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# MATHEMATICS COMPETITIONS

JOURNAL OF THE

WORLD FEDERATION OF NATIONAL MATHEMATICS COMPETITIONS

AUSTRALIAN MATHS TRUST

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# World Federation of National Mathematics Competitions

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For WFNMC Standing Committees please refer to ABOUT WFNMC section of the WFNMC website http://www.wfnmc.org/.

# From the President

Dear readers of *Mathematics Competitions* journal!

The outgoing 2021 will be remembered by the mathematics education community as an exceptional year in holding ICMEs. As you know, the last ICME-14 was postponed for a year and took place in 2021 in a hybrid mode, face-to-face form (in Shanghai) as well as an online form. Topic Study Group TSG-46 "Mathematical Competitions and Other Challenging Activities" at ICME-14 was the place where the WFNMC members could have actively participate. The Group had 3 sessions with 10 presentations altogether. Because of the COVID-19 pandemic there were not many participants. However, the discussions were interesting.

It is now time to look ahead to the 9th Congress of the Federation. It will take place from the 19th to the 25th of July 2022 in Sofia, Bulgaria and will be hosted by the Institute of Mathematics and Informatics of the Bulgarian Academy of Sciences (https://math.bas.bg/?lang=en). The Congress will be held in a hybrid mode with preference for in-person participation but with the possibility for online attendance as well.

The program of the Congress will include talks, workshops, mini-courses, problem posing sessions, plenary talks, and other events. Much of the work at the Congress will center on the following four themes:

- Building Bridges between Problems of Mathematical Research and Competitions;
- Creating Problems and Problem Solving;
- Competitions around the World;
- Technological Applications in Mathematics Competitions.

Participants are invited to submit proposals for talks, workshops, mini-courses, posters and other mathematical content. These should include titles, the names of authors or organizers, and clear concise descriptions. Proposals have to be sent to both the Chair of the WFNMC Program Committee Robert Geretschläger (robert@rgeretschlaeger.com) and WFNMC President Kiril Bankov (kbankov@fmi.uni-sofia.bg) by January 15th, 2022. Participants are also invited to submit original contest problems and mathematical puzzles along with solutions and a brief analysis of why this particular problem seems to the author to be interesting/useful/stimulating, by April 1st, 2022. The Program Committee intends to organize a problem-creating session and to include a problem chapter in the proceedings book of Congress talks. So, please send us your bright ideas.

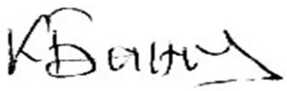
The congress website http://www.math.bas.bg/omi/wfnmc9/ is open. The announcement about the Congress is published on the WFNMC website (www.wfnmc.org).

As you know, every second year up to three nominees are awarded to receive the Paul Erdős Award. This is a unique WFNMC award in the sense that this is the only international award recognizing contributions in the field of mathematics competitions. It is time to nominate for the Paul Erdős Award 2022. Letters of nomination, supporting materials, curriculum vitae, list of publications, and the list of references with their titles, addresses, phones, and e-mails, should be forwarded to Prof. Alexander Soifer (asoifer@uccs.edu), Chair of Paul Erdős Awards Committee. Please refer to the Awards section of the WFNMC web site http://www.wfnmc.org/ for a detailed description of the requirements for nominations.

All complete nomination packages, received by November 30th, 2021, will be given a full consideration.

The awards would be presented at the 9th Congress of the Federation in July 2022 in Sofia.

My best regards,



Kiril Bankov
President of WFNMC
October 2021

# Editor's Page

Dear Competitions enthusiasts, readers of our *Mathematics Competitions* journal!

*Mathematics Competitions* is the right place for you to publish and read the different activities about competitions in Mathematics from around the world. For those of us who have spent a great part of our life encouraging students to enjoy mathematics and the different challenges surrounding its study and development, the journal can offer a platform to exhibit our results as well as a place to find new inspiration in the ways others have motivated young students to explore and learn mathematics through competitions. In a way, this learning from others is one of the better benefits of the competitions environment.

Following the example of previous editors, I invite you to submit to our journal *Mathematics Competitions* your creative essays on a variety of topics related to creating original problems, working with students and teachers, organizing and running mathematics competitions, historical and philosophical views on mathematics and closely related fields, and even your original literary works related to mathematics.

Just be original, creative, and inspirational. Share your ideas, problems, conjectures, and solutions with all your colleagues by publishing them here. We have formalized the submission format to establish uniformity in our journal.

Submission Format
FORMAT: should be LaTeX, TeX, or Microsoft Word, accompanied by another copy in pdf. However, the authors are strongly recommended to send article in TeX or LaTeX format. This is because the whole journal will be compiled in LaTex. Thus your Word document will be typeset again. Texts in Word, if sent, should mainly contain non-mathematical text and any images used should be sent separately.
ILLUSTRATIONS: must be inserted at about the correct place of the text of your submission in one of the following formats: jpeg, pdf, tiff, eps, or mp. Your illustration will not be redrawn. Resolution of your illustrations must be at least 300 dpi, or, preferably, done as vector illustrations. If a text is needed in illustrations, use a font from the Times New Roman family in 11 pt.
START: with the title centered in Large format (roughly 14 pt), followed on the next line by the author(s)' name(s) in italic 12 pt.
MAIN TEXT: Use a font from the Times New Roman family or 12 pt in LaTex.
END: with your name-address-email and your website (if applicable).
INCLUDE: your high resolution small photo and a concise professional summary of your works and titles.
Please submit your manuscripts to María Elizabeth Losada at
`director.olimpiadas@uan.edu.co`

We are counting on receiving your contributions, informative, inspired and creative. Best wishes,

Maria Elizabeth Losada
EDITOR

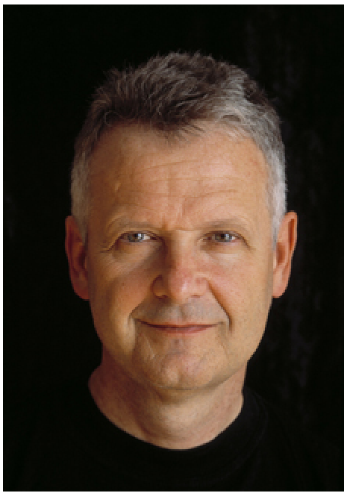
# The International Mathematical Olympiad, Age 62
## What we Know and What we would Like to Know

*Mark Saul and Paul Vaderlind*

Mark Saul has served as Executive Director of the Julia Robinson Mathematics Festival, as Director of Competitions for the Mathematical Association of America, and as director of the Center for Mathematical Talent at the Courant Institute of Mathematical Sciences, New York University. He grew up in the Bronx, got his BA from Columbia University and Ph.D. from New York University. He then spent 35 years in and around New York, teaching mathematics in classrooms from grades 3 through 12.

Saul has also served as Senior Scholar for the John Templeton Foundation, and as a program director for the National Science Foundation. His portfolio there included directing the Presidential Awards for Excellence in Mathematics and Science Teaching. He is a 1984 recipient of that award, the nation's highest honor for work in the classroom.

Internationally, he has given talks and led workshops in 24 countries, His publications include 13 books and numerous articles, including a elementary text on trigonometry, co-authored with I.M. Gelfand, and a translation and 'reader's companion' for Jacques Hadamard's Elementary Geometry.



Paul Vaderlind has been strongly engaged in mathematics competitions and in the education of gifted young people being a leader of the Swedish team participating at the IMO for 21 years. He has been well-known for his contributions to the Baltic Way competition with which he was strongly involved from the very beginning. A recognition for this involvement was the Gold Medal of the Latvian Mathematical Society given to him in connection with the fiftieth anniversary of the organization.

He has been involved with developing countries on a voluntary basis. This concerns mostly Master and PhD education but also mathematics competitions. Many projects, primarily in Africa, ranging from Madagascar through Rwanda to Senegal, are under the umbrella of SIDA (Swedish International Development Agency). As a result some new countries were introduced to PAMO (Pan African Mathematical Olympiad), a competition in which Paul has been involved since 2003.

Paul Vaderlind has an important role also in SNAP Mathematics Foundation of Edmonton. The Foundation was established to promote Math Fairs. He has held the world's largest Math Fair, with 147 booths, in Stockholm, and he has held two successful Math Fairs in Africa.

Paul was awarded the Paul Erdös Award by the World Federation of National Mathematics Competitions in 2012.

**Abstract**

The International Mathematical Olympiad (IMO) has been in existence for 62 years, and now includes more than 105 teams, representing more than 90% of the world's population. The IMO has taken center stage among national and international scientific competitions. Indications are that it has influenced national and regional contests, stimulated interest among mathematicians in pre-college education, and shaped the world's view of mathematics. We have much anecdotal evidence of this influence, but few deeper investigations into detail, or analysis of what we can learn from them going forward. The authors have prepared this article to review what we know, and suggest questions and methodologies for future work in understanding the role of the IMO in the areas of mathematics and mathematics education.

## Introduction

The International Mathematical Olympiad (IMO) has come of age. In its 62 years of existence, it has grown from the original seven East European countries to a global movement including more than 105 teams, representing more than 90

The IMO has taken center stage among national and international scientific competitions. Indications are that it has influenced national and regional contests, stimulated interest among mathematicians in pre-college education, and shaped the world's view of mathematics.

We have much anecdotal evidence of this influence, but few deeper investigations into detail, or analysis of what we can learn from them going forward. We give here an overview of how the IMO has influenced the mathematical professions, and particularly (pre-college) teaching and mathematical research. In our larger article (Zeitschrift Fur Didaktik der Mathematik, in preparation) we consider a wider range of questions: How has our annual event influenced the growth of local and regional competitions? How has it shaped services for students of high ability or motivation? Perhaps most important, what new opportunities have opened up as the IMO reaches a broader, more global audience?

## The IMO and classroom teaching

Interest in connections between classroom mathematics and IMO-level mathematics has grown along with the competition itself. Greitzer (1986)[9] wrote an early opinion piece about how curriculum for problem solving should be structured. Klamkin (1986)[13] gives examples of how contest problems can be used to encourage creativity in the classroom.

Surányi (2001)[29] recounts personal experience in relating competition mathematics to teaching. Tabesh and Mousavi (2012)[30] proposed an internet resource that would store and classify both problems and solutions. Based on the work of George Polya, it would use 'smart algorithms' to identify arguments within a proof that might be used elsewhere. Whether or not the system is implemented, the possibility of harnessing artificial intelligence to the analysis of problem solving is a tantalizing one.

It is not hard to find examples of contest problems that have entered into general texts. Here are two problems that were once considered contest material, but can now be found in numerous texts:

There are fourteen children. A van can hold five children. How many vans are needed to take all of the children to the zoo? (Zimmerman, 2015)[38]

How many ways can Bill walk to work if he lives 10 blocks south and 8 blocks west of where he works if the east-west block which is 4 blocks north of his house and between 2 and 3 blocks east of his house is flooded by the creek? (White, 2013, p. 43)[37]

A football is sewn up of 32 pieces of leather: white regular hexagons and black regular pentagons. Each pentagon borders only hexagons while each hexagon borders three hexagons and three pentagons. How many hexagons are used in this football? (Höjdal, 2009, p. 168)[11].

- These problems have 'trickled down' from contests into curriculum. Are there other such examples?
- On another level, have teachers integrated such problems into their teaching? The first problem, for example, gives students some idea of the relationship between quotient and remainder in a division problem. The second can be related more generally to binomial coefficients and Pascal's triangle. Are these relationships exploited in classrooms? In teacher education? To what extent do classroom teachers use competition problems, both IMO and lower-level competitions?
- Perhaps more importantly, how are contest questions used? Are they used for 'extra credit'? For motivation of core concepts? For solution by groups of students? How might their use be extended?
- Can contest problems become a source for teacher education? In particular, how might IMO problems be used to extend teachers' experiences in mathematics?
- Are there content areas that have been neglected? Certainly the IMO is constrained by the intersection of national curricula. Hence calculus is excluded, as well as vector methods and complex numbers. However, contestants are not so constrained, and often use techniques from these areas in their solution. But how often? Is the taboo on calculus, on vectors, on complex numbers outdated? One of the most serious constraints in problem creation is that the problem not be made significantly easier with the application of advanced methods. We note in passing that in some local and regional competitions, student solutions have actually been judged invalid if the tools used go beyond the usual secondary school curriculum.

Two areas of mathematics which are neglected in most pre-college curricula, but suddenly become important in higher mathematical education, are functional equations and inequalities. Functional equations are more or less untouched by traditional curricula. The essential concept that they introduce, which is largely missing in students' experience before college, is the notion of a function space. In terms introduced by Dubinsky (2002)[6], students are rarely given exercises that lead them to 'encapsulate' functions, to use them as objects to manipulate and relate just as they are trained to work with numbers or algebraic expressions. This concept is often lacking in early courses in calculus, which after all treat operators on functions.

Then suddenly, say in a course on differential equations, students are asked to conceive of a whole space of functions, whose elements are solutions to the differential equations.

- The IMO archive of problems includes a wide variety involving functional equations, and a body of 'professional competition' literature has grown up around it. Can this experience be drawn on to fashion problems and curricula which introduce functional equations in beginning undergraduate work? In high school curricula?

Inequalities are key to an understanding of analysis. Yet in most high school curricula, they are given short shrift. Linear inequalities (in one or two unknowns) are introduced as an adjunct to the study of the corresponding equations. Quadratic inequalities are introduced in order to set up problems in calculus (the sign of the derivative, etc.). But general theorems involving inequalities, methods of proof or solution that are not analogous to those used to solve equations or prove identities, are poorly developed. Yet another difficulty concerning inequalities is the use of common language terms as “larger” and “smaller” which invite misunderstandings. (See for example Blanco and Garrote (2007)[3].)

- The IMO archives offer numerous inequalities, both algebraic and geometric. Can these be used to structure experiences for pre-college students that give them a deeper understanding of the order relation on the real numbers?
- Teachers (of pre-college students) are sometimes involved in the IMO on several levels. Some are team leaders or deputies. Teachers of high ability students in various countries identify and prepare students for the IMO. But how often does this happen? Is there a difference in the kind of training students receive, depending on whether trainers are teachers, graduate students, or professors of mathematics?

Published accounts of particular competitions sometimes report teacher involvement. Rejali (2003)[22] performed a statistical analysis of a poll of teachers in Iran, concerning their attitudes towards mathematical competitions. The results were overwhelmingly positive: teachers saw the events as motivational and instructive, and expressed the desire to be more closely involved in their construction and administration

There is still more to learn about the relationship of the IMO to the teaching professions.

- Are IMO materials, or other contest materials, used in the preparation of teachers or in professional development? There has been much discussion in the field (Ball 2008[2], Shulman 1986[25]) about teacher content knowledge. IMO problems offer a source of advanced mathematics directly accessible to pre-college students. Is it being made accessible to pre-college teachers as well?

## The IMO and Research Mathematics

In this section we consider IMO content in the broader context of the mathematical community. What is the relationship of mathematical research to IMO content?

The central conundrum here was articulated by Jószef Pelikán, IMO medalist and longtime leader of the Hungarian IMO team and professor at Eötvös Loránd University in Budapest, who once remarked (Saul, 2011[23]) that “IMO problems are like animals in a zoo. Mathematical research is like studying animals in the wild.” (Page 414)

- Do IMO problems lead to the discovery of new mathematical knowledge?

A preliminary search of sources shows considerable interest in this last question. Schleicher and Lackmann (2011)[24] have compiled a set of articles about mathematical research questions, many starting with IMO problems.

In particular, Gowers (in Schleicher and Lackmann, 2011 pages 54-55[24]) remarks:

Many people have wondered to what extent success at the International Mathematical Olympiad is a good predictor of success as a research mathematician. This is a fascinating question: some stars of the IMO have gone on to extremely successful research careers, while others have eventually left mathematics (often going on to great success in other fields). Perhaps the best one can say is that the ability to do well in IMO competitions correlates well with the ability to do well in research, but not perfectly. This is not surprising, since the two activities have important similarities and important differences.

The main similarity is obvious: in both cases, one is trying to solve a mathematical problem. In this article, I would like to focus more on the differences, by looking at an area of mathematics, Ramsey theory, that has been a source both of Olympiad problems and of important research problems. I hope to demonstrate that there is a fairly continuous path from one to the other, but that the two ends of this path look quite different.

Later (page 68), Gowers writes:

"If you are an IMO participant reading this, it may seem to you that your talent at solving Olympiad problems has developed almost without your having to do anything: some people are just good at mathematics. But if you have any ambition to be a research mathematician, then sooner or later you will need to take account of the following two principles.

Principle 8.1. If you can solve a mathematical research problem in a few hours, then it probably wasn't a very interesting problem.

Principle 8.2. Success in mathematical research depends heavily on hard work."

In the same volume, Smirnov reifies some of Gowers' observations: "This article started with a problem that I solved at the 1986 IMO. It therefore seems appropriate to end with one of the problems that I am trying to solve now. . . " (Schleicher and Lackmann, page 79[24])

It has often been noted that successful mathematicians sometimes have not participated in competitions.

- What is the size of this cohort? Rather ratio than size
- What is the feeling among mathematicians about the relationship between timed competitions and preparation for a mathematical career?

The work of Victor Pambuccian (2014)[19] shows how an IMO problem can lead directly to serious mathematical insights. The article starts with a classic IMO problem, which has come to be known as the "Windmill Problem". It was first posed by Geoffrey Smith, the leader of UK team, for the 52nd IMO, held in Amsterdam. Pambuccian pares down the solution, creating a minimal axiomatic system in which the problem can be posed and solved.

The "Grasshopper Problem" (2009 IMO, problem 6) was another of the most difficult and most interesting problems posed at the IMO. It received much attention from the mathematical community (Kós, 2017[14], Polymath 2009, Tao, 2009b[32] Tsai, 2009[36]).

Taylor (2019)[34] gives several examples of mathematical content which crosses over from IMO problems to research mathematics. In some cases, the actual IMO problem motivated the more serious mathematical investigation. Heuberger (2020)[10] takes a lemma from number theory ('lifting the exponent'), which is often seen as a competition technique, and shows how it can be harnessed to prove a significant theorem about elliptic curves.

The work above relates the mathematical content of the IMO to research mathematics. Another aspect of the IMO is its convening an international community. This process is only scantily documented.

IMO materials also act indirectly to bring mathematicians together. An article by Pease, Aberdein, and Martin, (2019)[20] shows how researchers into the social processes of mathematics can use IMO-generated materials. The data for their investigation was from the Mini-Polymath projects. This is a blog started by Terrence Tao (https://terrytao.wordpress.com/tag/mini-polymath/) facilitating online collaboration on the solution of IMO problems. The concept of such a blog was first described by Timothy Gowers (2009)[8] in a blog entry.

Research such as that of Pease et al can yield very useful results. For example:

One of the popular myths about mathematics that does not survive the investigation of mathematical practice is that it is a solitary pursuit. Although there are celebrated incidents of 'solo ascents', such as Andrew Wiles's lengthy pursuit of the Taniyama–Shimura conjecture, successful mathematical practice is more characteristically collaborative: for example, single authored papers make up only 36% of the articles published in the leading research journal Annals of Mathematics between 2000 and 2010 [34, p. 11] (Section 3 Par. 2; the reference is to Sharvate, Wetzel, and Pattersin (2011))

The work of Pease et al shows how Tao's platform for mathematical discussion can be analyzed to yield useful results. There may be other such platforms available for analysis. For example, the Art of Problem Solving (www.aops.com) is a vast international website hosting a variety of fora. In particular, it includes in-depth discussions of problems by student contestants. There does not seem to be a serious analysis of these discussions. Such an analysis might shed light on how competition materials are perceived by students, what their effect is on student thinking, or how students naturally think of particular problem situations.

Tao (2009c)[33] himself comments on the format of his venue. Among seven points he makes, two are clearly researchable:

[3.] There is an increasing temptation to work offline as the project develops.

It is possible that future technological advances (e.g. the concurrent editing capabilities of platforms such as Google Wave) may change this, though; also a culture and etiquette of collaborative thinking might also evolve over time, much as how mathematical research has already adapted to happily absorb new modes of communication, such as email.

[4.] Without leadership or organisation, the big picture can be obscured by chaos.

There have been numerous historical and anecdotal accounts of the "Polymath" concept, such as that of Nielsen (2009)[17], or Teitelbaum (2010)[35].

- The blogs and discussion boards described above offer a rich source of data for the analysis of mathematical discovery. Is there a way to coordinate such blogs? To index them, preserve them, or relate them, for historical or psychological analysis?

This cursory overview of publications shows that the IMO seeds people's minds with ideas that may grow over decades. It also shows the interest of the community in an analysis of the creative process. Further, we can see that such analyses can lead to significant insights, such as those of Pease et al. Further still, serious investigation can validate and help shape new venues for mathematical discussion.

We have written above about the influence of the IMO on professional mathematics. But what about the opposite phenomenon: the contributions of mathematicians to the content of the IMO? In fact we don't know very much about how research mathematics generates IMO problems (or problems in other competitions). Part of the reason for this is that the construction of IMO problems is cloaked in mystery, a cloak required of us by the need to secure problems before the competition. After the competition, the problem has acquired a life of its own, and its genesis is rarely inquired after.

And yet there are several fascinating records of the development of competition problems, including IMO problems. Soifer (2008, 2012)[27],[28] traces the intellectual genesis of some problems.

Several writers have addressed the direct involvement of mathematicians in the IMO and related competitions. Some of these descriptive works shed light on the relationship between the mathematical research community and student competitions.

Domoshnitsky and Yavich (2011)[4] write about recruiting judges for their "Internet Mathematical Olympiad" from among research mathematicians. Their article ends with an observation particularly relevant to the IMO: "Students and teachers sit together and watch the students from other countries who are competing with them. You feel that you belong to some global club with no borders. And isn't that in itself worth something?" (Page 18)

The involvement of mathematicians in competition is often seen as a service by the field to a wider community. Lamar (2017)[16] describes a mathematical program run by mathematicians and inspired by the US competitions system, with a target audience of minority students in the Atlanta metropolitan area. Jarvis (2012)[12], in a news article, describes a program preparing US minority students for international Olympiads. Druck and Spira (2008)[5] describe their service to public schools in Brazil, including increased access to competitions, publication of problem solving materials, and improvement in teachers' mathematical content knowledge.

All these pieces are descriptive, rather than analytic.

- How do research mathematicians use their expertise in training or structuring mathematical competitions?
- What motivates mathematicians to work with pre-college competition mathematics?

## Conclusion

In this survey, we have striven to gauge the influence of the IMO on the mathematical teaching and research communities. We have then tried to indicate the limits of our knowledge.

The majority of published articles about the IMO (and other competitions) are descriptive in nature, and often recount personal experiences or observations. That is, they are basically anecdotal.

Our view is that anecdote fuels scientific inquiry in two important ways. First, it is a source of hypotheses to investigate. As many philosophers of science have argued (Polanyi 1958[21], Kuhn 1962[15]), scientific methods tell us how to investigate an hypothesis, but not how to formulate one. Hypotheses stem from creativity and imagination, are driven unconsciously.

The hypotheses we have suggested set the stage for a new stage of scientific inquiry into the IMO

Another important role of anecdote—and one which is much neglected in the study of education—is as a check into the veracity of findings. This phenomenon is clear, for example, in medicine. Clinical reports, which play the role of anecdota, generate hypotheses, but also reify, or sometimes negate, any research findings that have been put into practice. We seek the same sort of virtuous cycle in educational research, particularly about the IMO. The collection of anecdota should be viewed as a first stage in scientific inquiry, and not as antithetical to it.

But we do need to advance to a deeper stage of inquiry. We would like to see the methods of economics (Agarwal and Gaule, 2020[1]) or of psychometrics (Gleason, 2008[7]) applied more widely. We would also like to see the methods of cultural anthropology harnessed to ask questions about traditions and values across cultures. And we need the insights of political science to examine the role of government support.

In general, the data we have amassed about the IMO, both numerical and anecdotal, has not been utilized significantly, either to get answers to questions about the IMO itself or to stimulate the collection of more data on local and national levels. Much work awaits.

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# In search of lost time: Gifted Iranian Students and the Tuymaada Olympiad

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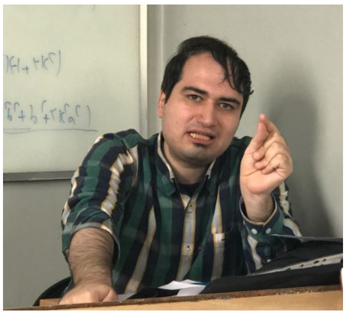


Navid is a senior researcher at the Sharif University of Technology, Tehran, Iran. His main research interest is the evolutionary theory for the socioeconomic domain. He has been the head of the Mathematical Olympiad Department at Salam High schools complex since 2009. He has participated as an instructor and problem proposer for the Iranian TST since 2014. He has been involved in training curricula for mathematics competitions since 2005. Since 2020, he has organized several joint online IMO training camps between the Iranian IMO team and several European countries.
He has written more than 20 books (in Persian) for high school mathematics including Calculus, Mathematics 9, 10, 11, published by Olgoo publication, Tehran, Iran. Navid is a local organizer in Iran for European Mathematical Cup (EMC), mathematical Naboj, and Silk Road Mathematical Competition (SRMC). Since 2019, He has been the Iranian team leader for the Romanian Masters of Mathematics (RMM) competition.
As an instructor, he has been invited to several training camps around the world. Amongst other training camps, he was the instructor of the IMO training camps of Azerbaijan, Bulgaria, Croatia, and Singapore. He also published many articles concerning methods and techniques for solving mathematical competitions problems in Mathematical Reflections (MR), Mathematics Competitions (WFNMC), Arhimede, Cibiti Matematiki, i. e., USM (in Ukrainian), Mathematika (in Bulgarian). He also proposed some problems for the American Mathematical Monthly (AMM) journal. Since 2015, Navid has been collaborating with XYZ-Press publishing house. His second book about polynomials came out in September 2021.

**Abstract**
Mathematical competitions have a lot of interesting properties. If we look at every competition as an event, it then would have some consequences and implications. Amongst them is the durable role of its problems in the skill uplift of the students working on them, either during or after the competitions. In this article, through a qualitative account of the author's experience along with a detailed analytic account of some outstanding problems from the International Tuymaada Olympiad in Mathematics, we try to shed more light on some unnoticed aspects of mathematical competitions.

