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MATHEMATICS COMPETITIONS

JOURNAL OF THE

WORLD FEDERATION OF NATIONAL MATHEMATICS COMPETITIONS

AUSTRALIAN MATHS TRUST

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Contents

World Federation of National Mathematics Competitions

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From the President

Dear readers of *Mathematics Competitions* journal!

It is my great pleasure to announce the recipients of the 2020 Paul Erdős Award. The Awards Committee chaired by Alexander Soifer collected and assessed the nominations. The recommended candidates were approved by the Executive Committee of WFNMC. They are (in alphabetic order):

Gangsong Leng (China)

Jaromír Šimša Czech Republic)

Jaroslav Svrcek (Czech Republic)

Congratulations to our distinguished colleagues for their outstanding achievements and meritorious national and international contributions!

We remind all that the Paul Erdős Award has been established to recognize contributions of persons who have played a significant role in the development of mathematical challenges with essential impact on mathematics learning. The following brief description of the main contributions of our awardees (taken from the report of the Awards Committee) shows that they all completely satisfy the requirements.

Gangsong Leng, Shanghai University, Shanghai, China has served as a member of the China Mathematics Olympiad Main Examination Committee for the past 20 years. He was coach of the National Training Team of the China Mathematics Olympiad, and director of the CWMO Main Examination Committee. He was the national team leader of the China Mathematics Olympiad in 2007; and the Vice-Leader of China Mathematics Olympiad National Team in 2006 and 2009. During this period, he contributed many test questions for the Chinese Mathematics Competition. In 2005 to 2006, the Mathematics Trainers' Guild (MTG) of the Philippines sought the expertise of Gangsong Leng to train Filipino students for their participation in the 2006 Southeast Asia Mathematical Olympiad (SEAMO) which was held in Penang, Malaysia. From 2006 to 2010, he was selected to be the special training facilitator of Hong Kong students for the International Mathematics Olympiad. From 2011 to 2014, he was the guest facilitator of the Macau Team for IMO.

Jaromír Šimša, Czech Republic, has served as Chairman of the National Committee of the Czech MO from 2000 through to the present. For 25 years he has been involved in training Czech contestants for the International Mathematical Olympiad. Several times he acted as the leader of the Czech Republic team at the IMO. He started the Junior Czech-Polish-Slovak Mathematics Competition, jointly with Waldemar Pompe of Poland. Many of his problems were used at the Czech-Polish-Slovak (CPS) Mathematics Competitions. All his problems are of a very high quality, serving not only as great competition challenges but also later as valuable teaching material. In fact, Jaromír Šimša is the author of more than 250 mathematics Olympiad problems used in the Czech Mathematical Olympiad over the past 30 years. In 2020, he will once again serve as the Leader of the Czech Team to the International Mathematical Olympiad.

Jaroslav Svrcek, Czech Republic, was for 12 years, 2004 to 2016, the Editor in Chief of the WFNMC journal *Mathematics Competitions*. From 1996 to 2014 he was one of the most active mathematicians in the training of the Czech team for the International Mathematical Olympiad and during most of those years he was either Team Leader or Deputy Team Leader. He founded the Mathematical Duel (together with Jozef Kalinowski of Poland). He worked for years on a national Czech journal for students. For the past 20 years, he served as Vice-Chairman of the Czech Mathematical Olympiad. In 2020, he will once again serve as the Deputy Leader of the Czech Team to the International Mathematical Olympiad.

These short biographies can also be found on the WFNMC web site: http://www.wfnmc.org/erdos2020ann.html

It was expected that the awards would be presented at ICME-14 in July 2020. Since the Congress has been postponed by one year, the ceremony will take place during the same Congress in July 2021.

My best regards,

Kiril Bankov
President of WFNMC
May, 2020

Editor's Page

Dear Competitions enthusiasts,
readers of our *Mathematics Competitions* journal!

Following the example of previous editors, I invite you to submit to our journal *Mathematics Competitions* your creative essays on a variety of topics related to creating original problems, working with students and teachers, organizing and running mathematics competitions, historical and philosophical views on mathematics and closely related fields, and even your original literary works related to mathematics.

Just be original, creative, and inspirational. Share your ideas, problems, conjectures, and solutions with all your colleagues by publishing them here.

We have formalized the submission format to establish uniformity in our journal.

Submission Format

Format: should be LaTeX, TeX, or Microsoft Word, accompanied by another copy in pdf.

Illustrations: must be inserted at about the correct place of the text of your submission in one of the following formats: jpeg, pdf, tiff, eps, or mp. Your illustration will not be redrawn. Resolution of your illustrations must be at least 300 dpi, or, preferably, done as vector illustrations. If a text is needed in illustrations, use a font from the Times New Roman family in 11 pt.

Start: with the title in BOLD 14 pt, followed on the next line by the author(s)' name(s) in italic 12 pt.

Abstract: Include a 40 - 100 word abstract of your paper.

Main Text: Use a font from the Times New Roman family in 11 pt.

End: with your name-address-email and your website (if applicable).

Include: your high resolution small photo and a concise professional summary of your works and titles.

Please submit your manuscripts to María Elizabeth Losada at
director.olimpiadas@uan.edu.co

We are counting on receiving your contributions, informative, inspired and creative.

Best wishes,

María Falk de Losada
Acting Editor, Mathematics Competitions

Past President, WFNMC

Editorial Board

Three Etudes in Mathematical Coloring

Alexander Soifer (asoifer@uccs.edu)

University of Colorado. Colorado Springs, USA

Born and educated in Moscow, Alexander Soifer has for over 38 years been a Professor at the University of Colorado, teaching math, and art and film history. He has published over 300 articles, and a good number of books. In the past several years, 7 of his books have appeared in Springer: The Scholar and the State: In the search of Van der Waerden; The Mathematical Coloring Book: Mathematics of Coloring and the Colorful Life of Its Creators; Mathematics as Problem Solving; How Does One Cut a Triangle?; Geometric Etudes in Combinatorial Mathematics; Ramsey Theory Yesterday, Today, and Tomorrow; and Colorado Mathematical Olympiad and Further Explorations: From the Mountains of Colorado to the Peaks of Mathematics. He has founded and for 32 years ran the Colorado Mathematical Olympiad. Soifer has also served on the Soviet Union Math Olympiad (1970-1973) and USA Math Olympiad (1996-2005). He has been Secretary of WFNMC (1996-2008), and Senior Vice President of the World Federation of National Mathematics Competitions (2008-2012); from 2012-2018 he was the president of the WFNMC. He is a recipient of the Federation's Paul Erdős Award (2006). Soifer's Erdős number is 1.

Abstract
Our original idea for the article was to create a braid of history and mathematics and present it in the form of three etudes. In retrospect, we see that these etudes also convey indivisibility of Time, interconnections between the Past, the Present, and the Future.

Etude 1
Issai Schur, His Theorem and His Numbers


1

Issai Schur was born on January 10, 1875, in the Russian city of Mogilyov (presently in Belorussia). Being a Jew, Issai could not enroll in any Russian university. At 13 he went to the German language Nicolai-Gymnasium (1888–1894). That prepared him for entering a German university in 1894. In Berlin, on September 2, 1906, Issai Schur married Regina Malka Frumkin. On the personnel form, on the line "Arian," Schur wrote "*nicht*" for himself and "*nicht*" for his wife. The happy and lasting marriage produced two children, Georg and Hilde.

Issai Schur gave most of his life to the University of Berlin, first as a student (1894–1901; Ph.D. in Mathematics and Physics summa cum laude, November 27, 1901), then as a *Privatdozent* (1903–1909), *ausserordentlischer Professor* (equivalent to an associate professor, December 23, 1909 – April 21, 1913 and again April 1, 1916 – April

[1] Photograph of young Issai Schur, compliments of his daughter, Hilde Abelin-Schur.

1, 1919) and *Ordinarius* (equivalent to a full professor, April 1, 1919 – September 30, 1935). The only three years away from Berlin, 1913–1916, Schur spent at the University of Bonn.

Issai Schur was elected to many academies of sciences. He was a legendary lecturer. Schur's student and friend Alfred Theodor Brauer (Ph. D. under Schur 1928) recalls [Bra2] that the number of students in Schur's elementary number theory courses often exceeded 400, and during the winter semester of 1930 even exceeded 500.

Hitler's appointment as *Reichskanzler* by President von Hindenburg on January 30, 1933 changed this idyllic life. Schur's former student Menahem Max Schiffer recalls in his talk at the 4th Schur conference in May 1986 at Tel Aviv University, which was subsequently published [Schi]:

> Now, the year 1933 was a decisive cut in the life of every German Jew. In April of that year [April 7, 1933 to be precise] all Jewish government officials were dismissed, a boycott of Jewish businesses was decreed and anti-Semitic legislation was begun.

