CHALLENGING MATHS IN AND BEYOND THE CLASSROOM

Proposal of contribution for the Congress in Norway 2006

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Contribution: **THREE PERSONAL INITIATIVES: THE PROBLEM SEMINAR OF VALLADOLID, THE MEDITERRANEAN MATHEMATICS COMPETITION, THE DIGITAL JOURNAL "REVISTA ESCOLAR DE LA OLIMPIADA IBEROAMERICANA DE MATEMÁTICAS"**

a)THE PROBLEM SEMINAR OF VALLADOLID

The Problem Seminar of Valladolid was established by myself in 1988, as a training for the two students from the town who were members of the Spanish team and were taking part in the 29th I.M.O. and in the 4th Iberoamerican Math Olympiad (La Havana, april 1989).

During the schoolyear 1989/90, the Seminar was established as permanent, from October to May. The purposes were to give the necessary theoretical and practical basis to the students who wished to solve mathematical problems and to participate in the mathematical contests. Many important items, as Number Theory and Geometry, have been disappearing from the regular curriculum of the Secondary students, although there are still some problems about these items in the contests.

The Seminar is developed on the basis of one session per week, usually on Wednesday from 17h-19h. For the last 3 years I have also developed some two hours'Seminary sessions for younger students (12-14 years old), also of 2 hours, on Tuesday from 17h-19h. The sessions take place in the Faculty of Sciences in the University of Valladolid, to which I am very grateful for the "free borrowing" of their facilities. Some weeks the Seminar is carried out simultaneously with the Seminars for solving problems organized by the Mathematics Department of the Faculty of Sciences. In that case, all the participants of my Seminar attend those organized by the Mathematics Department.

The Seminar is conceived as an *Open Seminar:* the students have no other duty but attending the Seminar if they wish; but I must say that they usually continue attending the Seminar up to the end, although there may be some fluctuations in the number of attendants due to constraints of examinations or some extra-school activities.

Although the training for the Spanish Mathematical Olympiad are very important in the Seminar activities, the main purpose is to give the students the opportunity to learn technicisms to solve problems which are impossible to be developed in a usual lesson. In Spain there are no establishments like in other countries, mostly from East Europe, specially devoted to the teaching of

Mathematics and Physics; so it is very hard to teach items like the principle of extreme, the pigeonhole principle or the geometrical transformations, in an ordinary class.

When it is possible (that is, taking advantage of some visiting Professor to the Algebra and Geometry Department), the students of the Seminar receive a special lesson by that visitor. I must also say that I have never to ask for financial aid for this activity.

As a natural sequel of this, the students who attend the Seminar usually have good qualifications in the contests in which they participate: Spanish Mathematical Olympiad, Kangaroo Mathematical contest, and others.

Some sample problems to "start" the Seminar :

1990-91: In a finite sequence of real numbers, the sum of any seven succesive terms is negative, and the sum of any eleven succesive terms is positive. Determine the maximum number of terms in the sequence.

(*Source: IMO 1977, problem proposed by VietNam)*

This problem is considered by Arthur Engel as a paradigm of a "good problem for a contest", that is, one such that a teacher has not advantage over an student to solve it...

1991-92: Two students, A and B are playing of the following manner: Each one of them writes down on a sheet of paper a positive integer and gives the sheet to the referee. The referee writes down on a blackboard two integers, one of which is the sum of the integers written by the players. After that the referee asks to student A : "Can you tell the integer written by the other student?". If A answer "NO", the referee puts the same question to the student B. If B answer "NO", the referee puts the same question to the student A, etc.

Assume that both students are intelligent and truthful.

Prove that after a finite number of questions, one of the students answer "YES".

(*Source: IMO 1991, proposed by Bulgaria but not used in the contest)*

This was, in my opinion, one of the best problems included in the "short list" of the IMO 1991, but it was not used in the contest...it is a problem which do not give any advantage to a well trained student versus a clever, but untrained participant.

Both problems are, in my opinion, interesting enough to motivate the students to try to solve them....taking into account that the students attending the Seminar are motivated enough to take part in it....