## Introduction

Since the early years of development, Mathematical Competitions have had many interesting properties. However many scholars consider their congenial atmospheres and their contents, which are at once playful and profound, there is another aspect that is hitherto hidden from view; the durable role of their problems in the skill uplift of the students working on them, either during or after the competitions. This component has been overshadowed by the above-mentioned components.

In this article, we take into account this component through a description of ten years of intensive work of my students, during, or after the exam, on the problems from the International Tuymaada Olympiad in Mathematics. This study can help the reader discover the value implicit in the discussion communities of problem-solving inside schools or training camps for gifted students.

After the interlude of enthusiasm, we begin to outline the theoretical background. The communities of students that are formed by the most talented ones from a school or an organization could be considered to be Public Spheres, in Jurgen Habermas's account. He defines Public Spheres as places in which dialogue, speech, debate, and discussion create a virtual or imaginary community that doesn't necessarily exist in any identifiable space (Soules, 2007). In these spheres, students and instructors communicate in the Rhizomatic way (Deleuze and Guattari, 1980). Indeed, this is a locus of competitive struggle that the cooperative culture of the school or organization will convert to the co-optation that is deeply rooted in the exchange of knowledge as a symbolic capital (Bourdieu, 1975) and production of value (Latour and Woolgar, 1986).

The portrayal resulting from the above combination of concepts has one decisive feature; during a situation where the students face new and challenging problems, after one or two fruitless tries, a circumstance which is not of their choosing emerges. It is then that components of their thought process - such as looking at the other side of the fact, arguing by contradiction, inductive reasoning, critical reasoning, etc. - can bring them some avail in finding an innovative solution even through recombination of already known facts or exaptation of in touch tools. Those components remain successful in this change and can find more strength in the process of informal or formal proof construction.

The article is divided into five parts. First, we shall outline a short literature review as a theoretical background of my designed curriculum. Second, we provide a brief history of the international Tuymaada Olympiad in mathematics. Third, we present, albeit concise, the history of the encounter of the Iranian students with the Tuymaada Olympiad. Fourth, we present six outstanding problems from the competition with multiple solutions from different angles. Fifth, we wind up the article with some remarks.

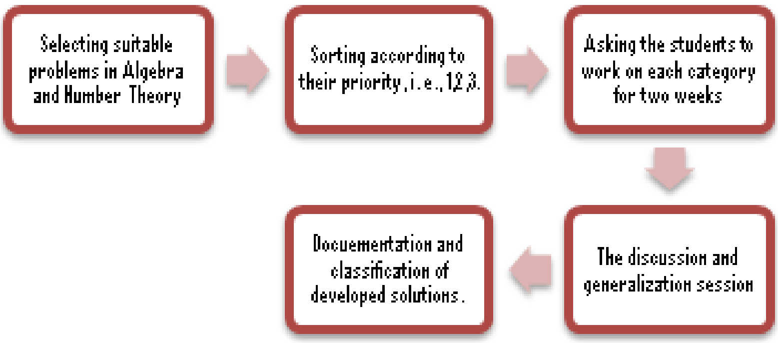
## Public sphere for mathematical problems solving

Starting in 2012, we decided to design a step-by-step program in problem-solving for our students who obtained a gold medal in the 3rd round of the Iranian Mathematical Olympiad. The chief reason for this design was helping our students to achieve more experiences through thinking about and solving problems from those competitions. Considering the fact that first-level competitions have several outstanding problems, the lessons which the students gained have been beneficial. In 2012, this program consisted of 4 competitions: the International Tuymaada Olympiad in Mathematics, the Romanian Masters of Mathematics (RMM), the USA Team Selection Test (USA TST), and the Chinese Team Selection Tests[1].

For this, we designed a five-step process outlined in the following figure. At first, we select suitable problems in Algebra and Number theory. Second, they undergo sorting according

[1]Now, in 2021, this program is more complex and consists of several journals in line with new competitions like the Metropolises Olympiad.

to their priorities and numbering as 1, 2, 3. Several factors such as level of difficulty, level of knowledge, and the relation of this problem with previous problems we solved, have a role in such numbering. Third, setting a deadline for the students to work in person, this deadline was about 2 weeks. Fourth, asking the students to send their mistakes or the problems that they need to know solutions for, then starting the discussion session that can continue for several hours until each mistake is resolved. During this session, we first explained our solution(s) and then asked them to evaluate our approaches and tell us about their approaches. Fifth, since we really appreciate knowledge management, immediately after the end of sessions, I documented newly generated solutions, to be adopted by future sessions.




The outcome of such a process was helping the students to rise to their highest potential or degree of perfection. During the discussions, some rhizomes in their minds found more vigor and brought other rhizomes into alignment with them. In recent years many of those students participated in the IMO and obtained medals in that competition. Now some of them are working as colleagues and assistants, participating in discussion sessions as mentors or instructors. This formation helped the teaching assistants to take apprenticeships through participating in fairly challenging situations.

According to its simplicity, the above model[2] can accommodate a variety of types. It can help us to design a more elaborate program for the training. Through the continuous improvement of our model, some supplementary elements have been added, like related articles, related problems, and even a session of problem construction based on some unfinished ideas to this model. These can be implemented according to the logic of the situation encountered.

## The International Tuymaada Olympiad

### A Brief History

In 1994 the International Tuymaada Olympiad was founded and since then has been hosted by the Ministry of Education and Science of the Republic of Sakha (Yakutia), a

[2]By the term "model" we mean a type of structure, or class of structures, all sharing certain general characteristics. For more details see Van fraassen (Van fraassen, 1980).

State-Funded autonomous institution of supplementary education of the Republic of Sakha (Yakutia). Other collaborators are the "Junior Academy of Sciences of the Republic of Sakha (Yakutia)", the federal-state autonomous educational institution of higher professional Education "M.K.Ammosov North-Eastern Federal University", the federal-state autonomous higher professional education institution, S.N.Donskoy – II education development and advanced training institute for teachers, and the "Far Eastern Mathematics Research Center" a regional mathematics education research center[3].

From the early years up to now, the Olympiad has been held in two categories, i. e., junior and senior categories. The mathematical competition takes place in two days, each paper contains 4 problems and the time allowed for each day is 5 hours. The level of difficulty of the competition is very high and at least one inter-disciplinary problem is presented in it. We have very good memories of solving Tuymaada problems since 2002. We learned a lot from its problems. Moreover, after 19 years of pursuing its problems, this competition definitely signifies my lost time - in Marcel Proust's sense, indeed.

## Iranian Students and the Tuymada Olympiad

In this section, we shall provide a relatively detailed account of our encounter with the Tuymaada Olympiad in mathematics. We divided this section into three parts as follows.

### Early Years

Starting from 2003 and through the medium of AOPS, we found that there is an Olympiad in the North East of Russia with relatively hard and challenging problems. Thus, we started to solve its problems and selected some of them for the training courses. Many students found the problems difficult and in some cases, technical. Notwithstanding this, the problems had three immense functions: First, they helped students to be overcautious at some points. Second, even when failing to solve them, they learned many interesting things from the problems. Third, they grasped the lesson each problem taught, and this per se, helped them with their self-esteem.

### Years 2014-2019

The profundity of the above mentioned gains made me develop a training stage for my students who achieved gold medals in our 3rd round of the national mathematical Olympiad. Thus, according to the model described in section 1.1, we formed discussion forums like a commute of practice to jointly work on the selected problems from the Tuymaada Olympiad. This commute of practice usually consists of 4-7 students from grades 11, 12, myself as a main instructor and one or two mentors selected from the former students.

The discussion sessions have been breath-taking since some of the problems need different, and sometimes quite diametrically opposite, approaches to be implemented simultaneously. But, after solving such a problem the students asserted that the learning outcome of this was invaluable and we reached a great milestone. Besides, we also observed that this

[3]For more details about the International Tuymaada Olympiad in Mathematics, please see reference [6]. In this precious book, you can find the problems and the solutions of the mathematical competition of the junior and senior categories from 1994 until 2012.

training stage helped them establish a much stronger link between their already existing instruction. Further, it also helped them to form small groups of people with similar, or even dissimilar but complementary, mental models. These small groups emerged through the course of this stage and some of them are still active, even after 5-6 years.

### Year 2020

The Covid-19 pandemic has deprived us from many opportunities and has also exerted a pressure for natural selection, i.e., selection pressure on some entities. Among these entities were mathematical competitions. Many of them have undergone some changes either in their procedures or their structures. We had the 61st IMO in virtual form and thus far all the competitions have gone online. Therefore, the Internet Communication Technologies (ICT) comes into play with mathematical competitions. This in itself has established a much stronger link between ICT and Mathematical competitions.

Starting on September 30th, the 27th version of the International Tuymaada Olympiad in mathematics is organized by the Ministry of education of Sakha republic. I availed myself of this opportunity and registered a team of 6 students who had been accepted for the third round of our national mathematical Olympiad, in the senior category[4]. The Ministry of education in the republic of Sakha did its best to organize the competition along with wonderful opening and closing ceremony, a 4-day seminar on mathematical education, and some technology-based excursions for the students.

The competition was held on October 2nd and 3rd in line with complete online proctoring[5]. Then the students sent their written solutions through specially developed software. However, I was responsible for translating the exam papers from Persian to English and it was a time-consuming process, nonetheless it was a nice and interesting experience in comparison with in-person mathematical competitions I have participated in as a leader. Indeed, the virtual math competitions are drawn heavily on the works of leaders.

Our students were happy to participate in a competition that they already knew, having solved and discussed its past years' problems. For them, this functioned as a sign of hope in the age of pandemic.

## The Tuymaada Olympiad: A competition with outstanding problems

In this section, we present some of the most interesting problems from the Tuymaada Olympiads. The solutions are the outcome of the discussion groups I formed in 2012. Some of the solutions are developed by the author and some are developed by his students. Especially, the 2020 solutions are developed from the exam papers of the Iranian students who participated in the competition.

[4]I should express my deep gratitude to: Prof. Vasili Pavlov, the president of the competition, for accepting one team from Iran. Prof. Evgeny Sharin, the head of the mathematical Olympiad jury and Prof. Alexander Golovanov, head of the problems selection committee of the mathematics competition that was in touch with me during the period of coordination. And Ms.Lyubov Popova, Ms.Galina Semenova, who both kindly answered my letters and guided me about the protocols of the competition.

[5]For more information, please browse here: http://tuymaada.lensky-kray.ru/en/

## On the Orbit of Polynomials

The first problem is a really interesting one from a strong mathematician, Prof. Fedor Petrov. The problem was modern by the year of the proposal, i.e., 2003 and is still challenging. There are several research papers concerning the orbit of polynomials with integer coefficients, these papers are indeed a combination of some ideas from dynamical systems and some ideas from number theory, see for example Gottesman and Tang (2010). As I remember, every year, there are only 2 or 3 students, out of 5-7 who completely solve this. Many had good ideas but couldn't go far with them.

**Problem 1.** *(Fedor Petrov, the Tuymaada Olympiad, 2003) Given a polynomial $P(x)$ with non-negative integral coefficients and a positive integer $a$. The sequence $(a_n)$ is defined by $a_1 = a, a_{n+1} = P(a_n)$. It is known that the set of primes dividing at least one of the terms of this sequence is finite. Prove that $P(x) = cx^d$ for some non-negative integers $c$ and $d$.*

*Solution.* If the terms of the sequence were not divisible by any prime number then $P(x) = 1 = x^0$. Now assume the set of prime divisors of the sequence consists of only $N$ different prime divisors. And let $P(x) = c_d x^d + \cdots + c_0$ take an arbitrary prime $p$ from the set of $N$ different prime divisors of the sequence, and assume $c_0 > 0, v_p(c_0) = r \geq 0$. Now obviously the sequence $a_n$ is indeed periodic modulo $p^{r+1}$ (that is, by the pigeon-hole principle, there are $s > m$ such that $a_s \equiv a_m \pmod{p^{r+1}}$ and hence $a_{s+1} = P(a_s) \equiv P(a_m) \equiv a_{m+1} \pmod{p^{r+1}}$ and so on.) Assume the length of the fundamental period is $T$. Then among the terms of the sequence modulo $p^{r+1}$ we have infinitely many non-zero remainders. Indeed assume $a_l \equiv 0 \pmod{p^{r+1}}$ then

$$a_{l+1} = P(a_l) \equiv P(0) = c_0 \not\equiv 0 \pmod{p^{r+1}}$$

Hence, from any two consecutive terms, at least one has a non-zero remainder modulo $p^{r+1}$. Whence, from the congruence $a_m \equiv a_{m+T} \not\equiv 0 \pmod{p^{r+1}}$ one can find that

$$v_p(a_m) = v_p(a_{m+T}) = \alpha \leq r.$$

Now we can construct an infinite sub-sequence $(b_n)$ of $(a_n)$ defined by $b_n = a_{m+nT}$ such that $v_p(b_n) \leq r$ are the same. Define $Q(x) = \underbrace{(P(P(\ldots P(x)\ldots))}_{T}$ then $b_{n+1} = Q(b_n)$ the polynomial $Q(x)$ has non-negative integers coefficients and the $v_p$ of the constant term of the polynomial $Q(x)$ is $r$. Hence when $x$ is a multiple of $p^\alpha$ then $Q(x)$ is again multiple of $p^\alpha$. If we define $d_n = b_n/p^\alpha$ and define $R(x) = (Q(p^\alpha x))/p^\alpha$ then $d_{n+1} = R(d_n)$ and the set of prime divisors of sequence $d_n$ decreases at least by one.

Now we use induction on $N$, in a way that for all positive integers less than $N$ the polynomial with the above properties is of the form $cx^d$ for some non-negative integers $c, d$, that is, $R(x) = cx^d$ and hence also the $Q(x)$ and the $P(x)$.

It remains to examine the case when $c_0 = 0$ and $P(x) = x^k(e_0 + \cdots + e_l x^l)$. Then $a_1|a_2|\ldots a_{n-1}|\ldots$ since the set of primes dividing at least one of the terms of this sequence is finite. Hence there is an index $M$ such that for all $n > M$, $a_n$ is divisible by all the primes in the set of prime divisors of the sequence. It is obvious that $a_n$ could be large enough that the exponent of at least one of primes (namely $p$) in the aforementioned set must be large (especially greater than $v_p(e_0)$) for any large $n > M$. Now if there is at least one nonzero term besides $e_0$ in the expression $e_0 + \cdots + e_l x^l$ then there is at least another prime between the divisors of $e_0 + \cdots + e_l a_n^l$ different than $p$. Continuing this way, we receive a number $a_{n'}$ for which, for all primes $p$ in the set of prime divisors of the sequence, we have $v_p(a_{n'}) > v_p(e_0)$. Now, there is at least another prime between the divisors of $e_0 + \cdots + e_l a_{n'}^l$ different from the primes dividing $a_{n'}$ and hence in $a_{n'+1} = P(a_{n'})$. Contradiction. Thus there is no other term in $e_0 + \cdots + e_l x^l$ except $e_0$ and hence $P(x) = e_0 x^k$. $\square$

*Second Solution.* Assume $P(0) \neq 0$. Write $f_0 = 0, f_1 = P(0), f_2 = P(P(0)) = P(f_1) \dots$. It is clear that both $a_n, f_n$ are monotonically increasing. Also that

$$a_{n+m} = \underbrace{(P(\dots(P(a_n)\dots))}_{m} \equiv f_m \pmod{a_n}$$

Hence $\gcd(a_n, a_{n+m}) \leq f_m$. Take an arbitrary integer $N$. Set $M = \lfloor N/2 \rfloor$ and assume $p_1, \dots, p_r$ are all prime divisors of $a_{M+1} \dots a_N$. Now for all $i \in M+1, \dots, N$ define the set $X_i$ as those indices $j \in M+1, \dots, N$ such that $p_j^{(v_{p_j}(a_j))}$ is the maximal value in the prime decomposition of $a_j$. Now there is an integer $s$ such that $|X_s| \geq (N-M)/r$. Since between any $r+1$ elements there are two elements with the same maximal prime, then there are $n_1, n_2 \in X_s$ such that $0 < n_1 - n_2 \leq r$. So one can find that $f_r \geq f_{n_1 - n_2} \geq \gcd(a_{n_1}, a_{n_2}) \geq \min(p_s^{v_{p_s}(a_{n_1})}, p_s^{v_{p_s}(a_{n_2})}) \geq \min(\sqrt[r]{a_{n_1}}, \sqrt[r]{a_{n_2}}) \geq \sqrt[r]{a_{M+1}}$. This implies that

$$f_r^r \geq a_{M+1}.$$

Since $r$ is constant, we find that $a_n$ is bounded from above, a contradiction. Thus $P(0) = 0$. Now we can continue the solution above (when $P(0) = 0$). $\square$

## Interesting Arithmetic Progressions

The second problem is about the prime decomposition of arithmetic progressions. The author of this problem is Prof. Alexander Golovanov, a mathematician with many interesting proposed problems. This problem was indeed modern by 2006 and in my opinion, is still modern. Many students tried to construct a geometric sequence which is a sub-sequence of one of the arithmetic sequences and impose the same criterion to the second arithmetic sequence. But, this idea needs further reasoning. Through these years, there were only 7 students, out of about 50-55 students who solved this problem completely. We provide a pure number-theoretic solution although there is also another approach that is rooted in a combination of enumerative approaches and number theory that will be presented at another instance.

**Problem 2.** *(Alexander Golovanov, the Tuymaada Olympiad 2006) For a positive integer, we define it's set of exponents as the unordered list of all the exponents of the primes in its decomposition. For example,* $18 = 3^2.2$ *has it's set of exponents* $1, 2$ *and* $300 = 2^2.3.5^2$ *has it's set of exponents* $2, 1, 2$. *Given two arithmetical progressions* $(a_n), (b_n)$ *such that for any non-negative integer* $n$, $a_n$ *and* $b_n$*have the same set of exponents. Prove that the progressions are proportional (that is, there is a* $k$ *such that,* $a_n = kb_n$*).*

*Solution.* Let us denote the set of exponents of a positive integer $r$, by $T(r)$. It is obvious that, if $n|m$ and $n < m$ , Then $T(m) \neq T(n)$. Let $a_n = a_0 + nd, b_n = b_0 + ne$, then $T(a_0) = T(b_0)$. If $d = 0$, then $T(a_n) = T(a_0)$, Now if $e \neq 0$,

$$T(b_{b_0}) = T(b_0(1+e)) \neq T(b_0) = T(a_0).$$

But $T(b_{b_0}) = T(a_{b_0}) = T(a_0)$, contradiction. This shows that $e = 0$ and hence $a_n/b_n = a_0/b_0 = k$.

Assume now $de \neq 0$ and set $n = ma_0^2 b_0^2$, let's define $D = da_0 b_0^2$, $E = eb_0 a_0^2$ . Then

$$a_n = a_0(1 + mD), b_n = b_0(1 + mE).$$

Since, $\gcd(a_0, 1 + mD) = \gcd(b_0, 1 + mE) = 1$, we find that $T(a_n) = T(b_n)$ reduces to $T(1+mD) = T(1+mE)$. Now if $E = D$, then $a_0/b_0 = d/e$ and we are done. Now assume $D < E$. Choose $m$ such that $1 + mD = (D + 1)^k$, i.e., $m = ((D + 1)^k - 1)/D$. This shows that all terms in $T(1 + mE)$, must be a $k$th power, that is, $1 + mE = a^k$ for some $a$. But

$$m = ((D + 1)^k - 1)/D = (a^k - 1)/E < ((E + 1)^k - 1)/E.$$

Hence, $D + 1 < a < E + 1$. Moreover, we must have $a^k \equiv 1 \pmod{E}$. Choose $k$ such that $\gcd(k, ord_E^a) = 1$, we find that $a \equiv 1 \pmod{E}$, but $1 < a < E + 1$, a contradiction, hence $D = E$, and we are done.

## Unexpected Problem about Polynomial Roots

The next problem is an interesting one; it needs the implementation of different approaches. No student was able to finish their ideas, in part because they did not believe that this problem is indeed very complex. The discussion session for this problem normally takes place for more than three hours. The problem is a product of a joint work of three great mathematicians. Here we present a solution with an approach more or less the same as the proposers', as I understand from my correspondences with Prof. Alexander Khrabrov[6].

**Problem 3.** *(Fedor Petrov, Dimitry Rostovsky, Alexander Khrabrov, the Tuymaada Olympiad, 2009) Determine the maximum number $h$ satisfying the following condition: for every $a \in [0, h]$ and every polynomial $P(x)$ of degree 99 such that $P(0) = P(1) = 0$, there are $x_1, x_2 \in [0, 1]$ such that $P(x_1) = P(x_2)$ and $x_2 - x_1 = a$.*

*Solution.* The answer is $1/50$. First we prove that for each polynomial $P(x)$ satisfying $P(0) = P(1) = 0$ and every $a, 0 \leq a \leq 1/50$, there are $x_1, x_2 \in [0, 1]$ such that $P(x_1) = P(x_2)$ and $x_2 - x_1 = a$.

Consider the polynomial $Q(x) = P(x + a) - P(x)$, we must prove that $Q(x)$ has a root on $[0, 1 - a]$. Assume that $Q(x)$ has no root in $[0, 1 - a]$. Then, we can assume that $Q(x) > 0$. That is, $P(x + a) > P(x)$ for each $x \in [0, 1 - a]$. Consider the points $z_k = ka, k = 1, 2, \ldots$ since $a \leq 1/50$, the points $z_1, \ldots, z_{50}$ belong to $[0, 1]$ and since $P(z_{50}) > P(z_{49}) > \cdots > P(z_1) > P(0) = 0$. We find that $P(x)$ is positive at these points. Further, consider the points $t_k = 1 - ka$ for $k = 1, 2, \ldots$ we find that $t_1, \ldots, t_{50}$ are lying in the interval $[0, 1]$. Moreover, $P(t_{50}) < P(t_{49}) < \cdots < P(t_1) < P(1) = 0$. Therefore, since $t_1 \leq z_1 \leq t_2 \leq z_2 \leq \cdots \leq t_{50} \leq z_{50}$, we find that $P(x)$ has 101 roots. This is impossible.

Lemma 1. There is a polynomial $P(x)$ of degree 99 and positive leading coefficients such that $0, 1/50, 2/50, \ldots, 49/50, 1$ are its roots, and $P'(1/50) = P'(2/50) = \cdots = P'(49/50) = -1$.

[6]This solution published with some additional remarks in [7].

*Proof.* We shall construct the polynomial of the form $P(x) = Q(x)R(x)$,

$$Q(x) = x(x - 1/50)(x - 2/50)\dots(x - 49/50)(x - 1),$$

and $R(x)$ is a polynomial of degree 48. Note that

$$P'(i/50) = Q(i/50)R'(i/50) + Q'(i/50)R(i/50) = Q'(i/50)R(i/50).$$

Therefore, $R(i/50) = (P'(i/50))/(Q'(i/50))$. Thus, we need a polynomial $R(x)$ of degree 48 such that $R(i/50) = -1/(Q'(i/50)), i = 1, 2, .., 49$. Such a polynomial could be uniquely determined by Lagrange's interpolation formula. We know that $\deg R(x) \leq 48$, we need to prove that $\deg R(x) = 48$. For this reason, note that in the sequence $Q'(1/50), \dots, Q'(49/50)$, we have 48 sign changes and hence $R(x)$ has 48 roots. At each point $i/50, i = 1, \dots, 49$, $P(x)$ changes its sign from positive to negative. Therefore, between each two consecutive points $P(x)$ has a root. However, the degree of $P(x)$ is 99, so these roots are not multiple, and there are no other roots. Hence $P(x)$ is negative on $(49/50, 1)$ and is positive on $(1, +\infty)$, that is, its leading coefficient is positive. Hence, the lemma is proved. □

Now, suppose $h > 1/50$ and we shall prove that $h$ can't satisfy the condition. Let $P(x)$ be the polynomial of Lemma 1 above. The polynomial $Q(x) = P(x + 1/50) - P(x)$ has degree 98 and positive leading coefficient. Thus, $Q(0) = Q(1/50) = Q(2/50) = \cdots = Q(49/50) = 0$, and, $Q'(1/50) = Q'(2/50) = \cdots = Q'(48/50) = 0$. In other words, $Q(x)$ has 48 double roots at $1/50, 2/50, \dots, 48/50$ and simple roots at $0, 49/50$. This implies that for all $x > 49/50$, we have $Q(x) > 0$ and for all $x \in (0, 49/50)$, we have $Q(x) \leq 0$. As $P'(i/50) = -1, i = 1, .., 49$, there is $delta > 0$ such that $P(x)$ is strictly decreasing on $(i/50 - 2\delta, i/50 + 2\delta)$. We now partition the interval $[0, 49/50]$ as the union of $A \cup B \cup (49/50 - \delta, 49/50]$. Where $A = [\delta, 1/50 - \delta) \cup \cdots \cup [48/50 + \delta, 49/50 - \delta]$ and

$$B = [0, \delta) \cup (1/50 - \delta, 1/50 + \delta) \cup \cdots \cup (48/50 - \delta, 48/50 + \delta).$$

Lemma 2. For each polynomial $P(x)$ there exists a constant number $C$ such that for all $x_1, x_2 \in [0, 1]$ the following inequality is satisfied:

$$|P(x_1) - P(x_2)| \leq C|x_1 - x_2|.$$

□

*Proof.* Let $P(x) = \sum_{i=1}^{d} a_i x^i$, take $C = \sum_{i=1}^{d} i|a_i|$ . Since $|x_1^i|, |x_2^i| \leq 1$, we have:

$$|P(x_1) - P(x_2)| = |\sum_{i=1}^{d} a_i x_1^i - \sum_{i=1}^{d} a_i x_2^i| \leq |x_1 - x_2| \sum_{i=1}^{d} |a_i||x_1^{i-1} + x_1^{i-2}x_2 + \cdots + x_1^{i-1}| \leq C|x_1 - x_2|.$$

Our proof is complete. □

The values of polynomial $P(x + 1/50) - P(x)$ for each $x \in A$ are negative, and there exists a maximum value. Thus for some $d > 0$, $P(x + 1/50) - P(x) < -d$ for all $x \in A$. Let $C$ be the constant of Lemma 2, choose $\epsilon > 0$ such that $\epsilon < \delta, a = 1/50 + \epsilon < h, C\epsilon < d$. Now, we shall prove that if $0 \leq x < x + a \leq 1$ then $P(x + a) < P(x)$. This means that $a = 1/50 + \epsilon < h$ doesn't satisfy the condition. If $x \in A$ then $P(x + 1/50) - P(x) < -d$ it follows from Lemma 2 that $P(x + 1/50 + \epsilon) - P(x + 1/50) < C\epsilon < d$. Therefore, $P(x + a) < P(x)$.