Issai Schur was a famous professor, a pride of his University and of his profession. Yet no achievement was high enough for a Jew in Nazi Germany. Following two years of pressure and humiliation, Schur,

faced with imminent expulsion, 'voluntarily' asked for resignation on August 29, 1935. On September 28, 1935, Reichs- and Prussian Minister of Science, Instruction and Public Education, replied on behalf of *Der Fürer und Reichskanzler*, i.e., Adolf Hitler himself (see facsimile next page):[2]

Fürer and *Reichskanzler* has relieved you from your official duties in the Philosophical *Facultät* of the University of Berlin effective at the end of September 1935, in accordance with your August 29 of this year request.

Schur was the last Jewish professor to lose his job at the University of Berlin. Schur was able to leave Germany in early 1939. Two years later, on January 10, 1941, on his 66th birthday, he passed away in Tel Aviv of a heart attack.

[2] Archive of Humboldt University at Berlin, document UK-Sch 342, Bd.I, Bl.25.
[3] Issai Schur, the collection of his daughter, Hilde Abelin-Schur

Der Reichs- und Preußische Minister für Wissenschaft, Erziehung und Volksbildung

W I p Schur.2 a

Es wird gebeten, dieses Geschäftszeichen und den Gegenstand bei weiteren Schreiben anzugeben.

Berlin W 8, den 28. September 1935
Unter den Linden 4
Fernsprecher: A 1 Jäger 0030
Postscheckkonto: Berlin 14402
Reichsbank-Giro-Konto
— Postfach —

Verw. Dir.
b. d. Univ. Berlin
Eing. - 4 OKT 1935
V.D. P.A. Schur /31
28.9.

25

Der Führer und Reichskanzler hat Sie auf Jhren Antrag vom 29. August d.Js. mit Ablauf des Monats September 1935 von den amtlichen Verpflichtungen in der Philosophischen Fakultät der Universität Berlin entbunden.

Jch übersende Jhnen anbei die hierüber ausgefertigte Urkunde.

(Unterschrift)

An Herrn Professor Dr. Jssai Schur in Berlin-Schmargendorf, Ruhlaerstr.14 - Einschreiben -.

Abschrift zur Kenntnis und weiteren Veranlassung.

Jn Vertretung

gez. K u n i s c h

Preußisches Ministerium für Wissenschaft, Erziehung und Volksbildung Kanzlei

Beglaubigt.
Ministerial-Kanzleisekretär.

An

den Herrn Verwaltungsdirektor
bei der Universität Berlin

hier C 2.

Letter relieving Issai Schur from his duties at the University of Berlin
Courtesy of the Archive of the Humboldt University at Berlin

Schur's 1916 Theorem appears as a useful tool, "a very simple lemma," and is immediately used for obtaining a number-theoretic result related to Fermat's Last Theorem. Nobody then asked questions of the kind Issai Schur posed and solved in this paper [Sch]. Consequently, nobody appreciated this result much when it was published. Now it shines as one of the most beautiful, classic theorems of mathematics.

Schur's Theorem 32.1 ([Sch])**.** For any positive integer n there is an integer $S(n)$ such that any n-coloring of the initial positive integer array $[S(n)]$ contains integers x, y, z of the same color such that $x + y = z$.

In this paper, Schur shows that the least such integer $S(n)$ has the upper bound $\lfloor n!e \rfloor$ where $\lfloor x \rfloor$ denotes the largest integer not exceeding x. Only in 1973, was Schur's upper bound improved, and only slightly, by Robert Irving [I] to $\lfloor n!(e-1/24) \rfloor$.

There are two definitions of the Schur Number, differing by 1. Let us define the Schur Number as the largest integer $S(n)$, such that the integers 1, 2, …, $S(n)$ can be colored in n colors in such a way that no color contains integers x,y,z such that $x + y = z$.

In his 1916 paper, Schur established the following lower bound: $S(n) \geq (3^n - 1)/2$. Having found a beautiful proof of it, I decided to offer this lower bound as the hardest, "problem 5", in the 36th Soifer Mathematical Olympiad in April 2019 (formerly The Colorado

Mathematical Olympiad). Of course, Schur's formula in n serves as a hint, making the problem easier than a good size particular case. For example, try to offer your students or professional mathematicians, who are not familiar with Schur's result a problem like this:

> Can each of the integers 1; 2; …; 581,130,733 be colored in one of 19 colors so that no color contains numbers x, y, z such that $x + y = z$?
>
> *Hint*: for $n = 19$, $(3^n - 1)/2 = 581{,}130{,}733$.

Schur's lower bound is sharp for $n = 1, 2, 3$, which is easy to prove: $S(1) = 1$, $S(2) = 4$, and $S(3) = 13$.

For $n = 4$, Schur's lower bound gives 40, but in 1965, using a computer, Leonard D. Baumert and Solomon W. Golomb showed [BG] that in fact $S(4) = 44$.

Finding the exact value of $S(5)$ appeared to be very hard. In the 1970s, the best known bounds for $S(5)$ were $157 \le S(5) \le 321$, the lower bound obtained in 1979 by Harold Fredricksen [F] and the upper bound in 1973 by Earl Glen Whitehead [W].

Only two decades later, in 1994, Geoffrey Exoo proved [Ex] that $S(5) \ge 160$. Moreover, he writes [Ex]:

> We have found approximately 10,000 different partitions [colorings] of [1, 160]; of these, four are symmetric [palindromal]. These 10,000 partitions are all 'close' to each other. In other words, one can begin with one of the partitions, move an integer from one set to another, and obtain a new partition. This can be contrasted with the situation for partitions of [1, 159] where we found over 100,000 partitions, most of which were not close in this sense. It is tempting to conclude that there are far fewer sum-free partitions of [1, 160] than of [1, 159].

Then Marijn J.H. Heule became interested in this problem. In 2018 his result appeared [H1] in AAAI (Submitted Tue, 21 Nov 2017 22:54:59): $S(5) = 160$. Why was it significant? Because until his recent publication, the upper bound of $S(5)$ stood at 315.

Thus, for $n = 5$, $\left(3^n - 1\right)/2$ gives us 121 whereas the exact value is $S(5) = 160$.

Marijn J.H. Heule

Marijn writes:

> We obtained the solution, $n = 160$, by encoding the problem into propositional logic and applying massively parallel satisfiability solving techniques on the resulting formula. We constructed and validated a proof of the solution to increase trust in the correctness of the multi-CPU-year computations. The proof is two petabytes in size and was certified using a formally verified proof checker, demonstrating that any result by satisfiability solvers—no matter how large—can now be validated using highly trustworthy systems.

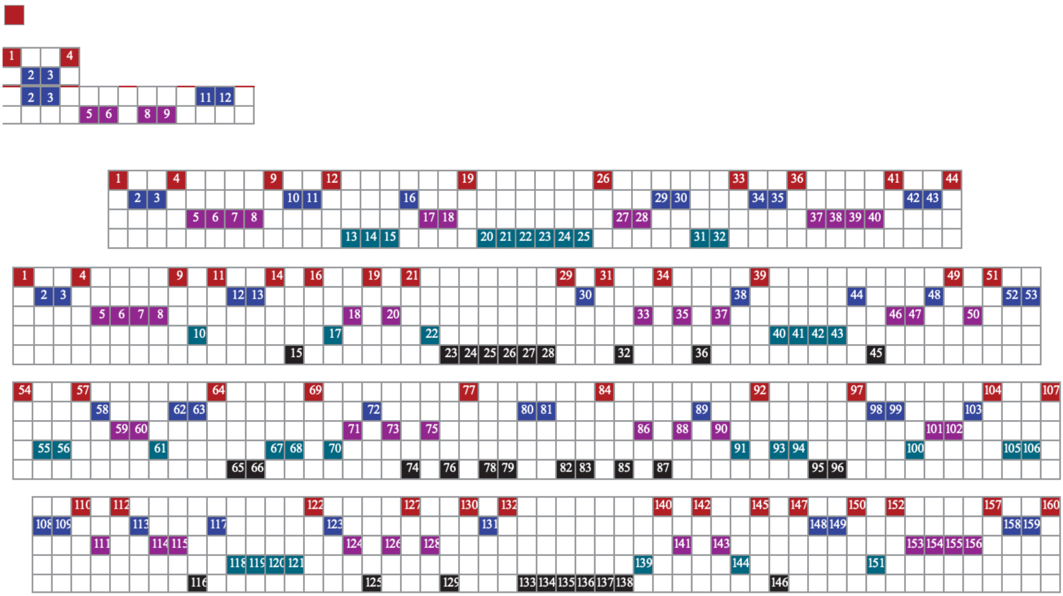
As we already know, the coloring of integers from 1 to 160 in 5 colors without a monochromatic pair and its sum, had been demonstrated first by Geoffrey Exoo. He even produced a palindromal coloring, i.e., coloring where numbers i and $160 - i$ are assigned the same color. I am showing here a palindromal coloring from Heule's paper simply because it is, yes, colorful (see this coloring below).

The asymptotic lower bound was slightly improved from Schur's exponential base 3. Following Abbott and Moser 1966 [AM], Abbott and Hanson 1972 [AH], Exoo's results allowed for the lower bound of $S(n) \geq c(315)^{n/5} \approx c(3.15981831)^n$ for $n > 5$ and constant

c [Ex]. Heule's result [H1] brought it a bit higher: $S(n) \geq c(321)^{n/5} \approx c(3.17176503)^{n}$.