The sessions of the Seminar are usually very participative. The students discuss and often solve the problems that are suggested. However, a theoretical background is sometimes needed, then the Seminar has the old aspect of the a "normal class": the teacher (me) speaks, the students write down their notes. But from some time up to now I have the feeling that one of the most appreciated characteristics of the Seminar is that the students come in expecting something different than an usual lesson. I also consider very fruitful to have a double session, one for younger and one for "old" students. But this is also "open": the young students (12-14 years old) who have succesfully attended the Seminar for two years are invited, the next year, to attend the Seminar for the "bigger ones", even in the case they are not in the 3 last years of the spanish scholarship yet.

The Seminar is planned to teach the questions of Theory of Numbers, Algebra, Geometry and Discrete mathematics which are never covered by the curriculum of the students, nevertheless they always appear in the problems proposed in the different contests. As I already said, the emphasis is made on proposing problems to be solved by the students, initiating them into the possible ways of solution, giving (if necessary) the useful theoretical background in the searching of ways of solution, and helping them step by step towards the solution.

Other sample problems discussed in the Seminar:
A problem from the Dutch Olympiad
Compute the sum

$$S = \sum_{n=1}^{2001} \frac{1}{\sqrt{n + \sqrt{n^2 - 1}}}.$$

Solution
As $n^2 - 1$ is not a perfect square except for $n = 1$, it seems necessary to express the denominator in some another way. We will see in what conditions it is possible to find numbers x, y such that

$$\sqrt{a + \sqrt{b}} = x + \sqrt{y}.$$

Squaring, one need to solve the system

$$\begin{cases} a = x^2 + y \\ \sqrt{b} = 2x\sqrt{y} \end{cases}$$

and, when one compute $a^2 - b$,results

$$a^2 - b = x^4 + y^2 + 2x^2y - 4x^2y = \left(x^2 - y\right)^2,$$

and so, if $a^2 - b$ is not a perfect square, it is useless try of continue. But if this were the case, calling it $a^2 - b = c^2$, the system becames

$$\begin{cases} a = x^2 + y \\ c = x^2 - y \end{cases}$$

and then we have

$$x = \sqrt{\frac{a + c}{2}}, y = \frac{a - c}{2}.$$

So,

$$\sqrt{a+\sqrt{b}}=\sqrt{\frac{a+c}{2}}+\sqrt{\frac{a-c}{2}}.$$

(The case $a-\sqrt{b}$ is similar).

Going back to the dutch problem, $a=n, b=n^2-1$, whence $a^2-b=1$, which is a perfect square. Then

$$\begin{aligned}\frac{1}{\sqrt{n+\sqrt{n^2-1}}} &= \sqrt{2}\cdot\frac{1}{\sqrt{n+1}+\sqrt{n-1}}\\ &= \frac{\sqrt{2}}{2}\left(\sqrt{n+1}-\sqrt{n-1}\right)\end{aligned}$$

and the sum is a telecoping one :

$$\frac{\sqrt{2}}{2}\left(\sqrt{2002}-\sqrt{2000}+\sqrt{2001}-\sqrt{1999}+\sqrt{2000}-\sqrt{1998}+\cdots\right)$$

which reduces to $\frac{\sqrt{2}}{2}\left(\sqrt{2002}+\sqrt{2001}-1\right)$.

Observation

If one tries to express $\sqrt[3]{a+\sqrt{b}}$ in the form $x+\sqrt{y}$, the system becomes

$$\begin{cases} a=x^3+3xy\\ a^2-b=(x-y)^3\end{cases}$$

from which it is only possible to follow if a^2-b is a perfect cube; in that case, we put $a^2-b=c^3$, we solve the equation $4x^3-3cx=a$, in x, and it is possible to compute y at $y=x^2-c$.

This problem was selected because it is a typical example of converting a finite sum into a telescoping one, and also of a clever use of the rationalization method.

A problem from the Czech Republic Olympiad 1995

Find all the real numbers p such that the equation

$$x^3-2p\left(p+1\right)x^2+\left(p^4+4p^3-1\right)x-3p^3=0 \qquad (1)$$

has three distinct roots, which are the lenghts of the sides of a right triangle.

