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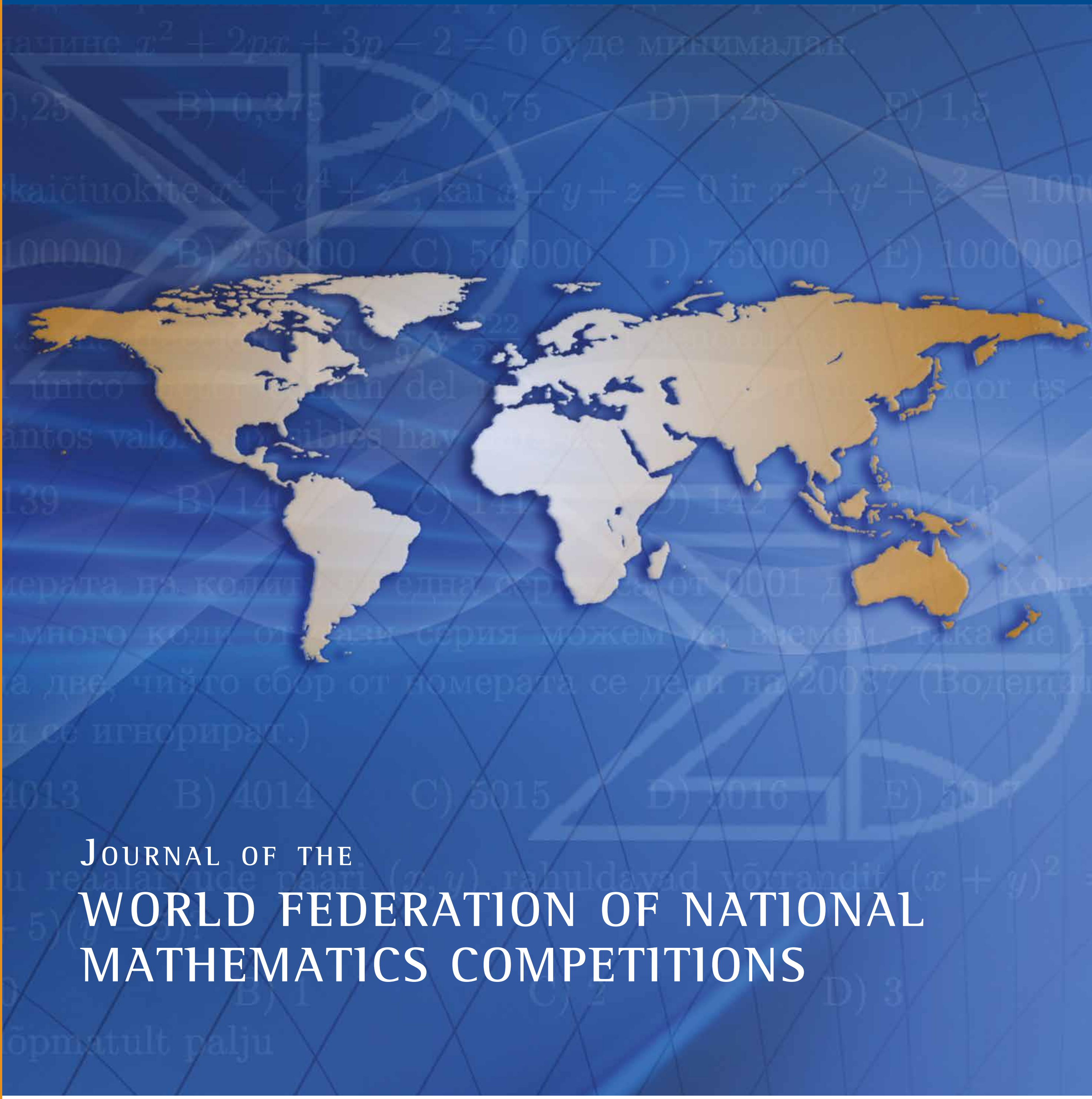
MATHEMATICS COMPETITIONS



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AUSTRALIAN MATHS TRUST

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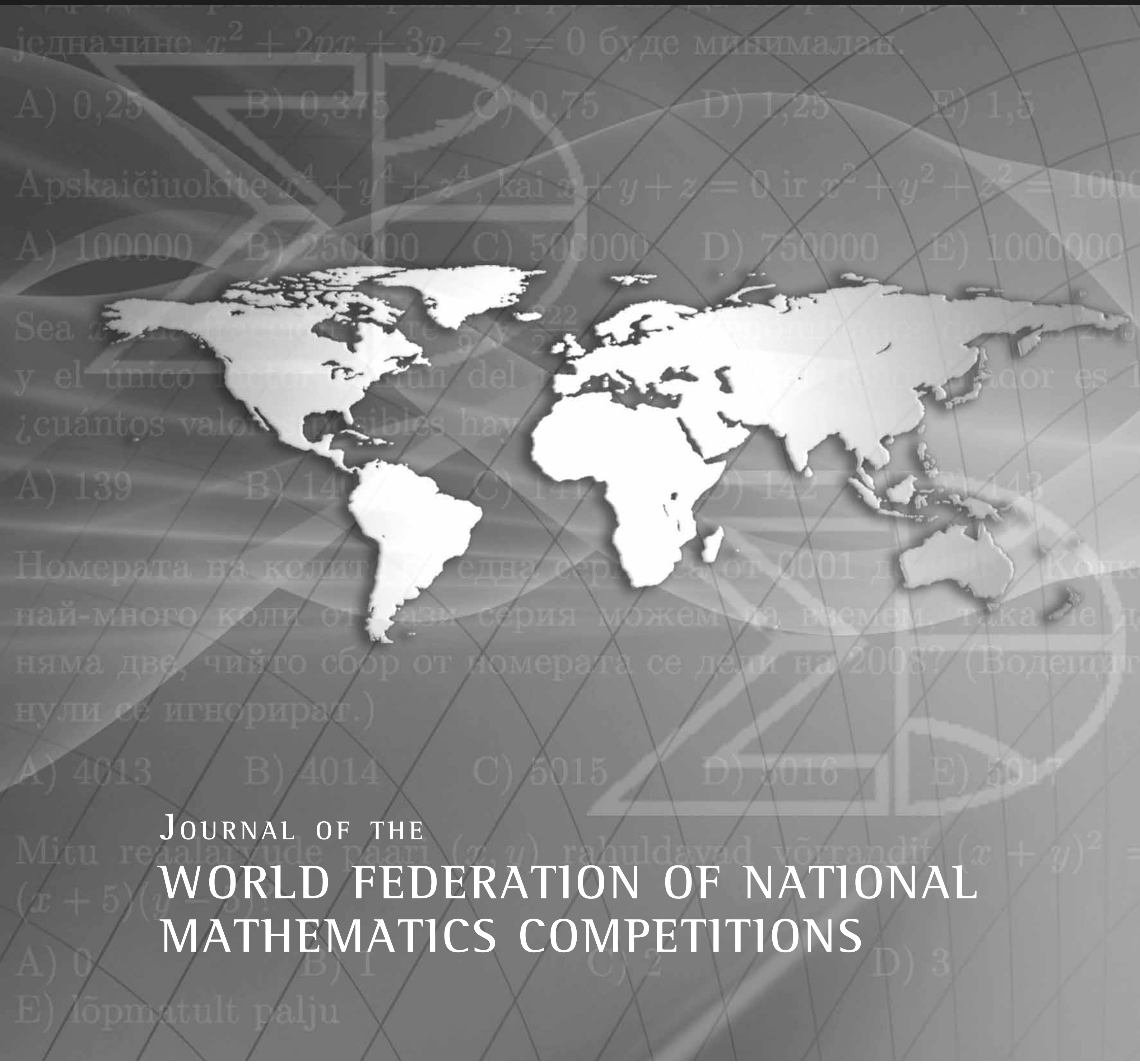

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From the President

Dear readers,
This is my first message as President of the WFNMC, and I cannot stress enough how honored and humbled I am to have this opportunity to follow in the footsteps of the wonderful mathematicians and educators that have held this position before me. I hope that I will be able to live up to the high standards set by these talented men and women.

Special thanks go to the outgoing President Kiril Bankov, whose duty it was to sail the Federation ship through the unbelievably rough seas of Covid. When he took over the helm (to continue the nautical metaphor for the moment), no one could have reasonably expected a disruption of this magnitude in the world of mathematics competitions, and therefore in the WFNMC. For instance, the mini-conference that was planned for ICME-14 in Shanghai had to be cancelled both in 2020 and again in 2021. Meanwhile, many national and international competitions were struggling to somehow redefine themselves on-line in order to keep up some sense of continuity.

Fortunately, despite Covid, it was possible for the ninth WFNMC congress to be held in Sofia, Bulgaria from July 19th to July 25th, 2022 with just a few restrictions. I was not personally able to attend, as I had tested positive for the virus and was stuck at home in quarantine. Due to the wonders of modern technology, however, I was still able to participate virtually, attending all the talks and contributing my own input online. Many thanks go out to the organisers of the conference, Borislav Lazarov and his team, for the great work they did to make it all possible.

There are a few small ways in which I hope to widen the scope of the activities of the WFNMC during my tenure as president. Already at the Sofia congress, the new topic group on technology in mathematics competitions showed how important various aspects of modern communications technology have become for mathematics competitions. Of course, there are many competitions being held online now, but even more traditional types of competitions now rely heavily on the organizational advantages of appropriate software applications. Cooperation in this area will certainly become ever more important in the years to come, and the WFNMC appears predestined as a forum for such international collaboration. I feel that this is a topic area we should give more thought to in the coming years.

Also, I hope to be able to find ways to motivate even more interested people from all around the world to become involved in the activities of the WFNMC. This would include math educators interested in the didactical background of mathematics competitions and research into the various aspects of this topic, a group that has grown appreciably in the last few years. I hope it will be possible for this group to find a home with us. Also, with mathematical competitions spreading across the whole world, it is my hope to widen the geographical reach of the Federation to include areas that have not traditionally been well represented in our group.

There are also a few other ideas floating around that may turn out to be quite exciting. We will have to wait and see.

For now, I invite you to be as active as you can be in our group. Perhaps you have an interesting idea for an article in this very journal. If so, you are very welcome to submit it! Please tell your friends and colleagues about us; perhaps they would like to join us as well. Of course, nominations for the Erdös Award are always welcome, and I hope to see you at our conferences. Remember, the Federation is what we make of it!

Robert Geretschläger

Editor's Page

Dear Competitions enthusiasts, readers of our *Mathematics Competitions* journal!

Mathematics Competitions is the right place for you to publish and read the different activities about competitions in Mathematics from around the world. For those of us who have spent a great part of our life encouraging students to enjoy mathematics and the different challenges surrounding its study and development, the journal can offer a platform to exhibit our results as well as a place to find new inspiration in the ways others have motivated young students to explore and learn mathematics through competitions. In a way, this learning from others is one of the better benefits of the competitions environment.

Following the example of previous editors, I invite you to submit to our journal *Mathematics Competitions* your creative essays on a variety of topics related to creating original problems, working with students and teachers, organizing and running mathematics competitions, historical and philosophical views on mathematics and closely related fields, and even your original literary works related to mathematics.

Just be original, creative, and inspirational. Share your ideas, problems, conjectures, and solutions with all your colleagues by publishing them here. We have formalized the submission format to establish uniformity in our journal.

Submission Format
FORMAT: should be LaTeX, TeX, or for only text articles in Microsoft Word, accompanied by another copy in pdf. However, the authors are strongly recommended to send article in TeX or LaTeX format. This is because the whole journal will be compiled in LaTex. Thus your Word document will be typeset again. Texts in Word, if sent, should mainly contain non-mathematical text and any images used should be sent separately.
ILLUSTRATIONS: must be inserted at about the correct place of the text of your submission in one of the following formats: jpeg, pdf, tiff, eps, or mp. Your illustration will not be redrawn. Resolution of your illustrations must be at least 300 dpi, or, preferably, done as vector illustrations. If a text is embedded in illustrations, use a font from the Times New Roman family in 11 pt.
START: with the title centered in Large format (roughly 14 pt), followed on the next line by the author(s)' name(s) in italic 12 pt.
MAIN TEXT: Use a font from the Times New Roman family or 12 pt in LaTex.
END: with your name-address-email and your website (if applicable).
INCLUDE: your high resolution small photo and a concise professional summary of your works and titles.
Please submit your manuscripts to María Elizabeth Losada at
`director.olimpiadas@uan.edu.co`

We are counting on receiving your contributions, informative, inspired and creative. Best wishes,

Maria Elizabeth Losada
EDITOR

Mathematics, its History, and Mathematical Olympiads: A Golden Braid[1]

Alexander Soifer

Alexander Soifer is a Russian-born American mathematician and mathematics author. His works include over 400 articles and 13 books.

Every spring since 1983, Soifer, along with other mathematician colleagues, sponsor the Colorado Mathematical Olympiad (CMO) at the University of Colorado in Colorado Springs. In May 2018, in recognition of 35 years of leadership, it was renamed the Soifer Mathematical Olympiad.

In 1991 Soifer founded the research quarterly Geombinatorics, and publishes it with the Geombinatorics editorial board.

In July 2006 at the University of Cambridge, Soifer was presented with the Paul Erdős Award by the World Federation of National Mathematics Competitions.

Soifer was the President of the World Federation of National Mathematics Competitions from 2012 to 2018. His Erdős number is 1.

Abstract
A beautiful braid of mathematics, history, and mathematical Olympiads will be presented 'in the flesh.' I will convey 4 stories, each featuring a braid of history, old mathematical papers that often contain unnoticed or little noticed treasures that, once dug out, lend themselves to creating new original problems for mathematical Olympiads. Some of these stories have appeared ([3], [5], [6]); others are waiting for an appearance in the new edition [4], and so you will be able to preview some pages of my future book.

Keywords: Mathematics, research, History, historical research, Soifer Mathematical Olympiad, Colorado, problem creating.

Story 1. Merry Go Round, or A Story of Colored Polygons and Arithmetic Progressions

The Story of Creation

I recall April of 1970. The thirty members of the Jury of the Fourth Soviet Union National Mathematical Olympiad, of which I was one of the two youngest (others included Konstantinov, Vasiliev, Gutenmacher, Egorov, Makar-Limanov, Tolpygo), stayed at a fabulous white castle with a white

[1]This essay was presented as a keynote lecture at the 9th Congress of the World Federation of National Mathematics Competitions, July 2022, Sofia, Bulgaria. It will appear in the volume of the Congress' Proceedings

watchtower, halfway between the cities of Simferopol and Alushta, nestled in the sunny hills of Crimea, surrounded by the Black Sea. This castle should be familiar to movie buffs: in 1934 the Russian classic film Vesyolye Rebyata (Jolly Fellows) was filmed here by Sergei Eisenstein's long-term assistant, director Grigori Aleksandrov. The problems had been selected and sent to the printers. The Olympiad was to take place a day later, when something shocking occurred.

Suddenly, a mistake was found in the only solution the jury had for the problem created by Nikolai (Kolya) B. Vasiliev, the Vice-Chair of the Olympiad and a fine problem-creator, head of the Problems Section of the journal Kvant from its inception in 1970 to the day of his untimely passing. What should we do? This question virtually monopolized our lives.

We could just cross out this problem on each of the six hundred printed problem sheets. In addition, we could select a replacement problem, but we would have to write it in chalk by hand in every examination room, as there would be no time to print it. Both options were pretty embarrassing, desperate solutions for the Jury of the National Olympiad, chaired by the great mathematician Andrej Nikolaevich Kolmogorov, who was to arrive the following day. The best resolution, surely, would have been to solve the problem, especially because its statement was quite beautiful, and we had no counterexample to it either.

Even today, half a century later, I can close my eyes and see how each of us, thirty members of the jury, all fine problem-solvers, worked on the problem. A few sat at the table as if posing for Rodin's Thinker. Some walked around as if measuring the room's dimensions. Andrei Suslin, who would later prove the famous Serre's Conjecture (Daniel Quillen proved it independently and got a Field's Medal primarily for that) went out for a thinking hike. Someone was lying on a sofa with his eyes closed. You could hear a fly. The intense thinking seemed to stop the time inside the room. We were unable, however, to stop the time outside. Night fell, and with it fell our hopes for solving the problem in time.

Suddenly, the silence was interrupted by a victorious outcry "I got it!" echoed through the halls and the watchtower of the castle. It came from Aleksandr "Sasha" Livshits, an undergraduate student at Leningrad (St. Petersburg) State University, and former winner of the Soviet and the International Mathematical Olympiads (a perfect 42 score at the 1967 IMO in Yugoslavia). His number-theoretic solution used the method of trigonometric sums. This, however, was, the least of our troubles: we immediately translated the solution into the elementary language of colored polygons.

Now we had options. A decision was reached to leave the problem in. The problem and its solution were too beautiful to be thrown away. We knew, though, that the chances of receiving a single solution from six hundred bright high school Olympians were very slim. Indeed, nobody solved it.

The Problem of Colored Polygons

Problem 1.1 (N. B. Vasiliev; IV Soviet Union National Olympiad, 1970). Vertices of a regular n-gon are colored in finitely many colors (each vertex in one color) in such a way that for each color all vertices of that color form themselves a regular polygon, which we will call a *monochromatic* polygon. Prove that among the monochromatic polygons there are two polygons that are congruent. Moreover, the two congruent monochromatic polygons can always be found among the monochromatic polygons with the least number of vertices.

Solution of problem 1.1 by Alexander Livshits (in 'polygonal translation'). Let me divide the problem into three parts: Preliminaries, Tool, and Proof.

Photo: Aleksandr "Sasha" Livshits with a USSR typical mathematical Olympiad first prize: a giant pile of math. books.

Preliminaries. Given a system S of vectors $\vec{v_1}, \vec{v_2}, \cdots, \vec{v_n}$ in the plane with a Cartesian coordinate system, all emanating from the origin O. We would call the system S *symmetric* if there is an integer $k, 1 \leq k < n$, such that rotation of every vector of S about O through the angle $\frac{2\pi k}{n}$ transforms S into itself.
Of course, the sum $\Sigma\vec{v_j}$ of all vectors of a symmetric system is $\vec{0}$, because $\Sigma\vec{v_j}$ does not change under rotation through the angle $0 < \frac{2\pi k}{n} < 2\pi$.

Place a regular n-gon P_n in the plane so that its center coincides with the origin O. Then the n vectors drawn from O to all the vertices of P_n form a symmetric system (Figure 4).

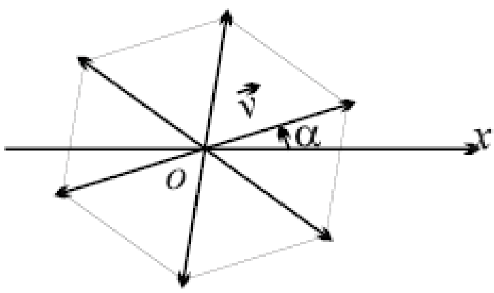

Figure 1

Let $\vec{v}$ be a vector emanating from the origin O and making the angle α with the ray OX (Figure 4).

Symbol T^m will denote a transformation that maps $\vec{v}$ into the vector $T^m\vec{v}$ of the same length as $\vec{v}$, but making the angle $m\alpha$ with OX (Figure 5).