If $x \in B$ then $x+1/50, x+1/50+\epsilon$ belong to one of intervals $(i/50-2\delta, i/50+2\delta)$ where $P(x)$ is strictly decreasing, therefore, $P(x+1/50+\epsilon) < P(x+1/50)$. Moreover, $P(x+1/50) - P(x) = Q(x) \leq 0$. Therefore, $P(x+a) < P(x)$. Finally, if $x \in (49/50-\delta, 49/50]$ then $x+1/50+\epsilon \in (1-\delta, 1]$. Hence, $P(x) > 0, P(x+a) < 0$, that is $P(x+a) < P(x)$, again! We are finally done! □

## A nice combination of Algebra and Number Theory

This problem is a magnum opus, i.e, a masterpiece. Prof. Golovanov constructed this problem.To solve this problem students need to implement different approaches. Indeed, all of these problems (i.e., problems 1, 2, 3, 4) have the same learning function: the deconstruction of stereotypical thought about problems of mathematical competitions. In fact, some ideas that are sufficient for finishing many other problems can only guide the students one step ahead. Thus, more courage along with a more complex mind is needed to bring out the full solution.

**Problem 4.** *4.(Alexander Golovanov, the Tuymaada Olympiad 2011) Let $P(x) = ax^2 + bx + c$, be a polynomial with integer coefficients, such that for any integer $n$ , $P(n)$ has a prime divisor $d_n$ with $1 < d_n < P(n)$. Further, $(d_n) = d_1, d_2, \dots$ is an increasing sequence. Prove that either $P(x)$ is the product of two linear polynomials with integer coefficients or there exists an integer $m$ such that $m$ divides $P(n)$, for each $n$.*

*Solution.* Write $P(n) = an^2 + bn + c = d_n.e_n$. Where $(d_n), (e_n)$ are two sequences of positive integers and $d_n$ is increasing. Therefore, we can say that $d_n \geq n$, therefore $e_n \leq 2an$, for all large enough $n$. Assume that $P(n)$ has no fixed positive divisor. Then, we shall prove following lemma.

Lemma 1. There is an integer $s$ such that $d_n \leq sn$, for all sufficiently large $n$.

Proof. We set $s = 2a$, assume that $d_n > 2an$, for all large enough $n$. Then,

$$e_{n+1} - e_n = \frac{P(n+1)}{d_{n+1}} - \frac{P(n)}{d_n} \leq \frac{P(n+1)}{1+d_n} - \frac{P(n)}{d_n} = \frac{P(n+1)-P(n)}{1+d_n} - \frac{P(n)}{d_n(1+d_n)}$$

$$= \frac{2an+a+b}{1+d_n} - \frac{P(n)}{d_n(1+d_n)}.$$

Now, if $d_n > 3an$, for some $n$ , then $0 < (2an+a+b)/(1+d_n), (P(n))/(d_n(1+d_n)) < 1$. Thus, $(2an+a+b)/(1+d_n) - (P(n))/(d_n(1+d_n)) < 1$. Since, $e_{n+1} - e_n$ is an integer, then we find that for all large enough $n$ , we shall have $e_{n+1} \leq e_n$. Now, if $2an < d_n < 3an$, then, $(P(n))/(d_n(1+d_n)) < 1 + 1/9a, 2/3 < (2an+a+b)/(1+d_n) < 1$. Hence, $e_{n+1} \leq e_n$ for all large $n$. This implies that:

$$\begin{aligned} d_{n+1} &= \frac{P(n+1)}{e_{n+1}} \geq \frac{P(n+1)}{e_n} = \frac{P(n)+2an+b}{e_n} \\ &= \frac{d_n e_n + 2an + b}{e_n} = d_n + \frac{2an+b}{e_n} \\ &= d_n + \frac{d_n(2an+b)}{P(n)} = d_n \frac{1+(2an+b)}{e_n}. \end{aligned}$$

Therefore,

$$d_{n+1} > d_n + (2an(2an+b))/(P(n)) > d_n + 3a > 2a(n+1).$$

Hence, there is a positive integer $n_0$, such that for all $n \geq n_0$, we have $e_{n+k} \leq e_{n+k-1} \leq \cdots \leq e_{n_0}$. Thus, after some certain point, $e_n$ must be constant. Hence, $P(n)$ should have a fixed divisor greater than 1, absurd. Thereby we prove the lemma.

Let us define $D_n = d_{n+1} - d_n, E_n = e_{n+1} - e_n$, we shall prove that for infinitely many $n$, $D_n, D_{n+1} \leq 2s$. Otherwise, for all large enough $n$,

$$D_n + D_{n+1} = d_{n+1} - d_{n-1} > 2s.$$

Whence, we find that:

$$d_{n+k} > 2sk + d_{n-1} > 2sk.$$

Putting $k = n$, we find that $d_2n > 2ns$, that is impossible. Now, we shall prove the following lemma.

Lemma 2. For a given value of $D_n, E_n$ can only take finitely many values.

Proof. Assuming $D_n = D$, then,

$$E_n = \frac{P(n+1)}{D+d_n} - \frac{P(n)}{d_n} = \frac{P(n)+2an+b}{D+d_n} - \frac{P(n)}{d_n} = \frac{2an+b}{D+d_n} - \frac{D(an^2+bn+c)}{d_n(D+d_n)}.$$

Since, $0 < (2an+b)/(D+d_n) < 3a$ and $0 < (D(an^2+bn+c))/(d_n(D+d_n)) < 2aD$, we find that $E_n$is bounded. This completes our proof.

Therefore there are infinitely many values of $n$, for which we have $D_n, D_{n-1} \leq s$. Hence, for infinitely many values of $n$, we have $d_n - d_{n-1} = D$. Then, by the Lemma 2, there are infinitely many values of $n$ such that $E_n$ has the same value. Now, for the quadruple $(D_{n-1}, E_{n-1}, D_n, E_n)$, there are infinitely many times that the quadruple $(u, v, w, z)$ occurs. Now, we will have two cases.

**Case-i.** $u/v \neq w/z$. Then, $P(n+1) - P(n) = d_{n+1}e_{n+1} - d_n e_n = (d_n + w)(e_n + z) - d_n e_n$. And,

$$P(n) - P(n-1) = d_n e_n - d_{n-1}e_{n-1} = d_n e_n - (d_n - u)(e_n - v).$$

Therefore,

$$\begin{cases} zd_n + we_n = 2an + b - wz + a \\ vd_n + ue_n = 2an + b - uw - a. \end{cases} \tag{0.0.1}$$

Solving this system with respect to $d_n, e_n$ we find that $d_n, e_n$ are linear polynomials with rational coefficients in $n$. Hence the polynomial $P(n) = d_n e_n$ for infinitely many $n$. Therefore, for all real numbers $x, P(x)$ is factored as a product of two linear polynomials with rational coefficients. Therefore, by Gauss's lemma, we are done.

**Case-ii.** $u/v = w/z$. In this case, $(P(n+1) - P(n))/(P(n) - P(n-1)) = (zd_n + we_n + wz)/(vd_n + ue_n - uv)$. Since $lim_{n\to\infty}(n \to \infty)(P(n+1) - P(n))/(P(n) - P(n-1)) = 1$. We find that $lim_{n\to\infty}(n \to \infty)(zd_n + we_n + wz)/(vd_n + ue_n - uv) = 1$. Hence $z = v, w = u$. That is,

$$P(t) = (d_n + u(t-n))(e_n + v(t-n)),$$

for $t = n-1, n, n+1$. Therefore, the quadratic equation has 3 distinct real roots. Hence, $P(t) = (d_n + u(t-n))(e_n + v(t-n))$, for each $t$. We are done. $\square$

## At the border of Combinatorial Geometry and Number Theory

The next problem is the last problem of day 01 of the 2020 Tuymaada Olympiad. This is again a problem constructed at the blurred border between Combinatorial Geometry and Number Theory. However, the official solution presented by the jury was purely combinatorial, two of my students solved it by number-theoretic approaches. Both solutions were awarded 6 points in the competition. Here we present the completed version of both solutions.

**Problem 5.** *5. (the Tuymaada Olympiad 2020) For each positive integer $k$ let $g(k)$ be the maximum possible number of points in the plane such that pairwise distances between these points have only $k$ different values. Prove that there exists a $k$ such that $g(k) > 2k+2020$.*

*Solution.* (Behdad Tabatabaei) We shall prove that the problem can be solved if we consider an $n\times n$ grid of lattice points, for a sufficiently large $n$. Indeed, this grid contains the points $(a,b)$ such that $1\le a,b\le n$ and the distance between the two distinct points $(a,b)$ and $(c,d)$ is $\sqrt{(a-c)^2+(b-d)^2}$ where $0\le |a-c|,|b-d|\le n-1$. Now, the total number of distances plus the number of duplicates is equal to the number of cases. Either $0\le |a-c| = |b-d| \le n-1$ or $0\le |a-c|\ne |b-d|\le n-1$. We have $n-1$ cases for $|a-c|=|b-d|$ and we have $(n(n-1))/2$ cases for $|a-c|\ne|b-d|$. Whence, the total number of distances are

$$S = n-1+(n(n-1))/2 - |\{\sqrt{(A^2+B^2)} : A^2+B^2 = C^2+D^2, A,B\ne C,D\}|.$$

We then want to prove that $g(S)\ge n^2$. It suffices to prove that

$$n^2 \ge 2(n-1+n(n-1)/2) - 2\cdot \#\text{ duplicates } + 2020.$$

That is, we should prove there are integers $n$ such that

$$\#\text{ duplicates } \ge n/2+1009.$$

On the other hand, if $p\equiv 1 \pmod 4$ is a prime number then there are distinct coprime positive integers $a,b$, with different parities such that $p=a^2+b^2$. Hence,

$$p^2 = p^2+0^2 = (a^2-b^2)^2+(2ab)^2.$$

Thus, for each positive integer $d$, $(pd)^2=(pd)^2+0^2=(da^2-db^2)^2+(2abd)^2$. Thus, for $n\ge pd$ there are at least $\lfloor n/p\rfloor$ pairs $(A,B),(C,D)$ such that $A^2+B^2=C^2+D^2, A,B\ne C,D$. Furthermore, if $p\equiv q\equiv 1 \pmod 4$ are two distinct prime numbers none of the $\lfloor n/p\rfloor$ and $\lfloor n/q\rfloor$ pairs coincide. Otherwise, consider

$$(pd)^2=(pd)^2+0^2=(da^2-db^2)^2+(2abd)^2,$$

$$(qD)^2=(qD)^2+0^2=(Dr^2-Ds^2)^2+(2rsD)^2.$$

Since $2Drs, D(r^2-s^2), 2dab, d(a^2-b^2)$ are non-zero, if the above-mentioned expressions are the same then $\{pd,0\}=\{qD,0\}$ and

$$\{2Drs, D(r^2-s^2)\} = \{2dab, d(a^2-b^2)\}.$$

It follows that $\gcd(2Drs, D(r^2-s^2)) = \gcd(2dab, d(a^2-b^2))$. On the other hand, since $\gcd(a,b)=1$ and they have different parities,

$$\gcd(2dab, d(a^2-b^2)) = d\gcd(2ab, a^2-b^2) = d.$$

This together with $\{pd, 0\} = \{qD, 0\}$ implies that $p = q, D = d$. We can therefore, find that these distances are pair-wise distinct. Now, let $p_1, \dots, p_t$ be all the primes less than $n$ that are congruent to 1 (mod 4). It follows that

$$\#\text{ duplicates } \geq \sum_{i=1}^{t} \lfloor n/p_i \rfloor \geq \sum_{i=1}^{t} n/p_i - 1 \geq n \sum_{i=1}^{t} 1/p_i - t \geq n(\sum_{i=1}^{t} 1/p_i - 1).$$

Since for all $\gcd(a, b) = 1$, $\sum_{\substack{p \equiv a \pmod{b} \\ p \leq n}} 1/p$ tends to[7] $\frac{1}{\phi(b)} \log \log n$, then there is a positive integer $t$ such that for all $n \geq p_t, \sum_{i=1}^{t} 1/p_i > 2$. Choosing this $n$, it could be found that

$$\#\text{ duplicates } > n$$

Thus, for all large enough $n$ ,

$$\#\text{ duplicates } > n/2 + 1009.$$

$\square$

*Second Solution.* ( Arvin Sahami) We can prove a more general statement. That is, for each constant $C > 0$ there is a positive integer $k$ such that $g(k) > 2k+C$. It suffices to prove there are $n$ points in the plane that don't produce more that $n/2 - C$ different distances. Firstly, we shall provide an example of such points: Consider $2n + 1$ (equidistant) points on a line, namely $l$. Putting $2n-1$ collinear points on it and continue this procedure until you reach a single point on the top. Now reflect all points with respect to $l$, now we have $2n + 1 + 2(1 + 3 + \cdots + 2n - 1) = 2n^2 + 2n + 1$ points designed by the following mesh:

We shall show that for all sufficiently large $n$, these points produce less than $n^2 + n - C$ different distances, for some constant $C$. For any two distinct points $A, B$ from this mesh we can shift both in such a way that at least one of them located on one of the 4 corners. Since the above figure is symmetric, without loss of generality, assume that $A$ is located on the leftmost corner. Therefore, instead of calculating all the distances, we can just consider the distances from the leftmost corner. Let us denote the leftmost corner by $M$. Since the figure is symmetric with respect to $l$, it only suffices to consider the distances between $M$ and the points located on or above $l$. The number of these points is $1 + 3 + \cdots + 2n + 1 = (n + 1)^2$. So, we should prove that the total number of duplicate distances are at least $n + C$. First, we provide following lemmas

Lemma 1. For each positive integer $k$ such that $2k + 1$ is not a prime, there are positive integers $a, b < k$ such that $a^2 + k^2 = b^2 + (k + 1)^2$.

Lemma 2. For each positive integer $k$ such that $k + 1$ is not a prime, there are positive integers $a, b < k$ such that $a^2 + k^2 = b^2 + (k + 2)^2$.

Considering any two adjacent columns, namely the $k$th and $(k + 1)$th columns, then by Lemma 1, we can deduce that there are two points with equal distance from $M$ and hence have been counted twice. Thus, we have counted the primes less than $2n$, twice. Analogously, in any two columns with only one column between them, namely columns $k$ and $k + 2$ there are points with the same distances from $M$. This also gives us another number of duplicates, which is equal to the numbers of primes less than $n$. Since the total number of primes less than $k$ are about $\frac{k}{\log k}$ we can argue as the following: Hence, we at least have $\frac{2n-n}{\log n} - \frac{2n}{\log 2n}$ distances that are counted twice. The last quantity is greater than $n + C$, for all large enough $n$. $\square$

[7]See for example Andreescu and Dospinescu (2012).

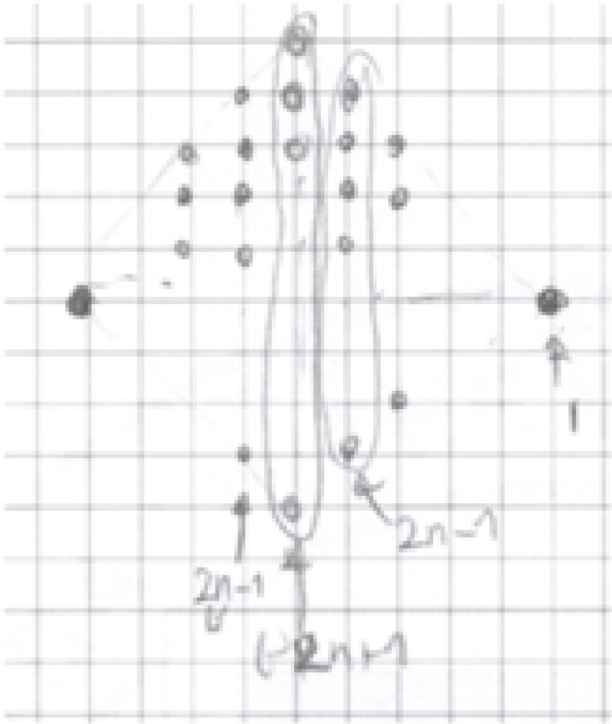



Figure 1:

## An Innovative Problem about Polynomials

The last problem is indeed the problem 4 of the day 02 of 2020 Tuymaada Olympiad. This problem needs deep insight into equations with unknown polynomials along with the uniqueness hypothesis. We present here 5 different solutions. The first is proposed by the jury and the next 4 were presented by our students during the exams. It was a great event for mthe author that 4 out of 6 of our students solved this problem with completely different approaches. Clearly and unequivocally this can be taken as proof of the fruitfulness of the discussion forum program.

**Problem 6.** *6. (K. Dilcher, M. Ulas, the Tuymaada Olympiad 2020) The degrees of polynomials $P$ and $Q$ with real coefficients do not exceed $n$. These polynomials satisfy the identity*

$$P(x)x^{n+1} + Q(x)(x+1)^{n+1} = 1.$$

*Determine all possible values of $Q(-1/2)$.*

*Solution.* We can prove that there are unique polynomials $P(x), Q(x)$ of degree at most $n$. That is, assume on the contrary that there are two pairs of polynomials namely, $(P_1(x), Q_1(x))$ and $(P_2(x), Q_2(x))$ such that:

$$(x)x^{n+1} + Q_1(x)(x+1)^{n+1} = 1 = P_2(x)x^{n+1} + Q_2(x)(x+1)^{n+1}.$$

Therefore,

$$(P_1(x) - P_2(x))x^{n+1} + (Q_1(x) - Q_2(x))(x+1)^{n+1} = 0.$$

Since the polynomials $x^{n+1}, (x+1)^{n+1}$ are co-prime, it follows that $(x+1)^{n+1}$ divides $P_1(x) - P_2(x)$ and $x^{n+1}$ divides $Q_1(x) - Q_2(x)$. This along with the degree condition implies that $P_1(x) = P_2(x)$ and $Q_1(x) = Q_2(x)$.

Setting $x = -1 - y$ yields

$$Q(-1-y)(-1)^{n+1}y^{n+1} + P(-1-y)(-1)^{n+1}(y+1)^n = 1$$

The uniqueness argument implies that

$$P(x) = (-1)^{n+1}Q(-1-x), Q(x) = (-1)^{n+1}P(-1-x).$$

Plugging $x = -1/2$ in we obtain

$$P(-1/2) = (-1)^{n+1}Q(-1/2)$$

and

$$(-1/2)(-1/2)^{n+1} + Q(-1/2)(1/2)^{n+1} = 1.$$

It follows that $Q(-1/2) = 2^n$. □

*Second solution.* (Parsa Naderian) This solution is the same, in essence, as the above solution. Let us denote by $R(x)$ the polynomial $P(x)x^{n+1} + Q(x)(x+1)^{n+1}$. Then the coefficients of $x, x^2, \dots, x^n$ in $R(x)$ are zero. Moreover, the coefficients of $1, x, x^2, \dots, x^n$ in $P(x)x^{n+1}$ are zero, thus we find that the constant term of $Q(x)$ is one and the coefficients of $x, \dots, x^n$ in $Q(x)(x+1)^{n+1}$ are zero. Now, we can prove that $Q(x)$ can uniquely be determined. Let $Q(x) = a_0 + \dots + a_d x^d, d \le n$. We can prove by inductive reasoning that the $a_0, \dots, a_d$ could uniquely be found. The base is true for $a_0$. Assume that $a_0, \dots, a_t$ are uniquely determined. Write $(x+1)^{n+1} = 1 + b_1 x + \dots + x^{n+1}$. Then the coefficient of $x^{t+1}$ is

$$a_{t+1} + \underbrace{b_1 a_t + \dots + b_{t+1}}_{Unique},$$

Since $t \le d - 1 \le n - 1$, it follows that $a_{t+1}$ is uniquely determined. This completes our proof. According to the above fact, the polynomial $P(x)$ would also be unique. Hence, we can continue our proof as the first proof. □

*Third Solution.* (Ilya Mahrooghi) Let $Q(x) = R(x+1/2), P(x) = S(x+1/2)$, for some polynomials $R(x), S(x)$. Then $R(x+1/2)x^{n+1} + S(x+1/2)(x+1)^{n+1} = 1$. Whence,

$$S(x)(x-1/2)^{n+1} + R(x)(x+1/2)^{n+1} = 1$$

where the degrees of $R(x), S(x)$ don't exceed $n$, and the question is converted to finding $R(0)$. We now divide the problem into two cases.

**Case I.** $n$ is odd. Then, $S(-x)(-x-1/2)^{n+1} + R(-x)(-x+1/2)^{n+1} = 1$. Therefore, $S(-x)(x+1/2)^{n+1} + R(-x)(x-1/2)^{n+1} = 1$. Combining this with our preceding fact, we find that

$$(R(x) - S(-x))(x+1/2)^{n+1} = (R(-x) - S(x))(x-1/2)^{n+1}.$$

Since $\deg R(x) - S(-x) \le n$ and $\deg R(-x) - S(x) \le n$, then from the fact that $(x-1/2)^{n+1}$ divides $R(x) - S(-x)$, we find that $R(x) = S(-x)$. That is,

$$R(-x)(x-1/2)^{n+1} + R(x)(x+1/2)^{n+1} = 1.$$

Putting $x = 0$ we find $R(0) = 2^n$.

**Case II.** $n$ is even. In this case, from $S(-x)(-x - 1/2)^{n+1} + R(-x)(-x + 1/2)^{n+1} = 1$, we find that

$$S(-x)(x + 1/2)^{n+1} + R(-x)(x - 1/2)^{n+1} = -1.$$

Hence, combining the fact that $S(x)(x - 1/2)^{n+1} + R(x)(x + 1/2)^{n+1} = 1$, it follows that

$$(x + 1/2)^{n+1}(R(x) + S(-x)) = -(x - 1/2)^{n+1}(R(-x) + S(x)).$$

Applying the same argument as above, we can find that $R(-x) = -S(x)$. Therefore,

$$R(x)(x + 1/2)^{n+1} - R(-x)(x - 1/2)^{n+1} = 1.$$

Again, we can find that $R(0) = 2^n$. □

*Fourth solution.* (Arvin Sahami). Plugging $x - 1$ instead of $x$, we obtain

$$P(x - 1)(x - 1)^{n+1} + Q(x - 1)x^{n+1} = 1.$$

It follows that $(x - 1)^{n+1}$ divides $T(x) = x^{n+1}Q(x - 1) - 1$ and $\deg T(x) \leq 2n + 1$. Now, we shall prove following lemma:

Lemma. If $\deg P(x) \leq d$ and for all $k \geq d$, if $(x-1)^{\alpha}|P(x)$ then $(x-1)^{\alpha}|x^k P(1/x)$. Proof. Write $P(x) = (x - 1)^{\alpha}R(x)$. Then,

$$x^k P(1/x) = x^k(1/x - 1)^{\alpha}R(1/x) = x^{k-\alpha}(1 - x)^{\alpha}(x^{d-\alpha}R(1/x)).$$

This completes our proof.

Now, setting $Q(x - 1) = T(x)$ it follows that $(x - 1)^{n+1}$ divides $x^{n+1}T(x) - 1$, putting $k = 2n + 1$ in the Lemma, it follows that $(x - 1)^{n+1}$ divides $x^n T(1/x) - x^{2n+1}$. Setting $S(x) = x^n T(1/x)$ then $\deg S(x) \leq n$. Moreover, $Q(-1/2) = T(1/2) = 2^{-n}S(2)$. Whence, $(x - 1)^{n+1}$ divides $S(x) - x^{2n+1}$. That is, $x^{n+1}$ divides $S(x + 1) - (x + 1)^{2n+1}$. Letting $A(x) = S(x + 1)$, it follows that $x^{n+1}$ divides $A(x) - (x + 1)^{2n+1}$. Since $\deg A(x) \leq n$, it follows that $A(x)$ is the remainder of the division of $(x + 1)^{2n+1}$ by $x^{n+1}$. Thus:

$$A(x) = \sum_{i=0}^{n} \binom{(2n + 1)}{i} x^i.$$

Thus, $A(1) = S(2) = \sum_{i=0}^{n} \binom{(2n+1)}{i} = 2^2 n$. Whence,

$$Q(-1/2) = T(1/2) = 2^{-n}S(2) = 2^n.$$

□

*Fifth solution.* (Behdad Tabatabei). Since $P(x)x^{n+1} = 1 - Q(x)(x+1)^{n+1} = 1$. It follows that $x^{n+1}$ divides $B(x) = Q(x)(x + 1)^{n+1} - 1$. Now, it is easy to find that $x^n$ divides $B'(x)$. That is, $x^n$ divides $Q'(x)(x + 1)^{n+1} + (n + 1)(x + 1)^n Q(x)$. Hence, $x^n$ divides $(x + 1)Q'(x) + (n + 1)Q(x)$. Assume now $Q(x) = a_d x^d + \cdots + a_0$. It is easy to find that $a_0 = 1$. We shall prove by induction that $a_i = (-1)^i \binom{(n+1)}{i}$. The base case is true. Assume that it also holds true for all $i = 0, \ldots, k$ considering the coefficient of $x^k$ in $B(x)$, we see it is zero, then

$$a_{k+1} = -((n + k + 1)/(k + 1))a_k = (-1)^{k+1} \binom{(n + k + 1)}{(k + 1)}.$$

Whence, $Q(x) = \sum_{i=0}^{n}(-1)^i\binom{(n+i)}{i}x^i = B_n(x)$. Assume that $B_m(x) = \sum_{i=0}^{m}(-1)^i\binom{(m+i)}{i}x^i$. We prove by induction that $B_m(-1/2) = 2^m$. The base case is true for $m = 1$, that is, $B_1(x) = -2x + 1$. Assume that it holds true for all m=1,...,k. We shall then prove it for $m = k + 1$. We then prove that for each $k$,

$$(k+1)B_k(x) + (x+1)B_k'(x) = (-1)^k(2k+1)\binom{2k}{k}x^k.$$

That is, for each $0 \leq s \leq k-1$ the coefficient of $x^s$ on the left hand side of the above identity is

$$(-1)^s(k+1)\binom{(k+s)}{s} + (-1)^{s+1}(s+1)\binom{(k+s+1)}{(s+1)} + s(-1)^s\binom{(k+s)}{s}$$

$$= (-1)^s\binom{(k+s)}{s}(k+1+s-(k+s+1)) = 0.$$

Moreover, the coefficient of $x^k$ is

$$(-1)^k(k+1)\binom{2k}{k} + (-1)^k k\binom{2k}{k} = \binom{2k}{k}(2k+1)\binom{2k}{k}.$$

On the other hand,

$$B_k'(x) = \sum_{i=0}^{k}(-1)^i i\binom{(k+i)}{i}x^{i-1} = \sum_{i=1}^{k}(-1)^i(k+1)\binom{(k+i)}{(i-1)}x^{i-1}$$

$$= -(k+1)(B_{k+1}(x) + (-1)^{k+1}\binom{(2k+1)}{k}x^k + (-1)^k\binom{(2k+2)}{(k+1)}x^{k+1}).$$

Therefore, $(k+1)B_k(x) + (x+1)B_k'(x) =$

$$(k+1)B_k(x) - (k+1)(x+1)B_{k+1}(x) - (-1)^k(k+1)(x+1)\left(\binom{(2k+2)}{(k+1)}x^{k+1} - \binom{(2k+1)}{k}x^k\right)$$

$$= (-1)^k(2k+1)\binom{2k}{k}x^k.$$

Putting $x = -1/2$ and using the fact that $B_k(-1/2) = 2^k$, it follows that $B_{k+1}(-1/2) = 2^{k+1}$. □

## Concluding Remarks

In this article, we first presented a relatively detailed account of one of the stages of the designed program for our students that achieved gold medals in the 3rd round of our national mathematical Olympiad, by 2012. We tried to throw some lights on the relatively unnoticed aspect of mathematical competitions i.e., the role of their problems in forging new horizons for those who worked on them after the exam. In order to obtain more fruits from the student's work on these problems, we designed a commute of practice for discussion, knowledge exchange, value construction, and evaluation of the different arguments and proofs. Through discussion in the public sphere, some arguments that at first sight seem to no avail recombined fruitfully; they either led to proofs or the construction of new problems. With empathetic sound, we prescribe the development of similar commutes of practices for the gifted students.

