire to capture in symbols the things we seem to know from our living experience (like time, space etc.)

The role and place of paradoxes in the process of cognitive development can be identified with the help of Piaget's concepts of *equilibration* and *reflective abstraction*, which rest on the Kantian epistemological proposition that *a knower constructs their knowledge of the world.*

"Implicit in Piaget's analysis is the idea that knowledge and understanding are acquired only as the epistemic subject applies its existing cognitive structures to that of which it becomes aware (cognitive aliment)... Equilibration refers to a series of cognitive actions performed by a knower seeking to understand cognitive aliment, which is experienced as novel, resistant, perturbing, disequilibrating. This experience of disequilibration motivates the knower to attempt to re-equilibrate... The most interesting form of equilibration is that in which particular cognitive structures re-equilibrate to a disturbance by undergoing a greater or lesser degree of re-construction, a process known as reflective abstraction. We would agree that in those cases in which successful learning occurs, reflective abstraction has taken place." [3]

Since a paradox (meaning in Greek "beyond belief") provides a disequilibration, it makes the subject realize the need to re-equilibrate. In this quality, paradoxes are valuable components which stimulate learning and discovery processes by means of restructuring existing schemata of the learner [9].

On the historical scale one can, for example, argue that paradoxes on infinity, attributed to Zeno (5th century B.C.), eventually led to the discovery of calculus by Newton and Leibnitz in the 17th century, contributing to the notion of the sum of infinite series and the notion of limit. For instance, the fact that the sum of the geometric series $\sum_{n=1}^{\infty}(1/2)^n = 1$ can be traced back to one of the most famous Zeno paradoxes on the subject of motion.

P2 *There is no motion, because that which is moved must arrive at the middle before it arrives at the end, and so on ad infinitum.*

Zeno concludes that the runner can never reach his destination because he must first run one-half of the distance, then an additional one-fourth, then an additional one-eighth, etc, always remaining short of his goal. And here is Aristotle's critique: "Zeno's argument makes a false assumption when it asserts that it is impossible to traverse an infinite number of positions in a finite time". The modern mathematical idea of the limit, and thus, whole mathematical analysis rests on such considerations. Note however that it was successfully used already by Archimedes who calculated the area of a circle by means of a sequence of inscribed regular polygons.

In a classroom perspective, a student, puzzled by a paradox, welcomes help and attends to the hints and related explanations to a greater degree, so the lesson becomes more productive and the theory more meaningful for such a student. The amount of hints and help critically depends on the student's background and abilities. " Students with talent for mathematics, once disequilibrated, do perform reflective abstraction on their own, spontaneously. Most student need help with this step." ([3] p.90). On the other extreme, "A student may persist in applying an incorrect understanding of a concept, even after a teacher pointed out the error."([3] p.88). If a teacher can come up with an example of how the error leads to a paradox (e.g. **P1**), that can help to correct the student's behaviour. The case of belief persistence is known to be difficult for pedagogical treatments due to the psychological nature of it: people continue to believe in a claim even after the basis for the claim has been discredited. Nevertheless, a paradoxical consequence may initiate some change in the learner's conceptual state, because she now has to live with both – the persistent concept and its nonsense conclusion.

Resolution of paradoxes is always an effort to overcome one's *mental set* or *functional fixedness*, the state when problem solvers rely too heavily on their previous concepts, expectations, strategies and approaches. One needs a fresh look, a new concept, a different method to show that either the contradiction is only apparent, or the paradox rests on invalid or unreasonable grounds.

It is crucial whether one believes that the nature is paradoxical[1] or that any paradox is such only due to restrictive frameworks in which one is trying to make sense of it. The later point assumes that the paradox can and should be resolved by embedding it in a wider context and possibly rethinking the meaning of the situation.

There is one more thing we wish to mention. " The suspense that accompanies an attempt to find a solution to a challenging puzzle, or anxiety that develops from not finding one right away, is a significant part of what makes the puzzle so fascinating and engaging... The peculiar kind of pleasure puzzles produce can be called an aesthetics of mind... They can never be characterized as sad or happy; they can only be called as ingenious or clever ... which produce a form of pleasure nonetheless... Puzzles with simple yet elegant solution, or puzzles that hide a nonobvious principle, have a higher aesthetic index. The aesthetic index is also high when demonstration produces a paradoxical result." ([2], p.227)

Different types and examples [2] of mathematical paradoxes.