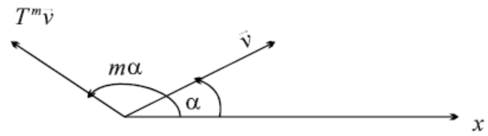

Figure 2

To check your understanding of these concepts, please prove the following tool on your own.
Tool 1.2. Let $\vec{v_1}, \vec{v_2}, \cdots, \vec{v_n}$ be a symmetric system S of vectors that transforms into itself under the rotation through the angle $0 < \frac{2\pi k}{n} < 2\pi$, $1 \leq k < n$, (you can think of $\frac{2\pi k}{n}$ as the angle between two neighboring vectors of S). A transformation T^m applied to S produces the system T^mS of vectors $T^m\vec{v_1}, T^m\vec{v_2}, \cdots, T^m\vec{v_n}$ that is symmetric if n does not divide km. If n divides km, then $T^m\vec{v_1} = T^m\vec{v_2} = \cdots = T^m\vec{v_n}$.
Solution of problem 1.1. We will argue by contradiction. Assume that the vertices of a regular n-gon P_n are colored in r colors and we got subsequently r monochromatic polygons: n_1-gon P_{n_1},n_2-gon P_{n_2} $\cdots$,n_r-gon P_{n_r}, such that no pair of congruent monochromatic polygons is created, i.e.,

$$n_1 < n_2 < \cdots < n_r.$$

We create a symmetric system S of n vectors going from the origin to all vertices of the given n-gon P_n. In view of tool 1.2, the transformation T^{n_1} applied to S produces a symmetric system $T^{n_1}S$. The sum of vectors in a symmetric system is zero, of course.
On the other hand, we can first partition S in accordance with its coloring into r symmetric subsystems $S_1, S_2, \cdots, S_r$, then obtain $T^{n_1}S$ by applying the transformation T^{n_1} to each system S_i separately, and combining all $T^{n_1}S_i$. By tool 1.2, $T^{n_1}S_i$ is a symmetric system for $i = 2 \cdots r$, but $T^{n_1}S$ consists of n_1 identical non-zero vectors. Therefore, the sum of all vectors of $T^{n_1}S$ is not zero. This contradiction proves that the monochromatic polygons cannot be all non-congruent.
□
Prove the last sentence of problem 1.1 on your own:

Problem 1.3. Prove that in the setting of problem 1.1, the two congruent monochromatic polynomials must exist among the monochromatic polynomials with the least number of vertices.
Readers familiar with complex numbers, may have noticed that in the proof of problem 1.1 we can choose the given n-gon P_n to be inscribed in a unit circle, and position P_n with respect to the axes so that the symmetric system S of vectors could be represented by complex numbers, which are precisely all n-th degree roots of 1. Then the transformation T^m would constitute raising these roots into the m-th power.

Translation into the Language of AP's

You might be wondering what this striking problem of colored polygons has in common with arithmetic progressions (AP), which are part of the section's title. Actually, everything! Problem 1.1 can be nicely translated into the language of infinite arithmetic progressions, or AP's for short.

Problem 1.4. In any coloring (partition) of the set of integers into finitely many infinite monochromatic AP's, there are two AP's with the same common difference. Moreover, the largest common difference necessarily repeats.
Equivalently:

Problem 1.5. Any partition of the set of integers into finitely many AP's can be obtained only in the following way: $\mathbb{N}$ is partitioned into k AP's, each of the same common difference k (where k is a positive integer greater than 1); then one of these AP's is partitioned into finitely many AP's of the same common difference, then one of these AP's (at this stage we have AP's of two different common differences) is partitioned into finitely many AP's of the same common difference, etc.
It was as delightful that our striking problem allowed two beautiful distinct formulations, as it was valuable: only because of that I was able to discover the prehistory of our problem.

Prehistory

A year after I first published the history of this problem in my 1994 Olympiad, I discovered that this unforgettable story actually had a prehistory! I became aware of it while watching a video recording of Ronald L. Graham's most elegant lecture Arithmetic Progressions: From Hilbert to Shelah. To my surprise, Ron mentioned our problem in the language of integers partitions into AP's. Let me present the pre-history through the original e-mails, so that you would discover the story the same way I have.

April 5, 1995; Soifer to Graham:
In the beginning of your video "Arithmetic Progressions," you present a problem of partitioning integers into AP's. You refer to Mirsky–Newman. Can you give me a more specific reference to their paper? You also mention that their paper may not contain the result, but that it is credited to them. How come? When did they allegedly prove it?

April 5, 1995; Graham to Soifer:
Regarding the Mirsky–Newman theorem, you should probably check with Erdős. I don't know that there ever was a paper by them on this result. Paul is in Israel at Tel Aviv University.

April 6, 1995; Soifer to Erdős:
In the beginning of his video "Arithmetic Progressions," Ron Graham presents a problem of partitioning natural numbers into arithmetic progressions (with the conclusion that two progressions have the same common difference). Ron refers to Mirsky–Newman. He gives no specific reference to their paper. He also mentions that their paper may not contain the result, but that it is credited to them … Ron suggested that I ask you, which is what I am doing.
I have good reasons to find this out, as in my previous book and in the one I am writing now, I credit Vasiliev (from Russia) with creating this problem before early 1970. He certainly did, which does not exclude others from discovering it independently, before or after Vasiliev.

April 8, 1995; Erdős to Soifer:
In 1950 I conjectured that there is no exact covering system in which all differences are distinct, and this was proved by Donald Newman and [Leon] Mirsky a few months later. They never published anything, but this is mentioned in some papers of mine in the 50s (maybe in the Summa

Brasil. Math. 11(1950), 113–123, but I am not sure).

April 8, 1995; Erdős to Soifer:
Regarding that Newman's proof, look at P. Erdős, On a problem concerning covering systems, Mat. Lapok 3(1952), 122–128.

I am looking at these early Erdős's articles. In the 1950 paper he introduces covering systems of (linear) congruences. Since each linear congruence defines an AP, we can talk about covering system of AP's and define it as a set of finitely many infinite AP's, all with distinct common differences, such that every integer belongs to at least one of the AP's of the system. In a 1952 paper Paul introduces the problem for the first time in print (in Hungarian!):

> I conjectured that if system [of k AP's with common differences n_i respectively] is covering, then
>
> $$\sum_{i-1}^{k} \frac{1}{n_i} > 1 \ \ (8)$$
>
> that is the system does not uniquely cover every integer. This, however, I could not prove. For (8) Mirsky and Newmann [Newman] gave the following witty proof (the same proof was found later by Davenport and Rado as well).

Wow, Leon Mirsky, Donald Newman, Harold Davenport and Richard Rado – quite a company of distinguished mathematicians, who worked on this bagatelle! Erdős then proceeds with presenting this group's proof of his conjecture, which uses infinite series and limits.

In viewing old video recordings of Paul Erdős' lectures at the University of Colorado at Colorado Springs, I found a curious historical detail that Paul mentioned in his March 16, 1989 lecture: he created this conjecture in 1950 while traveling by car from Los Angeles to New York!

Completing the Go-Round

In 1959 Paul Erdős and János Surányi published a book on the Theory of Numbers. In the 2003 English translation of its 1996 2nd Hungarian edition, Erdős and Surányi present the result from the Erdős's 1952 paper:

> In a covering system of congruences [AP's], the sum of the reciprocals of the moduli is larger than 1.

Erdős and Surányi then repeat Mirsky–Newman–Davenport–Rado proof from Erdős' Hungarian 1952 paper and call it Theorem 3. Then comes a surprise:

> A. Lifsic [sic] gave an elementary solution to a contest problem that turned out to be equivalent to Theorem 3.
> Based again on exercises 9 and 10, it is sufficient to prove that it is not possible to cover the integers by finitely many arithmetic progressions having distinct differences in such a way that no two of them share a common element.

Erdős and Surányi then repeat what we, the jury of the Soviet National Mathematical Olympiad, discovered in May 1970, the trick of converting the calculus problem into an Olympiad problem about colored polygons! Here is their text:

> Wind the number line around a circle of circumference d. On this circle, the integers represent the vertices of a regular d-sided polygon… The arithmetic progressions form the vertices of disjoint regular polygons that together cover all vertices of the d-sided polygon

Erdős and Surányi continue by repeating, with credit, Sasha Livshits's solution of Kolya Vasiliev's Problem of Colored Polygons that we have seen at the start of this story (they credit the 1988 Russian Olympiad book by Vasiliev and Andrei Egorov as their source). We have thus come a full circle, a Merry-Go-Round from the Soviet Union Mathematical Olympiad to Paul Erdős and back to the same Olympiad. I hope you have enjoyed the ride!

Story 2. Issai Schur and Problem 5 of the 36th Soifer Mathematical Olympiad

It is tempting to offer young mathematicians to solve lesser-known beautiful problems from the mathematical past. This time I chose a beautiful result, published in German in 1916. A particular case would have caused insurmountable difficulties:

SMO-36, Problem 5. Can You Color Integers?

Can each of the integers $1; 2; \cdots ; 581, 130, 733$ be colored in one of 19 colors so that no color contains numbers x, y, z such that $x + y = z$?
Are you scared?
You should be! :)

Hint: for $n = 19$, $\frac{3^n-1}{2} = 581, 130, 733$.

This is a good illustration that the general case may be easier than a particular one, because the general case may contain a conjecture. And so, I offered the easier, general case at the 36th Soifer Mathematical Olympiad:

Problem 36.5. Can You Color Integers? Given a positive integer n. Can each of the integers $1, 2, \cdots , \frac{3^n-1}{2}$ be colored in one of n colors so that no color contains numbers x, y, z such that $x + y = z$?

Solution. Coloring will be constructed by induction. The case $n = 1$ is trivial: one number requires one color. Assume that the statement is true for some n, i.e., there is a coloring of the numbers of the set $T: 1, 2, \cdots, \frac{3^n-1}{2}$ in n colors, not creating a monochromatic triple x, y, z such that $x + y = z$. Look now at the set $R: 1, 2, \cdots, \frac{3^n-1}{2}$ obtained for $n + 1$. I partition it into 3 subsets:

$$\begin{array}{cccc} 1 & 2 & \cdots & \frac{3^n-1}{2} \\ \frac{3^n-1}{2}+1 & \frac{3^n-1}{2}+2 & \cdots & 3^n \\ 3^n+1 & 3^n+2 & \cdots & \frac{3^{n+1}-1}{2}. \end{array}$$

The first subset can be properly colored due to the inductive assumption. The entire second subset we color in color $(n + 1)$. Since $\frac{3^n-1}{2}$–$(3^n + 1) + 1 = \frac{3^{n+1}-1}{2}$, the third subset has exactly the same number of elements as the first one, and we color it by the translation of the coloring of the first subset by 3^n. More precisely: if a of the first subset is colored in color m, we color $a + 3^n$ of the third subset in color m. Let us now prove that there is no monochromatic triple x, y, z with $x + y = z$.

If x, y both belong to the first subset, and their sum $x+y$ is in the first subset, then by the inductive assumption the triple is not monochromatic. If x, y both belong to the first subset and $x + y$ is in the second subset, then the sum is in color $n + 1$, and the triple is not monochromatic. If x, y both belong to the first subset, $x + y$ cannot belong to the third subset – in all cases we get no monochromatic triple.

The sums of any two numbers from the second subset belong to the third subset, thus again preventing a monochromatic triple.

If x belongs to the first subset and y and thus $x+y$ belong to the third subset, we do not get a monochromatic triple. Indeed, in this case y–3^n has the same color as y (by our definition of colors in the third subset). And if the triple $x, y, x + y$ is monochromatic, then the triple x, y–$3^n, x + (y$–$3^n)$ is monochromatic and entirely in the first subset, which contradicts our inductive assumption. Finally, if x, y both belong to the third subset, their sum $x + y$ lies outside of it. □

The result of problem 5 appeared in the 1916 German paper by the great mathematician Issai Schur.

Issai Schur as a young boy, courtesy of his daughter Hilde Abelin-Schur

Issai Schur and His 1916 Theorem

Issai Schur was born on January 10, 1875, in the Russian city of Mogilyov (presently in Belorussia). Being a Jew, Issai could not enroll in any Russian university. At 13 he went to the German language Nicolai-Gymnasium (1888–1894), then to the University of Berlin. On September 2, 1906, Issai Schur married Regina Malka Frumkin. On the personnel form, on the line “Arian,”

Schur wrote "nicht" for himself and "nicht" for his wife. The happy and lasting marriage produced two children, Georg and Hilde.
Issai Schur gave most of his life to the University of Berlin, as a student (1894–1901); a Privatdozent (1903–1909); ausserordentlischer Professor (associate professor, 1909–1913 and April 1, 1916–April 1, 1919); and Ordinarius (a full professor, 1919 –1935).
Hitler's January 30, 1933, appointment as Reichskanzler changed this idyllic life. Schur was a pride of his university. Yet no achievement was high enough for a Jew in the Nazi Germany. Following years of pressure and humiliation, Schur, faced with imminent expulsion, 'voluntarily' asked for resignation on August 29, 1935. On September 28, 1935, Reich's-Minister replied on behalf of Der Führer und Reichskanzler, i.e., Adolf Hitler himself:

> "Führer and Reichskanzler has relieved you from your official duties in the Philosophical Facultät of the University of Berlin effective at the end of September 1935, in accordance with your August 29 of this year request."

Der Reichs- und Preußische Minister für Wissenschaft, Erziehung und Volksbildung

W I p Schur.2 a

Es wird gebeten, dieses Geschäftszeichen und den Gegenstand bei weiteren Schreiben anzugeben.

Berlin W 8, den 28. September 1935
Unter den Linden 4
Fernsprecher: A 1 Jäger 0030
Postscheckkonto: Berlin 14402
Reichsbank-Giro-Konto
— Postfach —

Verw. Dir. b. d. Univ. Berlin Eing. - 4 OKT 1935

Der Führer und Reichskanzler hat Sie auf Ihren Antrag vom 29. August d.Js. mit Ablauf des Monats September 1935 von den amtlichen Verpflichtungen in der Philosophischen Fakultät der Universität Berlin entbunden.

Ich übersende Ihnen anbei die hierüber ausgefertigte Urkunde.

(Unterschrift)

An Herrn Professor Dr. Issai Schur in Berlin-Schmargendorf, Ruhlaerstr.14 - Einschreiben -.

Abschrift zur Kenntnis und weiteren Veranlassung.
In Vertretung
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Letter relieving Issai Schur from his duties at the University of Berlin. Courtesy of the Archive of the Humboldt University at Berlin

Schur got out of Germany in 1939. On January 10, 1941, he passed away in Tel Aviv of a heart attack.

The Schur 1916 Theorem appears as "a very simple lemma" and is used for obtaining a number-theoretic result related to Fermat's Last Theorem. Nobody appreciated Schur's result when it was published. Now it shines as one of the most beautiful theorems of mathematics.

The Schur Theorem (Schur, 1916). For any positive integer n there is an integer $S(n)$ such that any n-coloring of the initial positive integers array $[S(n)]$ contains integers x, y, z of the same color such that $x + y = z$.

In his paper Schur shows that the least such integer $S(n)$ has the upper bound $\lfloor n!e \rfloor$ where $\lfloor x \rfloor$ is the largest integer $\leq x$. In 1973 his upper bound was slightly improved by Robert Irving to $\lfloor n!(e - 1/24) \rfloor$.

Story 3. The Schur Numbers on the Frontier of Mathematics

Let us define the Schur Number as the largest integer $S(n)$, such that the integers $1, 2, \cdots, S(n)$ can be colored in n colors in such a way that no color contains integers x, y, z such that $x + y = z$.

As we have seen above, in his 1916 paper, Schur solved our problem 36.5 by establishing the lower bound $S(n) \geq \frac{3^n - 1}{2}$. This lower bound is sharp for $n = 1, 2, 3$, which is easy to prove:

$$S(1) = 1, S(2) = 4, \text{ and } S(3) = 13.$$

For $n = 4$, the formula of our problem 5 gives 40, but in 1965, using computer, Leonard D. Baumert and Solomon W. Golomb showed that in fact $S(4) = 44$.

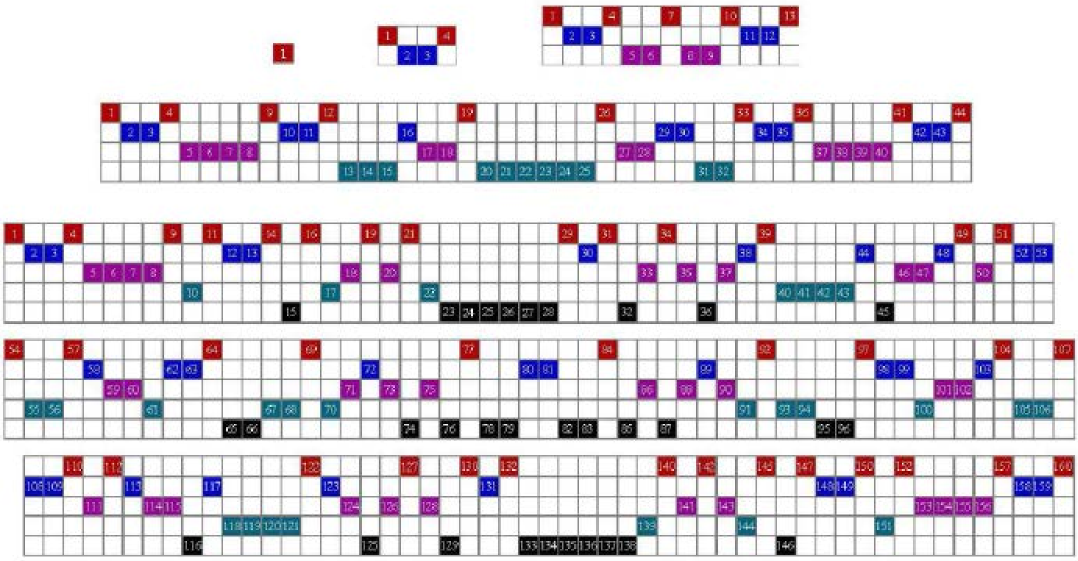
Finding the exact value of $S(5)$ appeared to be very hard. In the 1970s, best known bounds for $S(5)$ were $157 \leq S(5) \leq 321$, the lower bound obtained in 1979 by Harold Fredrickson and the upper bound in 1973 by Earl Glen Whitehead. Two decades later, in 1994, Geoffrey Exoo proved that $S(5) \geq 160$.

On November 21, 2018 Marijn J.H. Heule achieved the goal: $S(5) = 160$. Before his publication, the upper bound of $S(5)$ stood at 315. Thus, for $n = 5$, gives us 121 whereas the exact value is $S(5) = 160$. Marijn writes: "We obtained the solution, $n = 160$, by encoding the problem into propositional logic and applying massively parallel satisfiability solving techniques on the resulting formula ... The proof is two petabytes in size."

The coloring of integers from 1 to 160 in 5 colors without a monochromatic pair and its sum, was found first by Geoffrey Exoo. He even produced a palindromal coloring, i.e., coloring where numbers i and 160–i are assigned the same color.

In 2000, Harold Fredrickson and Melvin M. Sweet constructed colorings that proved new lower bounds $S(6) \geq 536$ and $S(7) \geq 1680$. Thus, a lot more of exciting research is waiting for you.

Here is this palindromal coloring from Heule's paper:



Story 4. The Van der Waerden Theorem about Monochromatic Arithmetic Progressions and the Old Japanese Bagatelle

Only now I may disclose the content of my 4th story, as it includes a problem that I was saving for the 37th Soifer Mathematical Olympiad that took place on October 1, 2021, after three epidemic-caused delays totaling 18 months.

Let me start by generously quoting from *The Mathematical Coloring Book* [3] (I changed results' numbers to fit the present exposition).

Bartel Leendert van der Waerden proved in 1926 and published a year later the following beautiful result:

Arithmetic Progressions Theorem 4.1 (Van der Waerden, 1927, [7], [3]). For any k, l, there is $W = W(k, l)$ such any k-coloring of the initial segment of positive integers $[W]$ contains a monochromatic arithmetic progression of length l.

Following my historical research that included letter exchange with Van der Waerden, I determined that two brilliant persons independently conjectured what Van der Waerden proved: Pierre Joseph Henry Baudet and Issai Schur. In view of this, I named this result The Baudet-Schur-Van der Waerden Theorem.

There was a pair of mathematicians, who published on Van der Waerden's 1927 proof very shortly after its publication, in 1930. Their result was cited in Paul Erdős and Ronald L. Graham's fine but hard to find 1980 problem book [1][2] as "an easy consequence of Van der Waerden's Theorem." In fact, the authors show [2] that this consequence is equivalent to the statement Van der Waerden proved.

[2] I own it only because Paul asked Ron to send me a copy.

Problem 4.2 [2]. If $A = \{a_1, a_2, \cdots\}$ is an increasing infinite sequence of integers with $a_{k+1} - a_k$ bounded, then A contains arbitrarily long arithmetic progressions.

The authors of [2] prove that in fact the statements of 4.1 and 4.2 are equivalent! In my opinion, 4.2 explains the essence of the celebrated theorem 4.1 better than anything ever has.

The great surprise is, [2] was published by the two Japanese mathematicians Sōichi Kakeya and Seigo Morimoto in 1930, much earlier than even Erdős and Turán's paper! How did they get a hold of the little-read Dutch journal where Van der Waerden published his result just 3 years earlier? The authors do misspell the name of Baudet everywhere, even in the title: *On a Theorem of MM. Bandet [sic] and van der Waerden*. But they were first to recognize that credit is due to both, Baudet for creating the conjecture, and to Van der Waerden for proving it. Without the Conjecture, Van der Waerden would have had nothing to prove!

