TUNING A MATH PROBLEM

Borislav Lazarov

Abstract

This article describes how a mathematics task can be presented to different groups of students while considering their level of mathematical ability. It is crucial to think of a mathematical problem as a triple, consisting of the task, the group of students and the nature of the help that is provided. The concept of *problem decomposition* is introduced as a natural extension of the concept of problem. Examples are given.

1 Introduction

Math-task-design is the favorite part of the work for a great part of the mathematicians who deal with mathematics for school students. But even the most beautiful mathematics constructions could be not efficient or even may collapse in the process of education if they are not properly presented to the target group of students. And here comes the challenge – to find an efficient way of exposing a math topic to a certain group of students. As a criteria for effectiveness of the process of teaching of mathematics we accepted it influence on the student's situative activeness (SAC).

By examining the SAC we found that it strongly depends on the ability of students to solve math tasks by themselves[Lazarov 2004]. Such an ability is specific for the pair task-student and changes during the learning process. Below we consider the impact of the task exposition mainly on the written materials.

2 Concepts

In the introduction we carefully avoided the use of the term *problem* replacing it by *task*. Further we reserve the word *task* for a math assertion with a known solution. We need to reserve word *problem* for the pair task-student meaning that a math task becomes problem for a specific student if the student does not have a direct way to solve the task [Tonov 2005]. In other words the concept of *problem* includes both a task and a person (student) who accepts the task and is challenged to perform a kind of discovery to solve it. Such a concept of problem justifies expressions like 'hard problem' and 'easy problem'. It also excludes the drill and practice type of exercise.

An integral part of any task is the solution. We can attach to any sophisticated solution a sequence of basic steps (simple calculations or transformations, implications based on basic statements, etc.) which are arranged along an axis [Lazarov and Tabov 1991]. Having this in mind a task could be represented as a chain of relatively simple steps leading to the main goal (the initial task). When composing a task for a large heterogeneous group of students the problem designer should take into account the different layers of students in the group. This is why some problems need additional tuning which we call *Problem Decomposition* (briefly ProDec). Thus in the context of challenges, we will envisage the ProDec as having a triple character that involves a task (i.e. what would be called "problem" in normal circumstances), the student to whom the task is assigned and a chain of steps that will enable the student to meet the challenge without destroying it: ProDec becomes a triple task-student-chain.

A brief psychological reasoning of the ProDec is given as an appendix.

3 Types of ProDec

A student comes to the completion of a task through a set of internalized facts and processes, but these may not be adequate since some specialized knowledge or preception of the situation may be needed. This may come in the form of a hint, a suggestion of intermediate steps, drawing the attention to an analogous or simpler problem, or a reminder for considering some particular things.

Below we group the types of ProDec in the following three basic forms:

- ProDec as giving a hint.
- ProDec as a description of steps.
- ProDec as a system of tasks.

3.1 Giving a hint

The hint is the simplest ProDec but often the most efficient one. In [Polya 1957] the teachers are recommended to help students by asking indirect leading questions. Unfortunately in a textbook such an approach could not be applied. Let us look at some examples for possible substitutions.

3.1.1 The 'how to start' hint.

Such kind of a hint is actually a part of the formulation of the task. Further we denote the tasks by **T** and the target group of students by **TG**. Let us consider the problem

T1. Solve the equation $\log_{x^2} 16 + \log_{2x} 64 = 3$.

Hint. Change the role of the base and the argument of the logarithms.[1]

TG1: Bulgarian secondary school, 11th grade, extended math curriculum.

Here the students are given a direct instruction for the first step of the solution and they are left to finish it with some routine calculations. So the problem T1(+ hint)&TG1 is nothing special. The hint is a ProDec since for a large amount of students this first step is insurmountable. Without the hint the situative activeness (SAC) of these students drops – the problem T1(no hint)&TG1 is quite difficult. However, for more experienced students, this step is the whole point of the problem which they could be expected to find on their own. Once this step is taken, then the rest of the solution is completely straightforward for them. So the hint faces the Hamlet's question: to put or not to put the hint? A partial solution of this paradigm for the textbooks could be in the proper location of the hint. If the hint does not follow immediately the assertion, the SAC in both target subgroups stays high. But a total solution of the question is given in the web-site [MCNMO] where hint is an option for every task which is supposed to contain a hint.

3.1.2 The hint as a 'milestone'.

A hint of this type can highlight an important part of the solution. Consider

T2. Solve the equation $1 + 2^x = 3^x$.