In 2000, Harold Fredricksen and Melvin M. Sweet [FS] constructed colorings that proved new lower bounds for $S(6)$ 536 and $S(7)$ 1680.

There are folks who do not appreciate proofs done with use of a computer. As my friend Paul Kainen wrote in *Geombinatorics*, "To reject the use of computers as what one may call "computational amplifiers" would be akin to an astronomer refusing to admit discoveries made by telescope."



Etude 2
The Chromatic Number of the Plane, Records and Prizes

Unfinished, a picture remains alive, dangerous. A finished work is a dead work, killed.
– Pablo Picasso

We do not need to worry about the chromatic number of the plane problem (CNP) dying any time soon, despite major breakthroughs reported in the Special Issue of *Geombinatorics* in July 2018. To make these lines self-contained, let me repeat the definition and statement of the problem in the general case. CNP was given birth by the 18-year old teenager Eddie Nelson [S1] in November 1950:

> *What is the smallest number of colors sufficient for coloring the plane in such a way that no two points of the same color are unit distance apart?*

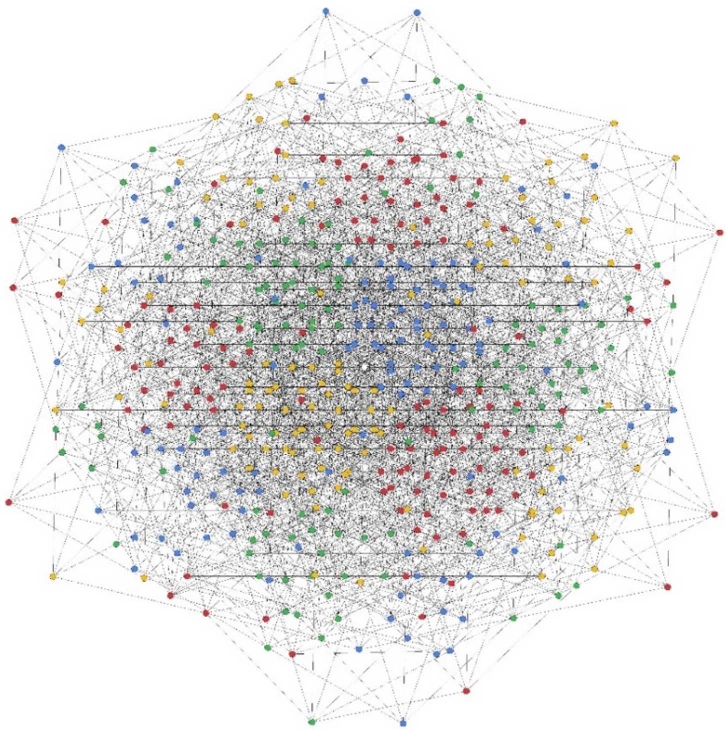
This number is called *the chromatic number of the plane*, or CNP, and is denoted by χ. In April 2018, Aubrey de Grey shrank the range of χ by raising the lower bound from 4 to 5 [G]: $5 \le \chi \le 7$. His first example of the unit-distance 5-chromatic graph included 20,425 vertices, which he then reduced to a graph on 1581 vertices. His result answered in the negative the May 4, 2002, $1000 problem of Ronald L. Graham, who asked whether it was possible to 4-color the plane to forbid a monochromatic distance 1 [S1]. The problem creator and the problem solver met in San Diego, where Aubrey de Grey received the $1,000 prize from the hands of Ronald L. Graham. On my request, they captured this landmark event, and thus the reader can in a sense participate in it.

Ronald L. Graham presents Aubrey de Grey the Prize:
$1,000. San Diego, September 22, 2018

“I will certainly be adhering to the convention of framing the check rather than cashing it,” wrote Aubrey to Ron and me.

Then came Marijn Heule, and in setting 7 consecutive world records for the minimal number of vertices in a unit-distance 5-chromatic graph, dramatically reduced that number to 529 vertices (the first 6 records you can see in [H1] and [S2]). I hope Marijn will present his latest record in form of an essay for *Geombinatorics*.

Below you can see a visualization of the Heule 529-vertex unit-distance 5-chromatic graph. Five colors are used for the vertices. Only the center uses the 5th color (white). In fact, all Heule's graphs are vertex critical. This implies that there exists a coloring in which every vertex can be the only one with the fifth color. On Aubrey's suggestion, Marijn let the central vertex be the only one of the 5th color.



"I spent a few thousand CPU hours to find it [the 529-vertex record]. This seems the best that I can do with the current methods," wrote Marijn to me on February 15, 2019 [H3].

Furthermore, Marijn Heule brought the record down to a 5-chromatic unit-distance 510-vertex graph. But the current world record stands at 509 and belongs to the Russian engineer Jaan Parts from the city of Kazan. It will appear in 2020 in the *Geombinatorics* quarterly. I will address his record construction in future publications.

---oOo---

Etude 3
The Chromatic Number of the Plane, Problems and Prizes

Of course, we are all interested in further reducing the size of the smallest 5-chromatic unit distance graph. We have a World Series of a mathematical kind.

5-Chromatic World Series. Find the smallest 5-chromatic unit-distance graph.

As we have said, the present record, at the time of submission of this paper, of 509 belongs to Jaan Parts.

The title of the next problem rhymes:

CNP – Triangle-Free. Construct a triangle-free 5-chromatic unit-distance graph.

Book-Prize 5-Chromatic Triangle-Free Competition. Find the smallest 5-chromatic triangle-free unit-distance graph.

Yes, as the prize for the smallest graph, constructed, say, by January 1, 2021, I am offering a copy of the second expanded edition of *The Mathematical Coloring Book* [S3].

Why do I pose this problem when the smallest size of a graph in the first problem (without a triangle-free condition) is lower or equal to the size of the graph with this condition?

The triangle-free condition makes the graph more embeddable. Moreover, the Exoo-Ismailescu result [EI] allows for a relatively small building block: the smallest unit-distance 4-chromatic triangle-free graph has only 17 vertices, whereas without a unit-distance requirement, the Grötzsch graph is not much smaller at 11 vertices. I therefore believe that in triangle-free unit-distance 5-chromatic graphs, we will succeed in lowering the order of the graph faster than in the general case. We could do it, I hope, 'in real time.'

Ron Graham believes that every talk ought to have at least one proof. I agree, at least one proof, but also at least one joke. And so, in my March 7, 2019, at Florida Atlantic University I proposed Ron's next $1,000 problem, subject to his approval, of course:

The Ronald L. Graham New $1,000 Problem. Prove or disprove the existence of a 6-chromatic unit-distance graph.

On March 16, 2019, Ron replied:

> *I approve of the new $1000 problem!*

To make this essay self-contained, let me repeat my old conjectures:

CNP Conjecture for the Plane [S1]. $\chi = 7$.

CNP Conjecture for E^3 [S1]. $\chi\left(E^3\right) = 15$.

CNP Conjecture for the Euclidean *n*-space E^n [S1]**:**

$$\chi\left(E^n\right) = 2^{n+1} - 1.$$

The exciting breakthroughs mentioned here and described in detail in the July-2018 Special Issue of *Geombinatorics*, and the popularity of *The Mathematical Coloring Book* (sold ca. 1000 hard copies and 64,000 downloads of the 2009 edition [S1]), inspired Springer to sign a contract for the new expanded edition [S4] of *The Mathematical Coloring Book.* I hope it will appear in 2021 or so with these and further advances on the theme of this colorful problem.

I am grateful to the Editor-in-Chief Maria Falk de Losada for reading my manuscript better than I have and sharing her suggestions on improving this essay.

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Some Identities Involving Sums of Consecutive Squares

Shawn Godin

Cairine Wilson S.S., Orleans, Canada

Shawn Godin teaches high school mathematics at Cairine Wilson S.S. in Orleans, Canada. He is currently the chair of the Fryer, Galois and Hypatia contest creation committee for the Centre for Education in Mathematics and Computing (CEMC) at the University of Waterloo. He is also involved with the Canadian Open Mathematics Challenge hosted by the Canadian Mathematical Society (CMS). Shawn has also been long involved with the CMS problem solving journal Crux Mathematicorum, as former Editor in Chief, as well as editing various sections and contributing articles. Shawn is also involved with mathematics teacher associations in Canada, writing articles, doing workshops as well as outreach programs for students.

Abstract

In this article we use three problems from the book *Five Hundred Mathematical Challenges* to generalize some sums involving squares of consecutive integers whose sum is also a square. In the end we invite the reader to investigate further identities of similar types.

1 Introduction

Identities involving sums of squares have been known for thousands of years. Pythagorean triples, that is ordered triples of positive integers (a, b, c) such that

$$a^2 + b^2 = c^2 \tag{1.1}$$

are a well known example. The complete solution to (1.1) is given by

$$a = 2xy, b = x^2 - y^2, c = x^2 + y^2, \text{ where } x, y \in \mathbb{Z}^+, x > y. \tag{1.2}$$

Most introductory number theory textbooks will include several theorems regarding sums of squares. Fermat's sum of two squares theorem [3] (theorem 2.15, p. 55) shows which positive integers are expressible as the sum of the squares of two integers; Gauss proved which integers can be written as the sum of three squares [3] (problem #12, p. 170); and Lagrange showed [3] (theorem 6.26, p. 317) that <u>every</u> positive integer can be expressed as the sum of squares of four integers. Since these theorems deal with the squares of integers, zero is allowed so we could get $9 = 3^2 + 0^2 + 0^2 + 0^2 = 2^2 + 2^2 + 1^2 + 0^2$ as two representations of 9 as a sum of "four" squares.