Problem 4.3(Kakeya–Morimoto, 1930, [2]). If $A = \{a_1, a_2, \cdots\}$ is an increasing infinite sequence of integers with $a_{k+1} - a_k$ bounded, then A contains arbitrarily long arithmetic progressions.

Proof. The differences $a_{k+1} - a_k$ are bounded by, say, the constant c. This suggests a $(c+1)$-coloring of the set of all positive integers in colors $0, 1, \cdots, c$ as follows: given a positive integer n, find the smallest term a in the sequence A such that $0 \leq a–n$. Obviously, $a–n \leq c$. We then color n in the color of $-n$. By the Baudet–Schur–Van der Waerden Theorem, for any length l there is a monochromatic arithmetic progression $b_1, b_2, \cdots, b_l$ of color, say, i. But then by the progression $b_1 + i, b_2 + i, \cdots, b_l + i$ is both arithmetic and is entirely contained in A. $\square$

Kakeya and Morimoto also construct a lovely simple example, showing that in theorem 4.1, the words "arbitrarily long arithmetic progressions" cannot be replaced by "infinite arithmetic progressions." Try to come up with a counterexample on your own. Then compare it to the following construction.

Counterexample 4.4.(Kakeya–Morimoto, 1930, [2]). There is an increasing infinite sequence $A = \{a_1, a_2, \cdots\}$ of integers with $a_{k+1} - a_k \leq 2$, such that A does not contain an infinite arithmetic progression.

Construction. An infinite arithmetic progression P of positive integers is defined by a pair (m, n) of integers, where m is the first term and the positive n is the common difference of P respectively. Therefore, the set of all such progressions is countable, i.e., can be enumerated by positive integers to look like $P_1, P_2, \cdots, P_k, \cdots$

Now we construct a sequence S as follows. For the first term s_1 of S we pick the first term of P_1. For the second term s_2 of S we choose a term of P_2, which is greater than $s_1 + 1$, and so on. Now consider the increasing sequence A of all integers from which we removed all the terms of the sequence S. Clearly, A does not contain any infinite arithmetic progression because it is missing a term from each of the infinite arithmetic progressions. And A satisfies the condition $a_{k+1} - a_k \leq 2$.

The construction of Kakeya and Marimoto's counterexample had such a Olympiad flavor that I decided to use it as the hardest problem 5 in the 37th Soifer Mathematical Olympiad. In the process, we found a much simpler counterexample than the two Japanese mathematicians published. The only person, whom I show the problems in advance has been Robert "Bob" Ewell, a Ph.D. and retired colonel. In interaction with Bob, the problem grew to contain 3 parts: **A, B**, and **C**. I will show you problem 5 as evolution of ideas.

Playing with Infinity

Problem 37.5A. Is there an increasing sequence $A : a_1, a_2, \cdots, a_k, \cdots$ of positive integers with $a_{k+1} - a_k \leq 2$ for every positive integer k, such that A does not contain an infinite arithmetic progression as a subset?

Preliminaries. An infinite arithmetic progression P of positive integers is defined by a pair (m, n) of integers, where m is the first term and the positive n is the constant difference of P respectively. Therefore, the set of all such progressions is countable, i.e., can be enumerated by positive integers to look like $P_1, P_2, \cdots, P_n, \cdots$. One way to enumerate, invented by Georg Cantor, the founder of the Set Theory, is shown in the figure 5 below. For each pair in the top table we assign the integer that is in exactly the same row and column in the bottom table. For example, to $(3, 4)$ we assign 18.

$$\begin{pmatrix} (1,1) & (1,2) & (1,3) & (1,4) & \dots \\ (2,1) & (2,2) & (2,3) & (2,4) & \dots \\ (3,1) & (3,2) & (3,3) & (3,4) & \dots \\ (4,1) & (4,2) & (4,3) & (4,4) & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 4 & 7 & \cdots \\ 3 & 5 & 8 & 12 & \cdots \\ 6 & 9 & 13 & 18 & \cdots \\ 10 & 14 & 19 & 25 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Figure 3

Solution of problem 37.5A. (Kakeya–Morimoto, 1930, [2]). We construct a sequence S as follows. For the first term s_1 of S we pick the first term of P_1. For the second term s_2 of S we choose a term of P_2, which is greater than $s_1 + 1$; for the third term s_3 of S we choose a term of P_3, which is greater than $s_2 + 1$ and so on. Now consider the increasing sequence A of all positive integers from which we removed all the terms of the sequence S. Clearly, A does not contain any infinite arithmetic progression because it is missing a term from each of them. And A satisfies the condition 2. $\square$

Robert "Bob" Ewell is a Ph.D. and Colonel, retired. He has been a senior member of the Soifer (formerly Colorado) Mathematical Olympiad ever since 1989. He beat the Japanese duo by finding a very short and simple solution.

Bob Ewell's idea, implemented by A.S. Start sequence S with one odd, follow by more than one consecutive evens starting with the prior odd +1, then more yet consecutive odds starting with the largest prior even +1, etc. Assume S contains an infinite AP of constant difference D. At some point S will have $2D$ consecutive odds, which gives AP two consecutive odds, making all AP terms odd. But further on AP will have an even, a contradiction. $\square$

I had to 'tighten the nuts' of the problem to disallow Bob's solution. Thus, problem 5B was born.
Problem 37.5B. For an increasing sequence A of positive integers, A_n denotes the number of terms of A that do not exceed n. We say that the sequence's density $D(A) = 1$ if the ratio A_n/n becomes as close to 1 as we please as n increases without bound. Is there a sequence A with $D(A) = 1$ that does not contain an infinite arithmetic progression?

Solution of 5B by Bob Ewell. There is such a sequence.
Let A be the sequence of all positive integers except:
- All of the integers between 1 and 10
- The first ½ of the integers between 11 and 100
- The first ¼ of the integers between 101 and 1000
- The first $\frac{1}{8}$ of the integers between 1001 and 10,000
- $\cdots$
- The first $\frac{1}{2k}$ of the integers between 10^k and 10^{k+1}.
- $\cdots$

Note that the number of integers removed at each power k of 10 (except the first 10) is $9 \times \frac{10k}{2k} = 9 \times 5^k$. That is, the "holes" increase without bound. Therefore, no matter where an arithmetic sequence starts and no matter how big its d is, the sequence will run into a hole too big to cross. The sequence will stop.

Is $D(A) = 1$? Yes. Let R_k be the total number of integers removed at each power, k, of 10. Then $D(A) = 1 - \frac{R_k}{10^{k+1}}$ (where $n = 10^{k+1}$). $R_k =$ sum of $9 \times 5^k < 10 \times 5^k = 10 \times \frac{10^k}{2^k}$. The first few terms of that sum are $50 + 250 + 1250 + 6250 + \cdots$. It is evident that at each stage, the sum of the 1st through k–1 terms is well less than the kth term. So we can say $R_k < 20 \times \frac{10^k}{2^k}$.

Therefore, $\frac{An}{n} = \frac{A(n)}{n} = \frac{A(10^{k+1})}{10^{k+1}} > 1–20 \times \frac{\frac{10^k}{2^k}}{10^{k+1}} = 1 - \frac{1}{2^{k-1}}$. $\frac{An}{n}$ becomes as close to 1 as we want.

Bob solved problem 37.5B thus forcing me to create problem 37.5C to hopefully stop Bob's wonderful successes. :)
Problem 37.5C. We call an increasing sequence A of positive integers *super dense* if for any positive integer n, A contains all integers from 1 through 10^n except at most n integers, and the differences between the consecutive integers excepted from A are strictly increasing. Is there a super dense sequence A that does not contain an infinite arithmetic progression?

Solution of 37.5B and 37.5C. As in the solution of 37.5A, we enumerate all infinite arithmetic progressions of positive integers to look like $P_1, P_2, \cdots, P_n, \cdots$ and construct a sequence S as follows. For the first term s_1 of S we pick the first term of P_1. For the second term s_2 of S we pick the term of P_2 that is no less than $s_1 + 10$. For the third term s_3 of S we choose a term of P_3, which is no less than $s_2 + 100$, and so on. Now consider the increasing sequence A of all positive integers from which we removed all the terms of the sequence S. Clearly, A does not contain any infinite arithmetic progression because it is missing a term from each of them. Its density is the limit of $\frac{(10^n - n)}{10^n} = 1 - \frac{n}{10^n}$ as n increases without bound, which is obviously 1. □

Notice: We can explicitly calculate the sequence A if, for example, we use the following fantastic mapping

$$f(a, b) = 2^{a-1}(2b - 1)$$

of ordered pairs of positive integers onto positive integers. Let now a be the first term and b the constant difference of an AP. Every positive integer can be uniquely expressed as a power of 2 times an odd integer, thus each positive integer has a unique pair that maps into it. This inverse function f^{-1} maps an integer $2^{a-1}(2b–1)$ into the pair (a, b), and we easily construct the terms of the sequence S:

$$\begin{array}{lllllll}
1 & = & 2^{1-1}(2 \cdot 1–1) & \to & (1,1) & s_1 & = & 1 \\
2 & = & 2^{2-1}(2 \cdot 1–1) & \to & (2,1) & s_2 & = & 11 \\
3 & = & 2^{1-1}(2 \cdot 2–1) & \to & (1,2) & s_3 & = & 111 \\
4 & = & 2^{3-1}(2 \cdot 1–1) & \to & (3,1) & s_4 & = & 1111 \\
5 & = & 2^{1-1}(2 \cdot 3–1) & \to & (1,3) & s_5 & = & 11113 \\
6 & = & 2^{2-1}(2 \cdot 2–1) & \to & (2,2) & s_6 & = & 111130 \\
7 & = & 2^{1-1}(2 \cdot 4–1) & \to & (1,4) & s_7 & = & 1111301
\end{array}$$

. . . . □

More beautiful mathematics of coloring is waiting for you on the pages of of my *The Mathematical Coloring Book*, Springer, New York, 2009. I'm working on expanding its first 640-page edition [3] to ca. 1,000+ oversized pages second edition [4], which will include new incredible breakthroughs. I hope it will appear in Springer in 2024.

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Scientific and World Affairs in the Soifer Mathematical Olympiad[3]

Alexander Soifer

Alexander Soifer is a Russian-born American mathematician and mathematics author. His works include over 400 articles and 13 books.
Every spring since 1983, Soifer, along with other mathematician colleagues, sponsor the Colorado Mathematical Olympiad (CMO) at the University of Colorado in Colorado Springs. In May 2018, in recognition of 35 years of leadership, it was renamed the Soifer Mathematical Olympiad.
In 1991 Soifer founded the research quarterly Geombinatorics, and publishes it with the Geombinatorics editorial board.
In July 2006 at the University of Cambridge, Soifer was presented with the Paul Erdős Award by the World Federation of National Mathematics Competitions.
Soifer was the President of the World Federation of National Mathematics Competitions from 2012 to 2018. His Erdős number is 1.

Abstract
The Soifer (formerly Colorado) Mathematical Olympiad differs from other Olympiads in a number of essential ways. We offer the same problems to all Olympians, who usually range from grade 6 through 12. This requires us to lean not on knowledge but instead on ingenuity and originality of our Olympians. Another distinction is our sources of inspiration. We often use old and new research mathematical and historical papers to squeeze out of them Olympiad-style gems of ideas. Then we dress them up to obtain exciting 'stories' that our Olympians enjoy. In this essay I will share with you the following story-problems:

- "In Order to Form a More Perfect Union..." [Minimizing Disagreements in the United Nations], problem 27.4, 2010 [2];
- A Dream for a Peaceful Ukraine, problem 31.3, 2014;
- (DNA of) Love and Death, problem 22.5 2005 [2];
- Stopping the Ebola Epidemics, problem 33.4, 2016.

Keywords: Mathematics, research, Soifer Mathematical Olympiad, Colorado, problem creating.

I started creating this problem by consulting Wikipedia, which informs: "The United Nations Organization (UNO) or simply United Nations (UN) is an international organization whose stated aims are facilitating cooperation in international law, international security, economic development, social progress, human rights, and the achieving of world peace... There are currently 192

[3]This essay was presented as a lecture at the 9th Congress of the World Federation of National Mathematics Competitions, July 2022, Sofia, Bulgaria. It will appear in the volume of the Congress' Proceedings

member states, including nearly every sovereign state in the world." And so the number 192 has entered my story-problem.

STORY 1: "In Order to Form a more Perfect Union. . . " (Soifer; 2010, Problem 4 of the 27th Colorado Mathematical Olympiad)

The United Nations Organization includes 192 Member States, every pair of which has a disagreement. In order to form a more perfect Union, a negotiation is introduced: if representatives of four Member States are seated at a round table so that each pair of representatives seated next to each other has a disagreement, the negotiation resolves one of these four disagreements. A series of consecutive negotiations reduces the total number of disagreements to n. What is the minimum of n?

1. Let each Member Country be represented by a vertex of a graph, in which we connect two vertices by an edge if and only if the corresponding countries have a disagreement. The Initial Disagreements Graph is the complete graph K_{192} on 192 vertices (a set of 192 vertices, every two of which are connected by an edge). A negotiation selects a 4-cycle C_4 of a graph ("representatives of four countries are seated at a round table so that each pair of neighbors has a disagreement") and removes one edge from it. The problem, translated into this language, asks to find the minimum number of edges in a Disagreements Graph obtained from the initial K_{192} by a series of consecutive removals of an edge from a 4-cycle.

2. Observe first that the removal of an edge in a C_4 subgraph preserves connectivity of a graph (i.e., ability to travel between any pair of points through a series of edges). If the series of consecutive negotiations were to eliminate all cycles, we would get a connected cycle-free graph, called a tree, on 192 vertices. Such a tree has exactly 191 edges (proof by an easy induction).
Note that for any two points of a tree we have a unique path through the edges (for otherwise we would have created a cycle in the union of two distinct paths). This observation allows us to show that any tree is 2-colorable (so that vertices of the same color are not adjacent). Indeed, color a point A in color 0, and any other point B in color 0 or 1 depending upon the parity of the edge distance from A to B.
Observe finally that the property of 2-colorability is preserved under the removal of an edge from a 4-cycle, and under the reverse operation of completing a 4-path to a 4-cycle. The Initial Disagreements Graph K_{192} is not 2-colorable (it requires 192 colors!), therefore we will never get a tree as a result of a series of negotiations! We proved that 191 is unreachable.

3. On the other hand, we can fly a kite and in the process get a Disagreements Graph with 192 edges.

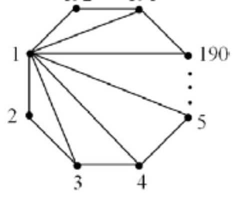

Figure 4: A Subgraph of Kite-0

Through the series of negotiations, we can get from the Kite-0 graph, which is K_{192}, to the Kite-1 graph, which consists of K_{191} with an attached 1-edge "tail." Indeed (see Figure 4), from the 4-cycle $\{1, 3, 4, 5\}$ we remove the edge $\{1, 3\}$; from $\{1, 4, 5, 6\}$ remove $\{1, 4\}$; $\cdots$, from $\{1, 190, 191, 192\}$ remove $\{1, 190\}$; from $\{1, 191, 192, 2\}$ remove $\{1, 191\}$. Finally, from the 4-cycle $\{1, 2, 3, 192$ we remove $\{1, 192\}$, getting the desired graph Kite-1 (see Figure 5).

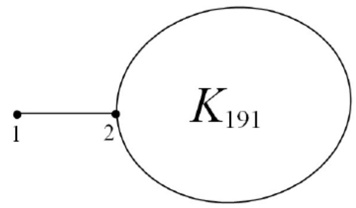

Figure 5: Kite-1

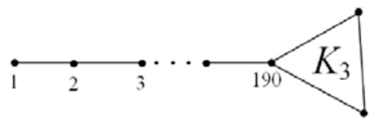

Figure 6: Kite-189

Continuing this process (you can formalize it by a simple mathematical induction), we will get to Kite-189 graph, which consists of K_3 with a tail of length 189 (Figure 7), which has exactly 192 edges as desired. $\square$

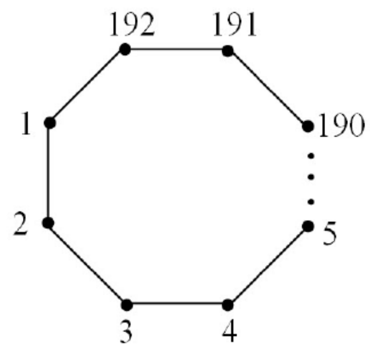

Figure 7

HOMEWORK. Determine which of the graphs in figures 7 and 8 can be obtained from the Initial Disagreements Graph K_{192} through a series of negotiations.

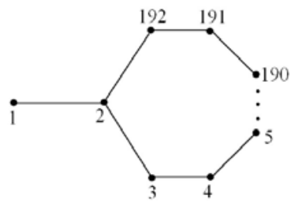

Figure 8

STORY 2: 31.3. A Dream for a Peaceful Ukraine (Soifer, 2014, problem 3 of the 31st Colorado mathematical Olympiad)

Each Ukrainian city flies one flag, Ukrainian or Russian, and connects by roads directly to 11 or 19 other Ukrainian cities, its *neighbors*. A city *lives in peace* if it flies the same flag as the majority of its neighbors, and *at war* otherwise. Each morning one city at war, if there is one, changes its flag. Will the day come when all Ukrainian cities will live in peace?

Solution. Create a graph with the Ukrainian cities as vertices and roads connecting them as edges. Denote by x the number of edges that connect cities flying opposite flags. With each change of a city flag, x reduces by at least 1 while remaining non-negative. Therefore, after finitely many steps we will achieve $x = 0$, there will be no flags to change, and peace will come to Ukraine. □

STORY 3: 22.5. Love and Death (Soifer; 2005, Problem 5 of the 22nd Colorado Mathematical Olympiad)

(A) The DNA of bacterium bacillus anthracis (causing anthrax) is a sequence, each term of which is one of 2005 genes. How long can the DNA be if no two consecutive terms may be the same gene, and no two distinct genes can reappear in the same order? That is, if distinct genes α, β, occur in that order (with or without any number of genes in between), the order $\alpha, \cdots, \beta$ cannot occur again.

(B) The DNA of bacterium bacillus amoris (causing love) is a sequence, each term of which is one of 2005 genes. No three consecutive terms may include the same gene more than once, and no three distinct genes can reappear in the same order. That is, if distinct genes α, β, and γ occur in that order (with or without any number of genes in between), the order $\alpha, \cdots, \beta, \cdots \gamma$ cannot occur again. Prove that this DNA is at most 12,032 long.

22.5. (A). First Solution. Let us prove that in a DNA satisfying the two given conditions, there is a gene that occurs only once. Indeed, let us assume that each gene appears at least twice and for each gene select the first two appearances from the left and call them a pair. The first gene from the left is in the first pair. This pair must be separated, thus the pair of the second gene from the left is nestled inside the first pair. The second pair must be separated, and thus the pair of the third gene from the left must be nestled inside the second pair, etc. As there are finitely many genes, we end up with a pair of genes (nestled inside other pairs) that is not separated, a contradiction.

We will now prove by mathematical induction on the number n of genes that the DNA that satisfies the conditions and uses n genes is at most $2n$–1 gene long. For $n = 1$ the statement is true, as the longest DNA is 2–1 $= 1$ genes long.

Assume that a DNA that satisfies the required conditions and uses n genes is at most $2n$–1 gene long. Now let S be a DNA sequence satisfying the conditions that uses $n + 1$ genes; we need to prove that it is at most $2(n + 1)$–1 $= 2n + 1$ genes long.

In the first paragraph of our solution, we proved that there a gene g that occurs only once in S; we throw it away. The only violation that this throwing may create is that two copies of another gene become adjacent – if so, we throw one of them away too. We get the sequence S' that uses only n genes. By the inductive assumption, S' is at most $2n$–1 genes long. But S is at most 2 genes longer than S', i.e., S is at most $2n + 1$ genes long. The induction is complete.

All that is left is to demonstrate that the DNA length of $2n$–1 is attainable. But this is easy: just pick the following sequence $1, 2, \cdots, n–1, n, n–1, \cdots, 2, 1$. □

22.5. (A). Second Solution. We will prove by mathematical induction on the number n of genes that the DNA that satisfies the problem conditions and uses n genes is at most $2n$–1 gene long. For $n = 1$ the statement is true, as longest DNA is 2–1 $= 1$ gene long.