In the last part of this article, I provided 6 nice and interesting problems from the International Tuymaada Olympiad, which has been one of the contents of those public spheres. The first four problems were selected from the early years of the Olympiad. The astute remarks of the members of the discussion sessions helped improve the solutions developed by the author. The last two problems were taken from the year 2020 and the solutions were taken from the exam papers of the Iranian team that participated in the 27th version of the Tuymaada Olympiad in mathematics. The variety of approaches in the last two solutions depicts the success of the described approach.

During the writing of this paper, all the good and bad memories I have faced had been conjured up. The profound influence of the public sphere upon my thought, teaching method, and problem construction will invariably continue. As an abiding recognition, I must express my extreme reverence for all the great authors of these problems. As I envisage on my past, in search of lost time, the public sphere for discussion and joint work signifies an uncased gain. As if that time is regained.

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# The Art of Proposing Problems in Mathematics Competitions I

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## Introduction

Problems are at the heart of mathematics.

Proposing good contest problems is crucial to mathematics competitions. What kind of contest problems is good? The criterion in mathematics is that the problems should be natural, reasonable and also elegant, that is, with unique perspective, original structure and brief description. Since mathematics competitions are a time-limited and space-restricted problem-solving activities, the educational criterion should be also taken into consideration. The problems should be of moderate difficulty, low threshold and adequate complexity. In short, a good math problem for high school students should be elementary mathematics problems, which are *natural and elegant, yet with educational function.*

Difficulty assessment is one important task in the problem-designing process. The test paper for mathematics competition is required to have three levels of problems: easy, intermediate and advanced, with a ratio of 2:2:2 in CMO (Chinese Mathematical Olympiad). Therefore, the test designer should assess the difficulty of every potential problem, especially from the perspective of students.

*The goal of problem designers is to develop innovative problems with high item discrimination that uncover the beauty of mathematics.*

We will provide some examples to share our experience in proposing problems in two parts. This paper is the first part.

# A problem originated from Tao's result

In 2009, the following result due to Terence Tao appeared in Romanian TST [3].

**Theorem 1.** *Let $A, B \subseteq \mathbb{Z}$ be two finite sets. Then there exists $X \subseteq \mathbb{Z}$ satisfying $|X| \leq \frac{|A+B|}{|B|}$ such that*

$$B \subseteq X + A - A,$$

*where $A + B = \{a + b \mid a \in A, b \in B\}$, $A - B = \{a - b \mid a \in A, b \in B\}$.*

During our reading, we found an interesting fact. Consider a special case: Let $A = \{-n, -n+1, \cdots, n-1, n\}$ and $B = \{x_1, x_2, \cdots, x_m\}$ be two sets of integers with $x_1 < x_2 < \cdots < x_m$. By Tao's theorem, there exists a set $X \subseteq \mathbb{Z}$ satisfying

$$|X| \leq 1 + \frac{1}{2n+1}\left(x_m - x_1\right),$$

such that $x_i = x + s$, where $x \in X,\ s \in [-2n, 2n]$.

However, we found that Tao's result is not strong enough for this special case, and realized that the range of $s$ can be improved to $[-n, n]$. Thus, we used Tao's frame to produce a problem which requires new methods.

**Problem 7.** *Let $m, n > 1$ be two given integers and let $a_1 < a_2 < \cdots < a_m$ be $m$ integers. Prove that there exists a subset $T$ of integers such that*

$$|T| \leq 1 + \frac{1}{2n+1}(a_m - a_1),$$

*and for each $i \in \{1, 2, \ldots, m\}$, $t \in T$ and $s \in [-n, n]$, one has $a_i = t + s$.*

*Solution.* Let $a_1 = a,\ a_m = b$ and

$$b - a = (2n+1)q + r,$$

where $q, r \in \mathbb{Z}$, $0 \le r \le 2n$.

Let

$$T = \{a + n + (2n+1)k \mid k = 0, 1, \cdots, q\}.$$

Then,

$$|T| = q + 1 \le 1 + \frac{b-a}{2n+1},$$

and the set

$$\begin{aligned} B &:= \{t + s \mid t \in T,\ s = -n, -n+1, \cdots, n\} \\ &= \{a, a+1, \cdots, a + (2n+1)q + 2n\}. \end{aligned}$$

Noticing that

$$a + (2n+1)q + 2n \ge a + (2n+1)q + r = b,$$

we have that each $a_i$ belongs to $B$. The conclusion follows. $\square$

It was actually the fourth problem of the 25th CMO in 2010 (see [7]). During the exam, nearly three quarters of the students got it right.

# The expansion property of pedal triangles

Let $P$ be an interior point of $\triangle ABC$, and $D, E, F$ be the projection of $P$ onto $BC, CA, AB$ respectively. We call the triangle $\triangle DEF$ the *pedal triangle about* $P$.

There are the following two famous theorems in elementary geometry.

**Theorem 2.** *The area of a parallelogram in any triangle is no more than half of the area of the triangle.*

**Theorem 3.** *Let $P$ be an interior point of $\triangle ABC$, then the area of the pedal triangle about $P$ is no more than $\frac{1}{4}S_{\triangle ABC}$.*

We found that Theorem 3 can be deduced from Theorem 2. This is because that we established an expansion property of pedal triangle. Its special version for acute triangles became one of the problems in Chinese Western Mathematical Olympiad in 2003 (see [5]).

**Problem 8.** *Let $P$ be an interior point of an acute triangle $\triangle ABC$, and let $\triangle DEF$ be the pedal triangle about $P$. Show that there is a parallelogram contained in $\triangle ABC$, such that its two adjacent edges are exactly two edges of $\triangle DEF$.*

*Proof.* Let $O$ be the circumcenter of $\triangle ABC$. Since $\triangle ABC$ is acute, $O$ lies in the interior of $\triangle ABC$. Without loss of generality, we may assume that $P$ lie in $\triangle AOB$. See Figure 2.

To prove the parallelogram $DFEG$ with adjacent edges $FE,\ FD$ lies in $\triangle ABC$, we only need to prove

$$\angle FEG \le \angle FEC, \tag{0.0.2}$$

$$\angle FDG \le \angle FDC. \tag{0.0.3}$$

Since

$$\angle FEG = \angle AFE + \angle BFD,\ \angle FEC = \angle AFE + \angle A.$$

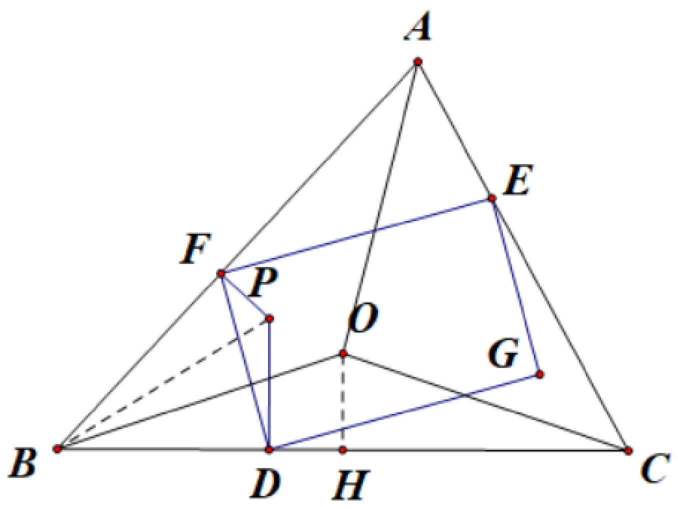


Figure 2:

So to pove (0.0.2), we only need to prove

$$\angle BFD \leq \angle A. \tag{0.0.4}$$

In fact, since that four points $B,\ F,\ P$ and $D$ are on a circle, we have

$$\angle BFD = \angle BPD. \tag{0.0.5}$$

Draw a line $OH \perp BC$ with $H$ the foot of perpendicular. By

$$\angle PBD \geq \angle BOH,$$

we have

$$\angle BPD \leq \angle BOH. \tag{0.0.6}$$

Moreover, since $O$ is the circumcenter of $\triangle ABC$, it follows that

$$\angle BOH = \angle BAC = \angle A. \tag{0.0.7}$$

From (0.0.5), (0.0.6), (0.0.7), (0.0.4), the proof of (0.0.2) is completed. The proof of (0.0.3) is similar. □

# The cardinal number of the maximal independent set

S. Fajtlowicz proved the following result in [4].

**Theorem 4.** *Suppose that $G$ is a simple graph with $n$ vertices and maximal degree $p$. Assume that $G$ does not contain the complete subgraph with $q$ vertices. If $p \geq q$, then the cardinal number $\alpha$ of the maximal independent set of $G$ satisfies*

$$\alpha \geq \frac{2n}{p+q}.$$

Furthermore, S. Fajtlowicz studied the equality conditions for the above inequality in another paper [2] and proved the following.

**Theorem 5.** *If $q \leq p$, then $\alpha = \frac{2p}{p+q}$ implies $3q - 2p \leq 5$. Moreover, for positive integers $p_1$ and $q_1$ with $3q_1 - 2p_1 = 5$, there exists an unique connected graph such that $p = p_1$, $q = q_1$ and $\alpha = \frac{2n}{p+q}$.*

After reading these two papers, we got an idea. Could we unite these results to yield a combinatorial extremal problem about the maximal independent subset of graphs? Testing some special values of $n, p, q$, we decided to set $n = 30$, $p = 5$, $q = 5$ and proposed the following problem.

**Problem 9.** *Let $G$ be a simple graph. Suppose that the degree of each vertex is at most 5, and for any 5 points of $G$, there are two points without any edge connecting them. Find the minimal cardinal number of the maximal independent subset.*

The answer to this problem is 6. The proof is not hard, but the construction is a challenge for us, because it would be tedious and non-intuitive if we perform the construction along with the original method. Finally, we found a simple but interesting combinatorial model. That is, we construct a graph $G$ that can be written as the disjoint union of 3 subgraphs, and each subgraph *"looks like a pentagonal prism"*.

When we have the intuitive construction, it would be appropriate for competition test for high school students. To make the problem more interesting, we verified the statement and proposed the fifth problem of the 30th CMO in 2015 (see [8]).

**Problem 10.** *There are 30 persons in a meeting. Each person has at most 5 acquaintances. For any 5 persons, there are at least 2 of them who do not know each other. Determine the maximal positive integer $k$ such that, for 30 persons satisfying the above conditions, there are $k$ persons who do not know each other.*

*Proof.* The desired value of $k$ is 6.

We can use 30 vertices to denote the persons. If two persons know each other, then we can draw an edge between the corresponding vertices. Thus, we get a simple graph $G$ with the vertex set of 30 points satisfying the following conditions:

(i) The degree of each vertex of $G$ is no more than 5.

(ii) For any 5 vertices of $G$, there are 2 vertices without connecting edges.

Denote the vertex set of $G$ by $V$. If $A \subseteq G$ and any 2 vertices of $A$ have no connecting edge, then we call $A$ an "independent set" of $G$. An independent set is called the maximal independent set if its number of elements is maximal.

(1) We first show that the cardinal number of the maximal independent set of $G$ that satisfies the conditions is no less than 6.

In fact, let $X$ be a maximal independent set of $G$. By the maximality of $|X|$, any vertex of $V \setminus X$ has an adjacent vertex in $X$. Otherwise, if $a \in V \setminus X$ does not have any adjacent vertices in $X$, then we can add the point $a$ into $X$ and get a bigger independent set. A contradiction. Thus, there are at least $|V \setminus X| = 30 - |X|$ edges between $V \setminus X$ and $X$. Notice that the degree of each vertex of $X$ is no more than 5. Hence, we have

$$30 - |X| \leq 5|X|, \tag{0.0.8}$$

which gives $|X| \geq 5$.

If $|X| = 5$, then by the equality condition of (0.0.8), these $30 - |X| = 25$ edges are distributed to the 5 vertices of $X$; i.e., the adjacent vertex set of each vertex of $X$ is formed by 5 vertices of $V \setminus X$. Since $|V \setminus X| = 25$, the adjacent vertex sets of any two vertices in $X$ do not intersect. Denote $X = \{a, b, c, d, e\}$. Consider the adjacent vertex set of $a$ and denote it by $Y_a = \{y_1, y_2, y_3, y_4, y_5\}$. By condition (ii), there are two vertices in $Y_a$ have no connecting edge, say $y_1, y_2$. Since the adjacent vertex sets of any two vertices do not intersect, $y_1, y_2$ are not the adjacent vertices of any point among $b, c, d, e$. Hence $\{y_1, y_2, b, c, d, e\}$ is an independent set of $G$, and the number of elements is greater than 5. A contradiction. Therefore, $|X| \geq 6$.

(2) Next we will prove that there exists a graph $G$ satisfying the conditions such that the cardinal number of each maximal independent set is no more than 6.

Divide $V$ into 3 sets $V_1, V_2, V_3$ such that $|V_i| = 10$, $i = 1, 2, 3$. Let $V_1 = \{A_1, A_2, A_3, A_4, A_5, B_1, B_2, B_3, B_4, B_5\}$. Connect the vertices of $V_1$ in the following way (as shown in Figure 3).

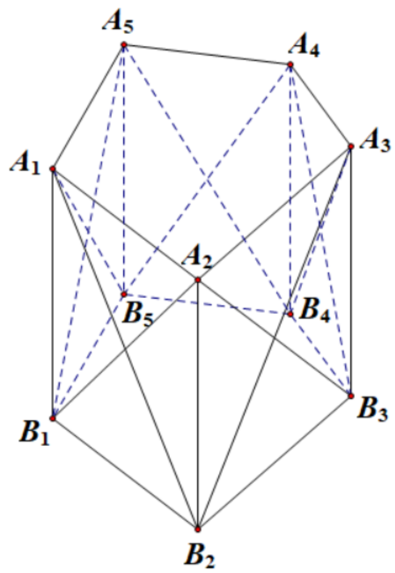


Figure 3:

(I) Connect edges $A_iA_{i+1}$, $i = 1, 2, 3, 4, 5$;

(II) Connect edges $B_iB_{i+1}$, $i = 1, 2, 3, 4, 5$;

(III) Connect edges $A_iB_i$, $A_iB_{i+1}$, $A_iB_{i-1}$, $i = 1, 2, 3, 4, 5$,

where $A_6 = A_1$, $B_6 = B_1$, $B_0 = B_5$.

The connecting ways of the vertex sets $V_2, V_3$ are completely the same as $V_1$, and for any $1 \leq i < j \leq 3$, there are no edges connecting $V_i$ and $V_j$. Then the degree of each vertex of $G$ is always 5 and there exist 2 vertices among any 5 vertices in $G$ without connecting edges.

For any maximal independent set $X$ of $G$, we will show $|V_i \cap X| \leq 2$.

In fact, since $A_i$ and $A_{i+1}$ are adjacent ($i = 1, 2, 3, 4, 5$), there are at most two sets among $A_1, \ldots, A_5$ belonging to $X$. Similarly, there are at most two sets among $B_1, \ldots, B_5$

belonging to $X$. If there are exactly two sets among $A_1, \ldots, A_5$ belonging to $X$, say $\{A_1, A_3\}$, then the union of the adjacent vertex sets of $A_1, A_3$ is precisely $\{B_1, B_2, B_3, B_4, B_5\}$. Thus, $B_1, B_2, B_3, B_4, B_5$ do not belong to $X$. Similarly, if there are exactly two sets among $B_1, \ldots, B_5$ belonging to $X$, then $A_1, A_2, A_3, A_4, A_5$ do not belong to $X$. This proves that $|V_1 \cap X| \leq 2$.

A similar arguement yields $|V_2 \cap X| \leq 2$, $|V_3 \cap X| \leq 2$. Thus,

$$|X| = |V \cap X| = |V_1 \cap X| + |V_2 \cap X| + |V_3 \cap X| \leq 6.$$

Therefore, $G$ has the desired property.

By the results of (1) and (2), it follows that $k = 6$.

□

During the exam, one fifth of the participants got it right, which indicates this problem is a hard problem.

# Finding isosceles trapezoids

In the 47th IMO (2007) in Vietnam, the leader of the Belorussian team presented some material to us, in which there were three problems for three grades in Belarus $(A, B, C)$:

**Problem 11.** *Color each point on the circle with red or blue. Each point is colored only once.*

*(1) Does there exist an equilateral triangle with vertices of the same color?*

*(2) Prove that there must be an isosceles triangle with vertices of the same color.*

Solution: (1) The answer is negative. We only need to color one semicircle red and the other blue. Thus, there does not exist an equilateral triangle with vertices of the same color.

(2) Consider a regular pentagon inscribed in the circle. It is easy to see that there are at least 3 vertices of the same color among all 5 vertices. On the other hand, any 3 vertices of the regular pentagon form an isosceles triangle. The conclusion follows. □

**Problem 12.** *Color each point on the circle with red or blue. Each point is colored only once.*

*(1) Is there always an inscribed rectangle with vertices of the same color?*

*(2) Prove that there must be an inscribed trapezoid with vertices of the same color.*

The answer of (1) is negative. It is easy to check by coloring two semicircles with different colors. In this case, there is no inscribed rectangles with vertices of the same color. The offprint from the manager of the Belorussian team did not provide the answer, but it claimed that (2) is a special case of the following problem.

**Problem 13.** *Color each point on the circle with one of the $N$ colors. Each point is colored only once. Prove that there exists an isosceles trapezoid with vertices of the same color.*

*Proof 1.* Let $A_1, A_2, \ldots, A_{N+1}$ be $n+1$ points on the circle. If the arc distances of any 2 adjacent points are all equal to a constant $a > 0$ which is independent of $N$, then we call these $N+1$ points a block. Now we choose $N^2+1$ blocks on the semicircle which don't meet each other (It can be true if $a$ is small enough). Notice that each block has two points of the same color, and then we let the color be $C$ with the arc distance $l$ between these two points. Thus, each block is associated with a pair $(C, l)$. Since $(C, l)$ has exactly $N^2$ different values, by the pigeonhole principle, these $N^2+1$ blocks must have two blocks with the same pair $(C_1, l_1)$. So the four points of the same color in the two blocks form an isosceles trapezoid. □

The advantage of Proof 1 is if we take such $N^2+1$ blocks from the semicircle, then we can find an isosceles trapezoid rather than a rectangle.

*Proof 2.* Let $KN+1$ be positive odd (where $K$ is an undetermined even number). Consider the vertices of a regular $(KN+1)$-polygon. By the pigeonhole principle, these must be $K+1$ vertices of the same color, whose arc distances have $\binom{K+1}{2}$ values. But the vertices of the $(KN+1)$-polygon yields $\frac{KN}{2}$ different arc distances. Thus, if $K$ satisfies

$$\binom{K+1}{2} > 2 \cdot \frac{KN}{2},$$

then we can find 3 arcs with endpoints of the same color, whose distances are identical when $K > 2N-1$. Hence there are two arcs of the same color without common endpoints, whose distances are identical. Because the diagonals of regular polygons with odd edges don't pass through the origin, such four vertices form an isosceles trapezoid with vertices of the same color. □

The well-known van der Waerden theorem states that for any given positive integer $N$ and $l$, there exists $W(l, N)$ such that when $n > W(l, N)$, the set $\{1, 2, \ldots, n\}$, after $N$-coloring, must have an $l$-term arithmetic progression with same color. (Another weaker but also very common version of van der Waerden's theorem says that the set of integers, after finite coloring, must have an arithmetic progression of the same color with arbitrary length.) The essence of Problem 13 is actually a special case of the van der Waerden theorem. In fact, let us divide the semicircle equally into $W(4, N)$ parts, and then denote the points of division by $1, 2, \ldots, W(4, N)$. Then there exists an 4-term arithmetic sequence with same color, and the corresponding four points form an isosceles trapezoid with same color.

All these problems and their background motivate us to design a new problem. They inspired us to find the isosceles trapezoid with same color in a regular odd $n$-polygon.

**Problem 14.** *Find the smallest possible of $n$ such that, after 2-coloring for the vertices of a regular odd $n$-polygon, there exists an isosceles trapezoid with vertices of same color.*

*Proof.* Obviously, when $n \leq 7$, we can color any 4 points among vertices red and the others blue, where these 4 points don't form a trapezoid. Then there does not exist a trapezoid of the same color. So we always assume $n \geq 9$.

When $n = 9$, we will show that after 2-coloring for regular 9-polygons, there exist trapezoids of the same color.

By pigeonhole principle, there are 5 points with same color. Assume that the color is red. For any vertices $A$ and $B$ of a regular 9-polygon, we connect $AB$ and consider the minor arc $AB$. If the minor arc $AB$ contains $r-1$ vertices ($r = 1, 2, 3, 4$) of the regular 9-polygon, then we say that the *span* of the the minor arc $AB$ is $r$. The 5 red points have $C_5^2 = 10$ spans. There are only 4 different spans, and thus there are $\lceil \frac{10}{4} \rceil + 1 = 3$ pairs have a same span.

If the points in the 3 pairs don't form any regular triangles, then they must form a red trapezoid. If they form a regular triangle, then the span is 3. Assume that these 3 red vertices are $1, 4, 7$. There are at least 2 other red points. Each point forms a minor arc with one of $1, 4, 7$ with span of 1. This yields a red isosceles trapezoid.

In conclusion, the smallest possible $n$ is 9. □

This problem is rather simple. To make it more difficult, we need to consider similar problems of 3-coloring for regular polygons with odd number of edges.

**Problem 15.** *Find the smallest possible odd number $n$ such that among the vertices of a regular polygon with $n$ edges, after 3-colored, there exists an isosceles trapezoid with vertices of same color.*

If we add another color, then the argument and construction become more difficult (Problem 13 didn't require any construction.)

Problem 15 is of great difficulty. Finally, we found that there is a counterexample for regular 15-polygon, and the minimum is 17.

Now let us go further: in the previous discussion we focused on regular polygons with odd number of edges, because in that case we believe it would be easier to find isosceles trapezoids with same color. But it is possible that regular polygons with even numbers of edges may have rectangles with vertices all having the same color. This will precisely be the starting point of our new problem. We studied regular 16-polygons and 18-polygons, and we found counterexamples for the problem we just proposed for both polygons. The counterexample for regular 18-polygons is accidentally misleading for the problem related to general regular polygons. Indeed, it seems reasonable to guess that $n = 19$ is the desired minimum based on the counterexample of $n = 18$. Based on our study of the problem with three colors, the fifth problem of the 23th CMO in 2008 (See [6]) is as follows.

**Problem 16.** *Find the smallest positive integer $n$ which has following properties: whenever we color each vertex of regular $n$-polygons arbitrarily with one of three colors (red, yellow, blue), there always exist four vertices with same color that are the vertices of an isosceles trapezoid.*

*Solution.* The smallest possible $n$ is 17.

We first show that the conclusion holds when $n = 17$.

By contradiction, we assume that there is a way of coloring the vertices of the regular 17-polygon with three colors such that there do not exist four vertices of the same color such that they are the vertices of an isosceles trapezoid.

Since $\left[\frac{17-1}{3}\right] + 1 = 6$, there must exist 6 vertices of same color, say, yellow. Connecting these 6 vertices with each other yields $\binom{6}{2} = 15$ segments. Since the lengths of segments have only $\lceil \frac{17}{2} \rceil = 8$ possible values, there must be one of the following two cases.

(1) There are 3 segments with the same length.

Notice that 17 is not a multiple of 3. It is impossible that any 2 segments among these 3 segments have a common vertex. So there exist two segments with different vertices. Thus, these 4 vertices of 2 segments satisfy the requirement, a contradiction.

(2) There are 7 pairs of segments with the same length.

By assumption each pair of segments must have a common yellow vertex. Otherwise we can find 4 yellow vertices satisfy the requirement. By the pigeonhole principle, there must be two pairs of segments that share the common yellow vertex. The other 4 vertices of these 4 segments must be the vertices of an isosceles trapezoid, a contradiction.

Therefore, when $n = 17$ the conclusion follows.

Next, when $n \leq 16$, we will construct a way of coloring, which does not satisfy the requirement. Denote the vertices of the regular $n$-polygon (clockwise) by $A_1, A_2, \ldots, A_n$. Let $M_1, M_2, M_3$ be the vertex sets of three colors, respectively.