In this article we will explore identities where a sum of a string of squares, most consecutive, is equal to a square. For the most part, we will use only high school algebra, a few congruences (to make the divisibility arguments less cumbersome) and the identity

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}. \qquad (1.3)$$

2 Three Problems

The following three problems appear in [1]:

Problem 4. *Observe that*

$$3^2 + 4^2 = 5^2,$$
$$5^2 + 12^2 = 13^2,$$
$$7^2 + 24^2 = 25^2,$$
$$9^2 + 40^2 = 41^2.$$

State a general law suggested by these examples and prove it.

Problem 226. *Let*

$$a_1 = 2^2 + 3^2 + 6^2, a_2 = 3^2 + 4^2 + 12^2, a_3 = 4^2 + 5^2 + 20^2,$$

and so on. Generalize these in such a way that the number a_n is always a perfect square.

Problem 326. *Observe that*

$$2^2 + 3^2 + 4^2 + 14^2 = 15^2,$$

$$4^2 + 5^2 + 6^2 + 38^2 = 39^2,$$

$$6^2 + 7^2 + 8^2 + 74^2 = 75^2,$$

$$8^2 + 9^2 + 10^2 + 122^2 = 123^2.$$

State and prove a general result suggested by these examples.

Finding the perfect squares from problem 226 and writing the results out in a similar manner to the other two problems, we get

$$2^2 + 3^2 + 6^2 = 7^2,$$

$$3^2 + 4^2 + 12^2 = 13^2,$$

$$4^2 + 5^2 + 20^2 = 21^2,$$

$$5^2 + 6^2 + 30^2 = 31^2.$$

We are lead to wonder if, for each n there exist families of sequences $a_1, a_2, a_3, \ldots, a_n, a_{n+1}, a_{n+2}$ such that

$$a_1^2 + a_2^2 + a_3^2 + \cdots + a_{n+1}^2 = a_{n+2}^2 \qquad (2.1)$$

where $a_1, a_2, a_3, \ldots, a_n$ are consecutive positive integers; and a_{n+1}, a_{n+2} are also consecutive positive integers.

3

If so, we can rewrite (2.1) as

$$a^2+(a+1)^2+(a+2)^2+\cdots+(a+(n-1))^2+(b-1)^2 = b^2 \quad (2.2)$$

where $b = a_{n+2}$ and $b-1 = a_{n+1}$. Equation (2.2) can be rearranged and rewritten as

$$\sum_{i=0}^{n-1}(a+i)^2 = 2b-1 \quad (2.3)$$

which simplifies to

$$na^2+n(n-1)a+\frac{n(n-1)(2n-1)}{6} = 2b-1 \quad (2.4)$$

Problems 4, 226 and 326 correspond to $n = 1, 2, 3$, respectively. We will look at each of these separately and then look at the generalized case.

3 Solutions to the Three Problems

Problem 4 corresponds to $n = 1$ where we are dealing with the square of one "consecutive" integer. Substituting into (2.4) yields

$$a^2 = 2b-1 \quad (3.1)$$

which implies that a must be odd, hence we can write $a = 2k+1$ which gives $b = 2k^2+2k+1$. Thus we get the family of solutions

$$(2k+1)^2+(2k^2+2k)^2 = (2k^2+2k+1)^2 \quad (3.2)$$

which can be easily verified. We can rewrite this relation in the whimsical form

$$n^2 + \left\lfloor \frac{n^2}{2} \right\rfloor^2 = \left\lceil \frac{n^2}{2} \right\rceil^2 \tag{3.3}$$

where n is an odd integer.

Problem 226 corresponds to $n = 2$ in which case (2.4) yields

$$b = a^2 + a + 1 \tag{3.4}$$

so the family of solutions are given by

$$a^2 + (a+1)^2 + [a(a+1)]^2 = [a(a+1)+1]^2. \tag{3.5}$$

for any positive integer a.

Finally, problem 326 corresponds to $n = 3$. Equation (2.4) becomes

$$2b = 3a^2 + 6a + 6 \tag{3.6}$$

which means that a must be even. Writing $a = 2k$, we get the identity

$$(2k)^2+(2k+1)^2+(2k+2)^2+(6k^2+6k+2)^2 = (6k^2+6k+3)^2 \tag{3.7}$$

which is true for all $k \in \mathbb{Z}^+$.

4 Values of n Which Yield Identities

From (2.4), it is evident that, to solve for b, we will need to know the parity of $\dfrac{n(n-1)(2n-1)}{6}$. This corresponds to determining when

$$n(n-1)(2n-1) \equiv 0, 6 \pmod{12}. \tag{4.1}$$

The solutions to $n(n-1)(2n-1) \equiv 0 \pmod{12}$ are $n \equiv 0, 1, 4, 5, 8, 9 \pmod{12}$ which is equivalent to $n \equiv 0, 1 \pmod 4$. Similarly the solutions to $n(n-1)(2n-1) \equiv 6 \pmod{12}$ are equivalent to $n \equiv 2, 3 \pmod 4$.

When $n \equiv 0, 1 \pmod 4$, we can rewrite (2.4) as

$$na^2 \equiv 1 \pmod 2 \tag{4.2}$$

which suggests that there are no identities of this type when n is a multiple of 4. When $n \equiv 1 \pmod 4$, a must also be odd.

Similarly, when $n \equiv 2, 3 \pmod 4$, we can rewrite (2.4) as

$$na^2 \equiv 0 \pmod 2 \tag{4.3}$$

which suggests that when $n \equiv 2 \pmod 4$, a can be any positive integer, while when $n \equiv 3 \pmod 4$, a must be even.

We can rearrange (2.4) to yield

$$b = \frac{na^2}{2} + \frac{n(n-1)a}{2} + \frac{n(n-1)(2n-1)}{12} + \frac{1}{2} \tag{4.4}$$

from which we will look at three different cases, based on n modulo 4 (recall there are no identities when $n \equiv 0 \pmod 4$).

Substituting $n = 4k+1$, $n = 4k+2$, and $n = 4k+3$ into (4.4) yields the three families of solutions.

When $n = 4k + 1$ we get solutions for all positive integers a, with

$$b = (2k+1)a^2 + (2k+1)(4k+1)a + 12k^2 + 1 + \frac{32k^3 + 13k}{3} \quad (4.5)$$

which, for example, yield the identities:

$$3^2 + 4^2 + 5^2 + 6^2 + 7^2 + 8^2 + 99^2 = 100^2 \quad (k = 1, a = 3),$$

$$1^2 + 2^2 + 3^2 + \cdots + 13^2 + 14^2 + 507^2 = 508^2 \quad (k = 3, a = 1).$$

When $n = 4k + 2$, a must be odd. We get

$$b = \frac{(4k+1)a^2}{2} + 2k(4k+1)a + \frac{(2k+1)(32k^2 - 4k + 3)}{6} \quad (4.6)$$

which holds for non-negative integers k and odd positive integers a.

In the final case $n = 4k + 3$ and a must be even. We get

$$b = \frac{(4k+3)a^2}{2} + (2k+1)(4k+3)a + \frac{(k+1)(32k^2 + 28k + 9)}{3} \quad (4.7)$$

for non-negative integers k and positive integers a. For example, one of the solutions ($k = 2$ so $n = 11$ and $a = 2$) is

$$2^2+3^2+4^2+5^2+6^2+7^2+8^2+9^2+10^2+11^2+12^2+324^2 = 325^2.$$

5 Next Steps

There are many different directions we can take our exploration. The most natural question would be: are there identities of the form

$$a^2 + (a+1)^2 + (a+2)^2 + \cdots + (a+n-1)^2 = k^2, \qquad (5.1)$$

that is, can a sum of consecutive squares be a square? There are solutions of this type, for example

$$18^2 + 19^2 + 20^2 + \cdots + 28^2 = 77^2 \qquad (5.2)$$

and, the impressive

$$3^2 + 4^2 + 5^2 + \cdots + 963^2 = 17\,267^2. \qquad (5.3)$$

The solution to this problem involves the theory of the Pell equation, which is beyond the scope of this paper. Interested readers can check [2], from which (5.2) and (5.3) came from entries in the table on page 439.

To keep the problem of the same difficulty, we could explore sums where the elements are not necessarily consecutive, but in some other finite arithmetic sequence. For example in

$$7^2 + 13^2 + 19^2 + 25^2 + 31^2 + 1082^2 = 1083^2 \qquad (5.4)$$

$7, 13, 19, 25, 31$ is an arithmetic sequence with common difference 6, while 1082 and 1083 are consecutive. We could make both arithmetic sequences have the same common difference to get something like

$$16^2+22^2+28^2+34^2+40^2+46^2+52^2+58^2+64^2+1377^2 = 1383^2 \quad (5.5)$$

where $16, 22, 28, 34, 40, 46, 52, 58, 64$ is arithmetic with common difference $6 = 1383 - 1377$. Have fun generating your own identities.