Assume that for any positive integer $k, k < n$, a DNA that satisfies the conditions and uses k genes, is at most $2k$–1 gene long. Now let S be the longest DNA sequence that satisfies the problem conditions and uses n genes; we need to prove that S is at most $2n$–1 gene long.

Let the first gene of S be 1, then the last term must be 1 as well, for otherwise we can make S longer by adding a 1 at the end. Indeed, assume that the added 1 has created a forbidden DNA. This means that we now have a subsequence $a, \cdots, 1, \cdots, a, \cdots, 1$ (with the added 1 at the end); but then the original DNA, that started with 1, already had the forbidden subsequence $1, \cdots, a, \cdots, 1, \cdots, a$.

Let us consider two cases.

Case 1. If there are no more 1's in the DNA, we throw away the first 1 and the last 1, and we get a sequence S' that uses n – 1 genes (no more 1's). By the inductive assumption, S' is at most $2n$–1 genes long. But S is 2 genes longer than S', i.e., S is at most $2n + 1$ genes long.

Case 2. Assume now that there is a 1 between the first 1 and the last 1. The DNA then looks as follows: $1, S', 1, S'', 1$. Observe that if a gene m appears in the sequence S', it may not appear in the sequence S'', for this would create the prohibited subsequence $1, \cdots, m, \cdots, 1, \cdots, m$. Let the sequence $1, S', 1$ use n' genes and the sequence $1, S'', 1$ use n'' genes. Obviously, $n' + n''$–1 $=$ n (we subtract 1 in the left side because we counted the gene 1 in each of the two subsequences).

By the inductive assumption, the lengths of the sequences $1, S', 1$ and $1, S'', 1$ are at most $2n'$–1 and $2n''$–1 respectively. Therefore, the length of S is $(2n'$–$1) + (2n''$–$1)$–1 (we subtract 1 because the gene 1 between S' and S'' has been counted twice). But $(2n'$–$1) + (2n''$–$1)$–$1 = 2(n' + n'')$–$3 = 2(n + 1)$–$3 = 2n$–1 as desired. The induction is complete.

This proof allows us to find a richer set of examples of DNAs of length of $2n$–1 (and even describe all such examples if necessary). For example:

$$1, 2, \cdots, k, k+1, k, k+2, k, \cdots, k, 2005, k, k\text{–}1, k\text{–}2, \cdots, 2, 1.\square$$

22.5.(B). Assume S is the longest DNA string satisfying the problem conditions. Partition S into blocks of 3 terms starting from the left (the last block may be incomplete and have fewer than 3 terms, of course). We will call a block extreme if a number from the given set of genes $\{1, 2, \ldots, 2005\}$appears in the block for the first or the last time. There are at most 2×2005 extreme blocks.

We claim that *there are no complete (i.e., 3-gene) non-extreme blocks.*

Indeed, assume the block B, which consists of genes α, β, γ in *some* order, is not extreme (in the original DNA string S these three genes do not have to be consecutive). This means that the genes α, β, γ each appears at least once before and at least once after appearing in B. We will prove that then the DNA would contain the forbidden subsequence of the type $\sigma, \tau, \omega, \sigma, \tau, \omega$. Let A denote the ordered triple of the first appearances of α, β, γ (these 3 genes may very well come from distinct 3-blocks). Without loss in generality we can assume that in A the genes α, β, γ appear in this order. Let C denote the ordered triple of the last appearances of α, β, γ in *some* order. Let us look at the 9-term subsequence ABC and consider three cases, depending upon where α appears in the block B.

Case 1. If α is the first gene in B (Figure 9), then we can choose β also in B and γ in C to form α, β, γ which with α, β, γ from A gives us the forbidden sequence $\alpha, \beta, \gamma, \alpha, \beta, \gamma$.

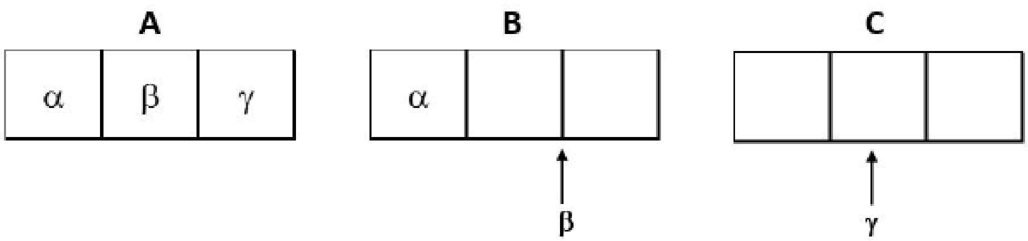

Figure 9

Case 2. Let α be the second gene in B (Figure 10). If β follows α, then with γ from C we get α, β, γ which with α, β, γ from A produces the forbidden sequence $\alpha, \beta, \gamma, \alpha, \beta, \gamma$. Thus, β must precede α in B. If the order of the genes β, γ in C isβ, γ, then we can combine an α from B with this β, γ to form α, β, γ which with α, β, γ from A gives us the forbidden $\alpha, \beta, \gamma, \alpha, \beta, \gamma$. Thus, the order in C must be γ, β. Now we can choose α, γ from A followed by β, α from B, followed byγ, β from C to get $\gamma, \beta, \alpha, \gamma, \beta, \alpha$ which is forbidden.

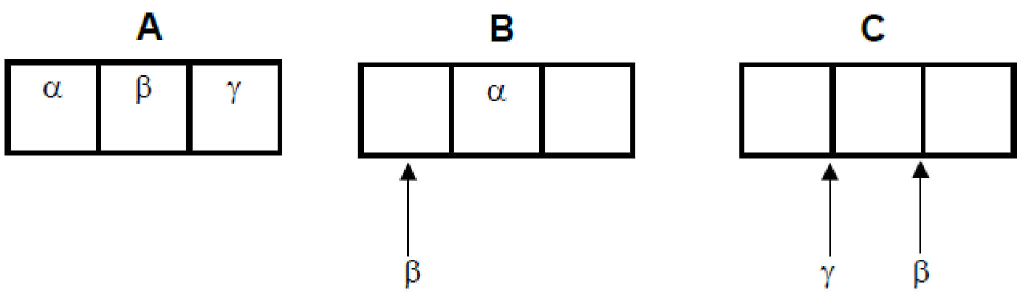

Figure 10

Case 3. Let α be the third gene in B (Figure 11), and is thus preceded by β in B. If the order in C is β, γ, then we get α, β, γ from A followed by α from B and β, γ from C to get the forbidden $\alpha, \beta, \gamma, \alpha, \beta, \gamma$. Thus, the order in C must be γ, β, and we choose α, γ from A, followed by β, α from B, and followed by γ, β from C to form the forbidden $\alpha, \gamma, \beta, \alpha, \gamma, \beta$.

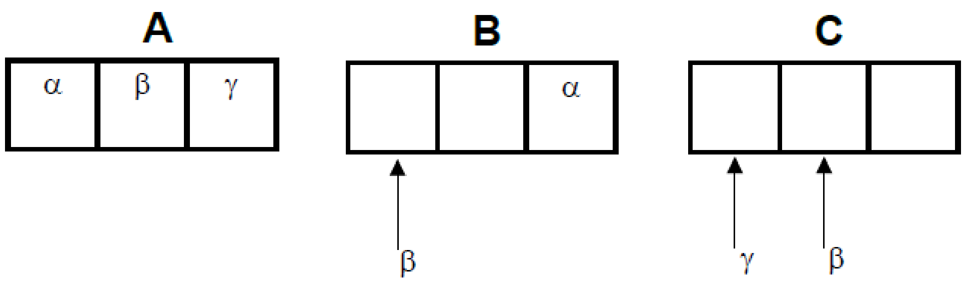

Figure 11

We are done, for the DNA sequence consists of at most 2×2005 extreme 3-blocks plus perhaps an incomplete block of at most 2 genes – or 12,032 genes at the most. □

Because this problem addressed gene of love and gene of death, and another talked about clones of convex figures, I received several invitations to give a talk and even organize a section at various symposia and congresses dedicated to genetics. The organizers somehow dug out the titles of my “Further Explorations” from *The Colorado Mathematical Olympiad* books (Springer, New York, 2011 and 2017) and interpreted the Olympiad problem titles literally:

> It is our great pleasure and privilege to welcome you to join the Annual World Congress of Food and Nutrition, which will be held in Singapore. On behalf of the Organizing Committee, we would be honored to invite you to be a chair/speaker at Session 405: Foodborne Diseases, Carcinogenic Food while presenting about *E23: More about Love and Death* at the upcoming WCFN. (December 16, 2017)

> It is our great pleasure and privilege to welcome you to join the World Gene Convention, which will take place in Macao. On behalf of the Organizing Committee, we would be honored to invite you to be a chair/speaker in Module 1: Breakthroughs in Gene while presenting about *E15: From Squares in a Square to Clones in Convex Figures*. (August 5, 2017)

One invitation came a few days ago, just as I was preparing this talk:

> From: Max Stevens
> Sent: Friday, May 27, 2022 4:41 PM
> To: Alexander Soifer asoifer@uccs.edu
> Subject: The Soifer (formerly Colorado) Mathematical Olympiad, why it — INFS-2022 at London, UK
> **Dear Alexander Soifer**,
> I hope you are doing well.
> On behalf of the scientific committee, we would like to invite you as a speaker for the **"6th Conference on Innovations in Nutrition and Food Science (INFS-2022)"** that will take place on Oct 06-08, 2022 at London, UK.
> We have gone through your recent article entitled **"The Soifer (formerly Colorado) Mathematical Olympiad, why it was founded, bridge between its problems and mathematics, and lives of its winners: an essay"** we believe that you will be an excellent speaker at our conference. We welcome you to disseminate your research findings at our conference and we hope that your talk would lend a valuable insight to our conference.
> For more details: https://nutrition-foodscience.org/
> To submit abstract online https://nutrition-foodscience.org/abstractsubmission
> We pleased to inform you that we have a shortage of funds for our conference. As an invited speaker you will be eligible to avail **3 nights of accommodation** at the conference venue We look forward to a positive response from you.
> **Best Regards**,
> Max Stevens
> INFS-2022

Further Exploration E23: More about Love and Death

I hope you did not take the DNA's featured in my problem 22.5 to faithfully reflect reality. Remember, we are in the Illusory World of Mathematics! To whet your appetite for the problem, I invented the bacterium *bacillus anthracis*, causing anthrax (death), in problem 22.5.(A). In problem 22.5.(B), I went even further by imagining the bacterium *bacillus amoris*, causing love. :). I was inspired by a talk by a Ph.D. student Martin Klazar that I attended during my long term visit of Charles University in the beautiful Prague, Czech Republic. Now Martin is a professor at that same university. The notes I took in 1996 during Klazar's talk, contained at the end the following remark:

> By overlapping the 3-gene blocks by their end terms and using the same argument, Martin showed that the upper bound can be reduced from $6n + 2$ (n is here the number of available genes) to $4n + 2$, and with clever observation of the starting and ending triples to even $4n$–4. It is possible to achieve the bound of $4n$–7, proof of which would require further cleverness.

These bounds, of course, are stronger than the ones I asked for in problem 22.5.(B). Their proofs were not presented during Klazar's talk. Now, Twenty Years After, as Alexandre Dumas named his sequel to The Three Musketeers, I asked Martin Klazar to enlighten us. Here is his reply, containing the proof of a much stronger upper bound, with my minimal stylistic editing.

From: Martin Klazar [mailto:klazar@kam.mff.cuni.cz]
Sent: Monday, September 12, 2016 12:53 PM
To: Alexander Soifer asoifer@uccs.edu
Subject: Re: Hello, Martin!

Dear Sasha,
Here is my proof that a 3-sparse word u [i.e., no three consecutive terms in u may include the same gene more than once] over n-element alphabet avoiding the pattern $abcabc$ as a subsequence has length at most $4n$–4 (for $n > 1$).

We denote by F the first occurrences (of a letter) in u, by L the last occurrences, and by S the intersection $F \cap L$. The intersection consists of exactly the letters that appear in u just once. We may assume that u has the length $|u|$ of at least six (else the bound holds) and split u into three words $u = u'vu''$ where $|u'| = |u''| = 3$. Note that each of the three terms of u' lies in F and those of u'' lie in L.

We look now for an upper bound of the length $|v|$ of the middle part of u. We cover v by k intervals $I_1, \cdots, I_k$ of length 3 each and by at most one residual term at the end, so that I_i and $I_i + 1$ share their endpoints (thus if $v = abcadeca$ then $I_1 = abc, I_2 = cad, I3 = dec$ plus the residual term a). If $k = 0$ then there may be two residual terms. Hence $|v|$ is at most $3 + 2(k–1) + 1 = 2k + 2$.

Consider one of these intervals $I = I_i = xyz$. By the sparseness condition for u, the x, y, z are of course distinct. If x is not in L, y is not in $F \cup L$, and z is not in F, then u has an $abcabc$ subsequence (for then y, z are forced to appear before I and x, y after I). Thus at least one of the following statements is true: (x is in L) or (y is in $F \cup L$) or (z is in F). I select one of these three elements of I (i.e., one for which the clause holds) and call it good (so all three terms in I may be good, or two of them, but certainly at least one term of I is good). I hope now it is clear what I meant then by "good" elements.

Let G be the set of good terms in v.

We bound k by the number $|G|$ of good terms in v. Since G is a subset of $F \cup L$, we have that $|G|$ is at most $2n$. Since the $I_i$are not disjoint, we may have chosen some g in G for two (but not more) intervals I_i. But if this happens then g is the last term in I_i, the first term in $I_i + 1$, and is in S. Thus k is at most $|F \cup L|'$ where the apostrophe means that each element of the subset S of $F \cup L$ is counted with the weight 2. But we still have that $|F \cup L|'$ is at most $2n$ (it is $< 2n$ only if some of the n letters do not appear in u at all), and so k is at most $2n$. But k is in fact at most $2n$–6 because the 6 terms in u' and u'' lie in $F \cup L$ and not in S, but not in v and are not used in any I_i.

Summarizing, $|u| = |u'| + |v| + |u''| = 6 + |v|$, which is at most $6 + 2k + 2$, which is at most $6 + 2(2n–6) + 2 = 4n–4$.
Best, Martin

P.S.: I do not know [a] better bound. I think I have somewhere stated and proved some lower bound and posed a problem to determine the extremal function $Ex(abcabc, n)$ exactly, which should be doable, but as far as I know, has not been done.

Let us formulate the results we in fact proved in problems 22.5.(A) and 22.5.(B) in the notations of Professor Martin Klazar's post scriptum.

Problem 22.5.(A). $Ex(abab, n) = 2n–1$.

Problem 22.5.(B). $Ex(abcabc, n) \leq 6n + 2$.

In this notation, Martin proved above the following result:

Upper Bound (Martin Klazar). $Ex(abcabc, n) \leq 4n–4$.

Open Problem (Martin Klazar). Find the exact value for $Ex(abcabc, n)$.

Did you get hooked on these sequences and would like to learn and solve more? Let me quote a paragraph from a relevant page of Wikipedia, so that you will know what to search for:

In combinatorics, a Davenport–Schinzel sequence is a sequence of symbols in which the number of times any two symbols may appear in alternation is limited. The maximum possible length of a Davenport–Schinzel sequence is bounded by the number of its distinct symbols multiplied by a small but non-constant factor that depends on the number of alternations that are allowed. Davenport–Schinzel sequences were first defined in 1965 by Harold Davenport and Andrzej Schinzel.

STORY 4: Stopping the Ebola Epidemic (Soifer, 2016, Problem 4 of the 33rd Colorado Mathematical Olympiad

A square region 2016×2016 miles is divided into 2016^2 cells each of which is a square of side 1 mile. Some cells are contaminated by the Ebola virus. Every month the virus spreads to those cells which have at least two sides in common with the contaminated cells. Find the maximum number of contaminated cells, such that no matter where they are located, the Ebola epidemic will not spread to cover the entire region.

Solution. As the epidemic spreads, the perimeter of the contaminated region cannot increase, for with each newly contaminated cell the perimeter loses at least two sides (shared with previously contaminated cells), and gains at most two new sides. If at most 2015 cells are contaminated initially, the starting perimeter is at most 2015×4, and thus the perimeter will never reach 2016×4, which is the perimeter of the entire region.

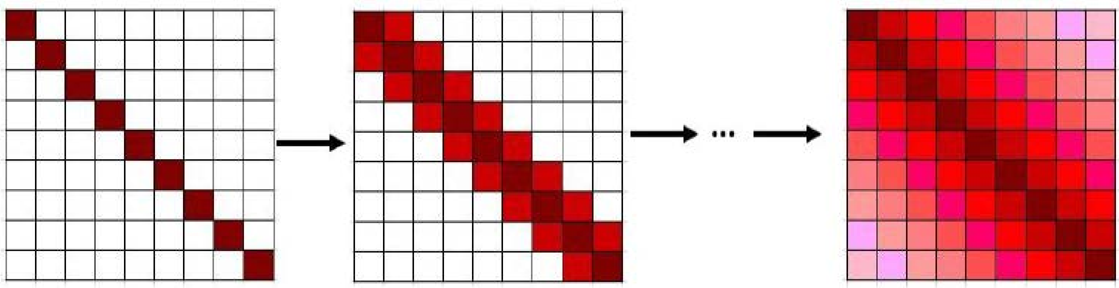

Figure 12

The contaminated main diagonal of the region (Figure 12) spreads to cover the entire region, thus showing that 2016 contaminated cells can possibly cause the spread of the Ebola on the entire region. The answer is thus 2015. □

Most of these and many other stories could be found in the Olympiad books [1] and [2]. Story 4 will have to wait until the completion of the 4th decade of the Soifer Mathematical Olympiad. I hope Springer will retain its kind attention to our Olympiad and publish the volume “The Fourth Decade and Further Explorations.”

References

[1] Soifer, A. The Colorado Mathematical Olympiad and Further Explorations: From the Mountains of Colorado to the Peaks of Mathematics. Springer, New York, 2011.

[2] Soifer, A. The Colorado Mathematical Olympiad, The Third Decade and Further Explorations: From the Mountains of Colorado to the Peaks of Mathematics. Springer, New York, 2017.

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Sets of Paths between Vertices of a Polygon

Ivaylo Kortezov

Ivaylo S. Kortezov is an Associate Professor at the Institute of Mathematics and Informatics of the Bulgarian Academy of Sciences. His research areas include Education in Mathematics and Math Contests, General Topology, Banach Spaces and Cardinal Invariants. He is a two-time Gold and Silver medalist at IMO and won the "Marin Drinov" Award of the Bulgarian Academy of Sciences for a young researcher in 2001. He has been part of the team preparing the national mathematics contests and training the national teams since 2001 and the Leader or Deputy of the Bulgarian JBMO or BMO teams since 2007.

Abstract
The paper deals with counting the sets of given magnitude each consisting of non-self-intersecting paths whose nodes are vertices of a given convex polygon. Some of the obtained formulae provide new properties of entries in the On-line Encyclopaedia of Integer Sequences, while others generate new entries therein.

Keywords: Enumerative combinatorics, Non-self-intersecting paths, Convex polygons, OEIS

This paper is inspired by a problem from the Winter Mathematics Contest 2020, a Bulgarian important and difficult high-school competition, asking for counting the number of non-self-intersecting paths whose nodes are vertices of a given convex n-gon. This problem turned out to generate new properties for two of the sequences in the On-line Encyclopaedia of Integer Sequences (OEIS) ([3], [4]); as a result the list of properties of these sequences was enriched. Similar combinatorial questions were answered in [1] and they generated new sequences for OEIS ([5], [6]). This paper finds further compact formulae when varying the number of paths whose nodes are disjoint subsets of the set of vertices of the polygon; they depend on whether one-node paths are allowed.

All variables in this paper denote positive integers.

Definition 1. Let A_1, A_2, …, A_k be different points in the plane such that no three of them are collinear. If the segments A_1A_2, A_2A_3, …, $A_{k-1}A_k$ have no common internal points then the union of these segments is called a *non-self-intersecting path (NSP)*; A_1, A_2, …, A_k are called *nodes* of the NSP.