If $n = 16$, let

$$\begin{aligned}
M_1 &= \{A_5, A_8, A_{13}, A_{14}, A_{16}\},\\
M_2 &= \{A_3, A_6, A_7, A_{11}, A_{15}\},\\
M_3 &= \{A_1, A_2, A_4, A_9, A_{10}, A_{12}\}.
\end{aligned}$$

For $M_1$, the distances from $A_{14}$ to any other verteices are unique, while the other 4 vertices are exactly the vertices of a rectangle. Similarly, one can verify that there do not exist 4 vertices in $M_2$, such that they are vertices of an isosceles trapezoid. For $M_3$, the 6 vertices are exactly the vertices of 3 diameters, so any 4 of them are either the vertices of a rectangle, or the vertices of a quadrilateral who is not an isosceles trapezoid.

If $n = 15$, let

$$\begin{aligned}
M_1 &= \{A_1, A_2, A_3, A_5, A_8\},\\
M_2 &= \{A_6, A_9, A_{13}, A_{14}, A_{15}\},\\
M_3 &= \{A_4, A_7, A_{10}, A_{11}, A_{12}\},
\end{aligned}$$

where each $M_i$ does not have 4 points who are the vertices of an isosceles trapezoid.

If $n = 14$, let

$$\begin{aligned}
M_1 &= \{A_1, A_3, A_8, A_{10}, A_{14}\},\\
M_2 &= \{A_4, A_5, A_7, A_{11}, A_{12}\},\\
M_3 &= \{A_2, A_6, A_9, A_{13}\}.
\end{aligned}$$

Each $M_i$ does not have 4 points satisfying that they are the vertices of an isosceles trapezoid.

If $n = 13$, let

$$\begin{aligned}
M_1 &= \{A_5, A_6, A_7, A_{10}\},\\
M_2 &= \{A_1, A_8, A_{11}, A_{12}\},\\
M_3 &= \{A_2, A_3, A_4, A_9, A_{13}\}.
\end{aligned}$$

Each $M_i$ does not have 4 points satisfying that they are the vertices of an isosceles trapezoid.

Deleting the vertex $A_{13}$, and then coloring other vertices as same as the case $n = 13$, we get the coloring method for $n = 12$. Next and similarly, deleting the vertex $A_{12}$ we get the coloring method for $n = 11$. At last, deleting the vertex $A_{11}$ we get the coloring method for $n = 10$.

When $n \leq 9$, we can let the number of vertices of same color be less than 4. Thus there are not 4 vertices of same color, such that they are vertices of an isosceles trapezoid.

The above constructions show that the case $n \leq 16$ does not satisfy the requirement.

In conclusion, the smallest possible $n$ is 17. □

During the exam, nearly a quarter of students got it right, which indicates that this problem is of intermediate difficulty with high item discrimination.

# Problems on convex sequences

In the 1990s, the USA TST (see [1]) used the following problem.

**Problem 17.** *Color each positive integer of* $1, 2, \cdots, \frac{n(n^2-2n+3)}{2}$ $(n \geq 2)$ *by one of two colors (red and blue). Prove that there exists an* $n$*-term sequence* $a_1 < a_2 < \cdots < a_n$ *of the same color satisfying* $a_2 - a_1 \leq a_3 - a_2 \leq \cdots \leq a_n - a_{n-1}$.

The solution depends on a strengthening induction.

*Proof.* Let $S_n = \frac{n\left(n^2-2n+3\right)}{2}$. If a sequence $a_1 < a_2 < \cdots < a_n$ satisfies

$$a_2 - a_1 \leq a_3 - a_2 \leq \cdots \leq a_n - a_{n-1} \leq m,$$

then we call it an *$n$-term $m$-sequence.*

By induction, we will prove a stronger proposition: after a 2-coloring for $\{1, 2, \cdots, S_n\}$, there must be contain an $n$-term $3\binom{n}{2}$-sequence of the same color.

In fact, the case $n = 2$ is trivial.

Assume that after a 2-coloring for $\{1, 2, \cdots, S_n\}$, there is a red $n$-term $3\binom{n}{2}$-sequence. Notice that

$$S_{n+1} - S_n = 3\binom{n}{2} + \binom{n}{1} + 1.$$

Consider the following $n + 1$ numbers

$$a_n + 3\binom{n}{2},\ a_n + 3\binom{n}{2} + 1,\ \cdots,\ a_n + 3\binom{n}{2} + n,$$

where

$$a_n + 3\binom{n}{2} + n < S_n + 3\binom{n}{2} + \binom{n}{1} + 1 = S_{n+1}.$$

If the terms are all blue, then we get a blue $(n+1)$-term 1-sequence, and the conclusion follows. Otherwise, there is at least one red term, say, $a_n + 3\binom{n}{2} + k$ $(0 \le k \le n)$. Let $a_{n+1} = a_n + 3\binom{n}{2} + k$. Then

$$a_{n+1} - a_n = 3\binom{n}{2} + k = 3\binom{n+1}{2} - 3\binom{n}{1} + k \le 3\binom{n+1}{2}.$$

Thus, we get a red $(n+1)$-term $3\binom{n+1}{2}$-sequence. The proof is completed by induction. □

The above question shows that after a 2-coloring for $1, 2, \cdots, \frac{n(n^2-2n+3)}{2}$, there exists an $n$-term convex sequence of the same color. It is a very interesting question, which often appeared in the frontier research in combinatorial mathematics. A natural question is whether the result $\frac{n(n^2-2n+3)}{2}$ could become smaller.

Now we consider the following problem.

**Problem 18.** *Find the minimum of positive integer $f(n)$ such that there exist a convex $n$-term sequence of the same color after 2-coloring for the sequence $1, 2, \cdots, f(n)$.*

Firstly, we concentrated on the solution of Problem 17 to improve the upper bound. We did find some ways to make the upper bound smaller. However, the improved upper bound is a polynomial of $n$ with degree 3, which means that we had not succeed in improving upon the order of $n$. With many unsuccessful attempts , we were stuck.

One day, a simple but natural idea occurred to us: a counterexample shows that the upper bound can not be better than $n^2 - n$.

In fact, we can do a 2-coloring in the following way: color one point red and another point blue, then color two points red and another two points blue, color three points red and another three points blue,......, and do this alternately. At last, we color $n-1$ points red and $n-1$ blue. For this coloring, there does not exist a convex $n$-term sequence with same color in $\{1, 2, \cdots, n^2 - n\}$.

With this counterexample, we may conjecture the minimum to be $f(n) = n^2 - n + 1$. Unfortunately, this conjecture remains open. However, this counterexample is sufficient to produce an intermediate contest problem as follows.

**Problem 19.** *We color $n^2 - n$ numbers: $1, 2, \cdots, n^2 - n$ $(n \ge 2)$ red or blue. Prove that there exists a way of coloring such that there don't exist $n$ numbers $a_1 < a_2 < \cdots < a_n$ with same color satisfying $a_k \le \frac{a_{k-1}+a_{k+1}}{2} (k = 2, 3, \cdots, n-1)$.*

Compared with the original problem, Problem 19 is totally new, which can be restated in the language of set classifying. This is the third problem of the 23th CMO in 2008.

**Problem 20.** *Given a positive integer $n$ with $n \ge 3$, prove that the set $X = \{1, 2, \cdots, n^2 - n\}$ can be divided into two disjoint nonempty subsets of $X$, such that these two subsets don't contain $n$ elements $a_1 < a_2 < \cdots < a_n$ satisfying $a_k \le \frac{a_{k-1}+a_{k+1}}{2} (k = 2, 3, \cdots, n-1)$.*

*Proof.* Define

$$S_k = \{k^2 - k + 1, k^2 - k + 2, \ldots, k^2\}, \quad T_k = \{k^2 + 1, k^2 + 2, \ldots, k^2 + k\},$$

where $k = 1, 2, \ldots, n-1$. Let $S = \bigcup_{k=1}^{n-1} S_k$, $T = \bigcup_{k=1}^{n-1} T_k$. We will show that the sets $S, T$ satisfy the requirement.

Firstly, $S \cap T = \emptyset$ and $S \cup T = X$.

Secondly, if the set $S$ has $n$ elements $a_1, a_2, \ldots, a_n (a_1 < a_2 < \cdots < a_n)$ satisfying

$$a_k \leq \frac{a_{k+1} + a_{k-1}}{2},$$

where $k = 2, 3, \ldots, n-1$, then

$$a_k - a_{k-1} \leq a_{k+1} - a_k. \tag{0.0.9}$$

Assume without loss of generality that $a_1 \in S_i$. Since $|S_{n-1}| < n$, it follows that $i < n-1$. Thus, for $n$ elements $a_1, a_2, \ldots, a_n$, there are at least $n - |S_i| = n - i$ elements in $S_{i+1} \cup \cdots \cup S_{n-1}$. By the pigeonhole principle, there must be a $S_j (i < j < n)$ containing at least two elements. Let $a_k \in S_j$ such that $k$ is smallest possible, then $a_k, a_{k+1} \in S_j$.

But $a_{k-1} \in S_1 \cup \cdots \cup S_{j-1}$, so $a_{k+1} - a_k \leq |S_j| - 1 = j - 1$, $a_k - a_{k-1} \geq |T_{j-1}| + 1 = j$. Thus, $a_{k+1} - a_k < a_k - a_{k-1}$, which contradicts against (0.0.9). It implies that the set $S$ does not have $n$ elements satisfying the requirement.

Similarly, the set $T$ does not have such $n$ elements, either.

This shows that the sets $S, T$ satisfy the requirement. □

We recalled that nearly one third of students gave the right answer, which shows this problem is of intermediate difficulty and has high item discrimination.

*Acknowledgements*

We would like to thank Ruixiang Zhang for their careful reading and suggestion to improve the original draft. Research of the second author is supported by a research grant from Shanghai Key Laboratory of PMMP 18dz2271000.

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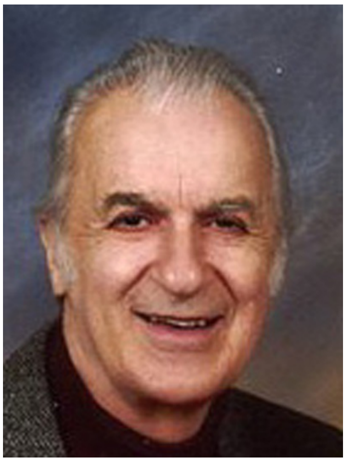
# Revisiting the Competition Corner

*George Berzsenyi*



Dr. George Berzsenyi is Professor Emeritus of Mathemactics at Rose-Hulman Institute of Technology (Terre Haute, IN, USA). He is a recipient of the WFNMC's Erdős Award and the MAA's Gung and Hu Award for Distinguished Service to Mathematics. As the first chair of the AIME (American Invitational Mathematics Examination) he co-authored *The Contest Problem Book V.* He also initiated the USAMTS and the IMTS (USA and the International Talent Searches), resulting in 4 more books of problems.

**Abstract**
The Competition Corner was a year-round talent-development program conducted in 1978-1981 via the Problem Section of the long defunct *Mathematics Student* journal of the National Council of Teachers of Mathematics (NCTM). The article describes the contents of the book devoted to that program. In addition to the problems and solutions featured in the program, the book revisits the career highlights of close to 150 former participants and their reflections on the program. The article also describes some of the efforts made to recreate a similar nationwide creative mathematical problem-solving program in the United States and Canada.

## Preliminaries

Almost exactly 45 years ago, when I attended the Fall Research Conference of the Fibonacci Association in Santa Clara, California, one evening I ended up as the dinner partner of Dave Logothetti. After I learned from him that he is also of Hungarian heritage, I told him about *KöMaL*[8], Hungary's famous high school mathematics journal and the fond memories I still had of the challenging problems featured in it. Subsequently, I sent a copy of KöMaL to Dave, wrote him more about the year-round problem-solving competitions conducted via the problems of KöMaL, and was confident that I had found in Dave another admirer of the 'Hungarian method' of talent development. Little did I suspect that with our dinner conversation I made even more of an impact on him.

Barely a year went by until Dave Logothetti was nominated to be the next editor of *The Mathematics Student*, a publication by the National Council of Teachers of Mathematics (NCTM). He accepted the position only after I enthusiastically accepted his offer to become the editor of its Problem Section,

[8] An abbreviation of Középiskolai Matematikai Lapok (in English, High School Mathematics Journal), whose full name includes physics and informatics nowadays.

## The Mathematics Student

While Hungary's KöMaL dates back to 1894, it was not until 1952 that the National Council of Teachers of Mathematics (NCTM) published the first issue of the *Mathematics Student Journal*. It was just a pamphlet of 4 pages, which, unfortunately didn't grow into more than 6 pages over the years. Its Problems Section was edited by Mannis Charosh until 1964, at which time he published a selection of its problems under the title *Mathematical Challenges*. Later the editor of the journal became Thomas Hill, who published *Mathematical Challenges Plus 6*, covering the problems proposed between 1965 and 1973 (the 'plus 6' in the title referred to 6 articles form the journal, 3 of which were authored by students).

By the time my friend, Dr. David (Dave) Logothetti took over the editorship in 1978, the journal was in its 26th year and its name was shortened to *Mathematics Student (MS)*; that's how I will refer to it in the future. By naming me editor of the Problem Section and giving me a free hand with it, Dave gave me my first opportunity to emulate KöMaL in America.

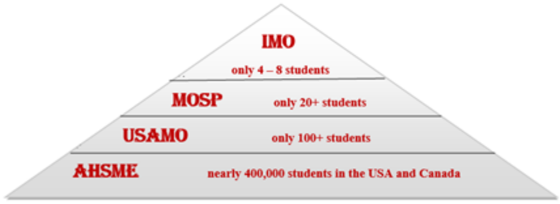
## The 'Competition Corner' in the *MS*

Fortunately, by the time I accepted Dave's invitation, I was a member of the Committee in charge of the American High School Mathematics Examination (AHSME, the forerunner of the present AMC-10 and AMC-12), as well as of the Subcommittee in charge of the USA Mathematical Olympiad (USAMO), and hence was able to get the home addresses of those students who made the National Honor Role on the AHSME and did not yet graduate from high school. To each of them I sent a personal invitation to take part in the Competition Corner.

The tabulation below shows the number of students taking the AHSME and the number of those who scored high on it, i.e., at least 100 out of 150 points. Of them the top 100 (+ a few) were invited to the USAMO, while the ones still in high school were invited to my program. Most of them accepted my invitation, and hence, along with those who were recommended by their teachers, as well as those whom I recruited otherwise, I had more than 500 students in the Competition Corner during its 3 years.

| Year | Schools | Students | Honor Role | Invitees | Participants |
|---|---|---|---|---|---|
| 1978 | 6342 | 370,414 | 531 | 170 | 194 |
| 1979 | 6425 | 377,764 | 830 | 266 | 241 |
| 1980 | 6887 | 416,024 | 256 | 82 | 103 |
| | | | **Total** | **518** | **538** |

## The competition-pyramid of the USA




I also show the pyramid of competitions in the United States, along with the number of students involved in them in the 1970s. It should be remembered that the USAMO was a brutally difficult competition, and that only the MOSP can be recognized as a talent development program. By inviting nearly 10 times that many students, including most of the MOSP participants, to the Competition Corner provided the much-needed talent development program for those who distinguished themselves on the AHSME.

## Returning to the *MS* and its Competition Corner

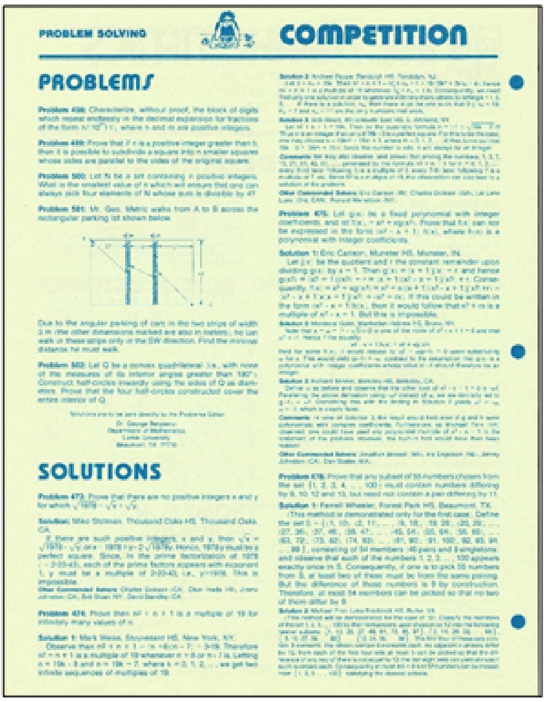
During my 3 years of service, the *MS* appeared a total of 20 times – 8 times covering 4 pages and 12 times covering 6 pages for a total of 104 pages, or 35 pages per year. That's less than 10% of 360, the number of pages devoted to mathematics by Hungary's KöMaL. Moreover, I must point out that Dave Logothetti's half of the *MS* was at most at the level of *Abacus*[9] and only in 'my half' could I aim to emulate KöMaL – hence, in reality, we were even worse off. Only by supplementing the printed materials in the *MS* by extensive correspondence and 'Furtherer Notes' with various attachments did I manage to be at the 10% level.

To launch the 'Competition Corner', I made a **Contest Announcement** in Issue 1, Volume 26, along with the **Contest Rules** and the first set of **5 Problems**. The contest rules and expectations were repeated at the beginning of Years 2 and 3, and I am happy to report that most of the students lived up to them.

The contest was to run throughout the school year with points gathered in each month (round) and summed up at the end of the year. Valuable prizes were promised to those students who accumulated the most points; that turned out to be a huge motivator. The contestants could earn 3 or more points for the solution of a problem. The deadline for submitting the solutions was set as the 20th of the month following the month of publication of the *MS*; that is, the 20th of November for the October problems, the 20th of December for the November problems, etc. The format of the submissions was specified, and it was explained to the students that they must provide derivations /proofs /verification and not just answers to the problems. I also encouraged extensions, generalizations, alternate solutions and references and was happy to see their enthusiasm for such extras, which also earned extra credit. Since the *MS* appeared 8 times in the 1978-79 school year, there were 8 rounds of 5 problems in that year, to be followed by 3 rounds of 5 and 3 rounds of 6 problems in 1979-1980 and 6 rounds of 5 problems in 1980-81.

Below I reproduced a couple of typical pages from the Competition Corner in the *MS*.

[9] Hungary's problem-solving journal for students in middle-school, launched in 1994 by Sándor Róka, a superb problemist at all levels. Abacus became the 'little sister' of KöMaL.



PROBLEM SOLVING COMPETITION

PROBLEMS

SOLUTIONS

CORNER

EDITED BY G. BERZSENYI

SUPER SOLVERS

SCOREBOARD

THE BOOKSHELF

SUBSCRIPTION RENEWAL

Being a one-man operation (with some essential help by my wife, Kay) in addition to a heavy teaching load at a state university, and not only did I have to come up with the problems to be posed (with the help of a 'Call for Problems' to mathematical friends), but I did all the grading and correspondence, and kept the scores in a 'hand-operated database'. After preparing the solutions, Kay typed them (since I didn't yet know how to type), and along with the new set of problems and often with additional materials I submitted them to Dave.

Reflecting upon them in my regular 'Problems, Puzzles and Paradoxes (PPP)' column in the Winter 1990 issue of *COMAP's Consortium*, I noted that "*On the average, 69 students submitted solutions to each of the 103 competition problems. In three huge boxes I still treasure the 7000+ solutions, along with copies of the evaluations and many-many letters by very faithful students, who greatly enjoyed and benefited from their extensive problem-solving activities.*"

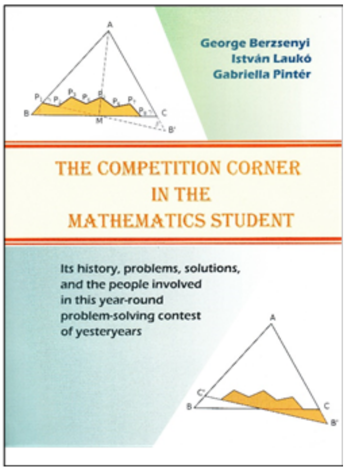
COMAP is the abbreviation of '*Consortium for Mathematics and Its Applications*'; my column in their publication began in January 1985. In 1989 I started another regular column entitled 'USA Mathematical Talent Search (USAMTS)' in Consortium, where I could announce the USAMTS, describe its progress, state its new problems and give some hints concerning the solutions for the previous round. Moreover, I could have more space at the beginning of each year, to give a complete 'Report on the Results of the Competition' in the previous year.

## 40 years later

Due to many other competition-related involvements, it was not until recently that I was finally able to put the materials of the 'Competition Corner' into a book with the able help of two younger Hungarian-born American mathematicians, Dr. Gabriella Pintér and her husband, Dr. István Laukó, professors at the University of Wisconsin in Milwaukee.

We self-published it on Amazon in July 2021 and maintain a website on it, https://www.ms-competitioncorner.com/

where much of the material is accessible to everyone[10].




More precisely, our book of more than 300 pages is much more than the statement and solution of 103 problems and 30 exercises which cover only 250 pages thereof. In addition to the detailed Historical Reflections and the Aftermath, which can be read on the website and describe my efforts to transplant the spirit of KöMaL (and Abacus) to the North American soil, the book's Epilogue is also accessible on the website. In it I collected the surprised reactions and recollections of 77 former participants when I contacted them last summer after a hiatus of 40 years with the idea of having a **Where are they now?**
section in the book concerning their careers and experiences. As a result, there are now sidebars on nearly every page with the 'then and now' photos of close to 100 former contestants and a brief introduction to them. The other sidebars summarize the results of 3 years of year-round competition in the MS, as well as some further accomplishments of the participants. The aforementioned Epilogue devotes 12 more pages to them and to their reflections on the program. Moreover, at my request, Manjul Bhargava and Natalie Wood, two of the multiple winners in the USAMTS program, added a few words of their own in support of such year-round programs. While I fully believe that our book should be on the bookshelf of everyone involved with creative problem-solving and mathematical competitions, in the present report I will limit my comments to those parts of the book that are accessible on its website even without purchasing the book. In the Preface, my co-editors describe the circumstances of our meeting and the decision to work together on the book. I continue to be amazed by the unexpected coincidence of living within a few minutes from one of the daughters of the late Lajos Pintér of Hungary, with whom I used to correspond, when I expanded the USAMTS to its global equivalent, the International Mathematical Talent Search (IMTS). And it is a wonder too, that his daughter, Gabriella, remembered me. Her contributions, as well as those of her husband, are most appreciated.

In the **Historical Reflections**, I write about the birth of KöMaL, and the year-round mathematics competitions conducted via KöMaL. Then I describe the year-round problem-solving competition in the Competition Corner, where I did my best to emulate KöMaL. I also express my gratitude to my supporters, describe the rest of the contents and define my intended audience. In particular, I wanted to say thanks to Hewlett Packard for providing state of the art calculators for the winners, to my idol, George Pólya for lending his name to the Pólya Prize, and to NCTM for letting me publish the present book independently.

In the **Dedication**, I write about indebtedness to my late friend, Dr. David (Dave) Logothetti, who gave me a free hand in transforming the Problem Section of the MS into a 'Competition Corner'. His widow and their children were pleased with the recognition.

In the **Kudos**, I single out some of the proposers of the problems for their contributions, share with my readers a few more of their challenges, and reflect on the over-all quality of the program. I also point out that other than the MOSP, the Competition Corner was the only talent-development program for the students who excelled on the AHSME. The same

[10]The rest of the material is available only to those who have purchased the book from Amazon.

is true for the USAMTS, since the USAMTS continues to invite each year the top 10,000 students of the AHSME. Presently, in its 33rd year, still under the auspices of the Art of Problem Solving (AoPS), and still supported by the National Security Agency (NSA), the USAMTS continues to be an excellent program.

Finally, the **Aftermath** and the **Final Remarks** are also accessible on the website. In the Aftermath, I describe the various attempts made by me to launch a proper journal for mathematically talented students at the high school level. I also introduce Abacus, Hungary's relatively new publication, which addresses the need for year-round problem-solving competition for middle schools. And most importantly, I also describe an on-line version of the proposed publication, and even address its costs. Admittedly, my views are somewhat utopian, but it is also true that for the United States to keep her position as a leader in technology, it will be necessary to pay proper attention to its greatest national treasure and develop the talents of her mathematically gifted youth to the fullest. The present book provides a possible method to accomplish that.

In addition to revisiting the problems and solutions of the Competition Corner (and improving a lot of the latter with some alternate solutions and insightful remarks), it was great to reconnect with many of the former contestants and learn about their struggles and accomplishments. They are recorded on the sidebars in the main body of the book, and available on the website only to those who purchased the book. In the Final Remarks I commented on some of the advantages of having access to the electronic version of the book.

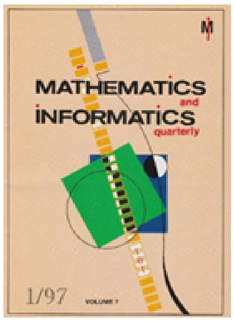
In closing, I want to remind my readers that I already wrote about the USAMTS and the IMTS in this publication back in the 1990s, when both of them were thriving[11].

To update the information therein, I should mention that upon my retirement in 1999, the USAMTS was not only kept alive but flourished under the joint management of COMAP and NSA, and that since 2004 it is administered by AoPS with continuing NSA support. I should also note that the problems and solutions of the first 20 years of the USAMTS were translated into Korean and appeared there under the joint authorship of the AoPS and myself in 2009. The resulting 2-volume book of 565 pages is shown on the right with my grandson Jackson holding it. Interestingly, Jackson's mother was born in Korea and was adopted by her American parents there.

The Korean publication prompted me to reconnect with my friend, Peter Taylor of Australia concerning the publication of the problems and solutions in English. Peter was not only kind enough to agree, but he prepared all of the figures in Part 1 of the resulting IMTS books. For Part 2 thereof, I managed to learn enough PiCTeX-ing to do my own drawings. The two books combined covered all 44 rounds of the IMTS, or equivalently, Years 3 to 13 of the USAMTS, along with the first 2 years of the USAMTS in one of the appendices. In the other appendix I have sample problems from Years 14 to 21 (when I was still in charge of the problems) of the USAMTS without solutions. The reader might recall that I initiated the IMTS upon the invitation to launch a Problem Section in the *Mathematics and Informatics Quarterly (M&IQ)* published by my friend, Willie Yong of Singapore with wide support by Bulgarian mathematicians.