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Olympiad Problems of Outstanding Beauty

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Krzysztof Ciesielski is professor at the Jagiellonian University in Kraków. For nine years he served as Vice Head (responsible for teaching) of the Mathematics Institute of this University. His mathematical specialty is topological dynamical systems. He is author and co-author (mainly with Z. Pogoda) of several books popularizing mathematics that became bestsellers in Poland and were awarded prestigious prizes, including the Steinhaus Prize and the Golden Rose Prize for the best book popularizing science published in that country. Since 1987 he has been a member of the Editorial Board (a Correspondent) of The Mathematical Intelligencer, from 1999-2012 he was an Associate Editor of The European Mathematical Society Newsletter, since 1996 he has been a member of the Editorial Committee of Polish monthly Delta (now its Chairman). Since 1980 he has been a member of the Kraków Committee of the Polish Mathematical Olympiad, holding its Chair since 2008. He is also a member of the Editorial Board of the journal *Mathematics Competitions*.

Abstract
During the Congress held on the occasion of the 100th anniversary of the Polish Mathematical Society a special session devoted to “beautiful” problems that appeared on some mathematical competitions was organized. In the article some information about the session and problems presented during the session are provided as well as a short note about Stefan Straszewicz, the creator the Polish Mathematical Olympiad.

The Polish Mathematical Society was created in 1919 in Kraków. Among the founding members of the Society there were well known mathematicians of that time, such as Stanisław Zaremba and Kazimierz Żorawski, as well as young mathematicians who had just started their scientific careers, such as Stefan Banach, Otton Nikodym and Franciszek Leja. Zaremba was elected to be the first President of the Society.

In September 2019, on the occasion of the 100th anniversary of the Society, the Jubilee Congress of Polish Mathematicians took place in Kraków. One of the Congress activities was a special session, organized and led by the author of this article[1]. The session was entitled Zadania olimpijskie niezwykłej urody (Olympiad problems of outstanding beauty).

Many mathematicians who have been involved with different mathematical competitions took part in this session and encountered a variety of outstanding problems. Some of these problems would be regarded as problems of special interest, mainly because of their original and nonstandard solutions. Sometimes just the formulation of the problem is fascinating.

The session proved to be of great interest, although as many as 21 other scientific sessions were taking place simultaneously, as well

[1] A modified version of this article was published in Polish in the journal of the Polish Mathematical Society *Wiadomo–ci Matematyczne*.

as panel discussions on teaching mathematics and the promotion of the Polish edition of the book by Dermot Turing *X, Y & Z, The Real Story of How Enigma Was Broken* with the participation of Dermot Turing himself.

The mathematicians who presented problems were: Dominik Burek, Bartłomiej Bzdęga, Andrzej Grzesik, Michał Krych, Barbara Roszkow-ska-Lech, Ryszard Rudnicki, Grzegorz Światek (who is now the chair of the Main Committee of the Polish Mathematical Olympiad), Edward Tutaj, Jakub Węgrecki and Michał Wojciechowski. In this article two extra problems are presented, one of them was discussed during the session (the problem presented by Krzysztof Oleszkiewicz) and one prepared by the organizer of the session.

As was mentioned above, two talks were exceptional. In one case, the talk was longer. Each year three main prizes of the Polish Mathematical Society are awarded and they are given to the recipients solemnly during the opening ceremony of the Congress of the Polish Mathematical Society. The prizes are: the Main Banach Prize (for achievements in pure mathematics), the Main Steinhaus Prize (for achievements in applied mathematics), and the Main Dickstein Prize (for achievements in mathematical culture, in particular concerning the history of mathematics, popularization of mathematics and teaching of mathematics). This year the Dickstein

Prize was given to Michał Krych, and consequently he was invited to present a longer lecture. The Olympic session was an appropriate place for this presentation, as Krych was awarded the Dickstein Prize chiefly (but not only) for his prolonged active work for the Polish Mathematical Olympiad. Krych has worked on the committees of the Polish Olympiad for 47 years! He was a member of the Warsaw Regional Olympiad Committee for 47 years, being its Chair for 27 of them; moreover, since 2007 he has been the Vice-chair of the Main Committee of the Polish Olympiad. Thus, Krych in his lecture presented two problems and made some more general remarks.

The first lecture, prepared by Danuta Ciesielska and Małgorzata Terepeta, was of a different nature. It was devoted to Stefan Straszewicz, the creator and the first chair of the Main Committee of the Polish Olympiad. Straszewicz was born in 1889 in Warsaw. He studied in Zurich and got his PhD in 1914; the supervisor of his PhD dissertation *Beiträge zur Theorie der konvexen Punktmengen* was Ernst Zermelo. In 1928 he was named professor of Warsaw Technical University. For some years he was a vice-rector of that university. In 1953-1957 he was the President of the Polish Mathematical Society. For many years he was the Polish representative in the International Commission on Mathematical Instruction, being from 1963-1966 the vice president of that Commission.

In 1949 in Poland it was decided that mathematical olympiads for secondary school students would be organized. Straszewicz alone was asked by the authorities to organize this competition. The form of the olympiad proposed by him did not change thereafter for 70 years. Straszewicz was the Chair of the Committee for 20 years.

He was the person who made the final decision concerning the choice of problems for the competition and the list of winners. It was always done perfectly and there was never any dissatisfaction. Following each olympiad a booklet was published, that included all the problems with solutions prepared by Straszewicz. After each five-year period, a book written by Straszewicz containing all the problems from the five most recent olympiads was published. Straszewicz prepared four volumes of such books, some collections were translated into English ([1]) and Russian ([2]). Straszewicz died in 1983.

The year 2019 marked not only the 100th anniversary of the Polish Mathematical Society but also the 70th anniversary of the Polish Mathematical Olympiad. As mentioned earlier, this year Michał Krych was awarded the Main Dickstein Prize. It is interesting to point out that in 1979 the first person to be awarded the Main Dickstein Prize was precisely Stefan Straszewicz.

A granddaughter of Straszewicz, a mathematician, Zofia Adamowicz, professor at the Mathematical Institute of the Polish

Academy of Sciences, also participated in the Congress and in the Olympiad Session.

As is generally known, very frequently only after some attempts at the solution may one feel the full taste of a mathematical problem. Therefore, this article is divided into two parts. In the first one the general description of the session and some associated topics have been presented and the problems are formulated. In the second part, which will appear in the forthcoming issue of *Mathematics Competitions*, the sources of the problems will be provided and the solutions will be presented, giving our readers ample time to work on them.

Problems

Problem 1 (Ryszard Rudnicki).

Assume that a polyhedron P is circumscribed about a sphere and it is possible to paint each side of P red or blue in such a way, that each two sides with a common edge are of different colours. Prove that the sum of the areas painted blue is equal to the sum of areas painted red.

Problem 2 (Michał Wojciechowski).

Baron Münchhausen says that pines and birches grow in his magic forest and at the distance 1 kilometer from each pine there are precisely 10 birches. Is Baron Münchhausen telling the truth?

Problem 3 (Edward Tutaj).

A $(x_n)_{n=1}^{+\infty}$ sequence is given by a recurrence relation

$$x_{n+3} = x_n + x_{n+1} \cdot x_{n+2}$$

with the initial conditions $x_1 = x_2 = x_3 = 1$. Prove that for each positive integer p some multiple of p is a term of (x_n).

Problem 4 (Barbara Roszkowska-Lech).

Let a and b be positive integers such that $ab+1$ divides a^2+b^2. Show that $\frac{a^2+b^2}{ab+1}$ is the square of an integer.

Problem 5 (Jakub Węgrecki).

Each positive integer was painted in one of k colours. Show that there exist four pairwise distinct integers a,b,c,d painted in the same colour satisfying the conditions:

$$ad = bc, \quad \frac{c}{a} = 2019^n, \quad \frac{b}{a} = 2020^m$$

for some positive integers m,n.

Problem 6 (Michał Krych).

Show that if integers a,b satisfy the equation $2a^2 + a = 3b^2 + b$, then $a - b$ and $2a + 2b + 1$ are the squares of integers.

Problem 7 (Michał Krych).

Assume that six points are given on edges of a tetrahedron $A_1A_2A_3A_4$, each one on a different edge. For each vertex of this tetrahedron we take a sphere containing this vertex and the three given points that are contained on the edges which have this vertex as an endpoint. Prove that those four spheres have a nonempty intersection.

Problem 8 (Bartłomiej Bzdęga).

We are given a convex pentagon $ABCDE$ with

$$|AB| = |BC| = |CD|, \quad |AE| = |EB| = |BD|, \quad |AC| = |CE| = |ED|.$$

Determine the measures of its angles.

Problem 9 (Dominik Burek).