Solution

Suppose that the roots are $0<a<b<c$, with $a^2+b^2=c^2$. By the relations of Cardano-Vieta,

$$a+b+c=2p\left(p+1\right),\quad ab+bc+ca=p^4+4p^3-1,\quad abc=3p^3 \qquad (2)$$

Therefore,

$$\begin{aligned} 2c^2 &= a^2+b^2+c^2 = (a+b+c)^2 - 2(ab+bc+ca) \\ &= 4p^2(p+1)^2 - 2\left(p^4+4p^3-1\right) = 2\left(p^2+1\right)^2, \end{aligned}$$

i.e., $c^2 = \left(p^2+1\right)^2$, or $c = p^2+1$.
From the two first Cardano formulae we have

$$\begin{aligned} a+b &= 2p(p+1) - c = p^2+2p-1 \\ ab &= p^4+4p^3-1-c(a+b) = 2p^3-2p. \end{aligned}$$

Then, the numbers a and b are the solutions of the equation

$$u^2 - \left(p^2+2p-1\right)u + 2p^3 - 2p = 0,$$

from which it results

$$\{a,b\} = \left\{2p, p^2-1\right\}.$$

This gives us a necessary condition : $p > 1$.

The numbers $2p, p^2-1, p^2+1$ are solution of (1) if and only if they satisfy the third one Cardano relation (2) :

$$2p\left(p^2-1\right)\left(p^2+1\right) = 3p^3 \Longleftrightarrow p\left(2p^2+1\right)\left(p^2-2\right) = 0.$$

But due to the necessary condition $p > 1$, we obtain the unique solution $p = \sqrt{2}$, and the roots are the numbers $1, 2\sqrt{2}, 3$.

The reason of choosing this problem for the Seminar is to show how a "cubic equation", studied in the curriculum, can be converted into a good Olympiad problem, using the relations between the coefficients and the roots.

b)**THE MEDITERRANEAN MATHEMATICS COMPETITION,** *Memorial Peter O'Halloran.*

During the Second Intenational Conference about students of high ability, held in Varna (Bulgaria) in 1997, I presented a paper in which I proposed the establishment of a new mathematical contest, on the basis of the Asian Pacific Mathematical Olympiad, with the participation of the countries from the Mediterranean Basin. I will quote some words from that paper:

The countries from the East of Europe have a long tradition in the organization of mathematical contests for students of high schools and Universities. The names of Kürschak Competition, Austrian_Polish Contest or Balkanic Olympiad are widely known and admired by the people who, as myself have been involved for many years in the work with students of high ability and also in the training of the olympic teams of our countries.

*The situation in the South West of Europe is, however, different. For example, France has not got an Olympiad (***the paper is from 1997***), although the Concours Géneral is almost bicentennial. In Spain, the Spanish Mathematical Olympiad is 33 years old, but a tradition of mathematical contest with really hard problems does not exists.*

So, in this way I presented the Project of a mathematical competition for the countries of the Mediterranean shore. In order to make the establishment of the contest easier, acting as an European representative of the WFNMC, I made a first drafting of the rules of the contest (almost identical to those of the Asian Pacific Olympiad) and I sent them to the leaders of the participanting countries at the IMO which are Mediterranean countries **or have a frontier with a Mediterranean country, from the Gibraltar Strait to the Black Sea.**

In 1998, four countries accepted the contest: Spain, Greece, Slovenia and Croatia. The next year, several more countries joined to the initiative: Bosnia-Herzegovina, Austria, Turkey, Bulgaria, Israel, Algeria...

The motto of the contest is always: *the problems move, the students don't.* It may seem unbelievely, but the things were easy to establish this international contest. Each country decides what their best date for run the competition is, and the problems are selected from the ones provided by the participants previously. It is clear that nowadays it is really easy to communicate via e-mail which simplifies thing a lot. In Spain, moreover, I have the help of a Mediterranean city, Requena, near Valencia, where one of the most enthusiastic teachers, and very good friend, Prof. Antonio Ledesma López always organizes the Competition very efficiently, usually taking advantage of the holidays on 1st of May.