[11]See Issues 9.2 (1996), pp. 40-45 and 10.1 (1997), pp. 74-78, respectively

I show here a typical copy of the *M&IQ*, which is still alive, but I discontinued the IMTS column in it when the Bulgarian mathematicians withdrew their official support from the publication. Nevertheless, I am most thankful for the opportunity of having the IMTS column is *M&IQ*, since it allowed space for the solutions as well. Moreover, many of the countries that took part in the IMTS stayed abreast via that column. I must also acknowledge the help of Béla Bajnok and Gene Berg in keeping that column alive after my retirement.




Next, I show below the 2-volume IMTS books published by the Australian Mathematics Trust in 2009 and 2011, respectively. In the Introduction to them, I give lots of information about the USAMTS also.



## Closing Comments

With the publication of the book on the Competition Corner in the Mathematics Student and the publication of the IMTS volumes, I am concluding my involvements with year-round competitions in creative problem-solving not only in the USA and Canada, but globally as well. I will still argue that smaller countries with a centralized university system should follow Hungary's and KöMaL's example, but for larger countries it might not be feasible to do so. In fact,

## Returning to the Competition Corner

It was, indeed, most pleasing to renew contact with my first-born mathematical children and learn about their careers, though it was very sad to learn that Gina Roberts and Lai-Lane Luey of Canada and Nadine Kowalsky and Bruce Brandt of the United States are no longer with us. Many of the participants became physicians – some practicing, others researching medicine. Some of them chose the law as a profession, yet others turned to physics to carve out a career for themselves. Many became engineers, software developers (some with their own company), or employees of giant companies in the Silicon Valley or elsewhere. Several remained with their primary love, mathematics, and enriched it with wonderful new results. I was pleased to see that many of the former contestants found a way to work with young and talented students, some even found it a calling to mentor others. And yet, there is still no journal for our mathematically talented high school students, as if we couldn't afford such an extravagance. And while I am thankful that at least NCTM had a Mathematics Student journal, I find it sad that during the first 26 years of its life, it made no progress whatsoever. It started as a pamphlet and remained just that, while KöMaL, as well as Abacus grew up to be professional publications much faster. And yet, I must be thankful that once there was such a pamphlet, since now even that is gone.

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# A Problem in Combinatorial Number Theory

*Yunhao Fu and Wei Luo*

Fu Yunhao participated in the IMO twice, in 2002 and 2003, achieving a perfect score on each occasion. He obtained a Doctor of Science degree in 2015 from Guangzhou University, and has recently joined the Southern China University of Science and Technology. He is on the coaching staff of the Chinese IMO team and a member of the Problem Committee for the Chinese Mathematical Olympiad. In 2016, he was a member of the Problem Committee for the IMO in Hong Kong.

Luo Wei finished his Ph.D of mathematics in MIT in 2003. He is now a math olympiad instructor in Hangzhou, China.

The following contest problem was posed in the 2020 Chinese Mathematical Olympiad.

**Problem 21.** *The positive integer $n$ is divisible by exactly 36 different primes. For $0 \leq k \leq 4$, let $c_k$ denote the number of integers in the interval $(\frac{kn}{5}, \frac{(k+1)n}{5}]$ which are relatively prime to $n$. If $c_0$, $c_1$, $c_2$, $c_3$ and $c_4$ do not all have the same value, prove that*

$$\sum_{0\leq j<k\leq 4} (c_j - c_k)^2 \geq 2^{36}.$$

The total number of integers in the interval $(0, n]$ which are relatively prime to $n$ is given by $c_0 + c_1 + c_2 + c_3 + c_4 = \phi(n)$. We can rewrite the desired sum in the form

$$\Omega(n) = 5\sum_{k=0}^{4} c_k^2 - \left(\sum_{k=0}^{4} c_k\right)^2 = 5\sum_{k=0}^{4} c_k^2 - (\phi(n))^2.$$

We have $\Omega(n) = 0$ if and only if $c_0 = c_1 = c_2 = c_3 = c_4$. Beyond this, it is not immediately clear how we may proceed.

A good way to approach this problem is to consider concrete examples. Of course, we are not going to choose $n$ with 36 different prime divisors. We downsize the problem by choosing $n = 72 = 2^3 3^2$ with only 2 different prime divisors. Then $\phi(72) = \phi(2^3)\phi(3^2) = 24$. We modify the expected lower bound from $2^{36}$ to $2^2 = 4$.

As it turn out, $\Omega(72) = 14 > 4$. This can be computed by brute force. The intervals are (0,14.4], (14.4,28.8], (28.8,43.2], (43.2,57.6] and (57.6,72]. The relevant numbers in them are {1,5,7,11,13}, {17,19,23,25}, {29,31,35,37,41,43}, {47,49,53,55} and {59,61,65,67,71} respectively. Thus we have $c_0 = c_4 = 5$, $c_1 = c_3 = 4$ and $c_2 = 6$, and $\Omega(72) = 5(5^2 + 4^2 + 6^2 + 4^2 + 5^2) - 24^2 = 14$.

Note that this value is not the ultimate objective of the problem. We need to express it in a form which we can manipulate so that we can establish the lower bound. An alternative approach is to make use of the Principle of Inclusion-Exclusion. We have

$$\begin{aligned}
c_0 &= \lfloor 14.4 \rfloor - \left\lfloor \frac{14.4}{2} \right\rfloor - \left\lfloor \frac{14.4}{3} \right\rfloor + \left\lfloor \frac{14.4}{6} \right\rfloor = 5,\\
c_1 &= (\lfloor 28.8 \rfloor - \lfloor 14.4 \rfloor) - \left(\left\lfloor \frac{28.8}{2} \right\rfloor - \left\lfloor \frac{14.4}{2} \right\rfloor\right) - \left(\left\lfloor \frac{28.8}{3} \right\rfloor - \left\lfloor \frac{14.4}{3} \right\rfloor\right) + \left(\left\lfloor \frac{28.8}{6} \right\rfloor - \left\lfloor \frac{14.4}{6} \right\rfloor\right) = 4,\\
c_2 &= (\lfloor 43.2 \rfloor - \lfloor 28.8 \rfloor) - \left(\left\lfloor \frac{43.2}{2} \right\rfloor - \left\lfloor \frac{28.8}{2} \right\rfloor\right) - \left(\left\lfloor \frac{43.2}{3} \right\rfloor - \left\lfloor \frac{28.8}{3} \right\rfloor\right) + \left(\left\lfloor \frac{43.2}{6} \right\rfloor - \left\lfloor \frac{28.8}{6} \right\rfloor\right) = 6,\\
c_3 &= (\lfloor 57.6 \rfloor - \lfloor 43.2 \rfloor) - \left(\left\lfloor \frac{57.6}{2} \right\rfloor - \left\lfloor \frac{43.2}{2} \right\rfloor\right) - \left(\left\lfloor \frac{57.6}{3} \right\rfloor - \left\lfloor \frac{43.2}{3} \right\rfloor\right) + \left(\left\lfloor \frac{57.6}{6} \right\rfloor - \left\lfloor \frac{43.2}{6} \right\rfloor\right) = 4,\\
c_4 &= (72 - \lfloor 57.6 \rfloor) - \left(\frac{72}{2} - \left\lfloor \frac{57.6}{2} \right\rfloor\right) - \left(\frac{72}{3} - \left\lfloor \frac{57.6}{3} \right\rfloor\right) + \left(\frac{72}{6} - \left\lfloor \frac{57.6}{6} \right\rfloor\right) = 5.
\end{aligned}$$

More generally, let $h$ be the number of different prime divisors of $n$. Let these prime divisors be indexed by the set $S = \{1, 2, \ldots, h\}$. For any subset $J$ of $S$, let $q_J$ denote the product of the primes in $J$. Then for $0 \le k \le 4$,

$$c_k = \sum_{J \subseteq S} (-1)^{|J|} \left( \left\lfloor \frac{(k+1)n}{5q_J} \right\rfloor - \left\lfloor \frac{kn}{5q_J} \right\rfloor \right).$$

Let $\{x\} = x - \lfloor x \rfloor$ for any positive real number $x$. Then

$$\begin{aligned}
c_k^2 &= \sum_{A,B \subseteq S} (-1)^{|A|+|B|} \left( \left\lfloor \frac{(k+1)n}{5q_A} \right\rfloor - \left\lfloor \frac{kn}{5q_A} \right\rfloor \right) \left( \left\lfloor \frac{(k+1)n}{5q_B} \right\rfloor - \left\lfloor \frac{kn}{5q_B} \right\rfloor \right)\\
&= \frac{\phi^2(n)}{25} + \frac{-2\phi(n)}{5} \left( \sum_{A \subseteq S} (-1)^{|A|} \left( \left\{ \frac{(k+1)n}{5q_A} \right\} - \left\{ \frac{kn}{5q_A} \right\} \right) \right)\\
&\quad + \sum_{A,B \subseteq S} (-1)^{|A|+|B|} \left( \left\{ \frac{(k+1)n}{5q_A} \right\} - \left\{ \frac{kn}{5q_A} \right\} \right) \left( \left\{ \frac{(k+1)n}{5q_B} \right\} - \left\{ \frac{kn}{5q_B} \right\} \right).
\end{aligned}$$

When summing from $k = 0$ to 4, the first term yields $\frac{(\phi(n))^2}{5}$ while cancellations reduce the second to 0. It follows that $\Omega(n)$ is equal to 5 times what results from the third term. Let $r(x)$ denote the remainder when an integer $x$ is divided by 5. Note that $r(x) = 0$, 1, 2, 3 or 4. Then

$$\Omega(n) = \frac{1}{5} \sum_{k=0}^{4} \left( \sum_{J \subseteq S} (-1)^{|J|} \left( r\left( \frac{(k+1)n}{q_J} \right) - r\left( \frac{kn}{q_J} \right) \right) \right)^2 = \frac{1}{5} \sum_{k=0}^{4} (\lambda_{k+1}(n) - \lambda_k(n))^2,$$

where $\lambda_0(n) = \lambda_5(n) = 0$ and $\lambda_k(n) = \sum_{J \subseteq S} (-1)^{|J|} r\left(\frac{kn}{q_J}\right)$ for $1 \le k \le 4$.

For $n = 72$, $h = 2$ and $S = \{1, 2\}$. The subsets of $S$ are $\emptyset$, $\{1\}$, $\{2\}$ and $S$ itself.

| $I$ | $q_J$ | $\frac{n}{q_J}$ | $(-1)^{\lvert J\rvert}r(\frac{n}{q_J})$ | $(-1)^{\lvert J\rvert}r(\frac{2n}{q_J})$ | $(-1)^{\lvert J\rvert}r(\frac{3n}{q_J})$ | $(-1)^{\lvert J\rvert}r(\frac{4n}{q_J})$ |
|---|---|---|---|---|---|---|
| $\emptyset$ | 1 | 72 | 2 | 4 | 1 | 3 |
| $\{1\}$ | 2 | 36 | -1 | -2 | -3 | -4 |
| $\{2\}$ | 3 | 24 | -4 | -3 | -2 | -1 |
| $S$ | 6 | 12 | 2 | 4 | 1 | 3 |
| $\lambda_k$ | | | $-1$ | 3 | $-3$ | 1 |

Here, $\lambda_1(72) = -1$, $\lambda_2(71) = 3$, $\lambda_3(72) = -3$ and $\lambda_4(72) = 1$. Hence

$$\Omega(72) = \frac{1}{5}((-1-0)^2 + (3-(-1)^2 + (-3-3)^2 + (1-(-3))^2 + (0-1)^2) = 14.$$

Things have become more systematic, but we still do not have anything which we can manipulate. We now utilize an isomorphism $\psi$ between the multiplicative group modulo 5 and the cyclic group $\{g, g^2, g^3, g^4 = 1\}$ generated by the element $g$. The isomorphism is $\psi(1) = 1$, $\psi(2) = g$, $\psi(3) = g^3$ and $\psi(4) = g^2$. We extend this definition to $\psi(0) = 0$ and $\psi(x) = \psi(r(x))$. The function $\psi$ is multiplicative. We define a linear function $\xi$ which acts like an inverse for $\psi$, in that $\xi(1) = 1$, $\xi(g) = 2$, $\xi(g^2) = 4$ and $\xi(g^3) = 3$.

Suppose that $n = \prod_{j=1}^{h} p_j^{t_j}$, where $p_j$ are distinct primes and $t_j$ are positive integers. We have $\phi(n) = \prod_{j=1}^{h} p_j^{t_j-1}(p_j - 1)$. We define $f(n, g) = \prod_{j=1}^{h} (\psi(p_j))^{t_j-1}(\psi(p_j) - 1)$. For $1 \le k \le 4$, we define $f_k(n, g) = \psi(k)f(n, g)$.

We now prove the important result that $\lambda_k(n) = \xi(f_k(n, g))$. We have

$$\begin{aligned}
\lambda_k(n) &= \sum_{J\subseteq S} (-1)^{|J|}\xi\left(\psi\left(r\left(\frac{kn}{q_J}\right)\right)\right)\\
&= \xi\left(\sum_{J\subseteq S} (-1)^{|J|}\psi\left(\frac{kn}{q_J}\right)\right)\\
&= \xi\left(\psi(k)\sum_{J\subseteq S}(-1)^{|J|}\prod_{j\in J}\psi(p_j^{t_j-1})\prod_{j\notin J}\psi(p_j^{t_j})\right)\\
&= \xi\left(\psi(k)\prod_{j=1}^{h}(\psi(p_j^{t_j}) - \psi(p_j^{t_j-1})\right)\\
&= \xi\left(\psi(k)\prod_{j=1}^{h}\psi(p_j^{t_j-1})\psi(p_j - 1)\right)\\
&= \xi(f_k(n, g)).
\end{aligned}$$

For $n = 72$, we have $f(72, g) = g^2(g-1)g^3(g^3-1)$. Hence

$$\begin{aligned}
f_1(72, g) &= -1 + 2g - g^2,\\
f_2(72, g) &= -g + 2g^2 - g^3,\\
f_3(72, g) &= 2 - g - g^3,
\end{aligned}$$

$$f_4(72,g) \quad = \quad -1-g^2+2g^3.$$

We then have $\lambda_1(72)=-1+2-2=-1$, $\lambda_2(72)=-2+8-3=3$, $\lambda_3(72)=2-2-3=-3$ and $\lambda_4(72)=-1-4+6=1$.

Suppose that $f_k(n,g)=a_0+a_1g+a_2g^2+a_3g^3$. Then $\lambda_k(n)=a_0+2a_1+4a_2+3a_3$. Note that we have $f_k(n,1)=a_0+a_1+a_2+a_3$, $f_k(n,i)=a_0+a_1i-a_2-a_3i$, $f_k(n,-i)=a_0-a_1i-a_2+a_3i$ and $f_k(n,-1)=a_0-a_1+a_2-a_3$. We now express $\lambda_k(n)$ in the form

$$\alpha f_k(n,1)+\beta f_k(n,i)+\gamma f_k(n,-i)+\delta f_k(n,-1).$$

Then

$$\begin{aligned}
\alpha+\beta+\gamma+\delta &= 1,\\
\alpha+\beta i-\gamma i-\delta &= 2,\\
\alpha-\beta-\gamma+\delta &= 4,\\
\alpha-\beta i+\gamma i-\delta &= 3.
\end{aligned}$$

Solving this system of equations, we have $\alpha=\frac{5}{2}$, $\beta=\frac{-3+i}{4}$, $\gamma=\frac{-3-i}{4}$ and $\delta=0$. Hence

$$\lambda_k(n)=\frac{5}{2}f_k(n,1)+\frac{-3+i}{4}f_k(n,i)+\frac{-3-i}{4}f_k(n,-i).$$

We always have $f_k(n,1)=0$. The imaginary parts in the above expression must cancel out, and the real part of $\frac{-3+i}{4}f_k(n,i)$ is equal to that of $\frac{-3-i}{4}f_k(n,-i)$. Hence $\lambda_k(n)$ is just the real part of $\frac{-3+i}{2}f_k(n,i)$.

Let $f(n,g)=b_0+b_1g+b_2g^2+b_3g^3$. Let $u_n=b_0-b_2$ and $v_n=b_1-b_3$. Then we have $f_1(n,i)=u_n+v_ni$, $f_2(n,i)=-v_n+u_ni$, $f_3(n,i)=v_n-u_ni$ and $f_4(n,i)=-u_n-v_ni$. Hence $\lambda_1(n)=\frac{-3u_n-v_n}{2}$, $\lambda_2(n)=\frac{3v_n-u_n}{2}$, $\lambda_3(n)=\frac{-3v_n+u_n}{2}$ and $\lambda_4(n)=\frac{3u_n+v_n}{2}$. It follows that

$$\begin{aligned}
\lambda_1(n)-\lambda_0(n) &= \frac{-3u_n-v_n}{2},\\
\lambda_2(n)-\lambda_1(n) &= u_n+2v_n,\\
\lambda_3(n)-\lambda_2(n) &= u_n-3v_n,\\
\lambda_4(n)-\lambda_3(n) &= u_n+2v_n,\\
\lambda_5(n)-\lambda_4(n) &= \frac{-3u_n-v_n}{2}.
\end{aligned}$$

Then $\Omega(n)$ is equal to $\frac{1}{5}$ of the sum of the squares of these five terms, which simplifies to

$$\Omega(n)=\frac{3u_n^2+2u_nv_n+7v_n^2}{2}.$$

For $n=72$, $f(72,g)=-1+2g-g^2$ so that $u_{72}=0$ and $v_{72}=2$. Once again, we have $\Omega(72)=\frac{7\times 2^2}{2}=14$.

We have $f(n,g)=\displaystyle\prod_{j=1}^{h}(\psi(p_j)^{t_j-1}(\psi(p_j)-1)$. The first factor $\displaystyle\prod_{j=1}^{h}(\psi(p_j)^{t_j-1}$, which comes from higher powers of the primes, contributes $\pm1$ or $\pm i$ to $\Omega(n)$ unless the prime is 5, in which case the contribution is 0. It follows that $n$ must not be divisible by 25. Thus the focus is on the contribution from the second factor $\prod_{j=1}^{h}(\psi(p_j)-1)$.

In the formula for $\Omega(n)$, $u_n^2$ and $v_n^2$ are always positive while $u_nv_n$ is positive if and only if $u_n$ and $v_n$ have the same sign. Thus changing the sign of the overall product has no effect, so that the contribution from the first factor, if non-zero, may be taken to be 1 or $i$. In the second factor, the prime 5 contributes 1. A prime of the form $5\ell+1$ contributes 0. A prime of the form $5\ell+2$ contributes $1-i$. A prime of the form $5\ell+3$ contributes $1+i$ while a prime of the form $5\ell+4$ contributes 2. It follows that $n$ must not be divisible by any prime of the form $5\ell+1$.

The combined contribution from two different primes of the form $5\ell+2$ is $(1-i)^2=-2i$. The combined contribution from two different primes of the form $5\ell+3$ is $(1+i)^2=2i$. The combined contribution from a prime of the form $5\ell+2$ and a prime of the form $5\ell+3$ is $(1-i)(1+i)=2$. The combined contribution from two different primes of the form $5\ell+4$ is $2^2=4$. Replacing two different primes of the form $5\ell+4$ by a prime of the form $5\ell+2$ and a prime of the form $5\ell+3$ clearly reduces $\Omega(n)$.

We are now in a position to tackle the contest problem. To minimize $\Omega(n)$, we should include the prime 5 to the first power, no prime of the form $5\ell+1$ and at most one prime of the form $5\ell+4$. There are two cases.

**Case 1.** There is a prime of the form $5\ell+4$.
The contribution from this prime is 2 and that from 5 is 1. The combined contribution from the remaining 34 primes is $2^{17}$ or $2^{17}i$. It follows that we have $(u_n,v_n)=(2^{18},0)$ or $(0,2^{18})$. In the former, $\Omega(n)=3\times 2^{35}>2^{36}$. In the latter, $\Omega(n)=7\times 2^{35}>2^{36}$.

**Case 2.** There are no primes of the form $5\ell+4$.
The contribution from 5 is 1. The combined contribution from the remaining 35 primes is $2^{17}(1\pm i)$. It follows that we have $(u_n,v_n)=(2^{17},\pm 2^{17})$. In the former, $\Omega(n)=6\times 2^{34}>2^{36}$. In the latter, $\Omega(n)=4\times 2^{34}=2^{36}$.

Equality may be attained if we take the prime 5, 18 different primes of the form $5\ell+2$ and 17 different primes of the form $5\ell+3$, all to the first power. Then the combined contribution is $2^{17}-2^{17}i$. Hence $(u_n,v_n)=(2^{17},-2^{17})$, and we have $\Omega(n)=2^{36}$.

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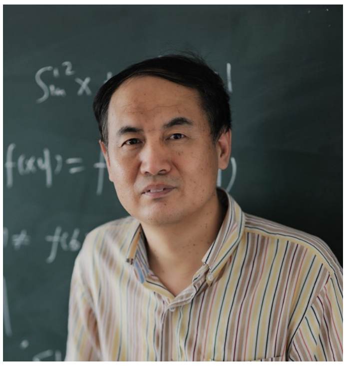
# Mixers and Sorters

*Zhichao Li and Huawei Zhu*

Zhichao Li, a mathematics and chess teacher in China, earned his PhD in Mathematics from the University of Texas at Dallas. His research interests are combinatorics and game theory.



Zhu Huawei has, since 2016, been the Principal of Shenzhen Middle School in Shenzhen, one of the top-ranked middle schools in China. His highest academic degree is a Doctor of Philosophy in Science Education, obtained from the Central China University of Science and Technology in 2005. His most recent award was as the Best Educator of the Year 2019 in China. He had served as the Leader of the Chinese national team in the International Mathematical Olympiad in 2009. He was on the Problem Committee of the Chinese Mathematical Olympiad from 2005 to 2016, and since 2011, he has been the Director of the International Mathematics Assessment Services. He has authored over one hundred books and over one hundred articles on mathematics competitions in Chinese. He also has publications in English outside China.

A switch receives two distinct input signals and produces two output signals. It is controlled manually. When it is off, the signals pass straight through the switch, as shown in Figure 1 on the left. When the switch is on, the signals cross over, as shown in Figure 1 on the right.

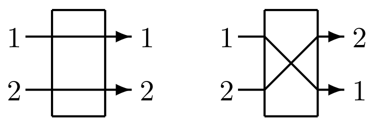



**Figure 1**

A single switch can be used to produce all 2! permutations of 2 input signals, and it may be called a 2-mixer. Figure 2 shows a network with three switches which can produce all 3! permutations of 3 input signals. We call this a 3-mixer.

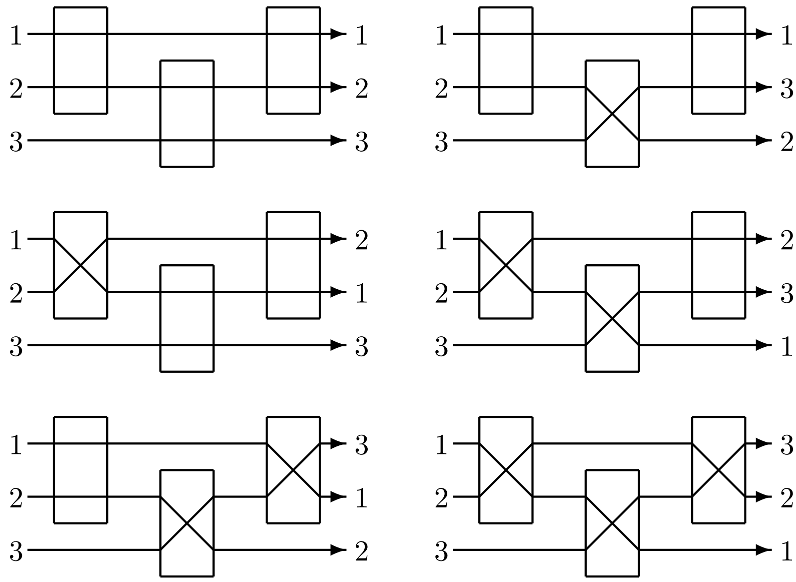



**Figure 2**

What is the minimum number $m$ of switches in a network which can produce all $n!$ permutations of $n$ input signals? Since each of the $m$ switches may be on or off, the network can handle at most $2^m$ input signals. Hence we must have $2^m \geq n!$. For $n = 3$, $2^2 < 3! < 2^3$. The minimum number of switches required is 3, so that the network in Figure 2 is optimal.

For $n = 4$, $2^4 < 4! < 2^5$. We seek a network with 5 switches which can produce all 4! permutations of 4 input signals. A 4-mixer is shown in Figure 3.

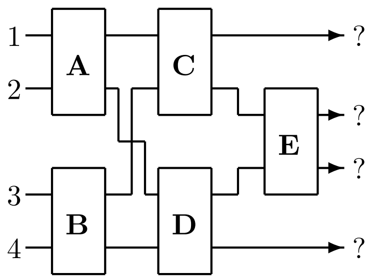



**Figure 3**

The chart below shows how each of the 4!=24 permutations of the 4 input signals may be obtained. A blank means that the particular switch is off. A cross means that the particular switch is on, so that a cross-over of signals results.

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **A** | | × | | | | | × | × | × | | | |
| **B** | | | × | | | | × | | | × | × | |
| **C** | | | | × | | | | × | | | | × |
| **D** | | | | | × | | | | | × | | × |
| **E** | | | | | | × | | | × | | × | |
| 1 | 1 | 2 | 1 | 3 | 1 | 1 | 2 | 3 | 2 | 1 | 1 | 3 |
| 2 | 3 | 3 | 4 | 1 | 3 | 2 | 4 | 2 | 1 | 4 | 2 | 1 |
| 3 | 2 | 1 | 2 | 2 | 4 | 3 | 1 | 1 | 3 | 3 | 4 | 4 |
| 4 | 4 | 4 | 3 | 4 | 2 | 4 | 3 | 4 | 4 | 2 | 3 | 2 |

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **A** | × | × | × | × | | | × | × | × | × | | × |
| **B** | × | × | × | | × | | × | × | × | | × | × |
| **C** | × | | | × | × | × | × | × | | × | × | × |
| **D** | | × | | × | × | × | × | | × | × | × | × |
| **E** | | | × | | | × | | × | × | × | × | × |
| 1 | 4 | 2 | 2 | 3 | 4 | 3 | 4 | 4 | 2 | 3 | 4 | 4 |
| 2 | 2 | 4 | 1 | 2 | 1 | 4 | 2 | 1 | 3 | 4 | 3 | 3 |
| 3 | 1 | 3 | 4 | 4 | 3 | 1 | 3 | 2 | 4 | 2 | 1 | 2 |
| 4 | 3 | 1 | 3 | 1 | 2 | 2 | 1 | 3 | 1 | 1 | 2 | 1 |

Like a switch, a sensor receives two distinct input signals and produces two output signals. It is automated. The signals pass straight through the switch if the top input signal is smaller than the bottom input signal. They cross over if the top input signal is larger than the bottom input signal.