The integers $a_1,a_2,\ldots,a_n$ satisfy the inequalities

$$1 < a_1 < a_2 < \ldots < a_n < 2a_1.$$

Show that if m is the number of different prime divisors of the product $a_1a_2\dots a_n$, then

$$(a_1a_2\dots a_n)^{m-1} \geq (n!)^m.$$

Problem 10 (Andrzej Grzesik).

Determine the maximal number of lines in three-dimensional space such that all of them have one point in common and the angles between each two of them are the same.

Problem 11 (Grzegorz Świątek).

A light ray moves in the region $U := \{(x,y) : x > 0, 0 < y < x^2\}$ reflecting from the boundary components according to the rule of equal angles of incidence and reflection. Prove that its trajectory will reflect only a finite number of times.

Problem 12 (Krzysztof Oleszkiewicz).

Assume that r is a positive integer. Show that for any real numbers $a_1, a_2,\dots,a_r$ the inequality

$$\sum_{m=1}^{r}\left(\sum_{n=1}^{r}\frac{a_m a_n}{m+n}\right) \geq 0.$$

holds. Determine for which numbers $a_1, a_2,\dots,a_r$ equality occurs.

Problem 13 (Krzysztof Ciesielski).

Assume that each three out of six points in the plane are vertices of a scalene triangle. Prove that the shortest side of one of the triangles is at the same time the longest side of another.

References

[1] S. Straszewicz, Mathematical problems and puzzles from the Polish Mathematical Olympiads, Pergamon Press, Oxford 1965.

[2] S. Straszewicz, J. Browkin, Polish Mathematical Olympiads (in Russian), Mir, Moscow 1978.

Math Kangaroo

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Meike Akveld completed her Ph.D. in symplectic geometry at ETH in 2000. She then taught at a Swiss high school for 11 years. Since 2007 she is a senior scientist at ETH Zürich, teaching mainly service and teacher education courses. Her research focus is first of all teaching knot theory (topology) to secondary and primary school kids and more generally secondary and tertiary education, e.g., curriculum reforms. Her main hobby, apart from climbing and skiing, is the Kangaroo, where she is the president of the Swiss and of the international association math competition, AKSF.

Luis Caceres completed his Ph.D. in Logic at The University of Iowa. He also has a MS in Applied Math from the University of Puerto Rico. He is professor at the University of Puerto Rico at Mayaguez. During the past 15 years he has been working in projects of Math Education at the school level and at the university level. He is the director of the Puerto Rico Math Olympiads Program.

Robert Geretschläger teaches Mathematics and Descriptive Geometry at BRG Kepler in Graz, Austria. Among other things keeping him busy, he is currently Austrian team leader at the International Mathematical Olympiad, treasurer of the Association Kangourou sans Frontières and Senior Vice-President of the WFNMC.

Abstract
In this paper we will outline the history of the mathematical competition Kangaroo, describe the structure of the organisation behind it and in particular show a sample of past questions to give a flavour of what this competition is about. It should be underlined that Math Kangaroo is a popularising maths competition which is organised on a non-profit basis.

1 Introduction

In the 1998 edition of the Journal of WFNMC, *Mathematics Competitions*, Robert Geretschläger, a long standing member of both AKSF and WFNMC, reported about the - in those days fairly new - competition Math Kangaroo [**RG**]. Since then a lot has happened. In October of 2019, Meike Akveld was elected as the new president of the Association Kangourou sans Frontières (AKSF) and it is our honour to inform you with this article about what has happened - in a nutshell - in the past 20 years.

2 Early history of AKSF

In the early 80's, Peter O'Halloran, a mathematics teacher from Sydney, invented a new kind of competition for Australian schools: multiple choice questions, corrected by computer, which meant that thousands of pupils could participate at the same time.

It was a tremendous success for the Australian Mathematical National Contest.

In 1991, two French teachers (André Deledicq and Jean Pierre Boudine) decided to start a similar competition in France under the name “Kangourou” to pay tribute to their Australian friends. In the first edition 120,000 children from France took part.

In June 1993 the Board of the French Kangourou organised a European meeting in Paris, to which many of the organizers of mathematical competitions in European countries were invited. All were impressed by the steadily increasing number of participants in the French Kangourou competition (120,000 in 1991, 300,000 in 1992 and 500,000 in 1993). Seven countries (Belarus, Hungary, The Netherlands, Poland, Romania, Russia and Spain) decided to adopt that same scheme and the competition was immediately a great success in all those countries.

In response to this success the General Assembly, consisting of the delegates of 10 European countries, decided to create the Association “Kangourou sans Frontières” (AKSF) in June 1994 in Strasbourg at the European Council.

3 AKSF – facts and numbers

Since the founding of the Association, Math Kangaroo has turned out to be an overwhelming success. From a small, friendly and familiar European Association, AKSF has grown to be one of the world players on the stage of mathematics competitions, always taking care to preserve the aspect of friendliness. After the founding in 1994, many more European countries quickly joined and in the year 2000 Mexico, the first over-seas country, became a member of the Association, soon followed by Venezuela in 2002 and many more Latin American countries in the following years. It didn't take much longer for the competition to jump over to Asia and recently Australia, the parent of Kangaroo, also joined AKSF, about which we are very pleased. A few African countries are also among our members and we can now truly and proudly say that the name "Kangourou sans Frontières" i.e. "Kangaroo without borders" is absolutely justified – see Figure 1. In the near future we hope to have some more members from the African continent.

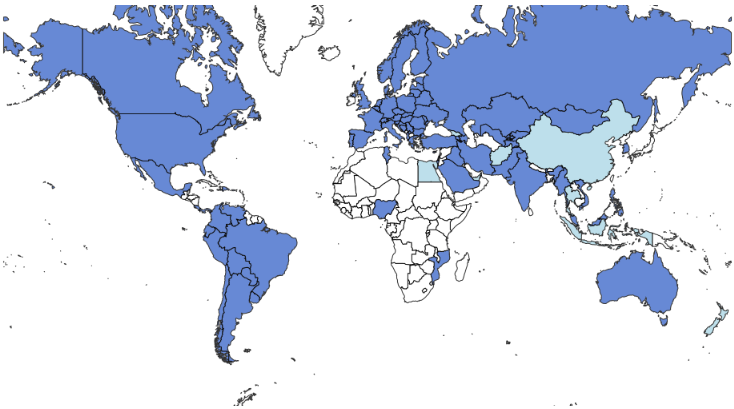


Figure 1: dark blue = members of AKSF; light blue = applicants of AKSF

Numberwise the story is no less impressive – see Figure 2. From about half a million participating children and youngsters in the early nineties, the competition has grown rapidly to surpass the 6 million participant limit in 2011 and has been fairly stable ever since, counting 6,293,071 participants in the latest competition in 2019 which was organised in 85 countries. At the moment AKSF counts 79 countries which are provisional or full members of the association and 9 countries in applicant status, eagerly waiting to become a member of AKSF soon. For more detailed information see the AKSF [1].

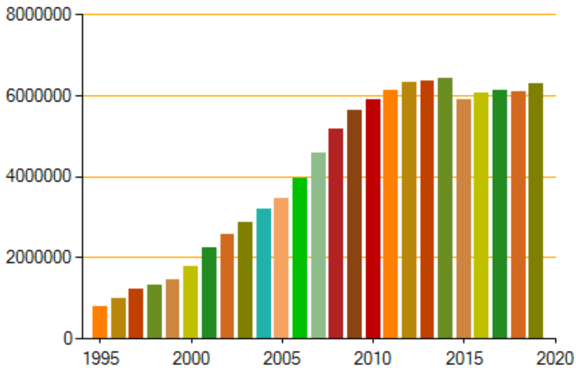


Figure 2. Development of the number of participants in the international Kangaroo competition

4 Goals of AKSF

In the founding documents of AKSF the very first article states:

The Association aims at spreading basic mathematical culture by all means and, in particular, by organizing an annual game contest to be held on the same day in all participating countries.

The purpose of the game contest is to stimulate and motivate the largest possible number of students (as a complement to other activities, competitions, Olympiads and rallies).

This is exactly how the competition should be understood i.e. it is not an elite competition to select for mathematical talent, but it is a competition which has as its main goal popularisation of mathematics in school, in particular AKSF wants to to spread the joy of mathematics, to support mathematical education in school and to promote a positive perception of mathematics in society. Over the past 20 years the role of mathematics in society has changed and it has become clearer and clearer that we need young people with a good knowledge of mathematics to study STEM subjects as these are the subjects that may solve the problems of the future. Math Kangaroo aims at showing young people that maths can be fun and so hopes to keep more people on board the ship called “mathematics”.

5 Structure of AKSF and of the competition

AKSF is an association governed under French law. Our members are entities represented by agents and our rules state that a country cannot be represented by more than one member i.e. only one entity per country can run the competition. This entity is then responsible for the entire organisation of the Kangaroo competition in its country, whereby it should be explicitly stated that our statutes require that the competition be run in a non-profit way. This autonomy of the individual country gives our members a lot of freedom and makes it possible to run the same competition on the same day (always the third Thursday in March) all over the world in many different countries with different (educational) cultures.