The countries often use the Mediterranean Math. Competition in the process of selection and/or training of the teams for the International Mathematical Olympiad. This is the case of Spain and Austria. The dates are most of the times selected between march and may, and of course there exists the "gentlemen agreement" for not giving publicity to the problems until the last country has run the contest. The participants are chosen by invitation, but following our rules every country can allow any number of participants to take part in the contest; although just the results of the top ten are considered to give the official results. I am quoting here the regulations concerning the Certificates of Merit:

2.4. All the MMC participants will receive a Certificate of Merit or Participation. The constraints for the issuing of Merit Award Certificates in general for a particular country are as follows:

a)to determinate the overall numbers and levels of gold, silver and bronze award certificates, the following guidelines are to be used:

i)Maximum total number of award certificates $= \left[\frac{n+1}{2}\right]$, *where* $n =$ *total number of MMC contestants.*

ii)scores for gold awards $\geq m + \sigma$,
scores for silver awards $\geq m + \frac{1}{3}\sigma$,
scores for bronze awards $\geq m - \frac{1}{3}\sigma$,

where m = mean MMC score, and σ = standard deviation of all the MMC scores.

b) For a particular country the number of

i)gold awards must be lesser or equal to one,

ii) gold + silver awards must be lesser or equal to three, and

iii) gold + silver + bronze awards must be lesser or equal to seven.

2.5. A Certificate of Honourable Mention will be awarded to any contestant who has not received a merit certificate but has obtained at least a perfect score of 7 for at least one question or has obtained scores of 5 or 6 points for at least two questions.

The background of the Certificate of Merit is this old Spanish map of the Mediterranean Sea, dated in the XIX Century, which I have hung on the wall of my office:

The level of difficulty of the problems can be appreciated by reading the following sample of problems from several years:

Problem 2, 2000 (proposed by Turkey) :Given n pairwise distinct positive numbers $a_1, \cdots, a_n$, prove that, for any ordered $n-$tuple $(\sigma_1, \cdots, \sigma_n)$, where σ_i is $+1$ or -1, there exists a permutation $b_1, \cdots, b_n$ of $a_1, \cdots, a_n$ and an ordered $n-$tuple $(\beta_1, \cdots, \beta_n)$, where each β_i is $+1$ or -1, such that the sign of the expression

$$\sum_{j=1}^{i} \beta_j b_j$$

coincides with $\sigma_i$for any $i \in \{1, \cdots, n\}$.

Problem 1, 2001(Proposed by Austria)Let k be a circle of center O, and P and Q points on k. Let M be the mid-point of PQ and A and C variable points on k such that AC passes through M. ABCD is a trapezoid with k as circumcircle and AB is parallel to CD and both parallel to PQ. Prove that AD and BC intersect in a point X independant of the choice of A on k.

Problem 4, 2001(proposed by Bulgaria) An equilateral triangle $\triangle ABC$ of side 1 is given, and Δ denotes the set of all points lying in its interior or on its boundary. For any $M \in \Delta$, a_M, b_M and c_M denote its distances to the sides BC, CA, AB, respectively. Let $f(M) = a_M^3(b_M - c_M) + b_M^3(c_M - a_M) + c_M^3(a_M - b_M)$.

a) Describe the set $\{M \in \Delta : f(M) \geq 0\}$ geometrically;

b) Find the maximal and the minimal values of $f(M)$ when $M \in \Delta$, and the points where they are attained.

Problem 4, 2004(proposed by Croatia)Let z_1, z_2, z_3 mutually different complex numbers such that

$$|z_1| = |z_2| = |z_3| = 1.$$

Prove that, if the equality

$$\frac{1}{2 + |z_1 + z_2|} + \frac{1}{2 + |z_2 + z_3|} + \frac{1}{2 + |z_3 + z_1|} = 1$$

holds, then the points $A(z_1), B(z_2), C(z_3)$ are the vertices of an equilateral triangle.

Problem 2,1998(proposed by Croatia)a) Show that the polinomial $z^{2n} + z^n + 1$, $(n \in \mathbb{N})$ is divisible by $z^2 + z + 1$ if and only if n is not a multiple of 3.

b) Find a necessary and sufficient condition on the natural numbers p, q for that the polinomial $z^p + z^q + 1$ would be divisible by $z^2 + z + 1$.