Note that, according to the definition, the NSP is direction-independent – e.g. $A_1A_2A_4A_3$ and $A_3A_4A_2A_1$ is the same NSP. Also, the definition allows a NSP to have just one node (and zero segments); in this case we will call it a *singleton*. It is not immediately clear whether it is reasonable to include the singletons among the NSPs, so below we will calculate the results both with and without them, obtaining outcomes of comparable compactness.

Definition 2. Let n, p be positive integers. Denote by:

- $nsp(n, p)$ the number of p-sets of non-singleton NSPs whose sets of nodes form a partition of the set of vertices of a given convex n-gon, with the natural extension for $n = 2$ (a segment);
- $nsp'(n, p)$ the number of p-sets of (possibly singleton) NSPs whose sets of nodes form a partition of the set of vertices of a given convex n-gon, with the natural extension for $n = 2$ (a segment);
- $NSP(n, p)$ the number of p-sets of non-singleton NSPs whose sets of nodes are disjoint subsets of the set of vertices of a given convex n-gon, with the natural extension for $n = 2$ (a segment) and $n = 1$ (a point);
- $NSP'(n, p)$ the number of p-sets of (possibly singleton) NSPs whose sets of nodes are disjoint subsets of the set of vertices of a given convex n-gon, with the natural extension for $n = 2$ (a segment) and $n = 1$ (a point).

The next statement has been proposed in [2]; below is one possible proof. The special case for $p = 1$ was part of the contest problem mentioned above; its result has been suggested by the author and accepted by `oeis.org` in the list of properties of A001792 [3]. The result for $p = 2$ been suggested by the author for publishing in `oeis.org` and accepted as A332426 [6].

Proposition 1. Let n, p be positive integers such that $n > p$. Then

$$nsp(n, p) = 2^{n-3p}\binom{n}{p}\sum_{i=0}^{p}\binom{p}{i}i^{n-p}(-1)^{p-i},$$

or equivalently, $nsp(n, p) = 2^{n-3p}V_n^{(p)}S_{n-p}^{(p)}$, with $V_n^{(p)}$ being the number of variations for n elements of p-th class and $S_{n-p}^{(p)}$ being the Stirling number of second kind for $n - p$ elements of p-th class.

Proof. Fix one of the end-nodes of each NSP; call it the *head* of that NSP; call the set of the rest of the nodes of the NSP the *body* of that NSP. There are $\binom{n}{p}$ choices for the set of heads among the n vertices of the polygon. We have to split the set of the remaining $n - p$ vertices into the p (nonempty) bodies[4]. For each of the $n - p$ vertices of there are p choices for the body (p^{n-p} variants); we have to exclude the variants where a body remains empty ($\binom{p}{1}(p-1)^{n-p}$ variants), then to include back the variants where two of the bodies remain empty ($\binom{p}{2}(p-2)^{n-p}$ variants), and continue further by the inclusion-exclusion principle to get

$$\binom{n}{p}\sum_{i=0}^{p}\binom{p}{i}i^{n-p}(-1)^{p-i}.$$

Let us now count the NSPs with a given head and body: starting from the head, for each subsequent vertex, except for the last one, there are 2 choices – the leftmost or the rightmost unused vertex from the set entitled to the body (in all other cases part of the vertices remain separated from the rest ones and there is no way to conclude without self-intersection). Thus the number of NSPs with a given head and body is 2^{x-1} where x is the magnitude of the body.

Among the n vertices there are p used for heads and $n - p$ used for the p bodies; summing up the above result for all these, we conclude that the number of ways to form p NSPs from a given decomposition of the set of vertices into p heads and bodies is $2^{(n-p)-p} = 2^{n-2p}$.

[4]Modulo the order, there are $S_{n-p}^{(p)}$ ways to to this, where $S_{n-p}^{(p)}$ is the Stirling number of second kind for $n - p$ elements of p-th class, but to take into account the order we do the details explicitly.

To conclude it remains to note that there are 2 possible choices for the head of each NSP, so we need to divide by 2^p. Thus

$$nsp(n,p) = 2^{-p}2^{n-2p}\binom{n}{p}\sum_{i=0}^{p}\binom{p}{i}i^{n-p}(-1)^{p-i}.$$

The statement regarding the Stirling numbers of the second kind is directly seen in the proof, since there are $V_n^{(p)}$ ways to choose the vertices for the heads (the order is now important, as we plan to connect each head with a specific body), $S_{n-p}^{(p)}$ ways to split the $n-p$ body vertices into the p (nonempty) bodies and 2^{n-2p} ways to form NSPs in the entitled bodies; lastly, each of the p NSPs is direction-independent, which is responsible for a multiplication by 2^{-p}.

We will also need the following propositions proven in [1] and [2], which we state here for convenience. The sequence generated by the first one has been suggested by the author for publishing in `oeis.org` and accepted as A332426 [6].

Proposition 2. If $n \geq 3$ then

$$nsp'(n,2) = n(n-1)2^{n-6}(2^{n-3}+3)$$

or equivalently

$$nsp'(n,2) = \binom{n}{2}2^{n-5}(2^{n-3}+3).$$

Proposition 3. If $n \geq 4$ then

$$nsp'(n,3) = n(n-1)(n-2)2^{n-10}(3^{n-4}+3\cdot 2^{n-3}+9)$$

or equivalently

$$nsp'(n,3) = \binom{n}{3}2^{n-9}(3^{n-3}+9\cdot 2^{n-3}+27).$$

Remark. As mentioned in [2], it can be shown that if $n > p$ then

$$nsp'(n,p) = 2^{n-3p}\binom{n}{p}\sum_{i=0}^{p}\binom{p}{i}i^{n-p}3^{p-i},$$

but we will not need this generality here.

Let us now discuss $NSP(n,1)$. Let $n > 2$ and $A_1A_2\ldots A_n$ be a convex n-gon. For example $NSP(3,1) = 6$ as the NSPs are A_1A_2, A_1A_3, A_2A_3, $A_1A_2A_3$, $A_2A_1A_3$, $A_1A_3A_2$. Also $NSP(4,1) = nsp(4,1) + 4nsp(3,1) + \binom{4}{2} = 8 + 12 + 6 = 26$ with the three summands corresponding to the NSPs with 4, 3 and 2 nodes, respectively.

The next proposition was also a part of the mentioned contest problem and its proof can be found in [1] (in Bulgarian); for completeness we include it here. The statement of the proposition has been suggested by the author and accepted by `oeis.org` in the list of properties of A261064 [4] together with a reference to the site of the Winter Math Tournament, Yambol 2020.

Proposition 4. If $n > 2$ then $NSP(n,1) = \frac{n}{4}\cdot(3^{n-1}-1)$.

Proof. If the NSP has k nodes ($k = 2, 3, \ldots, n$) then there are $\binom{n}{k} = \dfrac{n!}{k!(n-k)!}$ choices for the vertices. By Proposition 1 (for $p = 1$) there are $k \cdot 2^{k-3}$ choices for connecting them. Thus

$$NSP(n,1) = \sum_{k=2}^{n} \frac{n!}{k!(n-k)!} \cdot k2^{k-3} = \frac{n}{4} \cdot \sum_{k=2}^{n} \frac{(n-1)!}{(k-1)!(n-k)!} \cdot 2^{k-1} = \frac{n}{4} \cdot ((2+1)^{n-1} - 1)$$

$$= \frac{n}{4} \cdot (3^{n-1} - 1).$$

In particular, we get $NSP(3,1) = \frac{3}{4} \cdot (3^2 - 1) = 6$ and $NSP(4,1) = 3^3 - 1 = 26$ both in accord with the aforementioned observations.

Remark. Note that Proposition 4 remains valid under the natural definition for $n = 2$ (there is a single segment and $NSP(2,1) = \frac{1}{2}(3-1) = 1$) and for $n = 1$ (there are no non-singleton NSPs and $NSP(1,1) = \frac{1}{2}(1-1) = 0$); we will use these, too.

Let us now discuss $NSP'(n,1)$. For example $NSP'(3,1) = 9$: the NSPs are A_1, A_2, A_3, A_1A_2, A_1A_3, A_2A_3, $A_1A_2A_3$, $A_2A_1A_3$, $A_1A_3A_2$. Also $NSP'(4,1) = nsp(4,1) + 4nsp(3,1) + \binom{4}{2} + 4 = 8 + 12 + 6 + 4 = 30$ with the four summands corresponding to the NSPs with 4, 3, 2 and 1 nodes, respectively.

Proposition 5. If $n > 2$ then $NSP'(n,1) = \frac{n}{4} \cdot (3^{n-1} + 3)$.

Proof. We have to add the NSPs having only one node: $NSP'(n,1) = NSP(n,1) + n = \frac{n}{4} \cdot (3^{n-1} - 1) + 1 = \frac{n}{4} \cdot (3^{n-1} + 3)$.

In particular, we get $NSP'(3,1) = \frac{3}{4} \cdot (3^2 + 3) = 9$ and $NSP'(4,1) = 3^3 + 3 = 30$ both in accord with the initial observations.

Remark. Note that Proposition 5 remains valid under the natural definition for $n = 2$ ($NSP'(2,1) = \frac{1}{2} \cdot (3+3) = 3$) and for $n = 1$ ($NSP'(1,1) = \frac{1}{4} \cdot (1+3) = 1$); we will use these, too.

Let us now discuss $NSP(n,2)$. For example $NSP(5,2) = 45$ since there are $nsp(5,2) + 5nsp(4,2) = 30+15 = 45$ (since either all the 5 vertices are used or one is not) and $NSP(6,2) = nsp(6,2) + 6nsp(5,2) + 15nsp(4,2) = 210 + 180 + 45 = 435$ (since among the 6 vertices the used ones are 6, 5 or 4).

Proposition 6. If $n \geq 4$ then

$$NSP(n,2) = n(n-1)2^{-5}(5^{n-2} - 2 \cdot 3^{n-2} + 1)$$

or equivalently

$$NSP(n,2) = \binom{n}{2} 2^{-4}(5^{n-2} - 2 \cdot 3^{n-2} + 1).$$

Proof. If the number of used vertices is k ($k = 4, \ldots, n$), for which there are $\dfrac{n!}{k!(n-k)!}$ choices, then there are $nsp(k,2) = k(k-1)2^{k-6}(2^{k-3} - 1)$ variants for the unordered pair of NSPs, hence

$$NSP(n,2) = \sum_{k=4}^{n} \frac{n!}{k!(n-k)!} k(k-1)2^{k-6}(2^{k-3} - 1)$$

$$= \sum_{k=4}^{n} \frac{n!}{(k-2)!(n-k)!}(2^{2k-9} - 2^{k-6})$$

$$= n(n-1)\sum_{j=2}^{n-2} \frac{(n-2)!}{j!(n-2-j)!}(2^{2j-5} - 2^{j-4})$$

$$= n(n-1)\left(2^{-5}\sum_{j=2}^{n-2} \frac{(n-2)!}{j!(n-2-j)!}4^j - 2^{-4}\sum_{j=2}^{n-2} \frac{(n-2)!}{j!(n-2-j)!}2^j\right)$$

$$= n(n-1)(2^{-5}(5^{n-2} - 1 - 4(n-2)) - 2^{-4}(3^{n-2} - 1 - 2(n-2)))$$

$$= n(n-1)2^{-5}(5^{n-2} - 1 - 4(n-2) - 2 \cdot 3^{n-2} + 2 + 4(n-2))$$

$$= n(n-1)2^{-5}(5^{n-2} - 2 \cdot 3^{n-2} + 1).$$

Let us illustrate the validity of the above formula for small n. We have $NSP(4,2) = nsp(4,2) = 3$ and the formula gives $NSP(4,2) = 4 \cdot 3 \cdot 2^{-5}(25 - 18 + 1) = 3 \cdot 2^{-3} \cdot 8 = 3$. In addition, $NSP(5,2) = 5 \cdot 4 \cdot 2^{-5}(125 - 54 + 1) = 5 \cdot 2^{-3} \cdot 72 = 5 \cdot 9 = 45$ and $NSP(6,2) = 6 \cdot 5 \cdot 2^{-5}(625 - 162 + 1) = 15 \cdot 2^{-4} \cdot 464 = 15 \cdot 29 = 435$ in accord with the initial observations.

Remark. The above formula is valid also for $n = 1, 2, 3$ since in all these cases it yields 0, which is trivially true. We will use these, too.

Let us now discuss $NSP'(n,2)$. For example $NSP'(3,2) = 6$ since we have either two singletons or one singleton and one segment, and there are 3 variants in each case. Also $NSP'(4,2) = nsp'(4,2) + 4nsp'(3,2) + \binom{4}{2} = 15 + 12 + 6 = 33$ with the three summands corresponding to the cases of 4, 3 and 2 used vertices.

Proposition 7. If $n \geq 3$ then

$$NSP'(n,2) = n(n-1)2^{-5}(5^{n-2} + 6 \cdot 3^{n-2} + 9)$$

or equivalently

$$NSP'(n,2) = \binom{n}{2}2^{-4}(5^{n-2} + 6 \cdot 3^{n-2} + 9).$$

Proof. If both the NSPs are singetons, there are $\frac{n(n-1)}{2}$ choices. If the number of used vertices is k ($k = 3, \ldots, n$), for which there are $\frac{n!}{k!(n-k)!}$ choices, then there are $nsp'(k,2) = k(k-1)2^{k-6}(2^{k-3} + 3)$ variants for the unordered pair of NSPs, hence

$$NSP'(n,2) = \frac{n(n-1)}{2} + \sum_{k=3}^{n} \frac{n!}{k!(n-k)!}k(k-1)2^{k-6}(2^{k-3} + 3)$$

$$= \frac{n(n-1)}{2} + \sum_{k=3}^{n} \frac{n!}{(k-2)!(n-k)!}(2^{2k-9} + 3 \cdot 2^{k-6})$$

$$= \frac{n(n-1)}{2} + n(n-1)\sum_{j=1}^{n-2} \frac{(n-2)!}{j!(n-2-j)!}(2^{2j-5} + 3 \cdot 2^{j-4})$$

$$= n(n-1)\left(\frac{1}{2} + 2^{-5}\sum_{j=1}^{n-2}\frac{(n-2)!}{j!(n-2-j)!}4^j + 3\cdot 2^{-4}\sum_{j=1}^{n-2}\frac{(n-2)!}{j!(n-2-j)!}2^j\right)$$
$$= n(n-1)2^{-5}(16 + 5^{n-2} - 1 + 6(3^{n-2} - 1))$$
$$= n(n-1)2^{-5}(5^{n-2} + 6\cdot 3^{n-2} + 9).$$

To illustrate the validity of the above formula for small n, we calculate $NSP'(3,2) = 3\cdot 2\cdot 2^{-5}(5+18+9) = 6$ and $NSP'(4,2) = 4\cdot 3\cdot 2^{-5}(25+54+9) = 3\cdot 2^{-3}\cdot 88 = 33$ which both agree with our initial observations. In addition, $NSP'(5,2) = 5\cdot 4\cdot 2^{-5}(125+162+9) = 5\cdot 2^{-3}\cdot 296 = 5\cdot 37 = 185$ agrees with $NSP'(5,2) = nsp'(5,2)+5nsp'(4,2)+10nsp'(3,2)+10 = 70+75+30+10 = 185$, the summands corresponding to the cases of 5, 4, 3 and 2 used vertices.

Remark. The above formula is valid also for $n = 1, 2$; indeed it yields $NSP'(1,2) = 0$ and $NSP'(2,2) = 2\cdot 1\cdot 2^{-5}(1+6+9) = 1$, which is easily seen to correspond to the actual situation. We will use these, too.

Let us now discuss $NSP(n,3)$. For example $NSP(6,3) = nsp(6,3) = 15$, $NSP(7,3) = nsp(7,3) + 7nsp(6,3) = 315 + 7\cdot 15 = 420$ (since either all the 7 vertices are used or one is not) and $NSP(8,3) = nsp(8,3) + 8nsp(7,3) + 28nsp(6,3) = 4200 + 8\cdot 315 + 28\cdot 15 = 7140$ (since among the 8 vertices the used ones are 8, 7 or 6).

Proposition 8. If $n \geq 6$ then

$$NSP(n,3) = \frac{n(n-1)(n-2)}{384}(7^{n-3} - 3\cdot 5^{n-3} + 3^{n-2} - 1)$$

or equivalently

$$NSP(n,3) = \binom{n}{3}2^{-6}(7^{n-3} - 3\cdot 5^{n-3} + 3^{n-2} - 1).$$

Proof. If the number of used vertices is k ($k = 6, \ldots, n$), for which there are $\frac{n!}{k!(n-k)!}$ choices, then there are $nsp(k,3) = k(k-1)(k-2)2^{k-10}(3^{k-4} - 2^{k-3} + 1)$ variants for the unordered triple of NSPs, hence

$$NSP(n,3) = \sum_{k=6}^{n}\frac{n!}{k!(n-k)!}k(k-1)(k-2)2^{k-10}(3^{k-4} - 2^{k-3} + 1)$$

$$= \frac{1}{2^7}\sum_{k=6}^{n}\frac{n!}{(k-3)!(n-k)!}(\frac{6^{k-3}}{3} - 4^{k-3} + 2^{k-3}))$$

$$= \frac{n(n-1)(n-2)}{2^7}\sum_{j=3}^{n-3}\frac{(n-3)!}{j!(n-3-j)!}\left(\frac{6^j}{3} - 4^j + 2^j\right)$$

$$= \frac{n(n-1)(n-2)}{2^7}\left(\frac{1}{3}\sum_{j=3}^{n-3}\frac{(n-3)!}{j!(n-3-j)!}6^j - \sum_{j=3}^{n-3}\frac{(n-3)!}{j!(n-3-j)!}4^j + \sum_{j=3}^{n-3}\frac{(n-3)!}{j!(n-3-j)!}2^j\right).$$

We have

$$\sum_{j=3}^{n-3}\frac{(n-3)!}{j!(n-3-j)!}6^j = 7^{n-3} - 1 - 6(n-3) - 18(n-3)(n-4),$$

$$\sum_{j=3}^{n-3} \frac{(n-3)!}{j!(n-3-j)!} 4^j = 5^{n-3} - 1 - 4(n-3) - 8(n-3)(n-4)$$

$$\sum_{j=3}^{n-3} \frac{(n-3)!}{j!(n-3-j)!} 2^j = 3^{n-3} - 1 - 2(n-3) - 2(n-3)(n-4),$$

hence the quantity in the brackets equals

$$\frac{7^{n-3}-1}{3} - 5^{n-3} + 3^{n-3} + (-2+4-2)(n-3) + (-6+8-2)(n-3)(n-4)$$

and the last two summands equal 0. Thus

$$NSP(n,3) = \frac{n(n-1)(n-2)}{3 \cdot 2^7}(7^{n-3} - 3 \cdot 5^{n-3} + 3^{n-2} - 1).$$

Let us illustrate the validity of the above formula for small n. We have $NSP(6,3) = 15$ and the formula yields $NSP(6,3) = 6 \cdot 5 \cdot 4(343 - 375 + 81 - 1)/384 = 5 \cdot 48/16 = 15$. In addition $NSP(7,3) = 7 \cdot 6 \cdot 5(2401 - 1875 + 243 - 1)/384 = 210 \cdot 768/384 = 210 \cdot 2 = 420$ and $NSP(8,3) = 8 \cdot 7 \cdot 6 \cdot 8160/384 = 7 \cdot 8160/8 = 7 \cdot 1020 = 7140$ in accord with the initial observations.

Remark. The above formula is valid also for $n = 1, \ldots, 5$ since in all these cases it yields 0, which is trivially true.

Let us now discuss $NSP'(n,3)$. For example $NSP'(3,3) = nsp'(3,3) = 1$, $NSP'(4,3) = nsp'(4,3) + 4nsp'(3,3) = 6 + 4 = 10$ (since either all the 4 vertices are used or one is not) and $NSP'(5,3) = nsp'(5,3) + 5nsp'(4,3) + 10nsp'(3,3) = 45 + 5 \cdot 6 + 10 = 85$ (since among the 5 vertices the used ones are 5, 4 or 3).