The competition itself consists of six levels (Pre-Ecolier, Ecolier, Benjamin, Cadet, Junior and Student), according to the age or schooling of the participants, ranging from year 1 to year 12 (each level covers two years). Each level consists of 24 to 30 single-choice questions and takes between 60 and 90 minutes. In each level the competition starts with so-called 3-point-problems which are meant to be one-step-problems and should be solvable for the vast majority of the pupils. These simple, but beautiful warm-up problems are followed by 4-point and 5-point-problems. Set up this way the competition ensures that all participants can solve some of the problems and are succesful somewhere - a positive feeling is guaranteed for every one! In particular the 5-point problems are meant for the mathematically more gifted students and should give them something to think about. It is rare that a student solves all questions correctly and so a common feeling of managing some and failing some other problems is shared by all.

So how are these questions created? Each year the Association meets for four days at the so called Annual Meeting organised by one of its members. This means the meeting takes place in a different country every year and is a chance for our members to get to know other members' home and culture. Prior to this meeting each member is required to submit candidate questions for the next year's competition so that a collection of over 1000 single choice questions is created (to be precise 1,127 questions were submitted in 2019) which are then rated by the members before travelling to the annual meeting. Starting from this rating, the members work - divided up into working groups according to the various levels - on the selection of the problems for the next year's Kangaroo competition. Because of the enormous amount of expertise of both teachers, mathematicians and maths educators in our association the result is a high quality collection of original and fun problems. These problem sets are then translated by the individual countries into competition papers. Hereby each country has the freedom to exchange a certain but limited number of questions, e.g. if a particular question is not in accordance with the national curriculum.

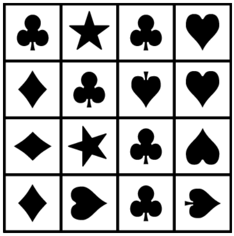
6 Sample of problems

In this section, we will take a look at a few examples of Kangaroo problems taken from the 2019 papers [2], chosen to reflect the style typical for this particular competition. Since the six age-levels of the Kangaroo require quite a range of levels of language skills and mathematical knowledge, problems posed at different age-levels can have quite a different flavour. In order to illustrate this, we have chosen problems from the 2019 competition in the very popular levels Ecolier (for grades 3 and 4) and Cadet (for grades 7 and 8). In each level, we have selected one three-point problem, one four-point problem and one five-point problem.

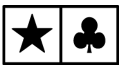
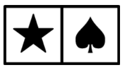
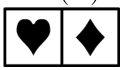
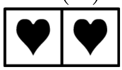
6.1 Ecolier 2019

Problem 6 (3 points)

Karina cuts out a piece of this form  from the following diagram:



Which of the following pieces can she cut out?

(A)  (B)  (C)  (D)  (E) 

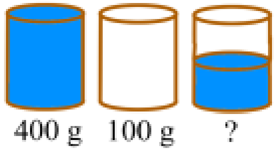
Solution: This is a three-point problem designed for 8- or 9-year-old students. As a three-point problem, it is meant to be accessible for all. It should give them something to think about and play with, without requiring any kind of school-based knowledge. This hefty restriction limits possible problems in this category to puzzles, and this visual puzzle is quite typical.

A close look at the various options reveals a section in the middle of the top row that looks exactly like tile (A). Once a student has found this, it is not really necessary for him or her to check that the other combinations are not present, but some participants in the competition may wish to do so, just to be sure. Taking a closer look at the symbols in the squares shows us that their orientations are not uniform. This is important to note, because (E) could otherwise also be considered a solution. The adjacent hearts in the upper right-hand corner are, however, not oriented in such a way that tile (E) could be considered to be "cut out" of the diagram.

This is certainly a problem that we would expect any active participant to solve correctly, assuming that they take it seriously. Finding a solution does not require any "mathematical knowledge", just a good eye and some concentration.

Problem 15 (4 points)

A full glass of water weighs 400 grams. An empty glass weighs 100 grams. How much does a half-full glass of water weigh?



(A) 150 g (B) 200 g (C) 225 g (D) 250 g (E) 300 g

Solution: This is a four-point problem, which is reflected in the fact that finding a solution is not just a one-step process. In order to solve this, we must note that the water in the full glass weighs $400 - 100 = 300$ g, and that the water in the half-full glass therefore weighs $\frac{1}{2} \cdot 300 = 150$ g. Adding this to the 100 g of the empty glass gives us $150 + 100 = 250$ g for the half-full glass, and therefore the correct answer (D).

Another way to solve this would be to take the average of the two measures, but averages are not things students of this age will typically think of. Of course, a very natural mistake for a student to make if he or she does not think about the problem very much, would be to say that a half-full glass weighs half as much as a full one, and therefore weighs 200 g. If this problem were to be used in the three-point section, this answer would typically not be included among the distractors, as the mistake involved will be made by many of the more distracted participants. In the four-point section, they are advised to be more careful, however.

A problem of this type should not be too difficult, but for some participants the time available in the competition may not be enough to solve all problems that would be accessible for them in principle. We would not expect everyone to solve this, although it should be possible for all students to understand the solution quite readily if it is explained to them.

Problem 21 (5 points)

Exactly 15 animals live on a farm, namely cows, cats and kangaroos. We know that exactly 10 animals are not cows and exactly 8 animals are not cats. How many kangaroos live on the farm?

(A) 2 (B) 3 (C) 4 (D) 10 (E) 18

Solution: As one of the last five-point problems, this one is meant to be a bit tricky. As was the case with the four-point problem presented, we would expect all participants to readily understand the solution if it was shown to them, but they may not find it on their own in the time they are given for the contest.

Of course, the solution comes down to opposites. If we know that 10 out of 15 animals are not cows, we also know that $15 - 10 = 5$ of them are. Similarly, if we know that 8 out of 15 animals are not cats, we know that $15 - 8 = 7$ of them are. Since there are 5 cows and 7 cats among the 15 animals, this leaves a total of $15 - 5 - 7 = 3$ kangaroos, and the correct answer is (B).

As becomes clear immediately, even the "hardest" problems in the Ecolier group are quite do-able, and it is not unusual to have a fairly large number of perfect scores in this category. This fits quite well with the stated intention of popularisation, of course.

6.2 Cadet 2019

Problem 6 (3 points)

Five friends get together. Each of them gives a cupcake to each of the others. They then eat all of the cupcakes that they have been given. As a result, the total number of cupcakes they originally had decreases by half. How many cupcakes did the five friends originally have?

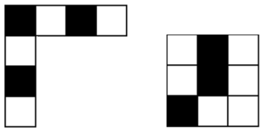
(A) 20 (B) 24 (C) 30 (D) 40 (E) 60

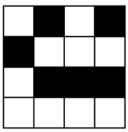
Solution: Since each of the five friends receives a cupcake from each of the four others, the total number of cupcakes eaten is equal to $5 \cdot 4 = 20$. Since this is half the number of cupcakes the friends had originally, the number they had before they started their feast was equal to $2 \cdot 20 = 40$, and the correct answer is (D).

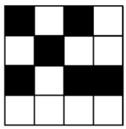
Note that this three-point problem is not really a one-step problem as should normally be the case. For this problem, it was decided that the two steps are both so tiny that they can essentially be treated as a single one. Anyone thinking logically about this and reading the question carefully should get this right.

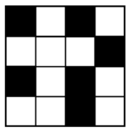
Problem 13 (4 points)

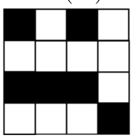
Which of the tiles shown cannot be formed by combining the two given pieces?

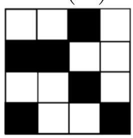


(A) 

(B) 

(C) 

(D) 

(E) 

Solution: This is a really nice puzzle. Three of the tiles are quite easy to form, namely the three in the middle, (B), (C) and (D). Each of these leaves the L-shaped piece in the orientation shown, and all it requires is to rotate the small square in an appropriate way. (That is, to get (B), it is rotated by 90° in a clockwise direction, for (C) by 180° and for (D) by 270°.) That leaves cases (A) and (E) to consider. Tile (E) is not possible, as there is no black corner which is two squares away from black squares in both directions, as must be the case with the black corner of the L-shaped piece. The correct answer is therefore certainly (E), but how can (A) be formed? In order to do this, both pieces must be rotated by 90° in a clockwise direction, and then placed together. This is not so obvious to see, and that makes this problem quite a bit harder for many students.

This is a nice medium-hard problem, well suited for four points. The answer cannot be found with a single step (this is a truism for problems asking which case out of several is *not* possible), but there is no formal mathematics and no calculation involved in the solution. Of course, competitors can draw the various cases, or rotate their papers to their hearts' content.

Problem 30 (5 points, very last problem)

A train is made up of 18 carriages. There are 700 passengers travelling on the train. In any block of five adjacent carriages, there are 199 passengers in total. How many passengers are in the middle two carriages of the train?

(A) 70 (B) 77 (C) 78 (D) 96 (E) 103

Solution: This is a very hard problem, and a very nice one.