One finds puzzles already in the Egyptian Rhind Papyrus written around 1650 B.C. The earliest known mathematical paradox is the *Cretan Paradox*, dating back to the 6th century B.C. attributed to the semi-mythical Cretan poet Epimenides.

P3 *Epimenides the Cretan says "All Cretans are liars."*

This statement, because it was uttered by a Cretan, is true if and only if it is false. Its 20th century restatement, the *Barber Paradox*, was given by the English philosopher and logician Bertrand Russell in 1903.

P4 *In a town, the barber shaves all and only those who do not shave themselves. Does the barber shave himself?*

Such sentences can be classified as *Semantic Paradoxes*, i.e. ones that are a simultaneous assertion of a statement and its negation or self-referential statements: an assignment of a truth value to the statement ultimately implies the assignment of a different value. Other examples of semantic paradoxes are:

P5 *This statement is false.*

[1] See e.g. works by Graham Priest on paraconsistency and dialetheic logic, and more discussion in [7].

[2] Most of the paradoxes in this paper are well known. Their formulations and further discussion can be found at Wikipedia.

P6 *Every rule has an exception.*

"Drawing Hands," composed by M. C. Escher in 1948, provides a visual analogy for this type of paradox.

Paradoxes involving limiting processes and infinite sets constitute another large category, which includes the Zeno paradox (P2) mentioned above. Mental games with *infinity* are proven to be *dangerous* and often lead to surprises.

P7 Galileo's paradox: *there are as many squares, n^2, as natural numbers, n, but there are also natural numbers which are not squares: 2,3,5,... .*

P8 *An infinitely long Gabriel's horn resulting from the revolution of the hyperbola $y = x^{-1}$, $x \in [1, \infty)$ about the x-axis, has finite volume and yet an infinite surface area.*

Fractals — objects resulting from an infinite self-similar reproduction — have some *strange* properties.

P9 *The fractal Koch snowflake has finite area and infinite perimeter at the same time.*

There are many *Paradoxes resulting from flawed reasoning, arithmetic error or faulty logic.* One of them, a proof that "All horses are the same colour", is analyzed in the Appendix. The following *Missing Dollar Paradox* may even be considered as an arithmetic joke, because it has so simple solution.

P10 *Three travelling salesmen break down and are forced to spend the night at a small town inn. They go in and the innkeeper tells them, "The rooms are \$30". Each man pays \$10 and they go up to the room. The husband of the innkeeper says to her, "Did you charge them the full amount? Why not give them five bucks back since their car is broken and they hadn't planned to stay here." She then brings the men five \$1 bills and each man takes one while the other \$2 rest on the table. Originally each man paid ten dollars* $(10 \times 3 = 30)$*; now each man has paid nine dollars* $(9 \times 3 = 27)$ *and there are \$2 sitting on the counter* $(27 + 2 = 29)$*. The last dollar had disappeared.*

Solution: the amount that the salesmen spent is $(10 - 1) \times 3 = 27$; amount that the innkeeper received is $(30 - 5) + 2 = 27$. There is no missing dollar.

Hovewer, the next one, known as an *Unexpected Exam Paradox* is more elegant.

P11 *A professor announces that he is giving an examination some day next week, and the exact day of the examination will surprise his students. One student reasons about the professor's statement as follows. Suppose the exam is on Friday (the last day of the week). Then on Thursday night, the students will know that the exam will be on the next day, so there is no surprise. Hence the exam will NOT be on Friday. Now suppose the exam is on Thursday. Then on Wednesday night, they will all know that the exam is on the next day: again there is no surprise and therefore the exam will not be on Thursday. The same argument applies to Wednesday, Tuesday and finally Monday, leaving the conclusion that there will not be any exam in the following week.*

The sequel to this story is that when this student receives the exam paper, he is very surprised.

True statements which contradict common intuition are often regarded as paradoxes. One needs to overcome his/her epistemological obstacles [10] in order to fully resolve such paradoxes.

For instance, one can never *come to believe* that $1 = 0.999999...$, even after learning the way the statement is understood. It is the same as Zeno thought about his runner.

Many probabilistic statements are found to be counter-intuitive.