Problem 3, 1999(proposed by Bosnia-Herzegovina)

Let a, b and c be non-zero real numbers and x, y and z positive real numbers with $x + y + z = 3$. Prove that

$$\frac{3}{2}\sqrt{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}} \geq \frac{x}{1+a^2} + \frac{y}{1+b^2} + \frac{z}{1+c^2}.$$

Problem 4,1999 (proposed by Bulgaria)

Let ABC a triangle with sides $BC = a, CA = b, AB = c$ and such that $\widehat{B} = 4.\widehat{A}$.

Prove that

$$ab^2c^3 = \left(b^2 - a^2 + ac\right)\left(a^2 - b^2 + ac\right)^2.$$

As I already said, it was surprinsingly easy coordinate the international work, and I am very grateful to the many colleagues in the different countries who agree on running the MMC, marking the problems and sending me their results in order to determine the Merit Certificates according the rules quoted before.

I would like quote here the names of Ted Bolis, Darjo Felda, Zeljko Hanjs, Shay Gueron, Robert Geretchsläger, Sefket Arslanagic, Semih Koray, Sava Grozdev, Mekki Daheche, among the full set of people who is working in this iniciative. And a, for me, sufficient proof that the work of selecting problems is made correctly, is that the MMC is one of the competitions included each year in a specialized publication of the Mathematical Association of America (Olympiads around the World).

Perhaps a comment about the surname I gave the Competition may be necessary. For me, the memory of Peter O'Halloran is always in my heart. I met Peter in several occassions, most of them related to Mathematics Competitions or the World Federation of National Mathematics Competitions Conferences and other Congresses. When I decided to propose the MMC, I included his name in the official title of the contest. It is my tribute to his memory.

c) **THE DIGITAL JOURNAL** *REVISTA ESCOLAR DE LA OLIMPIADA IBEROAMERICANA DE MATEMÁTICA*

The O.E.I. (Organization of Iberoamerican States for the Education, the Science and the Culture) is a overnational organization which supports, since its beginning, the Iberoamerican Mathematical Olympiad.Their members are the Ministries of Education from the Iberoamerican countries. The permanent site of the OEI is Madrid. So, in april 2002, I proposed to the O.E.I. to be the host of a new digital school journal for teachers and students of middle level, for which I was appointed as Editor. I reproduce here some of the words of the project :

Most of the countries with strong tradition and good results in the I.M.O. have excellent school journals, some of them with more than 100 years of life, as Romania or Hungary....

In the iberoamerican area, the situation is very different. During a short time (1996-1998), the journal SIPROMA, edited by the O.E.I., had a notable success among the professors and students of Olympiads, but the diffusion was very limited and the echo was small...

The project was accepted by the O.E.I. and so the issues of the digital journal *Revista Escolar de la O.I.M.,* began to appear, every two months, since May 2002.

In order to take advantage of Internet, the journal has no paper version , but it is possible to download the different issues (complete) or the relevant pages or sections in pdf format.

The sections of the journal are the following:

1.-Articles, Notes and Lessons of olympic training

2.- Problems for the youngest (proposed and sometimes solved)

3.- Problems of the Middle level and from Olympiads (proposed and solved)

4.-Problems of higher level - up to College level(proposed and solved)

5.- Mathematical Entertainements

6.- Comments of books and of web pages.

The languages of the journal are in Spanish and Portuguese, the two official languages of O.E.I. The articles submitted in other language are translated into either of them. The intention is that good articles can be read without additional work by people from countries who have no tradition in mathematics, like some from Center and South America. We are very conscious that some countries of the area do not need, strictly speaking, this sort of help, but others are in real need of any help that can be provided.

The subscription to the journal is free and it is possible to subscribe from any web page of the O.E.I.

In august of 2006, when this contribution is being written, we have more than 11500 suscriptors, and the tendency is growing up : each day, about 5 to 10 new subscriptions arrive at the O.E.I.

The URL of the journal is the following:

http://www.campus-oei.org/oim/revistaoim/