A single sensor may be considered as a 2-sorter. When two distinct input signals are put through it, the output signals will be sorted in ascending order from top to bottom. Figure 4 shows a 3-sorter which uses three sensors in three columns.

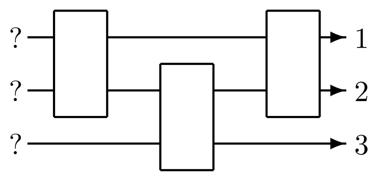



**Figure 4**

Figure 5 shows a 4-sorter which uses five sensors in three columns.

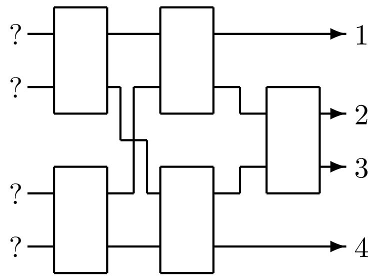



**Figure 5**

The similarlity of the two problems are striking, and yet they are very different. Readers are invited to construct $k$-mixers and $k$-sorters for $k > 4$.

The mixer problem came from a question in the 1971 Soviet Union Mathematical Olympiad [3]. In the last part of the question, the 8-mixer in Figure 6 was given, incorporating two 4-mixers. The students were asked to prove that it could indeed generate all 8! permutations of 8 input signals.

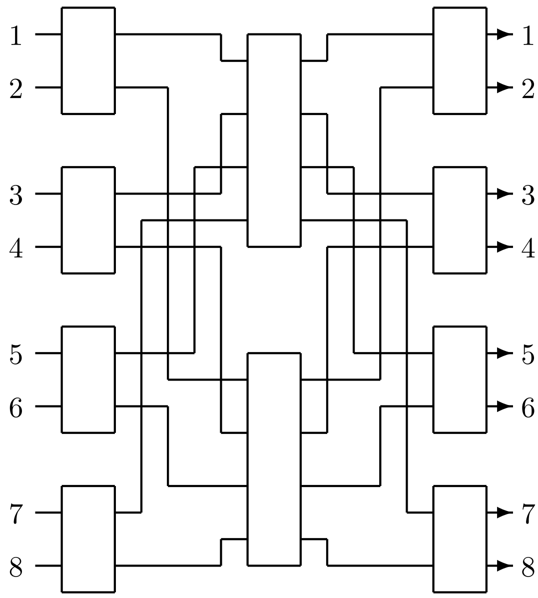



**Figure 6**

The sorter problem comes from a wonderful puzzle book [4]. Figure 7 shows an 8-sorter which incorporates two 4-sorters. This network is based on a parallel algorithm due to Batcher [1]. In the Spring 1991 International Mathematics Tournnament of the Towns [5], a question asked the students to construct a 16-mixer and a 32-mixer. Calvin Li, a contestant, came up with an alternative approach [2] to that of Batcher during the competition. Both constructions can be generalized.

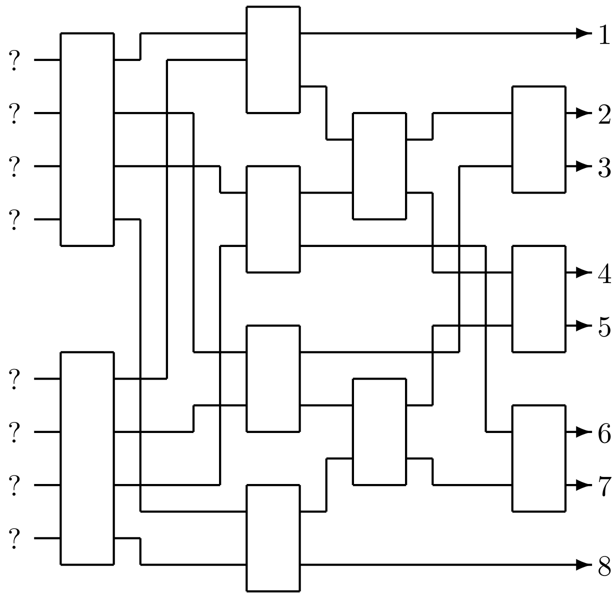



**Figure 7**

For $k = 2$, 3 and 4, a $k$-mixer and a $k$-sorter have the same structure. Yet the 8-mixer and the 8-sorter above have very different structures. Since the switches in a 4-mixer and the sensors in a 4-sorter are in three columns, the switches in an 8-mixer are in five columns while the sensors in an 8-sorter are in six columns. Readers are invited to tackle an open question [4] which asks whether an 8-sorter can consist of only 5 columns of sensors.

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# The Martian Citizenship Quiz

*David Te-Ching Liu*

Born and educated in Taipei, Te-Ching Liu is now a freshman in National Taiwan University. He has been participating in several international mathematics competitions since 2012. He was awarded a silver medal in BIMC 2013, a silver medal in KIMC 2014, a gold medal in CIMC 2015, a bronze medal in TIMC 2016, a silver medal in ITMO 2017, a gold medal in BIMC 2018, a bronze medal in APMO 2020, and a silver medal in IMO 2020. Initiated and inspired by Professor Wen-Hsien Sun, the Chairman of Chiu-Chang mathematics Education Foundation and Professor Andy Liu of Canada, Liu enjoys his exploration on the land of mathematics, a world of beauty and simplicity. He aspires to discover the most fundamental principles behind the superficially chaotic phenomena. David Te-Ching Li submitted his article when he was a grade 12 student in Jianguo High School, Taipei.

A story in [1] told of a spaceship crash landing on Mars and requiring local technical help. This was not forthcoming unless a crew member could pass a quiz and gain Martian citizenship. The quiz consisted of thirty True/False questions, and all thirty answers must be correct in order to pass.

After a failed attempt, the candidate would be informed of the number of answers that were correct, and allowed to take exactly the same test the next day. Naturally, the crew wanted to be able to head home as soon as possible. However, nobody on board the spaceship knew a single word of the Martian language.

The problem originally came from [3], asking for a solution in 25 attempts. This was improved to 22 attempts in [2], and was reported in [1]. In this paper, we further improve the result to 19 attempts.

Our plan consists of two parts. The strategy is to divide the first 24 questions into eight triples and consider the last 6 questions individually. We denote by $t(q_1, q_2, \ldots, q_k)$ the number of questions among $q_1$, $q_2$, $\ldots$, $q_k$ whose correct answers are True.

**Part One.**
This is enacted in three stages using a total of 9 attempts.

**Stage One.** 1 attempt.
**Action:** Answer True for all 30 questions.
**Result:** We will know $t(1, 2, \ldots, 30)$.

**Stage Two.** 3 attempts.
**Action:** In the second attempt, answer False only to questions 1 to 6. In the third attempt, answer False only to questions 4 to 9. In the fourth attempt, answer False only to questions 1, 2, 3, 7, 8, 9 and 25.
**Result:** We will know $t(1, 2, \ldots, 6)$, $t(4, 5, \ldots, 9)$ and $(1, 2, 3, 7, 8, 9, 25)$. Let $s$ denote their sum. Then $s = 2t(1, 2, \ldots, 9) + t(25)$.

| $s \equiv$ | $t(25) =$ |
|---|---|
| 0 (mod 2) | 0 |
| 1 (mod 2) | 1 |

In either case, we know $t(25)$ and $t(1,2,\ldots,9)$. We then have $t(1,2,3) = t(1,2,\ldots,9) - t(4,5,\ldots,9)$, $t(7,8,9) = t(1,2,\ldots,9) - t(1,2,\ldots,6)$ and $t(4,5,6) = t(1,2,\ldots,9) - t(1,2,3) - t(7,8,9)$.

**Stage Three.** 5 attempts.
**Action:** In the fifth attempt, answer False only to questions 10 to 21. In the sixth attempt, answer False only to questions 10 to 18, 22, 23 and 24. In the seventh attempt, answer False only to questions 10 to 15, 19 to 24 and 26. In the eighth attempt, answer False only to questions 10, 11, 12, 16 to 24 and 26. In the ninth attempt, answer False only to questions 13 to 24 and 27.
**Result:** We will know $t(10,11,\ldots,21)$, $t(10,11,\ldots,18,22,23,24)$, $t(10,11,\ldots,15,19,20,\ldots 24,26)$, $t(10,11,12,16,17,\ldots,24)$ and $t(13,14,\ldots,24,27)$. Let $s$ denote their sum. Then

$$s = 4t(10,11,\ldots,24) + 2t(26) + t(27).$$

| $s \equiv$ | $t(26) =$ | $t(27) =$ |
|---|---|---|
| 0 (mod 4) | 0 | 0 |
| 1 (mod 4) | 0 | 1 |
| 2 (mod 4) | 1 | 0 |
| 3 (mod 4) | 1 | 1 |

In any case, we know $t(26)$, $t(27)$ and $t(10,11,\ldots,24)$. As in Stage Two, we can deduce the values of $t(10,11,12)$, $t(13,14,15)$, $t(16,17,18)$, $t(19,20,21)$ and $t(22,23,24)$.

**Part Two.**
This is also enacted in three stages, using a total of 10 attempts. Note that each of $t(1,2,3)$, $t(4,5,6)$, $t(7,8,9)$, $t(10,11,12)$, $t(13,14,15)$, $t(16,17,18)$, $t(19,20,21)$ and $t(22,23,24)$ is 0, 1, 2 and 3. We wish to divide these eight triples into two groups of four so that neither group consists of four triples whose $t$ values consist of two 2s and two 1s. If at least two of the $t$ values is 0 or 3, put one triple in each group. If exactly one of the $t$ values is 0 or 3, then four of the remaining seven $t$ values must be the same, and we can put those triples in the same group. Suppose all the $t$ values are 2s and 1s. If there are six triples with the same $t$ value and the remaining two with a different $t$ value, put one of these two triples in each group. In all other cases, put four triples with the same $t$ value in the same group. Henceforth, we assume that the first group consists of the triples (1,2,3), (4,5,6), (7,8,9) and (10,11,12), and the second group consists of the triples (13,14,15), (16,17,18), (19,20,21) and (22,23,24).

**Stage Four.** 4 attempts.
We know $t(1,2,3)$, $t(4,5,6)$, $t(7,8,9)$ and $t(10,11,12)$, and these four values do not consist of two 2s and two 1s. We consider three cases.

**Case 1.** One of $t(1,2,3)$, $t(4,5,6)$, $t(7,8,9)$ and $t(10,11,12)$ is 0 or 3. We may assume that $t(10) = t(11) = t(12)$ are known.
**Action:** In the tenth attempt, answer False only to questions 1, 4 and 7. In the eleventh

attempt, answer False only to questions 2, 5, 6 and 7. In the twelfth question, answer False only to questions 8.
**Result:** We will know $t(1,4,7)$, $t(2,5,6,7)$ and $t(8)$, knowing already the values of $t(1,2,3)$ and $t(4,5,6)$. Let $s = t(1,4,7) + t(2,5,6,7) - t(1,2,3) - t(4,5,6) = 2t(7) - t(3)$.

| $s \equiv$ | $t(7) =$ | $t(3) =$ |
|---|---|---|
| 0 (mod 4) | 0 | 0 |
| 1 (mod 4) | 1 | 1 |
| 2 (mod 4) | 1 | 0 |
| 3 (mod 4) | 0 | 1 |

In any case, we know $t(3)$ and $t(7)$, and can deduce the values of $t(1,2)$ and $t(9)$. We consider two subcases.
**Subcase 1(a).** $t(1,2) = 0$ or 2 so we know the value of $t(1) = t(2)$.
**Action:** In the thirteenth attempt, asnwer False only to question 5.
**Result:** We will know $t(5)$. We can deduce the values of $t(4)$ and $t(6)$. We now know all of $t(1),\ t(2),\ \ldots,\ t(12)$.
**Subcase 1(b).** $t(1,2) = 1$.
We have $t(1,4) = t(1,4,7) - t(7)$. We divide this into two further subcases.
**Subcase 1(b$_1$).** $t(1,4) = 0$ or 2 so we know the value of $t(1) = t(4)$.
**Action:** In the thirteenth attempt, answer False only to question 5.
**Result:** We will know $t(5)$. We can deduce the value of $t(2)$ and $t(6)$. We now know all of $t(1),\ t(2),\ \ldots,\ t(12)$.
**Subcase 1(b$_2$).** $t(1,4) = 1$.
**Action:** In the thirteenth attempt, answer False only to questions 2, 4 and 5.
**Result:** We will know $t(2,4,5)$. Since $t(1,2) = t(1,4) = 1$, $t(2) = t(4)$.

| $t(2,4,5) \equiv$ | $t(2) = t(4) =$ | $t(5) =$ |
|---|---|---|
| 0 (mod 4) | 0 | 0 |
| 1 (mod 4) | 0 | 1 |
| 2 (mod 4) | 1 | 0 |
| 3 (mod 4) | 1 | 1 |

In any case, we can then deduce the value of $t(6)$. We now know all of $t(1),\ t(2),\ \ldots,\ t(12)$.

**Case 2.** All of $t(1,2,3),\ t(4,5,6),\ t(7,8,9)$ and $t(10,11,12)$ are the same. We may assume that they are all 2s.
**Action:** In the 10th attempt, answer False only to questions 1, 4, 7 and 10. In the eleventh attempt, answer False only to questions 2, 5, 8 and 11.
**Result:** We know $t(1,4,7,10)$ and $t(2,5,8,11)$. From these values, we can deduce $t(3,6,9,12)$. We may assume that $t(1,4,7,10) \geq t(2,5,8,11) \geq t(3,6,9,12)$. We consider four subcases.
**Case 2(a).** $t(1,4,7,10) = t(2,5,8,11) = 4$ and $t(3,6,9,12) = 0$.
No further actions are required. We know all of $t(1),\ t(2),\ \ldots,\ t(12)$.
**Case 2(b).** $t(1,4,7,10) = 4$, $t(2,5,8,11) = 3$ and $t(3,6,9,12) = 1$.
We know $t(1) = t(4) = t(7) = t(10) = 1$. Hence $t(2,3) = t(1,2,3) - t(1) = 1$. Similarly,

we have $t(5,6) = t(8,9) = t(11,12) = 1$.
**Action:** In the twelfth attempt, answer False only to questions 3 and 6. In the thirteenth attempt, answer False only to questions 3 and 9.
**Result:** We will know $t(3,6)$ and $(3,9)$.

| $t(3,6) =$ | $t(3,9) =$ | $t(3) =$ | $t(6) =$ | $t(9) =$ | $t(12) =$ |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 1 |
| 0 | 1 | 0 | 0 | 1 | 0 |
| 1 | 0 | 0 | 1 | 0 | 0 |
| 1 | 1 | 1 | 0 | 0 | 0 |

From these values, we can deduce $t(2)$, $t(5)$, $t(8)$ and $t(11)$.
**Case 2(c).** $t(1,4,7,10) = 4$ and $t(2,5,8,11) = t(3,6,9,12) = 2$.
We know $t(1) = t(4) = t(7) = t(10) = 1$. Hence $t(2,3) = t(1,2,3) - t(1) = 1$. Similarly, we have $t(5,6) = t(8,9) = t(11,12) = 1$.
**Action:** In the twelfth attempt, answer False only to questions 2 and 5.
**Result:** We will know $t(2,5)$. We divide this into two further subcases.
**Case 2(c$_1$).** $t(2,5) = 0$ or $2$ so we know the value of $t(2) = t(5)$.
No further actions are required because $t(2) = t(5) = t(9) = t(12) \neq t(3) = t(6) = t(8) = t(11)$. **Case 2(c$_2$).** $t(2,5) = 1$.
We can deduce that $t(8,11) = t(2,5,8,11) - t(2,5) = 1$ and $t(3) = t(5)$.
**Action:** In the thirteenth attempt, answer False only to questions 3, 5 and 8.
**Result:** We will know $t(3,5,8)$.

| $t(3,5,8) =$ | $t(2) =$ | $t(3) =$ | $t(5) =$ | $t(6) =$ | $t(8) =$ | $t(9) =$ | $t(11) =$ | $t(12) =$ |
|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 0 | 0 | 1 | 0 | 1 | 1 | 0 |
| 1 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 1 |
| 2 | 0 | 1 | 1 | 0 | 0 | 1 | 1 | 0 |
| 3 | 0 | 1 | 1 | 0 | 1 | 0 | 0 | 1 |

From these values, we can deduce $t(1)$, $t(4)$, $t(7)$ and $t(10)$.
**Case 2(d).** $t(1,4,7,10) = t(2,5,8,11) = 3$ and $t(3,6,9,12) = 2$.
**Action:** In the twelfth attempt, answer False only to questions 1, 2, 6 and 9.
**Result:** We will know $t(1,2,6,9)$ and $s = 2t(3) + t(12) = t(1,2,3) + t(3,6,9,12) - t(1,2,6,9)$.

| $s \equiv$ | $t(3) =$ | $t(12) =$ |
|---|---|---|
| 0 (mod 4) | 0 | 0 |
| 1 (mod 4) | 0 | 1 |
| 2 (mod 4) | 1 | 0 |
| 3 (mod 4) | 1 | 1 |

We divide this into four further subcases.
**Case 2(d$_1$).** $t(3) = t(12) = 0$.
We know that $t(1) = t(2) = t(6) = t(9) = t(10) = t(11) = 1$.

**Action:** In the thirteenth attempt, answer False only to question 4.
**Result:** If $t(4) = 0$, then $t(5) = t(7) = 1$ while $t(8) = 1$. On the other hand, if $t(4) = 1$, then $t(8) = 0$ while $t(5) = t(7) = 0$.

**Case 2(d$_2$).** $t(3) = t(12) = 1$.
We know that $t(1,2) = t(10,11) = 1$, $t(6) = t(9) = 0$ and $t(4) = t(5) = t(7) = t(8) = 1$.
**Action:** In the thirteenth attempt, answer False only to question 1.
**Result:** If $t(1) = 0$, then $t(2) = 4(10) = 1$ and $t(11) = 0$. On the other hand, if $t(1) = 1$, then $t(11) = 1$ and $t(2) = t(10) = 0$.
**Case 2(d$_3$).** $t(3) = 0$ and $t(12) = 1$.
We know that $t(1) = t(2) = 1$ and $t(10,11) = 1$.
**Action:** In the thirteenth attempt, answer False only to questions 6, 7, 8 and 10.
**Result:** We will know $t(6,7,8,10)$, which cannot be 0.

| $t(6,7,8,10) =$ | $t(4) =$ | $t(5) =$ | $t(6) =$ | $t(7) =$ | $t(8) =$ | $t(9) =$ | $t(10) =$ | $t(11) =$ |
|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| 2 | 1 | 1 | 0 | 0 | 1 | 1 | 1 | 0 |
| 3 | 1 | 0 | 1 | 1 | 1 | 0 | 0 | 1 |
| 4 | 0 | 1 | 1 | 1 | 1 | 0 | 1 | 0 |

We now know all of $t(1),\ t(2), \ldots, t(12)$.
**Case 2(d$_4$).** $t(3) = 1$ and $t(12) = 0$.
We know that $t(10) = t(11) = 1$ and $t(1,2) = 1$.
**Action:** In the thirteenth attempt, answer False only to questions 1, 5, 8 and 9.
**Result:** We will know $t(1,5,8,9)$, which cannot be 0.

| $t(1,5,8,9) =$ | $t(1) =$ | $t(2) =$ | $t(4) =$ | $t(5) =$ | $t(6) =$ | $t(7) =$ | $t(8) =$ | $t(9) =$ |
|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 1 | 1 | 0 | 1 | 1 | 1 | 0 |
| 2 | 0 | 1 | 1 | 1 | 0 | 1 | 0 | 1 |
| 3 | 1 | 0 | 0 | 1 | 1 | 1 | 1 | 0 |
| 4 | 1 | 0 | 1 | 1 | 0 | 0 | 1 | 1 |

We now know all of $t(1),\ t(2), \ldots, t(12)$.

**Case 3.** Three of $t(1,2,3)$, $t(4,5,6)$, $t(7,8,9)$ and $t(10,11,12)$ are the same and different from the fourth one. We may assume that $t(1,2,3) = t(4,5,6) = t(7,8,9) = 2$ and $t(10,11,12) = 1$.
**Action:** In the tenth attempt, answer False only to questions 1, 4, 7 and 10. In the eleventh attempt, answer False only to questions 2, 5, 8 and 11.
**Result:** We know $t(1,4,7,10)$ and $t(2,5,8,11)$. From these values, we can deduce $t(3,6,9,12)$. We may assume that $t(1,4,7,10) \geq t(2,5,8,11) \geq t(3,6,9,12)$. We consider four subcases.
**Case 3(a).** $t(1,4,7,10) = 4$, $t(2,5,8,11) = 3$ and $t(3,6,9,12) = 0$.
No further actions are required. We have $t(1) = t(2) = t(4) = t(5) = t(7) = t(8) = t(10) = 1$ and $t(3) = t(6) = t(9) = t(11) = t(12) = 0$.
**Case 3(b).** $t(1,4,7,10) = 4$, $t(2,5,8,11) = 2$ and $t(3,6,9,12) = 1$.
We know that $t(1) = t(4) = t(7) = t(10) = 1$ and $t(11) = t(12) = 0$. **Action:** In the

twelfth attempt, answer False only to question 6. In the thirteenth attempt, answer False only to question 9.
**Result:** We know $t(6)$ and $t(9)$. Then $t(3) = 1 - t(6) - t(9)$. We can deduce $t(2)$, $t(5)$ and $t(8)$ from $t(1,2,3)$, $t(4,5,6)$ and $t(7,8,9)$.

**Case 3(c).** $t(1,4,7,10) = t(2,5,8,11) = 3$ and $t(3,6,9,12) = 1$.
**Action:** In the twelfth attempt, answer False only to questions 3 and 6.
**Result:** We know $t(3,6)$, which is at most $t(3,6,9,12) = 1$. We divide this into two further subcases.
**Case 3(c$_1$).** $t(3,6) = 0$.
We know that $t(1) = t(2) = t(4) = t(5) = 1$.
**Action:** In the thirteenth attempt, answer False only to questions 8 and 10.
**Result:** We will know $t(8)$ and $t(10)$.

| $t(8,10) =$ | $t(7) =$ | $t(8) =$ | $t(9) =$ | $t(10) =$ | $t(11) =$ | $t(12) =$ |
|---|---|---|---|---|---|---|
| 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| 1 | 1 | 1 | 0 | 0 | 0 | 1 |
| 2 | 0 | 1 | 1 | 1 | 0 | 0 |

We now know all of $t(1),\ t(2),\ \ldots,\ t(12)$.
**Case 3(c$_2$).** $t(3,6) = 1$.
We know that $t(9) = t(12) = 0$ and $t(7) = t(8) = 1$.
**Action:** In the thirteenth attempt, answer False only to questions 3, 4, 5 and 11.
**Result:** We will know $t(3,4,5,11)$, which cannot be 0.

| $t(3,4,5,11) =$ | $t(1) =$ | $t(2) =$ | $t(3) =$ | $t(4) =$ | $t(5) =$ | $t(6) =$ | $t(10) =$ | $t(11) =$ |
|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| 2 | 1 | 0 | 1 | 0 | 1 | 0 | 0 | 1 |
| 3 | 1 | 1 | 0 | 0 | 0 | 1 | 1 | 0 |
| 4 | 0 | 1 | 1 | 1 | 0 | 0 | 1 | 1 |

We now know all of $t(1),\ t(2),\ \ldots,\ t(12)$.
**Case 3(d).** $t(1,4,7,10) = 3$ and $t(2,5,8,11) = t(3,6,9,12) = 2$.
**Action:** In the tenth attempt, answer False only to questions 1, 2, 9 and 12.
**Result:** We know $t(1,2,9,12)$ and $s = 2t(3) + t(6) = t(1,2,3) + t(3,6,9,12) - t(1,2,9,12)$.

| $s =$ | $t(3) =$ | $t(6) =$ |
|---|---|---|
| 0 | 0 | 0 |
| 1 | 0 | 1 |
| 2 | 1 | 0 |
| 3 | 1 | 1 |

We now know $t(3)$ and $t(6)$. We divide this into four further subcases.
**Case 3(d$_1$).** $t(3) = t(6) = 0$.
No further actions are required. We have $t(1) = t(2) = t(4) = t(5) = t(7) = t(9) = t(12) =$

1 and $t(3) = t(6) = t(8) = t(10) = t(11) = 0$.
**Case 3(d$_2$).** $t(3) = t(6) = 1$.
We know that $t(9) = t(12) = 0$ and $t(7) = t(8) = 1$.
**Action:** In the twelfth attempt, answer False only to questions 1 and 5.
**Result:** We will know $t(1, 5)$.

| $t(1,5) =$ | $t(1) =$ | $t(2) =$ | $t(4) =$ | $t(5) =$ | $t(10) =$ | $t(11) =$ |
|---|---|---|---|---|---|---|
| 0 | 0 | 1 | 1 | 0 | 1 | 0 |
| 1 | 1 | 0 | 1 | 0 | 0 | 1 |
| 2 | 1 | 0 | 0 | 1 | 1 | 0 |

We now know all of $t(1),\ t(2), \ldots, t(12)$.
**Case 3(d$_3$).** $t(3) = 0$ and $t(6) = 1$.
We know that $t(1) = t(2) = 1$.
**Action:** In the twelfth attempt, answer False only to questions 4, 7, 8, 11 and 12.
**Result:** We will know $t(4, 7, 8, 11, 12)$, which cannot be 0 or 5.

| $t(4,7,8,11,12) =$ | $t(4) =$ | $t(5) =$ | $t(7) =$ | $t(8) =$ | $t(9) =$ | $t(10) =$ | $t(11) =$ | $t(12) =$ |
|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 1 | 1 | 0 | 1 | 1 | 0 | 0 |
| 2 | 1 | 0 | 0 | 1 | 1 | 1 | 0 | 0 |
| 3 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| 4 | 1 | 0 | 1 | 1 | 0 | 0 | 0 | 1 |

We now know all of $t(1),\ t(2), \ldots, t(12)$.
**Case 3(d$_4$).** $t(3) = 1$ and $t(6) = 0$.
We know that $t(4) = t(5) = 1$.
**Action:** In the twelfth attempt, answer False only to questions 1, 7, 8, 11 and 12.
**Result:** We will know $t(4, 7, 8, 11, 12)$, which cannot be 0 or 5.

| $t(1,7,8,11,12) =$ | $t(1) =$ | $t(2) =$ | $t(7) =$ | $t(8) =$ | $t(9) =$ | $t(10) =$ | $t(11) =$ | $t(12) =$ |
|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 1 | 1 | 0 | 1 | 1 | 0 | 0 |
| 2 | 1 | 0 | 0 | 1 | 1 | 1 | 0 | 0 |
| 3 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| 4 | 1 | 0 | 1 | 1 | 0 | 0 | 0 | 1 |

We now know all of $t(1),\ t(2), \ldots, t(12)$.