P12 *Birthday paradox: if there are 23 or more students in a class then there is a chance of more than 50% that at least two of them will have the same birthday.*

P13 *Monty Hall Paradox: Suppose you're on a game show, and you're given the choice of three doors: Behind one door is a car; behind the others, goats. You pick a door, say No. 1, and the host, who knows what's behind the doors, opens another door, say No. 3, which has a goat. He then says to you, "Do you want to pick door No. 2?" Is it to your advantage to switch your choice? Answer: Yes, a player who has a policy of always switching will win the car on average two times out of the three.*

Paradoxes often result from an intentional error (algebraic, geometrical, logical) in a *model problem*. They teach us how to avoid similar errors in *real problem* solving. They warn us, for instance, that a proof by picture may be not reliable, that a word description may describe nothing, that formal manipulations may lead to an absurdness, and that something counter-intuitive may still be true. Paradoxes teach us to be sensitive and alert at every step we make in our derivations.

Teaching with paradoxes. Theory and practice.

There are two stages in the process of paradox understanding. First is the recognition of its ingredients — words, terms, graphs etc., as well as the underlying assumptions. Second is the realization that even if each part does make sense, the statement as a whole does not work. In order that the second stage produce the effect of confrontation (dis-equilibration), the first stage should be a simple cognitive act, presenting no difficulties, troubles or doubts. In other words, the learner should operate in a familiar clear environment to be able to analyse the situation and trace the fallacy.

Following [3], we will distinguish $\alpha-$, $\beta-$ and $\gamma-$ types of behaviour. $\alpha-$behaviour is the integration of a novelty in an unstable or inadequate way: "students thought that they understood and mastered a particular concept that they, in fact, had not." In contrast, $\beta-$behaviour signifies the re-organization of the cognitive system of the learner and the accommodating of the novel aliment within the reconstructed domain. A $\gamma-$behaviour consists in an accommodation of a novelty without a preceding reconstruction of the cognitive system.

• When does a student not learn from paradoxes?

Many students have developed (over years in high school) a habit of pseudo-learning, mostly in a declarative mode, using explicit rules, given formulae, memorized facts. This often leads to the $\alpha-$behaviour, which the learner has a tendency to substitute for an understanding. Such behaviour often results from a state of learners' minds in which a formal definition of a concept differs from its internal cognitive representations [11]. Consequently, a learner experiences confusion and paradoxes resulting from her own misconceptions and psychological obstacles to accept certain mathematical statements. At this point there is no room for critical learning of new material. Therefore such a learner is reluctant to be exposed to further contradictions. She does not have a way or habit to resolve them. For such a student, often individual assistance is required before she starts to accept the very idea of creative thinking.

• When does a student successfully learn from paradoxes?

The above classification suggests that when a paradox is introduced, a learner is expected to exhibit a γ-behaviour at the first stage, in other words, her cognitive system should be sufficiently rich to accommodate the parts by themselves. The second stage, the realization of the contradiction, then will initiate the β-behaviour targeting for the resolution. Such an interpretation is possible in a dialectic approach in which the learner's behaviour type is viewed relative to the task, experiencing changes due to the task performed. The $\beta-$ behaviour, being "the paradigm case of successful learning", is identified not only as an ability to produce a correct answer by reasoning, but as a readiness to review, analyse, and re-construct the existing cognitive system, if required. The ability to work with both concept image and concept definition, to create multi-linked representations, structure, and to fit ideas into a big picture are characteristics of advanced mathematical thinking. This type of thinking may be present at all grade levels and our point is that it is a prerequisite for beneficial learning through resolution of paradoxes.

During resolution of a paradox, a learner makes a revision of her understanding of the ingredients (terms, graphs, logical implications) and hidden assumption. The contradiction she faces is her motivation to localize the source and reason for it.

Everyone who manages to understand the essence of a paradox wants to know a resolution. This is a natural curiosity of mind. That curiosity forces the mind to make guesses. But if all attempts are unsuccessful and the ideas seems to be exhausted, a certain reluctance to proceed may be observed. (The reluctance is a sort of defensive reaction of the learner, unable to resolve the situation immediately. She may even convince herself that it is not important to complete the task.) Nevertheless, the process of the contradiction resolution still goes on internally, subconsciously, and the mind stays open for a possible hint from the environment. An *aha-moment* can occur any time. This state can be effectively used for teaching mathematical concepts related to the essence of the paradox. A desirable outcome is to make the learner to apply the concepts to resolve the paradox. A whole lesson can be designed to lead the discoverer to her goal. An example of such practice can be found in [5], where the entire course was built around discussions of paradoxes.