Hint. Take into account that the function $y = \left(\frac{1}{2}\right)^x$ is decreasing and that the function $y = \left(\frac{3}{2}\right)^x$ is increasing.

The ProDec given as a hint to the problem T2&TG1 is also the main step in a rigor solution of T2. It is easy to see that $x = 1$ is a root of the equation. For a part of TG1 guessing this root could

[1]Here and further the examples of tasks are taken from Bulgarian textbooks.

be a sufficient reason to think that the task is solved. The hint actually says they should give some reasons that there are no other roots. For other students the reasoning could be an estimation of the speed of increasing of $y = 2^x$ and $y = 3^x$ but the hint points that they should give a proof. In both cases the hint serves as a milestone that cannot be missed.

3.1.3 The hint as a 'back mirror'.

Sometimes a hint is put after the assertion to turn students back to the main text of the textbook. Such a hint inputs a ProDec described in the sample problems. Consider

T3. Find the values of the real parameter m for which the equation $9^{-x^2} - 3^{1-x^2} + m = 0$ has a real root.

Hint. If u_1 is the greater root of the equation $u^2 - 3u + m = 0$ then in virtue of $u_1 + u_2 = 3$ it follows $u_1 \geq 1.5 > 1$.

Out of the context of the lesson such a hint is unclear and a bit confusing. But since the main text contains a similar problem with a detailed solution the students are actually given a ProDec that clarifies the most important sequence of steps in the solution. Such a hint for TG1 could keep SAC at a high level by offering students an additional study of the disposition of the quadratic equation's roots during the exponential equations lesson. Let us note that a similar task without any hint is a hard problem for TG1 but would be an ordinary one for the entrance exams for the Bulgarian universities, i.e. for the layer of well performing applicants for university students.

3.1.4 The hint as an obstacle.

Sometimes the author's idea of how the student should solve a problem does not correspond to the most natural way of solving the task. Consider

T4. Given the cube $ABCDA_1B_1C_1D_1$ and the midpoints M, N, P and Q of the edges AB, BC, AA_1, and A_1D_1 respectively. Prove that M, N, P and Q are coplanar.

Hint. Prove that lines MN and PQ meet at a point of the line AD.

The problem T4&TG1 is easy (the solution is $MP \| BA_1 \| NQ$), but the ProDec T4(+hint)&TG1 turns this problem to quite harder (!?).

3.2 Description of steps

The most common ProDec in books for self-training is the description of steps that lead to the complete solution. It could be considered as an enlarged milestone-hint. Some items could contain intermediate results in explicit form which allows students to continue with solution even if a part of it is not done. Since the book contains detailed solutions of all the problems students can turn to the solution of a particular step or to skip it for a while and continue with the rest of the problem. Such a ProDec allows students to keep the inertia during the process of problem-solving. It also keeps SAC at a high level longer because of the possibility to consider the skipped parts subsequently.

ProDec of the description-of-steps form is often used in the contest problems. Here the reasons are to separate some parts of the solution for sharper criteria in the assessment. But such kind of problems from past contests play the role of the ProDec just described for the students who prepare themselves for the next issue of the contest. But ProDec in contest problems could be a risky business. Here is an example when the lack of ProDec is beneficial. The problem committee composing the entrance exam for the Higher Transport School (Sofia) decided to include the following

T5. A circle k of diameter $AB = c$ and two positive numbers a and b. are given. If the point C on k is such that the value of $a \cdot CA + b \cdot CB$ is maximal find the length of CA and CB.

TG2: Bulgarian secondary school average graduates.

The point in the problem is to examine the function $f(x) = ax + b\sqrt{c^2 - x^2}, \quad x \in [0; c]$. So a discussion was held whether to put this function as a description or not. Since the problem T5-TG3 is an ordinary one in the topic 'application of derivatives' no hint was needed. But for a large part of TG3 familiar with derivatives to compose a function could be an obstacle. The committee left the task without description and this was the chance for some students to solve T5 introducing $\angle CAB$ as an argument which allows to avoid derivatives at all. In this case pointing a way of solving T5 could check some important skills but could be a barrier for finding a more creative solution.

The next problem (taken from the entrance exam of University of Sofia) is analogous to the problem T5&TG2.

T6. The base $ABCD$ of the pyramid $ABCDM$ is a rhombus with side $AB = 1$ and $\angle BAD = 2\alpha$. The length of the edge MD is $2\sqrt{2}$ and $\angle MDA = \angle MDB = \angle MDC$. A plane parallel to AC passes through B and the midpoint of MD.

a) Prove that the area of the section of the plane with the pyramid equals $\frac{2}{3}\cos\alpha\sqrt{2 + 4\sin^2\alpha}$.

b) Find the value of α for which the area of the section is maximal.