Proposition 9. If $n \geq 6$ then

$$NSP'(n,3) = \frac{n(n-1)(n-2)}{384}(7^{n-3} + 9 \cdot 5^{n-3} + 3^n + 27)$$

or equivalently

$$NSP'(n,3) = \binom{n}{3} 2^{-6}(7^{n-3} + 9 \cdot 5^{n-3} + 3^n + 27).$$

Proof. If the number of used vertices is 3 then there are $\frac{n(n-1)(n-2)}{6}$ choices. If the number of used vertices is k ($k = 4, \ldots, n$), for which there are $\frac{n!}{k!(n-k)!}$ choices, then there are $nsp'(k,3) = k(k-1)(k-2)2^{k-10}(3^{k-4} + 3 \cdot 2^{k-3} + 9)$ variants for the unordered triple of NSPs. Hence

$$NSP'(n,3) = \frac{n(n-1)(n-2)}{6} + \sum_{k=4}^{n} \frac{n!}{k!(n-k)!} k(k-1)(k-2)2^{k-10}(3^{k-4} + 3 \cdot 2^{k-3} + 9)$$

$$= \frac{n(n-1)(n-2)}{6} + \frac{1}{2^7} \sum_{k=4}^{n} \frac{n!}{(k-3)!(n-k)!}(\frac{6^{k-3}}{3} + 3 \cdot 4^{k-3} + 9 \cdot 2^{k-3}))$$

$$= \frac{n(n-1)(n-2)}{6} + \frac{n(n-1)(n-2)}{3 \cdot 2^7} \sum_{j=1}^{n-3} \frac{(n-3)!}{j!(n-3-j)!}(6^j + 9 \cdot 4^j + 27 \cdot 2^j)).$$

The sum to the right can be split in the following three:

$$\sum_{j=1}^{n-3} \frac{(n-3)!}{j!(n-3-j)!} 6^j = 7^{n-3} - 1;$$

$$9 \sum_{j=1}^{n-3} \frac{(n-3)!}{j!(n-3-j)!} 4^j = 9(5^{n-3} - 1);$$

$$27 \sum_{j=1}^{n-3} \frac{(n-3)!}{j!(n-3-j)!} 2^j = 27(3^{n-3} - 1).$$

Substituting them yields:

$$NSP'(n,3) = \frac{n(n-1)(n-2)}{6} + \frac{n(n-1)(n-2)}{3 \cdot 2^7}(7^{n-3} + 9 \cdot 5^{n-3} + 3^n - 37)$$

$$= \frac{n(n-1)(n-2)}{3 \cdot 2^7}(64 + 7^{n-3} + 9 \cdot 5^{n-3} + 3^n - 37)$$

$$= \frac{n(n-1)(n-2)}{384}(7^{n-3} + 9 \cdot 5^{n-3} + 3^n + 27).$$

Let us illustrate the validity of the above formula for small n. We have $NSP'(3,3) = 1$ and the formula yields $NSP'(3,3) = 3 \cdot 2 \cdot 1(1 + 9 + 27 + 27)/384 = 64/64 = 1$. In addition $NSP'(4,3) = 4 \cdot 3 \cdot 2(7 + 45 + 81 + 27)/384 = 160/16 = 10$ and $NSP'(5,3) = 5 \cdot 4 \cdot 3(49 + 225 + 243 + 27)/384 = 5 \cdot 544/32 = 5 \cdot 17 = 85$ in accord with the initial observations.

Remark. The above formula is valid also for $n = 1, 2$ since in both these cases it yields 0, which is trivially true.

Let us wrap up the obtained results:

p	$NSP(n,p)$	$NSP'(n,p)$
1	$\frac{n}{4} \cdot (3^{n-1} - 1)$	$\frac{n}{4} \cdot (3^{n-1} + 3)$
2	$\binom{n}{2} 2^{-4}(5^{n-2} - 2 \cdot 3^{n-2} + 1)$	$\binom{n}{2} 2^{-4}(5^{n-2} + 6 \cdot 3^{n-2} + 9)$
3	$\binom{n}{3} 2^{-6}(7^{n-3} - 3 \cdot 5^{n-3} + 3^{n-2} - 1)$	$\binom{n}{3} 2^{-6}(7^{n-3} + 9 \cdot 5^{n-3} + 3^n + 27)$

From the above it can be conjectured that

$$NSP(n,p) = \frac{1}{2^{2p}} \binom{n}{p} \sum_{i=0}^{p} \binom{p}{i} (2i+1)^{n-p}(-1)^{p-i},$$

$$NSP'(n,p) = \frac{1}{2^{2p}} \binom{n}{p} \sum_{i=0}^{p} \binom{p}{i} (2i+1)^{n-p} 3^{p-i}.$$

A proof could be carried out using induction on p following the above scheme.

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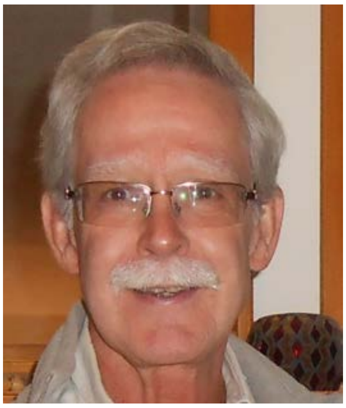
Further reflections on a particular hexagon

Michael de Villiers



Michael de Villiers has worked as mathematics and science teacher and researcher at institutions across the world. From 1983-1990, he was at the University of Stellenbosch, and from 1991-2016 part of the University of Durban-Westville (the University of KwaZulu-Natal from 2005). Though retiring in 2016, he has held an honorary position at the University of Stellenbosch since then. He was editor of Pythagoras, the research journal of the Association of Mathematics Education of South Africa, and was vice-chair of the SA Mathematics Olympiad (SAMO) from 1997-2016, and is still Coordinator of the Senior Problem Solving Committee for SAMO. His main research interests are Geometry, Proof, Applications and Modeling, Problem Solving, and the History of Mathematics.
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Dynamic Geometry Sketches: http://dynamic-mathematicslearning.com/JavaGSPLinks.htm

Abstract
This paper follows on a previous paper about a particular hexagon and proves additional properties. For example, proving that the hexagon in question is tangential, i.e. has an incircle, formulating & proving a converse, as well as exploring the conditions under which the hexagon becomes cyclic. Generalizations to particular $2n$-gons are included.

Introduction

In a recent paper by De Villiers & Hung (2022) some concurrency, collinearity & other properties of a hexagon $ABCDEF$ with $AB = BC, CD = DE, EF = FA$, and $\sphericalangle A = \sphericalangle C = \sphericalangle E = \theta$ were explored. However, shortly after publication the following additional properties were discovered upon 'looking back' at the results in the style of Pólya (1945). These additional properties should also be of interest not only to talented mathematics olympiad students, but since the proofs are quite elementary, possibly also suitable as enrichment for average high school geometry classes.

Incircle

Since the main diagonals of the hexagon above are concurrent, as proven in De Villiers & Hung (2022), it was obvious from the converse of Brianchon's theorem that this particular hexagon had an inscribed conic. Somewhat surprisingly though, it turns out on further investigation that the inscribed conic is a circle! This gives us the first additional theorem below. An interactive dynamic geometry sketch for this result, and those further on, is available for the reader to explore at: http://dynamicmathematicslearning.com/further-hexagon-properties.html

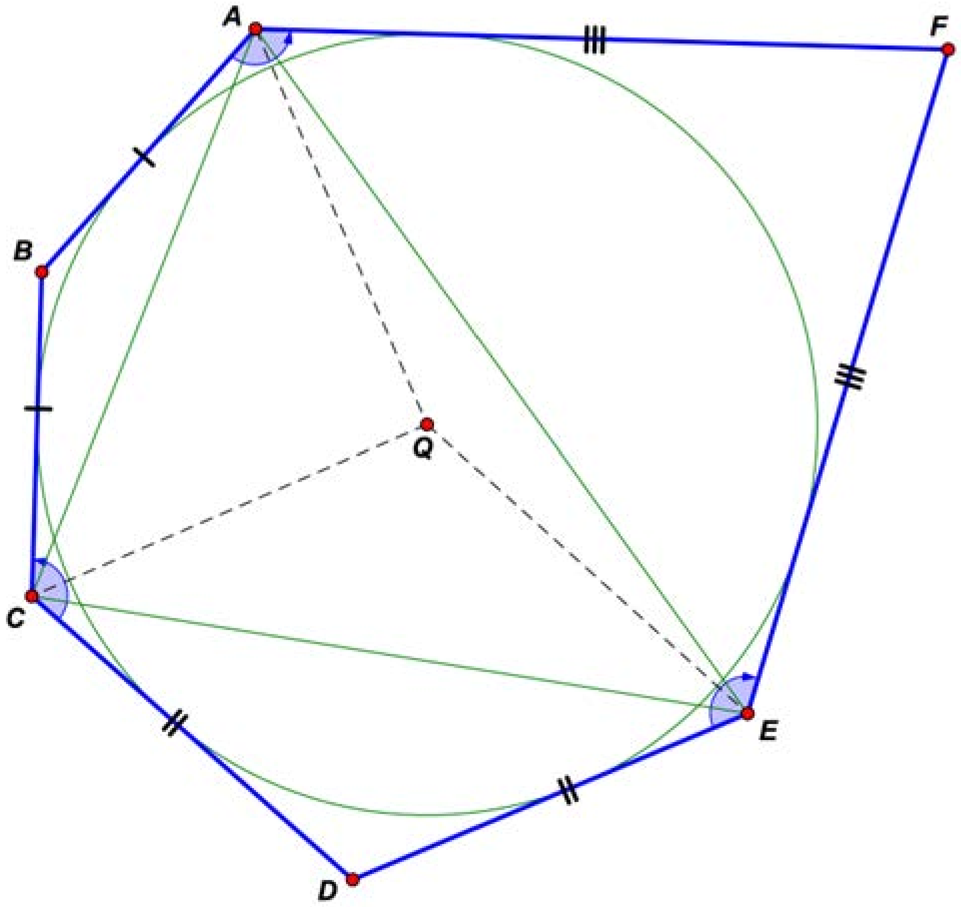

Figure 13: Incircle of hexagon

Theorem 1

Given a hexagon $ABCDEF$ with $AB = BC, CD = DE, EF = FA$, and $\sphericalangle A = \sphericalangle C = \sphericalangle E$, then $ABCDEF$ has an incircle.

Proof. Note that the angle bisectors of the angles at B, D and F are concurrent at the circumcentre, Q, of $\triangle ACE$. Hence, to prove the existence of an incircle, it suffices to show that the angle bisectors of the angles at A, C and E are also concurrent at Q. Connect A, C and E with Q. Now note that $ABCQ, AFEQ$ and $CDEQ$ are kites. Therefore, $\sphericalangle BAQ = \sphericalangle BCQ, \sphericalangle FAQ = \sphericalangle FEQ$ and $\sphericalangle DCQ = \sphericalangle DEQ$. But it is given $\sphericalangle BCQ + \sphericalangle DCQ = \sphericalangle FEQ + \sphericalangle DEQ$. Therefore, $\sphericalangle BCQ = \sphericalangle FEQ$, which implies that $\sphericalangle BAQ = \sphericalangle FAQ$, and there AQ bisects the angle at A. In the same way, we can show that the other two angles at C and E are respectively bisected by CQ and EQ. Since all six angle bisectors are concurrent at Q, it shows that Q is equidistant from all six sides, and therefore completes the proof that an incircle exists.

Alternative concurrency proof

In De Villiers & Hung (2022) we proved that the main diagonals of the hexagon $ABCDEF$ are concurrent by using a theorem by Anghel (2016). However, since the hexagon has an incircle as shown in the theorem above, the concurrency of the main diagonals AD, BE, and CF follows immediately from the application of Brianchon's paper, and provides much easier proof.

It has also come to my attention that this hexagon concurrency result is apparently attributed to A. Zaslavsky, and a diagram (without proof) of it is given in Akopyan (2011, problem 4.9.26, p. 53). It also appeared earlier as a problem in the Third Sharygin Olympiad in Geometry (2007, Final Round, Grade 9, Problem 3). Though in Russian, it's easy to see that the given solution on p. 6 to Problem 3 of the Third Sharygin Olympiad Solutions (2007), is via Theorem 1 above (see p. 6, Fig. 9.3).

Converse of Theorem 1

An equivalent formulation of Theorem 1 is the following: Given a hexagon $ABCDEF$ with $AB = BC, CD = DE, EF = FA$, and $\sphericalangle A = \sphericalangle C = \sphericalangle E$, then the angle bisectors of $\sphericalangle A, \sphericalangle C$, and $\sphericalangle E$ are concurrent at the circumcentre, Q, of $\triangle ACE$. This formulation now gives us the following neat converse: Given a hexagon $ABCDEF$ with $AB = BC, CD = DE, EF = FA$, and the angle bisectors of $\sphericalangle A, \sphericalangle C$, and $\sphericalangle E$ are concurrent at the circumcentre, Q, of $\triangle ACE$, then $\sphericalangle A = \sphericalangle C = \sphericalangle E$.

Proof. Again consider Figure 1. It is given that AQ and CQ respectively bisect the angles at A and C; thus $\sphericalangle BAF = 2 \sphericalangle BAQ$ and $\sphericalangle DCB = 2 \sphericalangle BCQ$. But as before $ABCQ$ is a kite. Therefore, $\sphericalangle BAQ = \sphericalangle BCQ$; thus $\sphericalangle BAF = \sphericalangle DCB$. Therefore, the two angles at A and C are equal. In the same way, we can show that the angle at E is equal to either one of the angles at A or C, to complete the proof that $\sphericalangle A = \sphericalangle C = \sphericalangle E$.

It's also interesting to explore when $ABCDEF$ is cyclic. A little exploring with the aid of a dynamic geometry sketch, quickly gave the following additional theorem.

Theorem 2.

Given a hexagon $ABCDEFwithAB = BC, CD = DE, EF = FA$, and $\sphericalangle A = \sphericalangle C = \sphericalangle E$, then $ABCDEF$ is cyclic only when $\triangle ACE$ is equilateral, and the hexagon is regular. Proof. For $ABCDEF$ to be cyclic the points B, D and F have to lie on the circumcircle of $\triangle ACE$. Assume that B lies on the circumcircle of $\triangle ACE$ as shown in Figure 2. Label $\sphericalangle BAC = x, \sphericalangle CAQ = p, \sphericalangle EAQ = r$ and $angleFAE = z$. Then determine the other angles in the diagram through some straightforward angle chasing.

From Theorem 1, we have the following equation:

$$x + p = z + r \tag{1}$$

Since $ABCE$ is a cyclic quadrilateral (by assumption/construction), $\sphericalangle ABC$ is supplementary to $\sphericalangle AEC$. Hence,

$$90^\circ - p = 2x \rightarrow 2x + p = 90^\circ \tag{2}$$

Similarly, for $ACEF$ to be cyclic, $\sphericalangle ACE$ must be supplementary to $\sphericalangle AFE$. Hence,

$$90^\circ - r + 180^\circ - 2z = 180^\circ \rightarrow 2z + r = 90^\circ \tag{3}$$

Equating Equations 2 and 2, gives $2x+p = 2z+r$. Subtracting Equation 1 from the corresponding sides of the preceding equation, gives $x = z$. Substitution of $x = z$ back into Equation 1, also

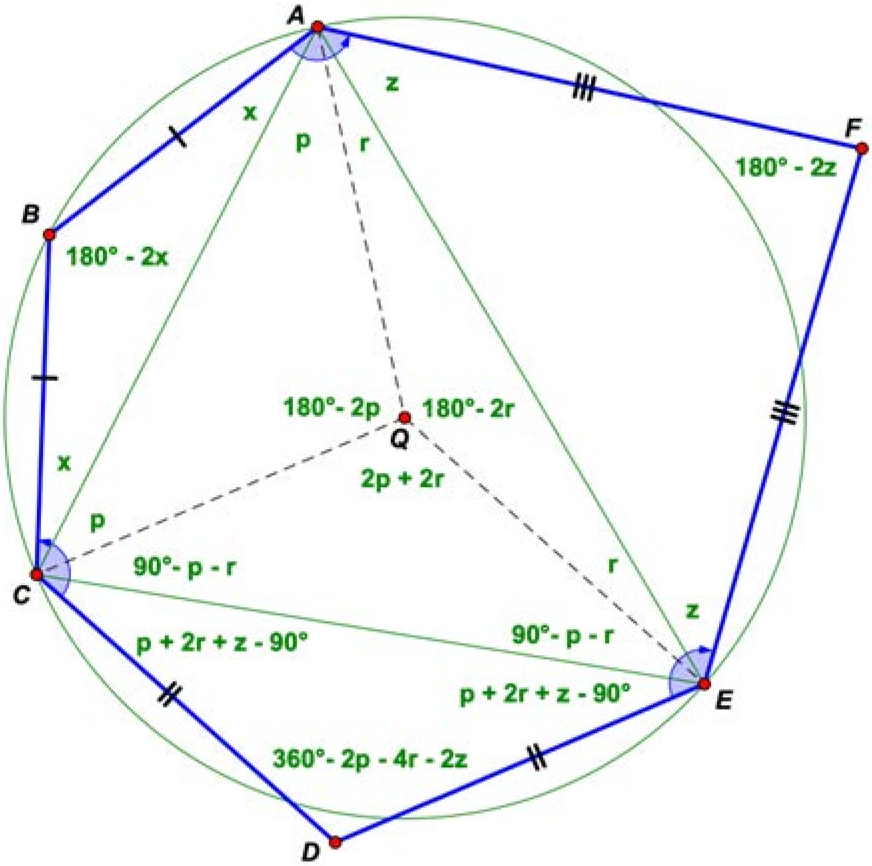

Figure 14: Angles in hexagon

implies that $p = r$. Similarly, for $ACDE$ to be cyclic, $\sphericalangle CAE$ must be supplementary to $\sphericalangle CDE$. Hence, $360° - 2p - 4r - 2z + p + r = 180° \rightarrow p + 3r + 2z = 180°$. But substituting $p = r$ from the above into this equation, gives

$$2r + z = 90° \tag{4}$$

Equating Equations 2 and 4, gives $2x + p = 2r + z$. Again subtracting Equation 1 from the corresponding sides of the preceding equation, gives $x = r$. Substituting $x = r$ and $p = r$ into Equation 1, gives $3r = 90° \rightarrow r = 30° = x = p = z$. From the symmetry of the problem, it's obvious that the same relationships between the four angles at each of the vertices C and E would also hold. Therefore, if $ABCDEF$ is cyclic, $\triangle ACE$ will be equilateral, and the isosceles triangles on its sides, congruent to each other (with apex angles of 120°). Thus, $ABCDEF$ will be a regular hexagon.

Alternative Construction

Theorem 1 and its converse provide an alternative, easier way to construct a dynamic version of $ABCDEF$ than the one implied by the results in De Villiers & Hung (2022). From Theorem 1, one can easily construct $ABCDEF$ by starting with an arbitrary $\triangle ACE$ and its circumcentre, Q. Connect Q with each the vertices A, C, and E. Choose an arbitrary point B on the perpendicular bisector of AC, and reflect line AB around AQ. The point F is then located at the intersection of the reflected line with the perpendicular bisector of AE. Repeat the same reflection with line

FE around EQ to locate point D at the intersection of the reflected line with the perpendicular bisector of CE. The formed hexagon $ABCDEF$ can then be dynamically changed by dragging any of the vertices of $\triangle ACE$, or the variable point B.

Further Generalization

It is not hard to see, and prove in the same way as before, that the converse of Theorem 1 generalizes as follows to an octagon: Given an octagon $ABCDEFGH$ with $ACEG$ cyclic, $AB = BC, CD = DE, EF = FG, GH = HA$, and the angle bisectors of $\sphericalangle A$, $\sphericalangle C$, $\sphericalangle E$ and $\sphericalangle G$ concurrent at the circumcentre, Q, of $ACEG$, then $\sphericalangle A = \sphericalangle C = \sphericalangle E = \sphericalangle G$(see Figure 3). From the argument it's easy to see that the converse of Theorem 1 would further generalize in the same way to a decagon, and in general, to a $2n$-gon with $n \geq 3$. Note that to construct a dynamic $2n$-gon with this property, one can use the alternative construction described above. For example, for an octagon, one again starts with a cyclic quadrilateral $ACEG$, and an arbitrary point B, on the perpendicular bisector of AC, and then reflect line AB around AQ, etc. Also note that since this construction produces a $2n$-gon with all the angle bisectors concurrent at Q, it follows that Q is equidistant from all the sides, and therefore the $2n$-gon has an incircle.