My experience of talking about paradoxes with university students has significant variations. There are students who definitely and undoubtedly learn a great deal from thinking about contradictions, students who this way invent and rediscover for themselves important mathematical constructions and notions. There are students who understand the essence of a puzzling statement, who become curious about the true reasons for it and appreciative of its various interpretations and explorations. Not being exposed to paradoxes would be to disadvantage of such students.

There is another group of students, for whom the craft of learning from paradoxes is not fully accessible. They still can be good students, hard working, good solvers of standard tests, interested in mathematics, enjoying simple puzzles, but surprisingly helpless with problems like the Missing Dollar Paradox (P10). Such students need special assistance to be able to benefit from paradoxes.

There is a third group of students who do not benefit from being directly exposed to paradoxes, they are getting misled, confused, frustrated. Their presence almost make a case against using paradoxes.

In order to estimate the relative sizes of those groups and the effect of working with paradoxes a survey study has been conducted among second and third year undergraduate students. The students were asked to resolve few paradoxes, including (P1), and then answer questions about their experience with paradoxes.

It would be worth mentioning that among the group of students I interviewed, 35% take math courses because they are required in their program, 32% take them because math is interesting, 38% — because they believe it is useful, and 15% admitted that they always liked mathematics. In this and following responses some students gave more then one answer, thus the total percentage may exceed 100.

It was observed that 62% of the students were able to resolve and explain paradox P1, identifying the wrong step and the forbidden operation; 15% of the students were able to come up with some explanations, but didn't give a clear resolution. For example, "Once you eliminate $(a-b)$ on both sides you are left with $a+b=b$, so something wrong is here." Another student wrote: " The only possible solution is $a=b=0$. Then $2b=b$ makes sense". 23% of the students could not say anything relevant, except " 2 is not equal 1", or "This can't be right", or "Magic!".

The students were also asked whether or not there is a notion or fact in mathematics which they regard as counter-intuitive. Most of the students didn't find anything of that nature, although a few examples were listed: " $(\#)^0 = 1$", "$0! = 1$", "The notion of infinity", "Harmonic series diverges", "$\sum 1/n^2$ is related to π". One student gave a long list of paradoxes and concluded that he would be willing to take a course about paradoxes.

Finally, the questions of the survey were answered in the following way:

1. Resolution of paradoxes is challenging (30%), interesting (65%), emotional (30%). No one said that it is boring. One student said that it can be annoying.

2. 66% of the students were in favor of discussion of paradoxes in class, and 34% didn't welcome this opportunity.

44% of the students thought that it would improve their understanding, 26% felt that it would be fun, 12% wished to learn about historical development of mathematics.

On the other end, 12% have enough confusion about math already, and 32% feel that they would rather spend time on regular problems. Nevertheless, some would like to discuss paradoxes if they are relevant to the area of their study.

These observations suggest that an instructor must be careful and sensitive while implementing paradoxes into a teaching process. It is a challenge and a risk, but it can be a valuable exercise in the creative thinking and problem solving orientations, if used deliberately and cleverly.

We can summarize some recommendations suggested by this study.

1. Build a collection of paradoxes and include them in the teaching portfolio as a specific challenging activity in the classroom. (New paradoxes are supplied by periodic editions e.g. [4]).

2. Teach paradoxes to illustrate a specific point. Make sure that the essence of paradox is understood in the context of the material being studied.

3. Allow the students some time to think on their own.

4. Engage the students into a discussion. Make them develop their conclusions, leading to a resolution.

5. State a resolution clearly.

6. Remember that the goal is to dis-equilibrate the learner in an aesthetically valuable manner, and then make use of her natural curiosity.

Will paradoxes serve that purpose better then other teaching approaches? It may well depend on the instructor's attitude towards paradoxes. But the same is true about any other tools and techniques, and there is no unique best one. Successful teaching is a mixture of many approaches, and we hope to bring paradoxes to the teachers' attention as one of them.

4. Conclusion.

In this article we attempted to show that mathematical paradoxes carry a special challenge, a hidden message, thinking of which provides the learner with an opportunity to refine their incomplete understanding. The degree of incompleteness of knowledge defines how productive such an experience can be and how much assistance is required. At the present time, the practice of using paradoxes in mathematical courses is not common, despite the fact that the subject intrigues many learners. Therefore, of great interest to cognitive sciences would be a more detailed investigation of the influence of instructor's extensive and creative use of mathematical paradoxes on students' attitudes and performance. Such research is not only of academic interest but is called for by pedagogical practices. It would allow building a portfolio of challenging activities incorporating paradoxes in a way the learners would fully benefit from them.

