The Role of Challenges in Mathematical Learning

Ed Barbeau

Tuesday, May 2, 2006

1. The Goals of the ICMI Study Conference

All of us in mathematics have an interest in sharing our understanding and enjoyment of the subject as far as possible. Even the researcher in the most abstruse areas needs to have some group that can understand his work and continue it, and a larger group that can at least grasp the significance of the achievement. Also, one could argue that most of our progress in science, technology and social organization is due to mathematics, so a person who is to be connected into the world should have some understanding of the processes of abstraction, organization and analysis in mathematics.
But it is not only in a utilitarian way that mathematics looms so large in our culture; games, puzzles and other recreations have diverted people since time immemorial, and so mathematics can lead to an enrichment of one's life experience.

There is a term in evangelical Christianity that it relevant to the exposition of mathematics, whether by the teacher or the curator in a museum: "witness". At its best, teaching and curating is a sharing of one's own love and experience of the subject. They are witnesses for mathematics. A major goal of the Study Conference is the examination of one way in which this witnessing might be done.

So our goal is to help people understand what mathematics is, and what its processes are. Thus, as educational researchers, we should have some vision of mathematics and some theories of how people learn and can be effectively taught that will inform our discussions.
Next, we need to focus on our immediate topic - challenges - and decide what constitutes a challenge, what role they play in our dialogue with various publics and how essential they are in our portrayal to the world of our discipline.

We now have many papers from the participants to begin with.
Some provide examples of challenges; others deal with classroom situations; some have put forward a theoretical framework for the use of challenges in pedagogy, while others discuss programs on whose effectiveness research is underway. So it seems to me that two main things should come out of the conference: what the use of challenges relate to our views of what mathematics is all about and how it is learned, and what research and experience has told us or will tell us about how they can be effectively employed. In the Study Volume, I hope that we will have some agreement about useful categories for the study of challenges in education and also a list of important questions and directions for possible future research.

Despite the concerns of some that competitions might dominate the proceedings, the bulk of the papers deal with classroom practice.

I think that this is all to the good, but I would have preferred a slightly larger contingent to talk about contests, exhibitions and applications.

In this essay, I will look at some assumptions and possible questions.
In a second essay, I have attempted a classification of challenges.
The word "learner" is meant to apply to, not only pupils in schools, but also their teachers, amateurs and members of the general public.

2. Assumptions and orientation

I begin with a number of statements about challenges whose truth could be affirmed or questioned and rejected or modified, and which might serve as a backdrop for our discussions.
Some of these are suggested by the submitted papers.

(a) Challenges force the recipient to engage mathematics and arouse a more authentic view of the nature of mathematics. They involve the ingredients of investigation, hypothesizing, analysis, and lead the learner to behave like a mathematician. They help the learner move beyond a mechanistic approach to the subject towards a more focussed use of technique and process. Are challenges uniquely suited to this purpose? What sort of challenge?

(b) The mastery of mathematics, in common with other areas of human achievement, such as music, involves an internalization process. At first the learner is presented with procedures and ideas from the outside in a teaching situation.
But with time and practice, the learner often can come to own the mathematics. As with other activities, this arouses a feeling of autonomy and of being able to control the situation. At this stage, the learning becomes robust and the learner feels that what has been mastered cannot be forgotten. This transformation might happen quickly with a sudden sense of things falling into place.
If this phenomenon actually occurs, how is it described and assessed? Importantly for the study, what role do challenges play in bringing about the transition from alien to familiar knowledge and understanding? Are challenges particularly or uniquely suited to this process?

(c) In the teaching and learning of mathematics, two processes seem to be occurring simultaneously. First, there is the linear presentation of the mathematics, where the material is organized and presented in a certain order.

But secondly, there is a more holistic phenomenon in which the learner takes in the mathematics more haphazardly, building up a cloud of connections, heuristics and associations from which they operate.
Indeed, the learner may come to the mathematics with some preconceptions or even misconceptions that need to be identified.

These will be affected by the examples and exercises that the learner is exposed to, and we should examine the role of challenges in this regard. One example of this might be with the concept of function.
This is formalized quite late in the syllabus, but surely even at the elementary level pupils see and deal with in an informal way with many instances of independent and dependent variables through formulas, lists and scientific relationships, that they come to the formalism with some ideas about what is under discussion.
Perhaps the difficulty some have with this concept is due to some dissonance between their experiences and what they perceive the formalism to be saying.

It is not clear to me that one necessarily precedes the other.
Even if the student has no background and is first faced with an orderly presentation of material, he will unlikely appreciate its significance but will try to make whatever sense of it he can. Some kind of intuition and context has to inform the formalism.
However, a broader undertanding can grow only if, from time to time, definitions, enunciation of results and descriptions of processes are made precise, and students see how some mathematics is contingent on others. In this way, through a "packing down" process, students can avoid being overwhelmed.

Challenges, by focussing the mind on a particular situation, induce the learner to review what has been learned and turn over in the mind facts and impressions that might possibly be relevant.