TG3: Bulgarian secondary school high-achiever graduates.

A ProDec in the problem T6&TG3 is absolutely indispensable. The given ProDec separates the task into two independent parts and the second part could be solved directly. But the ProDec is necessary for the assessment. It is extremely difficult to manage a sharp scale for assessment without the given description.

Going further we can try to answer a question risen in a university setting 'How to teach calculus when the students cannot perform simple algebraic transformations?' Of course such a fundamental question cannot be answered in a paragraph. But a partial solution could be given by ProDec. A complex task such as integration of rational functions can be decomposed into two parts: the first one – algebraic, in which the polynomial part is separated and then the proper fraction is presented as a sum of basic fractions; the second one – analytical, in which the integration is done by a simple implementation of the table of integrals. If a student has difficulties in the first part he/she can solve the second part skipping the details from the first part. The author's experience is that such an approach allows students to continue studying mathematics with possible gaps instead of slipping into a topic for a long time.

The same idea can be implemented in other complex math topics:

- A problem dealing with trigonometric equations could be decomposed into algebraic transformations, application of basic trigonometric formulas and finding solutions of basic trigonometric equations;
- in a geometry problem the parts that need proof could be detached from the calculations;
- in a solid geometry problem the plain geometry parts that appear in different planes could be separated;

A proper ProDec allows the student to concentrate on the topic under consideration and in the same time points out what the student should remember from the previous topics. 'While problem-based learning is sometimes effective, it is very time-consuming' says Theodore Nutting (in K-12 e-mail circle). A proper ProDec could be both time-saving keeping the effectiveness of the problem-based learning.

But sometimes the descriptions could themselves be confusing.

T7. In a tetrahedron $ABCD$ the points M, N, P and Q are midpoints of the edges AC, BC, AD, and BD respectively.

a) Prove that M, N, P and Q are coplanar.
b) Prove that $MNPQ$ is parallelogram.

It is clear that a student that can solve the sub-task a) would solve the sub-task b) immediately using the same idea. So the TG of such a decomposition is unclear.

3.3 System of auxiliary tasks

Another widely spread way of ProDec is to introduce foregoing auxiliary task(s) – something like lemmas in a proof of a theorem. Sometimes an auxiliary task is virtually a lemma, e.g.

T8a. The lateral faces of a pyramid form angles of the same magnitude with the base and the projection P of the vertex in the plane of the base lies inside the base. Proof that the base is a circumscribed polygon and P is its incenter.

T8b. Consider a pyramid $ABCM$ with altitude of length 3 and lateral faces forming angles of $60°$ with the base ABC. The length of the edge AB is 14 and the distance between C and AB is $\frac{15\sqrt{3}}{7}$. Determine the angle that MC is forming with the base of the pyramid.

For a great number of students belonging to a heterogeneous target groups such as

TG4: Bulgarian 11th grade students, potential applicants for university.
the sub-task T8a would be a well known result and T8b would be a routine problem. But for the rest of the students (possibly TG2) the sub-task T8b does not contain any indication about T8a which turns T8b into a very hard problem. The tasks T8a and T8b are taken from a Bulgarian weekly addressed to TG4 which makes such a ProDec reasonable.

More often the auxiliary tasks are based on a fragmentation of the solution of the main task. Here we will omit an example because such examples are too long. However we will discuss the method. The difference between organized-in-lemmas proof and the organized-in-auxiliary-tasks ProDec is that the students in general are not told which is the goal of the series of auxiliary tasks. They can see the final problem only after reaching the end of the series, which is a small disadvantage of the method. But the real advantage is in the small steps of approximately the same level of difficulty that are easily done without seeing the monster-problem. So the challenge appears to be in determining the size of the steps: how small (or how large) should they be in order to keep the intrigue but also to stay every time into the so called 'zone of actual development'.[2]

This is an open question for the author: a lot of examples show that a priori stated difficulty of an auxiliary problem appears quite different a posteriori. Perhaps a possible solution for textbooks could be to design a more detailed ProDec in which some problems are marked to be skipped so as to prevent an advanced student from getting bored.

Let us point that the ProDec by auxiliary tasks does not suppose a student to go further with gaps in contrast to the ProDec by description. If the student fails in one of the auxiliary tasks he/she in general is not ready to attack the main one.