References

[1] Anne Carson, *Essay on what I think about most*, in "Men in Off Hours", Alfred A. Knopf: New York, 2000.

[2] Marsel Danesi, "The Puzzle Instinct", Indiana University Press, 2002.

[3] Ed Dubinsky and Philip Lewin, *Reflective abstraction and mathematics education: the genetic decomposition of induction and compactness*, The Journal of Mathematical Behaviour, **5**, 55-92 (1986)

[4] *Fallacies, flaws,and flimflam*, column edited by Ed Barbeau, The College Mathematics Journal.

[5] Brigitte Lahme and Elaine McDonald, *Infinity and Beyond - A Mathematics Class for Life Long Learning*, Focus, **26:1**, 16-18 (2006).

[6] George Lakoff and Rafael E. Nuvez, "Where mathematics comes from", New York, NY : Basic Books, 2000.

[7] Doris Olin, "Paradox", McGill-Queen's University Press, 2003.

[8] Roger Penrose, "Shadows of the mind", Oxford University Press, 1994.

[9] D.E. Rumelhart and D.A.Norman, *Accretion, tuning and restructuring: three modes of learning.* In "Semantic factors of cognition", Hillsdale, N.J.:Lawrence Erlbaum Associates, 37-52 (1978).

[10] Anna Sierpinska, "Understanding in mathematics" London; Washington, D.C. : Falmer, 1994.

[11] D.Tall and S.Vinner, *Concept image and concept definition in mathematics with particular reference to limit and continuity*, Educational Studies in Mathematics, **12**, 151-169 (1981).

Appendix: Teaching with paradoxes: a classroom analysis.

After teaching the method of mathematical induction and giving several examples of its use, the teacher announces that she is going to prove that *All horses are the same colour.* The goal is to make the students discover the flaw in the proof.

Teacher: We use the principle of mathematical induction. As the basis case, we note that in a set containing a single horse, all horses are clearly the same colour. Now assume the truth of the statement for all sets of at most N horses. Let there be N+1 horses in a set. Remove the first horse to get a set of N horses. By the induction assumption, all horses in this set are the same colour. It remains to show that this colour is the same as that of the horse we removed. But this is easy: put back the first horse, take out a different horse and apply the induction principle to this set of N horses. Thus all horses in any set of N+1 horses are the same colour. By the principle of induction, we have established that all horses are the same colour.

Student A: This is clearly impossible.

Student B: But she proved it!

Student A: (repeats the argument of the implication from N to N+1 set of horses, confirms it, gets puzzled): I do not understand it anymore...

Teacher: Let's try some values, say $N = 10$. (Intentionally takes bigger number).

Student A: Any 10 horses are of the same colour, and any 11 horses can be reduced to two sets of ten, which intersect; thus any eleven are of the same colour. There is no way to be mistaken about this.

Teacher: And it work for N greater that 10.

Student B: Yah, I never fully trusted this method!

Student A: Hold on, let me check smaller N. Let me check all over again. One horse is of its colour. True. But two are not. They may be not of the same colour. Aha, the induction is gonna get stuck here. It does not work for N=1! Here is the hole in the proof!

Teacher: Please, elaborate.

Student A: Your proof makes the implicit assumption that the two subsets of horses to which we apply the induction assumption have a common element, but this fails when N=1. So the induction step does not work for N=1, and thus the conclusion is wrong!

Student B: So the falling chain of dominos has a gap at the very beginning! (Referring to the visual analogy the teacher used to describe the induction as the chain of dominos each forcing to fall the next one.)

Teacher: Exactly! Thus this is not a paradox, but merely the result of flawed reasoning; it exposes the pitfalls arising from failure to consider special cases for which a general statement may be false.

A few remarks are in order. Both students are puzzled by the proof, but Student A is clearly the one who found the flaw, whereas Student B is more like not analyzing the problem, but making supportive claims from what he remembers. At some point he even doubts the method's validity, because he didn't develop trust in it yet. It is likely that student A would find the resolution on his own, but it is student B who needs the discussion and help.