(d) While much is made of the utility of mathematics, many are attracted by its recreational and cultural qualities, particularly its elegance. A good challenge captures the imagination and allegiance of the solver, and the best deserve to be embedded in the human corporate memory along with the finest works of art.
There is something for the laity, witness the current popularity of puzzles such as Sudoku, and the eternal appeal of magic squares.
Some challenges are quite ancient, and link us to people in other places and times. If we conceive of education as connecting young into the whole panorama of human experience, how do challenges play into this, and to what extent should such ideas be part of the Study?

(e) Challenges promote depth of mathematical thinking, and foster analytical and expository skill. They foster a research mentality.

(f) Challenges help secure curricular knowledge, by highlighting the significance of what was supposed to be learned. If one teaches to cover a list of topics in some syllabus, there is the risk of presenting pupils with a great deal of undifferentiated knowledge that simply has to be memorized. But some mathematical facts and processes are more important and deeper than others. For example, a pupil might be taught techniques of factoring and expanding algebraic expressions, but not appreciate their role in laying bare information that might be latent in them. When they are presented with a problem whose solution requires some particular form, say the factored form or sum or squares form of a quadratic, then mathematical artifacts have greater meaning and there is an incentive to master them.

(g) Challenges, properly calibrated, ease the transition between different levels of education, and between school and professional life. They foster a ever more mature apprehension of mathematics.
For the lay person, they engender an appreciation of the work of mathematics and an understanding of mathematics they may come across in everyday life.

3. Questions

Here are some things that might inform our discussions:

(a) The pedagogical dimension:
What makes a good challenge? Is it one that helps the student realize significance? Is it one that attunes the student to detail and brings about a more alert and discriminating state of mind? What are the attributes that we want to encourage in the learner? What are the pedagogical goals that challenges can help us achieve?

(b) Appropriateness: What makes a challenge right for a situation?

(i) Comeliness: what is there that can draw people in? how can it motivate students?

(ii) Continuity: A challenge should not be a mere exercise for the parroting of known procedures, but should entice the recipient into new territory. At the same time, it should not require the crossing of an overly great procedural or conceptual gap. It must be a natural extension of the recipient's knowledge and skills that the recipient can be reasonably expected to handle.

(iii) Universality: In general, all students and even members of the general public can be appropriately challenged and gain satisfaction from surmounting the challenge, regardless of ability and situation.

This raises the issue of assessing a situation and designing challenges appropriately.

(c) Making silk purses out of sow's ears: Much of the material that appears in textbooks is mundane. How can exercises and standard classroom material be used to generate challenges?
How can we extend material beyond the predictable? For example, instead of simply providing a page of subtraction exercises to be done by a class of ten-year-olds, one

might begin by asking them to construct two numbers from the digits 1, 2, 3, 4, 5, 6 (using each digit exactly once) and subtract the smaller from the larger; follow this up with the challenge to make the difference as small as possible. In this way, one can achieve some practice while providing the pupils with a goal. Or one might see in a text book the exercise to factor x^2 - 5x - 6, and note that x^2 - 5x + 6 can also be factored over the integers and ask for more examples of a similar type, or whether such a change of sign preserves factorability in general. (On my website, www.math.utoronto.ca/barbeau, there are some problems on factoring differences of squares and on quadratics that take students beyond the usual fare.)

(d) The aesthetic dimension: What are the characteristics of a challenge that deserves to live? What makes a challenge popular?
What do such challenges teach us about the nature of mathematics?

(e) Psychological effect: What encourages and discourages recipients of challenges? What are indicators?

(f) Measuring the value of a challenge: What do we look for?

(g) The role of technology: Technology impacts on the Study in a number of ways. First, it expands the range of techniques that can be used in addressing challengings, by allowing the learner to focus on the substance rather than computational issues, and allowing for the rapid investigation of examples and instances. Secondly, it allows for more ready contact between peers and between teachers and students, who can now operate over wide distances and communicate rapidly. Thirdly, it allows access to a wide range of materials. There are a number of issues that involve the use of technology, including

(i) effective use (active rather than passive, expansive rather than contractive);

(ii) breaking isolation (bringing together people with common interests and similar situations, mentoring and support networks);

(iii) sources of resources and feedback (technological assistance for working on challenges, software, access to the literature, suggestions).

(h) Professional development of teachers in the use of challenges:

(i) How do we induce a state of mind and an experience whereby teachers themselves will accept mathematical challenges?

(ii) How can teachers work with textbook material, exercises and examples, and turn them into challenging situations?

(iii) How can teachers adapt challenges to suit the circumstances? How do they ensure that pupils are encouraged and that pedagogical goals are reached?

(iv) How do teachers plan for the use of challenges, and assess performance of pupils?

(i) Challenges in the curriculum

(i) How can challenges be incorporated as a valued part of the curriculum?

(ii) What part do challenges play in assessment, particular in examinations?

(j) Challenges outside of the classroom: What are appropriate ways of engaging the public (exhibitions, magazines, newspapers, contests)?

4. I am indebted to Derek Holton and Mark Saul for their comments on earlier versions of this essay.