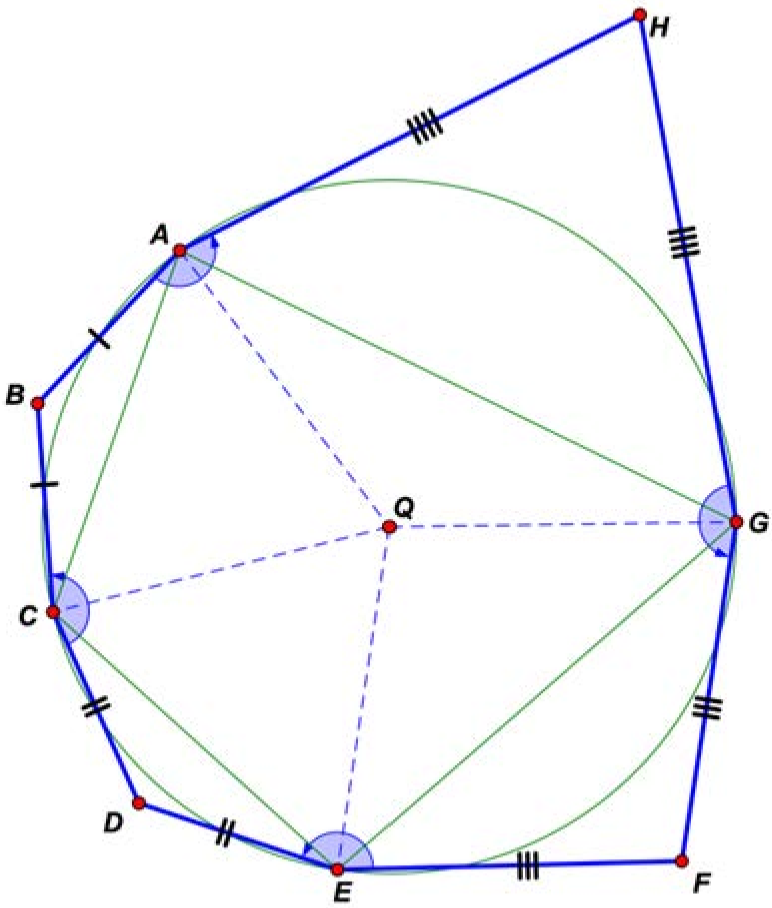

Figure 15: Octagon generalization of converse of Theorem 1

Perhaps unexpectedly, Theorem 1 does not likewise generalize to an octagon. For example, Figure 4 provides a counter-example to the statement: Given an octagon $ABCDEFGH$ with

$ACEG$ cyclic, $AB = BC, CD = DE, EF = FG, GH = HA$, and $\sphericalangle A = \sphericalangle C = \sphericalangle E = \sphericalangle G$, then the angle bisectors of $\sphericalangle A, \sphericalangle C, \sphericalangle E$ and $\sphericalangle G$ are concurrent at the circumcentre, Q, of $ACEG$. The figure clearly shows that $\sphericalangle BAQ \neq \sphericalangle HAQ$, and therefore AQ is not the angle bisector of $\sphericalangle A$.

However, Theorem 1 does generalize to a decagon as follows: Given a decagon $ABCDEFGHIJ$ with $ACEGI$ cyclic, $AB = BC, CD = DE, EF = FG, GH = HI, IJ = JA$, and $\sphericalangle A = \sphericalangle C = \sphericalangle E = \sphericalangle G = \sphericalangle$ then the angle bisectors of $\sphericalangle A, \sphericalangle C, \sphericalangle E, \sphericalangle G$ and $\sphericalangle I$ are concurrent at the circumcentre, Q, of $ACEG$ (see Figure 5).

With this arrangement of the kites and the equal angles at vertices A, C, E, G, I, the same proof of Theorem 1 can again be used and is left to the reader to complete. Note that Theorem 1 can therefore be generalized to a $2n$-gon where n is odd and $n \geq 3$. In addition, since all the angle bisectors are again concurrent at Q, these $2n$-gons will all have incircles.

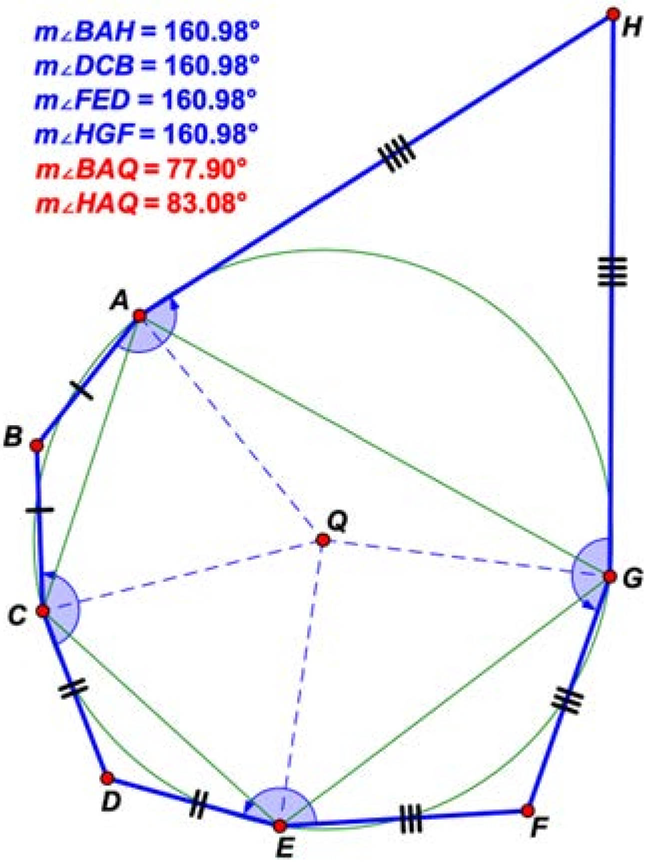

Figure 16: Octagon counter-example for generalization of Theorem 1

Concluding Remarks

The proof of Theorem 1 needs some modification for the cases when the circumcentre, Q, of $\triangle ACE$ lies outside the triangle. However, these modifications can be avoided by stating and consistently using directed angles through-out. Further reflection and investigation of the hexagon

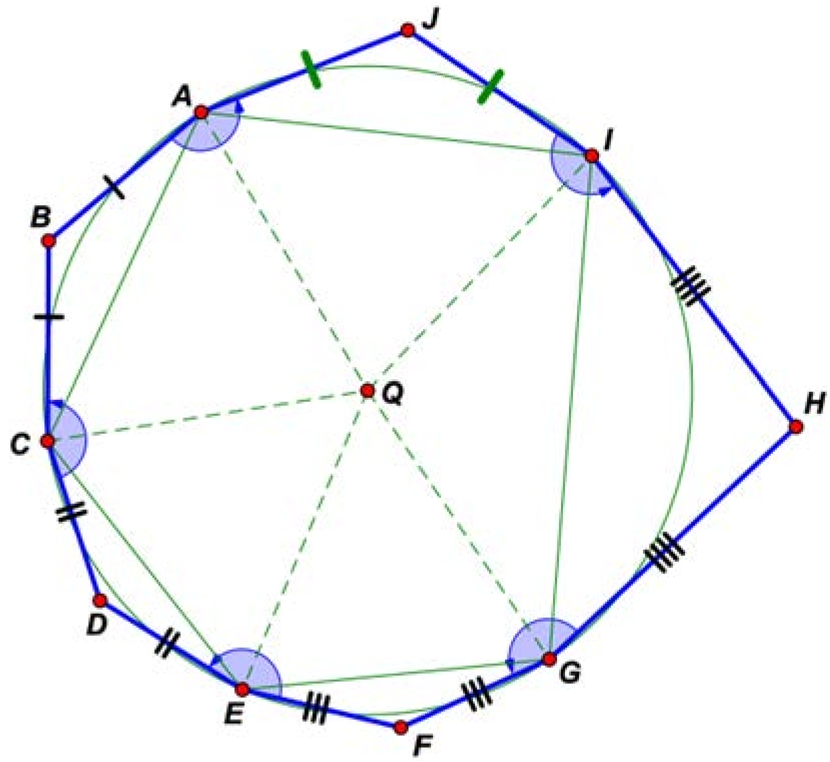

Figure 17: Decagon generalization of Theorem 1

in question not only produced some other interesting properties, but also a simpler proof of the concurrency of the main diagonals, as well as some generalizations to $2n$-gons. This demonstrates the value of ‘looking back’ as advocated by Polyá (1945). Over-all, the problems are relatively straight forward and quite suited for use in a problem solving course with novice learners and students or for some basic practice for a mathematics competition at an introductory level.

Web Supplement.
http://dynamicmathematicslearning.com/further-hexagon-properties.html

Disclaimer. No potential competing interest was reported by the authors.

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On some trigonometric inequalities

Józef Kalinowski

Józef Kalinowski completed a PhD in mathematics in 1977 defending the dissertation on ordinary differential equations with deviated arguments at the Silesian University in Katowice, Poland. The thesis supervisor was Tadeusz Dłotko. Since 1973, for 50 years, Józef Kalinowski has been a member of the Silesian Regional Olympiad Committee in Katowice. Since 1981 he has been a vice-chair of the organization. He worked for 27 years in the Department of Mathematics at the Silesian University. He is the author of two books of problems of the Czech and Slovak Mathematical Olympiads (in Polish). He is also (together with Jaroslav Švrček of Czech Republic) one of the founders and organizers of all twenty nine international mathematical competitions Mathematical Duel. Currently, Józef Kalinowski is retired.

Abstract
In the paper two expressions are considered in a triangle with angles α, β, γ: $\cos\alpha + x\cos\beta - y\cos\gamma$ and $\cos\alpha - x\cos\beta - y\cos\gamma$ for positive constants x, y. The best upper and lower estimates of these expressions are found.

First we prove the following

Lemma. In a triangle with angles α, β, γ (that is, $\alpha \in (0,\pi), \beta \in (0,\pi)$ and $\gamma = \pi - (\alpha+\beta) \in (0,\pi)$) the following inequalities hold for any positive constants u, v, w

$$\min\{u - v - w;\ -u + v - w\} < u\cos\alpha + v\cos\beta - w\cos\gamma < u + v + w. \qquad (1)$$

The constant $\min\{u - v - w;\ -u + v - w\}$ cannot be enlarged and constant $u + v + w$ cannot be reduced. The inequalities in (1) are always strong.

Proof. Since $\alpha + \beta + \gamma = \pi$, then $\gamma = \pi - (\alpha + \beta)$ and

$$u\cos\alpha + v\cos\beta - w\cos\gamma = u\cos\alpha + v\cos\beta + w\cos(\alpha+\beta).$$

Put $D := \{(\alpha,\beta) \in [0,\pi]^2 : \alpha + \beta \leq \pi\}$ and define the function $f : D \longrightarrow \mathbb{R}$ as the right hand side of the above equality, i.e.

$$f(\alpha,\beta) = u\cos\alpha + v\cos\beta + w\cos(\alpha+\beta).$$

Observe that f is defined in the triangle D. Also, f is a continuous function in its domain. The set D in the Cartesian coordinate system may be represented as the triangle OAB (see Figure 1). The points on the edge of the triangle D represent degenerate triangles in which at least one of the angles α, β, γ is equal to zero. The interior points of D represent the non-degenerated triangles, i.e. where each of α, β, γ with $\alpha \in (0,\pi)$, $\beta \in (0,\pi)$, $\gamma \in (0,\pi)$ and $\alpha + \beta + \gamma = \pi$ may be obtained.

Observe that if we fix $\beta \in [0, \pi]$ then $f(\cdot, \beta)$ is a strictly decreasing function. To see it, take the angles α_1, α_2, such that $0 \le \alpha_1 < \alpha_2 \le \pi - \beta$. Since cosine is decreasing function in the segment $[0; \pi]$ then

$$f(\alpha_2, \beta) - f(\alpha_1, \beta) = u(\cos\alpha_2 - \cos\alpha_1) + w[\cos(\alpha_2 + \beta) - \cos(\alpha_1 + \beta)] < 0.$$

This means that $f(\cdot, \beta)$ is strictly decreasing on each horizontal segment in D.

Similarly, using analogous reasoning, we prove that $f(\alpha, \cdot)$ is strictly decreasing for each $\alpha \in [0, \pi]$. This means that on each vertical segment in D the function f is strictly decreasing.

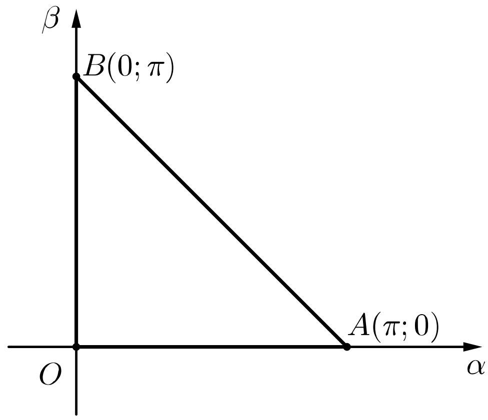

Figure 1

Calculate now the values of the function f on the sides of the triangle OAB.

On the side OA we have $\beta = 0$ and for any $\alpha \in [0; \pi]$ we obtain

$$f(\alpha, 0) = u\cos\alpha + v\cos 0 + w\cos(\alpha + 0) = v + (u + w)\cos\alpha.$$

On the side OB we have $\alpha = 0$ and for any $\beta \in [0; \pi]$ we obtain

$$f(0, \beta) = u\cos 0 + v\cos\beta + w\cos(0 + \beta) = u + (v + w)\cos\beta.$$

On the side AB we have $\beta = \pi - \alpha$ for any $\alpha \in [0; \pi]$ and we obtain

$$\varphi(\alpha) := f(\alpha, \pi - \alpha) = u\cos\alpha + v\cos(\pi - \alpha) + w\cos[\alpha + (\pi - \alpha)] =$$

$$= u\cos\alpha - v\cos\alpha + w\cos\pi = (u - v)\cos\alpha - w,$$

whence the function $\varphi : [0, \pi] \longrightarrow \mathbb{R}$ is strictly monotone for each $u \neq v$. More exactly, it is strictly decreasing if $u > v$, and strictly increasing if $u < v$. In other words, the function $f|_{AB}$ is strictly decreasing (from $u - v - w$ to $v - u - w$, if $u < v$) and strictly increasing (from $u - v - w$ to $v - u - w$, if $u > v$) when the point $(\alpha, \varphi(\alpha))$ moves from B to A. If $u = v$ then $\varphi(\alpha) = -w$ is constant on the segment AB.

From the above considerations on monotonicity of the function f it follows that in D the function f reaches its maximum at the point $O = (0, 0)$ and $\max f(D) = f(0, 0) = u + v + w$. Note that this is a strict maximum, i.e. for every $(\alpha, \beta) \neq (0, 0)$ we have $f(\alpha, \beta) < u + v + w$. Moreover, by the continuity of f the estimation $u + v + w$ cannot be reduced.

The function f obtains its minimal value on D at the point A if $u > v$, at the point B if $v > u$, or at any point of the side AB if $u = v$. Because $f(A) = -u + v - w$, $f(B) = u - v - w$

and $f(p) = -w$ for any $p \in AB$ if $u = v$, we get $\min f(D) = \min\{u - v - w;\ -u + v - w\}$. Similarly as above, this is a strict minimum and the estimation cannot be enlarged (by continuity of f).

From the above reasoning it follows that for any positive u, v, w and $\alpha, \beta \in (0, \pi)$ with $\alpha + \beta < \pi$ we get

$$\min\{u - v - w;\ -u + v - w\} < f(\alpha, \beta) < u + v + w,$$

and the inequalities in (1). $\square$

Using the Lemma we can prove

Theorem 1. In a triangle with angles α, β, γ the following inequalities hold for any positive constants x, y

$$\min\{1 - x - y;\ -1 + x - y\} < \cos\alpha + x\cos\beta - y\cos\gamma < 1 + x + y. \qquad (2)$$

The estimate $\min\{1 - x - y;\ -1 + x - y\}$ cannot be enlarged and the estimate $1 + x + y$ cannot be reduced. The inequalities in (2) are always strong.

Proof. In equation (1) from the Lemma take $u = 1$, $v = x$ and $w = y$. We obtain the inequalities in (2). Also by the Lemma the estimate $\min\{1 - x - y;\ -1 + x - y\}$ cannot be enlarged and the estimate $1 + x + y$ cannot be reduced. $\square$

Theorem 2. In a triangle with angles α, β, γ the following inequalities hold for any positive constants x, y

$$-1 - x - y < \cos\alpha - x\cos\beta - y\cos\gamma < \max\{1 - x + y;\ 1 + x - y\}. \qquad (3)$$

The estimate $-1 - x - y$ cannot be enlarged and the estimate $\max\{1 - x + y;\ 1 + x - y\}$ cannot be reduced. The inequalities in (3) are always strong.

Proof. Multiplying the inequalities in (1) by -1 we obtain

$$-u - v - w < -u\cos\alpha - v\cos\beta + w\cos\gamma < \max\{u - v + w;\ -u + v + w\}.$$

If we put $w = 1$, $u = x$, and $v = y$ and replace the angles γ, α, β by the angles α, β, γ, respectively, we get the estimate in (3). By the Lemma the estimate $-1 - x - y$ cannot be enlarged and the estimate $\max\{1 - x + y;\ 1 + x - y\}$ cannot be reduced. $\square$

Final remarks.

The results obtained in Theorems 1 and 2 are probably new. The partial cases of inequalities (2) and (3) were published in the book [1] at point 2.20, pages 23-24 using results from the paper [2]. The partial cases of inequalities (3) were published in the book [1] at point 2.18, pages 22-23. The special case of inequalities (2) was examined in the book [3], pages 87-88.

Acknowledgements.

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International Mathematics Tournament of the Towns

Andy Liu

In 1976, Andy Liu received a Doctor of Philosophy in mathematics and a Professional Diploma in elementary education, making him one of very few people officially qualified to teach from kindergarten to graduate school. He was heavily involved in the International Mathematical Olympiad. He served as the deputy leader of the USA team from 1981 to 1984, and as the leader of the Canadian team in 2000 and 2003. He chaired the Problem Committee in 1995, and was on the Problem Committee in 1994, 1998 and 2016. He had given lectures to school children in Canada, the United States, Colombia, Hungary, Latvia, Sweden, Tunisia, South Africa, Sri Lanka, Nepal, Thailand, Laos, Malaysia, Indonesia, the Philippines, Hong Kong, Macau, Taiwan, China and Australia. He ran a mathematics circle in Edmonton for thirty-two years, and continued his book-publishing after his retirement from the University of Alberta in 2013. He is currently writing his twentieth mathematics book, which is based on Greek Mythology.

Selected Problems from the Spring 2022 Papers

1. Two friends walked towards each other along a straight road at different constant speeds. At the same moment, each friend released his dog to run at equal constant speed to meet the other friend. As soon as that happened, each dog returned to its owner. Which dog returned to its owner first, the one owned by the slower walker or the one owned by the faster walker?

2. All genuine coins weigh the same. All counterfeit coins also weigh the same, but are lighter.

   (a) Among seven yellow coins and four green coins, seven are genuine and four are counterfeit. Are two weighings on a balance be sufficient for determining whether all four green coins are genuine?

   (b) Among five yellow coins and three green coins, five are genuine and three are counterfeit. Are two weighings on a balance be sufficient for determining whether all three green coins are genuine?

3. Among seven red coins, seven yellow coins and seven green coins, all are genuine except one. All genuine coins of the same color have the same weight, but different for each color. If the counterfeit coin is red, it is lighter than a genuine red coin. If the counterfeit coin is green, it is heavier than a genuine green coin. If the counterfeit coin is yellow, it can be either way. Identify the counterfeit coin in three weighings on a balance.

4. Each of Alice and Bob independently covers a 20×21 board with 1×3 pieces. Bob wins one dollar for each piece which is in the same position in both coverings. What is the maximum number of dollars Bob can guarantee to win?

5. Counters are placed on the squares of a 100×100 board. In each row which has an odd number of counters, the middle one is painted red. In each column which has an odd number of counters, the middle one is painted green. All the red counters lie in different columns and all the green counters lie in different rows.

(a) Prove that there exists a painted counter which is both red and green.

(b) Is it true that every painted counter is both red and green?

6. A quadrilateral $ABCD$ is inscribed into a circle ω with center O. The circumcircle of triangle AOC intersects the lines AB, BC, CD and DA again at M, N, K and L respectively. Prove that there is a circle tangent to KL, MN and the tangents to ω at A and C.