4 ProDec for Advanced Students

The ideas discussed above focus on large target groups. In such groups different types of ProDec provide opportunities for some students to perform a bit of discovery by themselves. Dealing with small groups of advanced students one should take into account their specific needs of challenges by more conceptual problems. Such kind of problems call different ProDec that goes out of our sight in this article. However we point the two main ideas.

[2]About zone of actual development (ZAD), as well as ZCD, K_A, K_D, in the following we refer to the appendix.

The concept of *Ladder* is introduced by Kenderov [Kenderov 2003] (applied in [Grozdev 2005] and [Bilchev 2005]). In some sense the Ladder could be considered as a far gone evolution of the ProDec by auxiliary problems: it is a self-contained math text, focused on a specific topic, organized as a sequence of problems with explanations and open problems ordered in slowly increasing degree of difficulty. The aim is to give a student a medium to enlarge his K_D topic area as far as possible. Indeed the benchmarks of ProDec are tight to include the Ladder. Nevertheless we will compare briefly some points of the two methods.

A difference between the Ladder and the auxiliary-tasks-organized-ProDec is in the type of steps. One can think about the auxiliary-tasks-organized-ProDec as a relatively constant (in terms of difficulty) system of problems. The Ladder is structured as an increasing sequence. Another conceptual feature of the Ladder is that the target group which as a rule consists of advanced students becomes a pyramid-like structure with respect of how high the student can climb. So the method allows also gifted students to be identified. Let us point that a Ladder-like structure has been implemented in a large number of booklets published in Bulgaria in the 80's of the last century. In the last years this tradition is slightly revived.

In a Ladder the direction of the sequence of problems is topic oriented. It is completely different from the approach-structured math units where we can see a decomposition of a certain method in problems like in [Tabov and Taylor]. Probably a study on this method will give another view point on SAC.

5 Conclusions

'However, the solution of a mathematical problem cannot begin until the problem has been translated into appropriate mathematical terms. This first and essential step presents very great difficulties to many pupils - a fact, which is often too little appreciated.' [Cockcroft 1982] (Paragraph 249). The idea of ProDec corresponds to the 4th phase of the famous Polya's model of problem solving – looking back[Polya 1957]. Above we tried to argue that the central question for a student 'How to solve it?' should be preceded by the question 'How to expose it?' to be answered by the educator. The way of decomposing a problem from the K_C into K_A is widely exploited in mathematics texts. But as it was shown the ProDec is target-group-oriented process. It is not a shell and needs a kind of flexibility in any particular case. So ProDec stands somewhere between the didactics, mathematics and psychology and becomes a bit of applied art. The concept of the problem assumes the task should be potentially available for the students since only such kind of tasks could become a challenge for them. Two benchmarks point the range of the concept of the PROBLEM: below are the exercises where the solution of the math task is a simple calculation or a direct implication of a statement; above the problem there are some statements the solution of which is beyond the student's knowledge and abilities at that moment (beyond its ZAD). The great challenge for the passive educator (author of math texts) and for the active educator (teacher using these texts) is to determine this interval for a certain TG and then to tune the problem. If this happens a necessary condition for high SAC is satisfied. A proper ProDec could be a powerful tool for tuning a task in an appropriate form. By applying different forms of ProDec the educator should stimulate students' activity without preventing students from showing their creativity.

6 Appendix

Imre Lakatos proposed a cyclic model of the logic of mathematical discovery [Lakatos 1976] in which any stage is of the form

stage in progress: {problem set (conjecture); informal proof or refutation of the conjecture} $\rightarrow$ next stage.

Since the problem requires a bit of math discovery (by our definition) we can step on the Lakatos' model to analyze some phenomena in student behavior (situative activeness) caused by the external factor for the solution axes. The idea of ProDec has its psychological background in Vygotsky's works on higher psychological functions operationalized for the purposes of the math education by Ivan Ganchev [Ganchev et al. 1996]. The set of acquired higher psychological functions of a person is called by Ganchev *zone of actual development* (ZAD), the set of higher psychological functions of a person which are still in process is called *zone of close development* (ZCD). The knowledge that induces ZAD and ZCD is denote by K_A and K_C, respectively, and T_A and T_C stands for the corresponding math tasks. Technically ProDec includes a main T_C-task and a chain of T_A-tasks or instructions, which are analogous to the stages in the Lakatos' model. The paradigm is *if there is a need of ProDec in a problem which one could induce higher student situative activeness.*

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