**Stage Five.** 5 attempts.
We know $t(13, 14, 15)$, $t(16, 17, 18)$, $t(19, 20, 21)$ and $t(22, 23, 24)$, and that these four values do not consist of two 2s and two 1s. We consider three cases.

**Case 1.** One of $t(13, 14, 15)$, $t(16, 17, 18)$, $t(19, 20, 21)$ and $t(22, 23, 24)$ is 0 or 3. We may assume that $t(22) = t(23) = t(24)$ are known.
**Action:** In the fourteenth attempt, answer False only to questions 13 and 28. In the

fifteenth attempt, answer False only to questions 14 and 28. In the sixteenth attempt, answer False only to questions 16, 19 and 29. In the seventeenth attempt, answer False only to questions 17, 20, 21 and 29.
**Result:** Consider first the values of $t(13, 28)$ and $t(14, 28)$. Suppose that $t(13, 28) = 0$. Then $t(13) = t(28) = 0$. We also know $t(14) = t(14, 18)$ and $t(15) = t(13, 14, 15) - t(14)$. Similarly, if $t(13, 28) = 2$, $t(14, 28) = 0$ or $t(14, 18) = 2$, we will also know all of $t(13)$, $t(14)$, $t(15)$ and $t(28)$. Finally, if $t(13, 28) = t(14, 28) = 1$, then $t(13) = t(14)$.

| $t(13, 14, 15) =$ | $t(13) = t(14) =$ | $t(15) =$ | $t(28) =$ |
|---|---|---|---|
| 0 | 0 | 0 | 1 |
| 1 | 0 | 1 | 1 |
| 2 | 1 | 0 | 0 |
| 3 | 1 | 1 | 0 |

We also know the values of $t(13)$, $t(14)$, $t(15)$ and $t(28)$. We now turn our attention to the values of $t(16, 19, 29)$ and $t(17, 20, 21, 29)$. Let $s = t(16, 19, 29) + t(17, 120, 21, 29) - t(16, 17, 18) - t(19, 20, 21)$. This simplifies to $s = 2t(29) - t(18)$.

| $s \equiv$ | $t(18) =$ | $t(29) =$ |
|---|---|---|
| 0 (mod 4) | 0 | 0 |
| 1 (mod 4) | 1 | 1 |
| 2 (mod 4) | 0 | 1 |
| 3 (mod 4) | 1 | 0 |

In any case, we know $t(18)$ and $t(29)$. Now $t(17) - t(19) = t(17, 20, 21, 29) - t(19, 20, 21) - t(29)$ can only be $-1$, 0 or 1. We consider two subcases.
**Subcase 1(a).** $t(17) \neq t(19)$.
**Action:** In the eighteenth attempt, answer False only to question 20.
**Result:** We know $t(20)$ along with $t(17)$ and $t(19)$. We already know $t(18)$, $t(28)$ and $t(29)$, so that we can deduce the values of $t(16)$, $t(21)$ and $t(30)$. Thus we know all of $t(1)$, $t(2)$, …, $t(30)$. **Subcase 1(b).** $t(17) = t(19)$.
**Action:** In the eighteenth attempt, answer False only to questions 17, 19 and 20.
**Result:** We will know $t(17, 19, 20)$.

| $t(17, 19, 20) =$ | $t(17) = t(19) =$ | $t(20) =$ |
|---|---|---|
| 0 | 0 | 0 |
| 1 | 0 | 1 |
| 2 | 1 | 0 |
| 3 | 1 | 1 |

As in subcase 1(a), we know all of $t(1)$, $t(2)$, …, $t(30)$.

**Case 2.** All of $t(13, 14, 15)$, $t(16, 17, 18)$, $t(19, 20, 21)$ and $t(22, 23, 24)$ are the same. We may assume that they are all 2s.
**Action:** In the fourteenth attempt, answer False only to questions 13, 16, 19, 22 and

28. In the fifteenth attempt, answer False only to questions 14, 17, 20, 23 and 29. In the sixteenth attempt, answer False only to questions 15, 18, 21, 24 and 29.
**Result:** We will know $t(13,16,19,22,28)$, $t(14,17,20,23,29)$ and $t(15,18,21,24,29)$. Let

$$\begin{aligned} s &= t(13,16,19,22,28)+t(14,17,20,23,29)+t(15,18,21,24,29) \\ &\quad -t(13,14,15)-t(16,17,18)-t(19,20,21)-t(22,23,24) \\ &= t(28)+2t(29). \end{aligned}$$

| $s=$ | $t(28)=t(29)=$ | $t(20)=$ |
|---|---|---|
| 0 | 0 | 0 |
| 1 | 1 | 0 |
| 2 | 0 | 1 |
| 3 | 1 | 1 |

From the values of $t(28)$ and $t(29)$, we can deduce $t(13,16,19,22)$, $t(14,17,20,23)$ and $t(15,18,21,24)$. We can now continue as in Case (2) in Stage Four, using the seventeenth and the eighteenth attempts in place of the twelfth and the thirteenth attempts. We will know all of $t(1)$, $t(2)$, $\ldots$, $t(30)$.

**Case 3.** Three of $t(13,14,15)$, $t(16,17,18)$, $t(19,20,21)$ and $t(22,23,24)$ are the same and different from the fourth one. We may assume that $t(13,14,15)=t(16,17,18)=t(19,20,21)=2$ and $t(22,23,24)=1$.
This can be handled in essentially the same way as Case (2) above.

**Stage Six.** 1 attempt.
**Action:** In the nineteenth and last attempt, we give the correct answer to each question.
**Result:** We pass the Martian Citizenship Quiz.

The readers are invited to either find an improvement over 19, or alternatively find a proof that 19 is best possible.

# References


[1] Andy Liu, "The S.M.A.R.T. Circle: Overview", Springer, ISBN 978-3-319-56823-2, (2018), 8-17.

[2] Yen-Kang Fu and Te-Cheng Liu, The Martian Citizenship Quiz, Mathematics Competition **29** (2016) #2 67–70.

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# International Mathematics Tournament of the Towns

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Andy Liu is a Canadian mathematician. He is a professor emeritus in the Department of Mathematical and Statistical Sciences at the University of Alberta. Liu attended New Method College in Hong Kong. He then did his undergraduate studies in mathematics at Mc-Gill University, and earned his Ph.D. in 1976 from the University of Alberta, under the supervision of Harvey Abbott, with a dissertation about hypergraphs. He was the leader of the Canadian team at the International Mathematical Olympiad in 2000 (South Korea) and 2003 (Japan) and acts as vice-president of the Tournament of Towns.

**Selected Problems from the Spring 2021 Papers**

1. In triangle $ABC$, $BE$ and $CF$ are angle bisectors while $BH$ and $CK$ are altitudes. Must $ABC$ be isosceles if $\angle HBE = \angle KCF$?

2. (a) There are four tokens of weights 1001, 1002, 1004 and 1005 grams respectively. Is it possible to determine the weight of each token using a normal balance at most four times, with the result available immediately after each weighing?

   (b) There are four tokens of weights 1000, 1002, 1004 and 1005 grams respectively. Is it possible to determine the weight of each token using an abnormal balance at most four times, with the result available immediately after each weighing? In an abnormal balance, the left pan is 1 gram lighter than the right pan.

3. Is it possible to dissect a polygon into four isosceles triangles no two of which are congruent, if the polygon is

   (a) a square;

   (b) an equilateral triangle?

4. For which positive integers $n$ do there exist $n$ consecutive positive integers whose product is equal to the sum of $n$ other consecutive positive integers?

5. The diameter through $A$ of the circumcircle of an acute triangle $ABC$ intersects the altitude from $B$ at $H$ and the altitude from $C$ at $K$. The circumcircles of triangles $BAH$ and $CAK$ intersect again at $P$. Prove that the line $AP$ bisects $BC$.

6. In a room are 1000 candies on a table and a line of children. Each child in turn divides the current number of candies on the table by the current number of children in the line. If the quotient is an integer, the child takes that number of candies. Otherwise, a boy rounds the quotient up and a girl rounds it down to the next integer and then takes that number of candies. After taking the candies, the child leaves the room. The process continues until all the children have left the room. Prove that the total number of candies taken by the boys is independent of their positions in the line.

7. $E$ and $F$ are points on the sides $CA$ and $AB$, respectively, of an equilateral triangle $ABC$, and $K$ is a point on the extension of $AB$ such that $AF = CE = BK$. Let $P$ be the midpoint of $EF$. Prove that $PK$ is perpendicular to $PC$.

8. Fifty natives stand in a circle. Each announces the age of his left neighbour. Then each announces the age of his right neighbour. Each native is either a knight who says both numbers correctly, or a knave who increases one of the numbers by 1 and decreases the other by 1. Each knave chooses which number to increase and which to decrease independently. Is it always possible to determine which of the natives are knights and which are knaves?

9. A hotel has $n$ unoccupied rooms upstairs, $k$ of which are under renovation. All doors are closed, and it is impossible to tell if a room is occupied or under renovation without opening the door. There are 100 tourists in the lobby downstairs. Each in turn goes upstairs to open the door of some room. If it is under renovation, she closes its door and opens the door of another room. If it is not under renovation, she moves in and then closes the door. Each tourist chooses the doors she opens. For each $k$, determine the smallest $n$ for which the tourists can agree on a strategy while in the lobby, so that no two of them move into the same room.

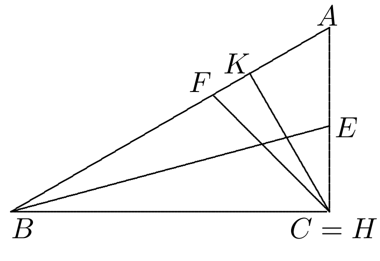
10. Does there exist a positive integer $n$ such that for any real numbers $x$ and $y$, there exist $n$ real numbers such that $x$ is equal to their sum and $y$ is equal to the sum of their reciprocals?

11. There is a row of $100n$ tuna sandwiches. A boy and his cat take alternating turns, with the cat going first. In her turn, the cat eats the tuna from one sandwich anywhere in the row if there is any tuna left. In his turn, the boy eats the first sandwich from either end, and continues until he has eaten 100 of them, switching ends at any time. Can the boy guarantee that, for every positive integer $n$, the last sandwich he eats contains tuna?

12. Find a real number $r$ such that for any positive integer $n$ the difference between $\lceil r^n \rceil$ and the nearest square of an integer is equal to 2.

**Solutions**

1. $ABC$ does not have to be isosceles. We may have $\angle B = 30°$ and $\angle C = 90°$. Then $\angle ACF = 45°$ and $\angle ACK = 30°$, so that $\angle KCF = 15°$. Now $H$ coincides with $C$, and we have $\angle HBE = 15°$ also.




2. (a) Let the four tokens be A, B, C and D. We first weigh A and B against C and D. There are two cases.
**Case 1.** We have equilibrium.
We weigh A against B and C against D. By symmetry, we may assume that

A is heavier than B and C is heavier than D. Finally, we weigh A against C. By symmetry, we may assume that A is heavier than C. Then the weights of A, B, C and D are 1005, 1001, 1004 and 1002 grams respectively.

**Case 2.** We do not have equilibrium.

By symmetry, we may assume that A and B are heavier than C and D. We weigh A against B and C against D. By symmetry, we may assume that A is heavier than B and C is heavier than D. We know that the weight of A is 1005 grams and the weight of D is 1001 grams. Finally, we weigh B against C. The heavier of them has weight 1004 grams and the lighter one has weight 1002 grams.

(b) Let the four tokens be A, B, C and D. In the first weighing, put A and B on the lighter left pan and C and D on the heavier right pan. There are three cases.

**Case 1.** There is equilibrium.

The weights of A and B are 1005 grams and 1000 grams while the weights of C and D are 1004 grams and 1002 grams. Comparing A with B and C with D in the next two weighings settles the issue.

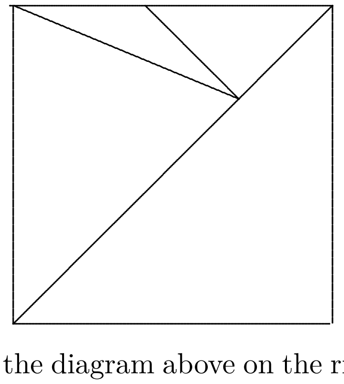
**Case 2.** The left pan sinks.

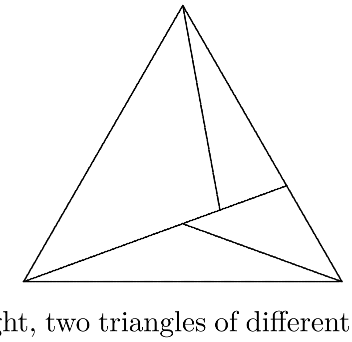
The weight of one of A and B is 1005 grams and the weight of one of C and D is 1000 grams. These can be determined in the next two weighings. The final weighing determines which token weighs 1004 grams and which token weighs 1002 grams.

**Case 3.** The right pan sinks.

The weight of one of A and B is 1000 grams. In the second weighing, compare A with B. By symmetry, we may assume that A weighs 1000 grams. In the third weighing, put C in the left pan and D in the right pan. If there is equilibrium, the issue is completely settled. Otherwise, whichever pan sinks contains the heavier token. By symmetry, we may assume that D is heavier than C. In the final weighing, put D in the left pan and B in the right pan. If there is equilibrium, D weighs 1005 grams and B weighs 1004 grams. If the left pan sinks, D weighs 1005 grams and B weighs 1002 grams. If the right pan sinks, B weighs 1005 grams and D weighs 1004 grams.

3. (a) It is possible. In the diagram below on the left, two triangles of different sizes have equal angles of 45°, a third triangle has equal angles of $22\frac{1}{2}^\circ$ and the last triangle has equal angles of $67\frac{1}{2}^\circ$.

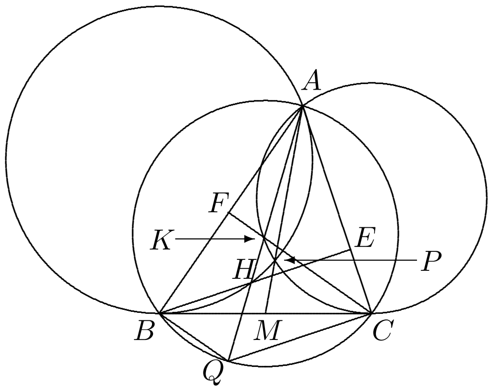




(b) It is possible. In the diagram above on the right, two triangles of different sizes have equal angles of 40°, a third triangle has equal angles of 20° and the last triangle has equal angles of 80°.

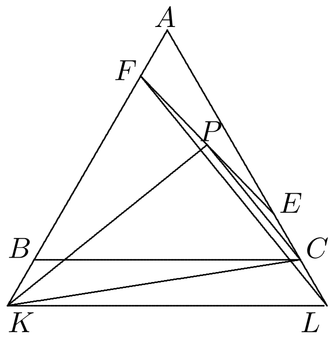
4. Consider the product of any $n$ consecutive positive integers. Since one of them is divisible by $n$, so is their product. Then the sum of the other $n$ consecutive positive integers is also divisible by $n$. Hence their average is an integer. This is not possible if $n$ is even. If $n$ is odd, take $n$ copies of the average. Keep the middle copy. Increase those on one side by 1, by 2 and so on, and decrease those on the other side by 1, by 2 and so on. This will result in $n$ consecutive integers with the correct sum. Since the minimum value of the average is $(n-1)!$, all of these integers are positive.

5. Let the diameter $AQ$ intersect the altitudes $BE$ and $CF$ at $H$ and $K$ respectively. Then $\angle AQB = \angle ACB$. Hence $QAB$ and $CBE$ are similar right triangles, so that $\angle QAB = \angle CBE$. It follows that $BC$ is tangent to the circumcircle of triangle $BAH$. We can prove in an analogous manner that $BC$ is also tangent to the circumcircle of triangle $CAK$. Let $P$ be the second point of intersection of these two circles, and let the line $AP$ intersect $BC$ at $M$. Then $MB^2 = MP \cdot MA = MC^2$, so that $M$ is indeed the midpoint of $BC$.




6. Note that the last child leaving the room takes all remaining candies. Let $k$ be the number of children initially. If $k$ is a divisor of 1000, then every child gets the same number of candies and there is nothing to prove. Otherwise, by the Division Algorithm, there exist integers $q$ and $r$ such that $1000 = kq + r$ with $0 < r < k$. Let there be $kq$ chocolates and $r$ caramels. Every child is guaranteed $k$ chocolates. If the current number of caramels is less than the current number of children, a boy will get a caramel while a girl will not. If the current number of caramels is equal to the current number of children, this situation will remain until the end, and every child from this point on will get a caramel. It follows that every boy gets a caramel, wherever he may be in the line up. The desired conclusion follows.

7. Extend $AC$ to $L$ so that $EC = CL$. Then $AKL$ is also equilateral. Since $EP = PF$, $FL = 2PC$ and is parallel to $PC$. Since $KL = AK$, $FK = FK - BK + AF = AB = CA$ and $\angle FKL = 60° = \angle CAK$, $FKL$ and $CAK$ are congruent triangles. It follows that we have $CK = FL = 2PC$. Moreover, triangles $FAL$ and $CLK$ are also congruent, so that $\angle FLA = \angle CKL$. Now $\angle KCP = 60° - \angle PCA + \angle BCK = 60° - \angle FLA + \angle CKL = 60°$. Hence $KPC$ is half an equilateral triangle, so that $PK$ is perpendicular to $PC$.




8. The sum of the two announcements of each native, knight or knave, is equal to the sum of the ages of the two neighbours. Let the natives be #1 to #50 in cyclic order. If we add up the sums from #2, #6, #10, ..., #46 and #50, we have the total age of the 25 odd-numbered natives plus the age of #1. If we add up the sums from #4, #8, ..., #48, we have the total age of the 24 odd-numbered natives other than #1. Subtracting the second total from the first and dividing the difference by 2, we have the age of #1. The age of every native can be determined in a similar manner. Knowing the age of each native, we can tell whether any particular native is a knight or a knave.

9. We consider two separate cases.
**Case 1.** $k = 2t$ for some $t \geq 1$.
We assign $t + 1$ rooms to each tourist. When a tourist goes upstairs, she opens the rooms assigned to her in order. If each finds a room not under renovation, all is well. Note that since $2(t+1) > 2t$, the number of tourists whose assigned rooms are all under renovation is at most one. There are only $2t - (t+1) = t - 1$ other rooms under renovation, and she can move into the last room assigned to any of the other tourists. Thus $100(t+1) = 50k + 100$ rooms are sufficient. We now show that these many rooms are necessary. If $k = 0$, clearly 100 rooms are necessary. Let $k > 0$. Any strategy requires each tourist to open $k + 1$ rooms on a list. We can organize these lists as a $2(t+1) \times 100$ table. Suppose there are at most $100(t+1) - 1$ rooms. The top $t + 1$ rows of the table contains $100(t + 1)$ entries. Since the number of rooms is 1 less, there exists two equal entries in different columns. For definiteness, say that the $i$th entry from the top in the first column and the $j$th entry from the top in the second column both indicate Room $m$. Since $i \leq t+1$ and $j \leq t+1$, the total number of entries in the two columns above $m$ is at most $t + t = k$. Let all the rooms listed above Room $m$ in these two columns be renovated. Then the first tourist and the second tourist will both move into Room $m$.
**Case 2.** $k = 2t + 1$ for some $t \geq 0$.
Again, we assign $t+1$ rooms to each tourist, but there is one other unassigned room. When a tourist goes upstairs, she opens the rooms assigned to her in order. If each finds a room not under renovation, all is well. Since $2(t+1) > 2t+1$, the number of tourists whose assigned rooms are all under renovation is at most one. This tourist now opens the door of the unassigned room. If it is not under renovation, she moves in. This will be the case if $t = 0$. If $t > 0$ and this room is under renovation, then there are only $(2t + 1) - (t + 1) - 1 = t - 1$ other rooms under renovation, and she can move into the last room assigned to any of the other tourists. Thus $100(t+1)+1 = 50k+51$ rooms are sufficient. We now show that these many rooms are necessary. Any strategy requires each tourist to open $k + 1$ rooms on a list. We can organize these lists as a $2(t + 1) \times 100$ table. The top $t + 1$ rows of the table

contains $100(t+1)$ entries, equal to the number of rooms. If there exists two equal entries in different columns, we can argue as in Case 1 that two tourists will move into the same room. Otherwise, one of the entries must be identical to the entry in the $(t+2)$nd row of another column. Since $t+(t+1) = 2t+1 = k$, the same argument can also be applied.

10. Such a value is $n = 6$. If $x = 0 = y$, take three copies of 1 and three copies of $-1$. If $x = 0 \neq y$, take four copies of $\frac{3}{y}$ and two copies of $-\frac{6}{y}$. We have $\frac{4\times3}{y} - \frac{2\times6}{y} = 0$ while $\frac{4y}{3} - \frac{2y}{6} = y$. Similarly, if $x \neq 0 = y$, take four copies of $\frac{3}{x}$ and two copies of $-\frac{6}{x}$. Finally, suppose that $x \neq 0 \neq y$. Take two copies of each of $\frac{2x}{3}$ and $\frac{3}{2y}$ along with $-\frac{x}{3}$ and $-\frac{3}{y}$. We have

$$\frac{4x}{3} - \frac{x}{3} + \frac{2\times 3}{2y} - \frac{3}{y} = x \text{ and } \frac{2\times 3}{2x} - \frac{3}{x} + \frac{4y}{3} - \frac{y}{3} = y.$$

11. The cat can win if $n = 3^{100}$, which is the total number of moves. Number the tuna sandwiches 1 to $100 \times 3^{100}$ from left to right. Note that

$$1 + 2(1+3+3^2+\cdots+3^{99}) = 1 + 2\left(\frac{3^{100}-1}{3-1}\right) = 3^{100}.$$

The cat's strategy is divided into 100 stages. For $1 \leq k \leq 100$, the $k$th stage consists of $2\times 3^{100-k}$ moves, reducing the number of sandwiches from $100\times 3^{101-k}$ to $100\times 3^{100-k}$. There is a final move after stage 100. In stage 1, the cat divides the sandwiches into a left part, a middle part and a right part, each consisting of $100\times 3^{99}$ sandwiches. In the first $3^{99}$ moves, the cat eats all the tuna from the sandwiches in the middle part whose numbers are congruent to 1 modulo 100. Meanwhile, in each move, the boy eats 100 sandwiches from one end or the other, or both. In any case, the numbers of these sandwiches are not congruent to one another modulo 100, and none of them are from the middle part. At the midway point of stage 1, the boy has eaten $100\times 3^{99}$ sandwiches, leaving behind $200\times 3^{99}$ sandwiches. The tuna in all sandwiches in the middle part whose numbers are congruent to 1 modulo 100 has been eaten. In the next $3^{99}$ moves, the cat eats the tuna from the remaining sandwiches whose numbers are congruent to 1 modulo 100. At the end of stage 1, there are $100\times 3^{99}$ sandwiches. The tuna from all those whose numbers are congruent to 1 modulo 100 has been eaten.
The remaining stages are conducted in exactly the same manner. At the end of stage $k$, $1 \leq k \leq 100$, there are $100\times 3^{100-k}$ sandwiches, and the tuna from all those whose numbers are congruent to 1, 2, $\ldots$, $k$ modulo 100 has been eaten. At the end of stage 100, there are 100 sandwiches, and the tuna from all of them has been eaten. The cat wins on the next move.

12. Define $a_0 = a_1 = 2$ and $a_{n+2} = 2a_{n+1} + a_n$ for $n \geq 0$. The characteristic equation of this recurrence relation is $x^2 - 2x - 1 = 0$, and the characteristic roots are $x = 1 \pm \sqrt{2}$. Hence $a_n = \alpha(1+\sqrt{2})^n + \beta(1-\sqrt{2})^n$ for some real numbers $\alpha$ and $\beta$. From the initial conditions, we have $2 = \alpha+\beta = \alpha(1+\sqrt{2}) + \beta(1-\sqrt{2}) = (\alpha+\beta) + \sqrt{2}(\alpha-\beta)$. Hence $\alpha - \beta = 0$ so that $\alpha = \beta = 1$. Now

$$\begin{aligned} a_{2n} &= (1+\sqrt{2})^{2n} + (1-\sqrt{2})^{2n} \\ &= ((1+\sqrt{2})^n + (1-\sqrt{2})^n)^2 - 2(1+\sqrt{2})^n(1-\sqrt{2})^n \end{aligned}$$

$$= \quad a_n^2 - 2(-1)^n.$$

Note that $(1 \pm \sqrt{2})^2 = 3 \pm 2\sqrt{2}$ so that $a_{2n} = (3 + 2\sqrt{2})^n + (3 - 2\sqrt{2})^n$. Since $0 < 3 - 2\sqrt{2} < 1$, $0 < (3 - 2\sqrt{2})^n < 1$ for any $n$, so that $\lceil (3 + 2\sqrt{2})^2 \rceil = a_{2n}$. Hence $3 + 2\sqrt{2}$ has the desired property since $a_{2n}$ is alternatingly 2 more or 2 less than $a_n^2$.

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