7. Two triangles intersect in a hexagon, forming six triangles outside this hexagon. If they have equal inradii, prove that so do the original triangles.

8. The graph of a function $y = f(x)$ is drawn in the coordinate plane. Then the y-axis and all the scale marks on the x axis are erased. Find a Euclidean reconstruction of the y-axis if

   (a) $f(x) = 3^x$;

   (b) $f(x) = \log_a x$, where $a > 1$ is an unknown number.

9. What is the maximum number of roots on the interval (0,1) for a monic polynomial of degree 2022 with integer coefficients?

10. Let n be a positive integer. Consider all n-tuples $(a_1, a_2, \ldots, a_n)$ in which $a_i = i$ or $i + 1$ for $1 \le i \le n$. Alice computes the product $a_1 a_2 \cdots a_n$ whenever the sum $a_1 + a_2 + \cdots + a_n$ is odd, while Bob computes the product whenever the sum is even. For each n, determine whether the sum of Alice's products or the sum of Bob's products is greater, and compute the difference between them.

11. On the head of each of 300 wizards is placed a hat. Each wizard can see the color of the hat of any wizard except his own. The wizards are to declare the colors of their hats simultaneously. They are informed that the hats come in 25 different colors, and the number of hats of each color used is different. Can the wizards come up with a strategy in advance so that at least 150 of them would make correct declarations?

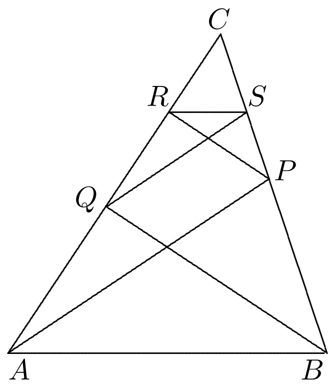
12. A starship is lost in a halfspace and the crew tries to reach the boundary plane. They know that they are at a distance a away, but not in which direction. The starship may travel through space along any path. The crew may measure the distance it has travelled, and will know when the boundary plane is reached. Is it possible to guarantee that this will happen after travelling through a distance of no more than

    (a) $14a$;

    (b) $13a$?

Solutions

1. In the diagram below, the horizontal dimension is distance while the vertical dimension is time. At time 0, the faster boy Alf releases his dog Fido at A while Bob releases Rover at B. The boys eventually meet at C. Fido meets Bob at P and returns to Alf at R. Rover meets Alf at Q and returns to Bob at S. Since the dogs have the same speed, AP is parallel to QS and BQ is parallel to PR. From similar triangles, $\frac{CQ}{CA} = \frac{CS}{CP}$ and $\frac{CR}{CQ} = \frac{CP}{CB}$. Multiplication yields $\frac{CR}{CA} = \frac{CS}{CB}$. It follows that RS is parallel to AB, so that the reunions at R and S also occur at the same time.



2. (a) Two weighings are sufficient. In the first weighing, put two green coins in each pan. If there is no equilibrium, at least one coin among them is counterfeit and the answer is No. Suppose we have equilibrium. Then all four of them are genuine, all four are counterfeit, or two are genuine while the other two are counterfeit. In the second weighing, put the four green coins in one pan and four yellow coins in the other. If we have equilibrium, then there are two counterfeit coins in each side and the answer is No. Otherwise, the green coins are all genuine if they are on the heavier side, and the answer is Yes, If they are on the lighter side, then they are all counterfeit, and the answer is No.

   (b) Two weighings are sufficient. Let G_1, G_2 and G_3 be the green coins and let Y be one of the yellow coins. The first weighing is G_1 and G_2 versus G_3 and Y. The second weighing is G_3 versus Y. The results are summarized in the following chart.

Answer	$G_3 >$Y	$G_3 =$Y	$G_3 <$Y
G_1+G_2 $>$$G_3$+Y	Yes	No	No
G_1+G_2 $=$$G_3$+Y	No	Yes	No
G_1+G_2 $<$$G_3$+Y	No	No	No

3. Label the red coins R1 to R7, the yellow coins Y1 to Y7 and the green coins G1 to G7. The first weighing is (R1,R2,R3,Y1,Y2,Y3) versus (R4,R5,R6,Y4,Y5,Y6). We have two cases.
Case 1. There is equilibrium.
The counterfeit coin is among R7, Y7, G1, G2, G3, G4, G5, G6 and G7. The second weighing is (G1,G2,G3) versus (G4,G5,G6). If there is equilibrium again, the counterfeit coin is among R7, Y7 and G7. The third weighing is (R6,G6) versus (R7,G7). If we still have equilibrium, the counterfeit coin is Y7. If (R6,G6) are heavier, the counterfeit coin is R7. If (R7,G7) are heavier, the counterfeit coin is G7. Suppose there is no equilibrium in the second weighing. We may assume by symmetry that (G1,G2,G3) are heavier. The counterfeit coin is among G1, G2 and G3. The third weighing is G1 versus G2. If we have equilibrium, the counterfeit coin is G3. Otherwise, whichever of G1 and G2 is heavier is the counterfeit coin.
Case 2. There is no equilibrium.
We may assume by symmetry that (R1,R2,R3,Y1,Y2,Y3) are heavier. The counterfeit coin is among R4, R5, R6, Y1, Y2, Y3, Y4, Y5 and Y6. The second weighing is (R4,Y1,Y4) against (R5,Y2,Y5). If there is equilibrium, the counterfeit coin is among R6, Y3 and Y6. The third weighing is (Y2,Y5) versus (Y3,Y6). If there is equilibrium again, the counterfeit coin is R6. If (Y2,Y5) is heavier, the counterfeit coin is Y6. If (Y3,Y6) are heavier, the counterfeit coin is Y3. Suppose there is no equilibrium in the second weighing. We may assume by symmetry that (R4,Y1,Y4) are heavier. The counterfeit coin is among

R5, Y1 and Y4. The third weighing is (Y1,Y4) versus (Y2,Y5). If there is equilibrium, the couterfeit coin is R5. If (Y1,Y4) are heavier, the counterfeit coin is Y1. If (Y2,Y5) is heavier, the counterfeit coin is Y4.

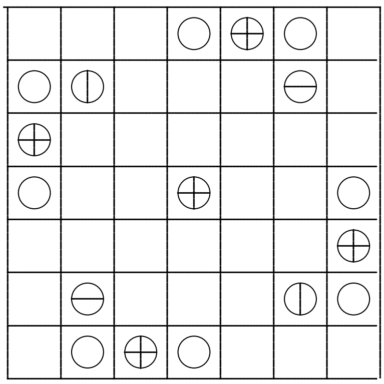
4. Bob can guarantee winning at least 14 dollars by covering his board entirely with horizontal pieces. Consider now Alice's board. Label the columns 0 to 20 from left to right. Since each column has 20 squares, the number of horizontal pieces with a square in that column is congruent to 2 modulo 3. If that square is the leftmost one, the piece is said to start in that column.

   Let $r(k)$ be the remainder when the number of horizontal pieces which start in the kth column is divided by 3. A horizontal piece with a square in the 0th column must start there. Hence $r(0) = 2$. A horizontal piece with a square in the 1st column must either start there or start in column 0. Hence $r(0) + r(1) = 2$. Similarly, we have $r(0) + r(1) + r(2) = 2$. It follows that $r(1) = r(2) = 0$. Similarly, $r(k-2) + r(k-1) + r(k) = 2$ for $k \geq 3$. Hence $r(k) = 2$ if $k \equiv 0 \pmod 3$ and $r(k) = 0$ if $k \not\equiv 0 \pmod 3$. Now a horizontal piece which starts in column k for $k \equiv 0 \pmod 3$ matches a piece in Bob's board. Hence Bob can indeed guarantee winning at least $7 \times 2 = 14$ dollars.

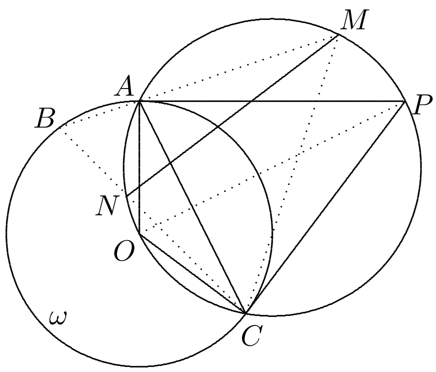
   By peeking at Bob's board, Alice can limit his winning to at most 14 dollars. She covers the top two rows of her board entirely with horizontal pieces, conceding 14 dollars if necessary. She then divides the remaining part of the board into 3×3 subboards. If Bob has a horizontal piece in a subboard, Alice covers the subboard with three vertical pieces. If Bob has a vertical pieces in a subboard, Alice covers it with three horizontal pieces. If Bob does not have either a horizontal or a vertical piece in the subboard, Alice can cover it with three horizontal pieces or three vertical pieces. None of the pieces Alice places below the top two rows can possibly match any of Bob's pieces, so that Bob cannot win more than 14 dollars.

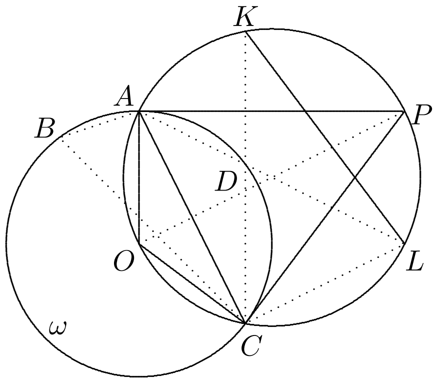
5. (a) Since the red counters are in different columns, the number of columns is at least the number of rows. Since the green counters are in different rows, the number of rows is at least the number of columns. Hence these two numbers are the same, and there is exactly one counter of each color in each row and column. Consider the red counter in the leftmost column. Since it is the middle counter in its row, there is only one counter in that row. However, that row contains a green counter, which must coincide with this red counter.

   (b) This is not necessarily true. In a 100×100 board, only a 7×7 subboard has counters, and they are placed as shown in the diagram below. A red counter is marked with a vertical segment and a green counter with a horizontal segment. A counter which is both red and green is marked with a cross. Four of the counters are painted in only one color.



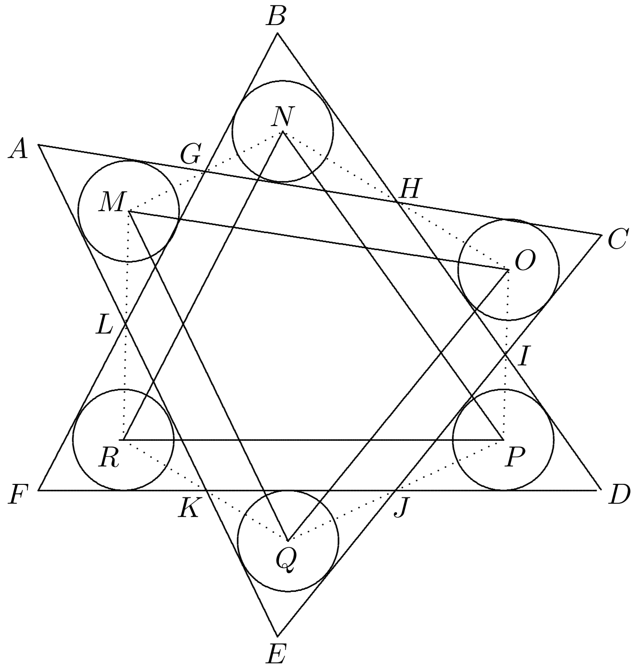
6. The tangents to ω at A and C meet at a point P on the circumcircle of triangle OAC. We claim that $MN = KL = PA = PC$. Then a circle concentric with the circumcircle with be tangent to all four of them. In the diagram below on the left, $\sphericalangle APC = \sphericalangle BMC$ while $\sphericalangle PAC = \sphericalangle POC = \frac{1}{2} \sphericalangle AOC = \sphericalangle MBC$. It follows that $\sphericalangle ACP = \sphericalangle BCM$ so that $PA = MN$.





In the diagram above on the right, $\sphericalangle ACP = \sphericalangle AOP = \frac{1}{2} \sphericalangle AOC = \sphericalangle ABC = \sphericalangle CDL$ while $\sphericalangle APC = \sphericalangle DLC$. It follows that $\sphericalangle CAP = \sphericalangle DCL$ so that $PC = KL$.

7. Label the points as shown in the diagram below.



Since the circles with centers M and N respectivelt are symmetric about G, the tangents from G to these circles are equal. This also applies to the two circles symmetric about each of H, I, J, K and L. It follows that $GH + IJ + KL = LG + HI + JK$. Since G and H are the respective midpoints of NM and NO, $MN = 2GH$. Similarly, $OQ = 2IJ$ and $QM = 2KL$. Hence $MO + OQ + QN = 2(GH + IJ + KL) = 2(LG + HI + JK) = RN+NP+PR$. In other words, triangles MOQ and NPR have the same perimeter. Let r be the common inradii of triangles ALG, BGH, CHI, DIJ, EJK and FKL. The N is at a distance r from GH, and at a distance $2r$ from MO. Hence the area of triangle MNO is rMO. Similarly, the areas of triangles OPQ and QRM are rOQ and rQM, respectively. Hence their total area is $r(MO + OQ + QM) = r(RN + NP + PR)$, which is the total area of triangles NOP, PQR and RMN. Removing them from the hexagon $MNOPQR$

leaves behind triangle NPR, while removing triangles MNO, OPQ and QRM leaves behind triangle MOQ. Hence MOQ and NPR have the same area.

Since they have equal perimeters, their inradii are also equal. The inradius of triangle BDF is r greater than that of triangle NPR while the inradius of triangle ACE is r greater than that of triangle MOQ. It follows that triangles QCE and BDF also have equal inradii.

8. (a) Let P be any point on the graph. Draw a vertical line through P, intersecting the horizontal x-axis at the point N. Draw A horizontal line at a distance $3NP$ above the x-axis, intersecting the graph at the point Q. Draw a vertical line through Q, intersecting the x-axis at the point M. Let the x-coordinate of N be x. Then the y-coordinate of P is 3^x, the y-coordinate of Q is 3^{x+1} and the x-coordinate of M is $x + 1$. If follows that $MN = 1$. Draw a horizontal line at a distance 1 above the x-axis, intersecting the graph at some point. The vertical line through this point is the y-axis.

   (b) We obtain an equivalent problem by taking the function as $f(x) = a^x$ and erasing the x-axis instead of the y-axis. We have the point K of intersection of the graph with the vertical y-axis, and its y-coordinate is 1. Once we have constructed a segment of length 1, the x-axis will be the horizontal line at a distance 1 below K. Let P be any point on the graph to the right of the y-axis. Draw a vertical line at twice the distance of P to the right of the y-axis, intersecting the graph at the point Q. Project P and Q respectively to N and M on the y-axis. Let the x-coordinate of P be x. Then the x-coodinate of Q is $2x$, the y-coordinate of N is a^x and the y-coordinate of M is a^{2x}. We have $ON = a^x - 1$ and $OM = a^{2x} - 1$. Hence $MN = a^x(a^x - 1)$ and $MN - ON = (a^x - 1)^2$. We can construct $\frac{MN \cdot ON}{MN - ON} = a^x$. Subtracting ON from it yields the desired segment of length 1.

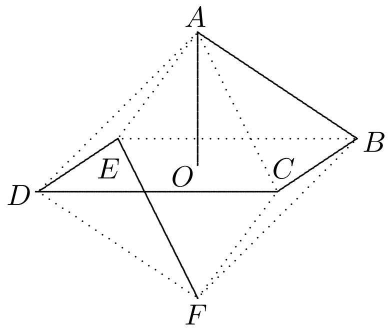
9. A polynomial of degree 2022 has at most 2022 real roots. Since it is monic and has integral coefficients, the absolute value of the products of the roots is an integer. Hence the absolute value of at least one of them must be greater than 1. It follows that the number of roots in (0,1) is at most 2021. We now construct a polynomial with all the desired properties. Choose rational numbers $0 = b_0 < a_1 < b_1 < a_2 < \cdots < a_{2021} < b_{2021} = 1$. The monic polynomial $Q(x) = (x-a_1)(x-a_2)\cdots(x-a_{2021})$ has rational coefficients. The numbers $Q(b_0),\ Q(b_1), \ldots, Q(b_{2021})$ alternate in sign. Hence it has a root in each of the intervals $(b_0, b_1),\ (b_1, b_2), \ldots, (b_{2020}, b_{2021})$. Let k be the greatest of these roots. Let m be the least common multiple of the denominators of $a_1,\ a_2, \ldots, a_{2021}$. Let n be a positive multiple of m which is greater than $\frac{1}{k}$. Define $P(x) = x^{2022} + nQ(x)$. It is monic of degree 2022 and has integral coefficients. For $0 \leq i \leq 2001$, the signs of $P(b_i)$ and $Q(b_i)$ coincide. Hence $P(x)$ also has a root on each of the intervals (b_i, b_{i+1}), yielding 2021 roots on (0,1).

10. Let O_n be the sum of the products of the n-tuples with odd sums and E_n be the sum of the products of the n-tuples with even sums. We claim that $|O_n - E_n| = 1$. This is true for $n = 1$ since $O_1 = 1$ while $E_1 = 2$. Suppose this is true for some $n \geq 1$. The 2^{n+1} $(n+1)$-tuples may be divided into 2^n pairs $(a_1, a_2, \ldots, a_n, n+1)$ and $(a_1, a_2, \ldots, a_n, n+2)$. One of them contributes its product to O_{n+1} while the other contributes its product to E_{n+1}. The difference between the contributions is $a_1 a_2 \cdots a_n$. Now this product was contributed to O_n if the sum $a_1 + a_2 + \cdots + a_n$ is odd, and to E_n if this sum is even. It follows that $|O_{n+1} - E_{n+1}| = |O_n - E_n| = 1$. The claim is now justified by mathematical induction. We have $E_1 > O_1$. Since 2+1 is odd, we switch to $O_2 > E_2$. Since 3+1 is even, we still have $O_3 > E_3$. Since 4+1 is odd, we switch to $E_4 > O_4$. This pattern will continue. It

follows that $E_n - O_n = 1$ if $n \equiv 0$ or 1 (mod 4), while $O_n - E_n = 1$ if $n \equiv 2$ or 3 (mod 4).

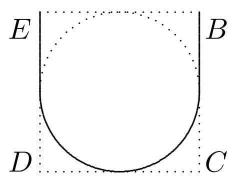
11. This is possible. Let the colors be numbered from 0 to 24. Since $0 + 1 + 2 + \cdots + 24 = 300$, we may assume that there are k hats of color k, $0 \leq k \leq 24$. The actual color distribution among the hats is a permutation of the numbers from 0 to 24, which may be an odd permutation or an even permutation. For each wizard, the colors of the 299 hats he observes come from $1 + 2 + \cdots + (k-1) + (k-1) + (k+1) + \cdots + 24$. Hence he knows that his own hat is either of color 0 or of color k. Now a transposition changes the parity of the permutation. Hence he can make the permutation odd or even by choosing his hat color. The strategy is for 150 of the wizards to choose his hat color to make the permutation odd, and the other 150 to make it even. Since it is either odd or even, exactly 150 wizards will be making the correct choice.

12. (a) This is possible. Let the current position of the starship be O. Then the sphere with center O and radius a is tangent to the boundary plane. Let it be the insphere of a regular octahedron $ABCDEF$. Then at least one of its vertices lies on or beyoond the boundary plane. Following the path $O - A - B - C - D - E - F$, as shown in the diagram below, the starship will reach the boundary plane along the way.



Let $AB = BC = CD = DE = EF = 2s$. Then $OA = \sqrt{2}s$. The volume of the square pyramid $ABCDE$ is given by $\frac{1}{3}\sqrt{2}s(2s)^2 = \frac{4\sqrt{2}}{3}s^3$. Hence the volumn of the tetrahedron $OABC$ is $\frac{\sqrt{2}}{3}s^3$. It is also given by $\frac{1}{3}a\frac{\sqrt{3}}{4}(2s)^2 = \frac{\sqrt{3}}{3}as^2$. Hence $s = \frac{\sqrt{6}}{2}a$. It follows that $OA + AB + BC + CD + DE + EF = (\sqrt{2} + 10)s = (1 + 5\sqrt{2})\sqrt{3}a < 14a$.

(b) This is still possible. We may replace part of the path from B to E with an arc of the incircle of $BCDE$, as shown in the diagram below. The saving is $(4 - \pi)\frac{\sqrt{6}}{2}a > a$.



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