As the last problem on the paper, number 30 is often chosen as a grand separator, with the intent of only being accessible to very few of the participants. Such problems are generally avoided at the very young age-levels, but prize winners in the upper age-levels should be given a chance to earn their laurels. On the other hand, such problems should also be very nice, not just hard and dull, but hard and interesting. This problem certainly fits the bill.

So, how do we solve it? At first glance, it is not obvious that the situation described is at all possible. And even if it is, we do not know whether the solution is unique.

First of all, let us assume that a solution does exist. For a competitor, this is certainly a legitimate assumption. If there were no solution, the problem would not be on the paper. We do not know what the number of people in each of the carriages is, but we can start out by letting a, b, c, d and e denote the numbers in the first five carriages, respectively. Since the sum in any five adjacent carriages is always the same, the number of passngers in the sixth must be the same as the number in the first, and this must therefore once again be a. By the same argument, the number in the seventh must be b, the number in the eighth must be c, and so on. For the 18 carriages, we get the following numbers:

$$a, b, c, d, e, a, b, c, d, e, a, b, c, d, e, a, b, c$$

Now, the magic of elementary algebra will do the rest of the work for us. The total number of passengers on the train is equal to 700, and we therefore have

$$4a + 4b + 4c + 3d + 3e = 700.$$

Since the number in five adjacent carriages is always equal to 199, we have

$$a + b + c + d + e = 199,$$

which we can multply by 4 to obtain

$$4a + 4b + 4c + 4d + 4e = 796.$$

Subtracting the first equation from this one gives us $d + e = 96$, and this is precisely the number of passengers in the two middle carriages. The correct answer is therefore (D).

Now, it is also easy to see that such a situation is indeed possible. Numbers d and e can be chosen in any way, such that $d + e = 96$ holds. If we then choose a, b and c in such a way that $a + b + c = 101$ holds, all conditions of the problem are certainly met. The situation is certainly possible, and the solution is, perhaps somewhat surprisingly, unique.

7 Conclusion

We would like to conclude this article with a few summarising words formulated as answers to questions you may have had before reading this article.

- What is Math Kangaroo? Math Kangaroo is the world's largest international math classroom competition.
- What is the goal of Math Kangaroo? Math Kangaroo helps to popularize math between students, their families, their teachers and within the society.
- Why should my students participate in Math Kangaroo? Because the problems are original and fun as they are the product of the collective thinking of a very experienced group of teachers, mathematicians and math educators from different cultures all over the world.
- Why should I, as a teacher, participate with my class in Math Kangaroo? Because the Math Kangaroo problem collection gives teachers around the world an extraordinary tool to improve their teaching and these problems allow the teachers to present Maths in a recreational way.
- Why should society care about Math Kangaroo? Because the Math Kangaroo problems motivate mathematical logical thinking from an early age, which is an absolute necessity for being succesful in today's society.

We hope to have given you an insight in the association KSF which organises the annual Math Kangaroo competition and a better understanding of the structure and the philosophy of the competition. Feel free to contact us with questions, comments or if you wish to join!

References

[1] Website AKSF, http://www.aksf.org

[2] AKSF selected problem set for the competition 2019 at the annual meeting in Vilnius, Lithuania 2018.

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Problems relating Geometry and Number Theory

Robert Bosch (bobbydrg@gmail.com)

Robert Bosch was Coach and Coordinator of Mathematics Competitions during the period (2014-2016) at Archimedean Schools, Miami, FL, USA, culminating with the 2016 Edyth May Sliffe Award for Distinguished Teaching from the Mathematical Association of America (MAA). Involved in problem solving since 2000, he won a Bronze Medal in VIII Iberoamerican Mathematical Olympiad for University Students. The author of 20 articles and 5 books, he received his B.Sc. in Mathematics from the University of Havana, Cuba. Robert has given lectures in different math circles, universities, and summer programs, such as the Metroplex Math Circle, Awesome Math Summer Program, IDEA Math Summer Program, University of Texas at Dallas, Florida Atlantic University, and University of Chicago (USA).

Abstract
We present three problems and their solutions relating Geometry and Number Theory. We refer to problem 6 from the International Mathematical Olympiad (IMO) 2001, proposed by A. Ivanov from Bulgaria, where an alternative solution by Law of Cosines and Ptolemy's theorem is shown. We also explore problem 9 from Cuban Math Olympiad 2007, and finally an original problem of the author involving angle bisectors, collinearity and divisibility by 6.

1 Introduction

One of the characteristics of many math Olympiad problems is the way in which they target unsuspected or unexplored links between different fields of elementary mathematics. In this article we pursue problems sparked by relations between geometry and number theory.

2 Problems

Problem 1. [1].

Let a, b, c, d be integers with $a > b > c > d > 0$. Suppose that

$$ac + bd = (b + d + a - c)(b + d - a + c).$$

Prove that $ab + cd$ is not prime.

Solution:

The equality

$$ac + bd = (b + d + a - c)(b + d - a + c)$$

is equivalent to

$$a^2 - ac + c^2 = b^2 + bd + d^2. \tag{1}$$

Let $ABCD$ be the quadrilateral with $AB = a, BC = d, CD = b, AD = c$, $\measuredangle BAD = 60^\circ$, and $\measuredangle BCD = 120^\circ$. Such a quadrilateral exists in view of (1) and the Law of Cosines. Let $\measuredangle ABC = \alpha$, so that $\measuredangle CDA = 180^\circ - \alpha$. Applying the Law of Cosines to triangles ABC and ACD gives

$$a^2 + d^2 - 2ad\cos\alpha = AC^2 = b^2 + c^2 + 2bc\cos\alpha.$$

Hence

$$2\cos\alpha = \frac{(a^2 + d^2 - b^2 - c^2)}{(ad + bc)},$$

and

$$AC^2 = a^2 + d^2 - ad \cdot \frac{a^2 + d^2 - b^2 - c^2}{ad + bc} = \frac{(ab + cd)(ac + bd)}{ad + bc}.$$

Because $ABCD$ is cyclic, Ptolemy's theorem gives

$$(AC \cdot BD)^2 = (ab + cd)^2.$$

It follows that

$$(ac + bd)(a^2 - ac + c^2) = (ab + cd)(ad + bc). \tag{2}$$

Next, observe that

$$ab + cd > ac + bd > ad + bc. \tag{3}$$

The first inequality follows from $(a-d)(b-c)>0$, and the second from $(a-b)(c-d)>0$. Now assume that $ab+cd$ is prime. It then follows from (3) that $ab+cd$ and $ac+bd$ are relatively prime. Hence, from (2), it must be true that $ac+bd$ divides $ad+bc$. However, this is impossible by (3). Thus $ab+cd$ must not be prime.

Note. Examples of 4−tuples (a,b,c,d) that satisfy the given conditions are $(21,18,14,1)$ and $(65,50,34,11)$.

Problem 2. [3].

Let O be the circumcenter of a triangle ABC, with $AC=BC$. The line AO intersects BC in D. If BD and CD are integers and $AO-CD$ is a prime number, find these three numbers $(BD, CD, AO-CD)$.

Solution:

Let H be on AB such that $CH \perp AB$. Let $AO=R$, $BD=x$, $DC=y$, $AO-CD=p$, $\measuredangle OAC=\measuredangle OCA=\alpha$. Applying the

Law of Sines to $\triangle AOC$ and $\triangle ACD$ we get

$$\frac{R}{\sin\alpha} = \frac{x+y}{\sin 2\alpha} \quad \Rightarrow \quad \cos\alpha = \frac{x+y}{2R},$$
$$\frac{AD}{\sin 2\alpha} = \frac{y}{\sin\alpha} \quad \Rightarrow \quad AD = 2y\cos\alpha = \frac{y(x+y)}{R}.$$

By the Angle Bisector Theorem applied to $\triangle ACD$, we have

$$\frac{AO}{OD} = \frac{x+y}{y} \quad \Rightarrow \quad \frac{AD}{AO} = \frac{x+2y}{x+y} \quad \Rightarrow \quad AD = R\cdot\frac{x+2y}{x+y}.$$

Therefore

$$\frac{y(x+y)}{R} = R\cdot\frac{x+2y}{x+y} \quad \Rightarrow \quad R^2 = \frac{y(x+y)^2}{x+2y}.$$

This equation can be written as a quadratic in x:

$$yx^2 + (2y^2 - R^2)x + y^3 - 2R^2y = 0 \tag{4}$$

with discriminant

$$\Delta_1 = (2y^2 - R^2)^2 - 4y(y^3 - 2yR^2) = R^2(4y^2 + R^2).$$

To have integer solutions in the quadratic (4), we need Δ_1 to be a perfect square, i.e., $(2y)^2 + R^2 = z^2$. The solution to this last equation is the Pythagorean triple

$$\begin{aligned} y &= mnv \\ R &= v(m^2 - n^2) \\ z &= v(m^2 + n^2) \end{aligned}$$

where m, n are relatively prime and of different parity. It follows that $\Delta_1 = v^4(m^4 - n^4)^2$, and therefore the solutions to (4) are

$$\begin{aligned} x_1 &= \frac{vm(m^2 - 2n^2)}{n}, \\ x_2 &= \frac{vn(n^2 - 2m^2)}{m}. \end{aligned}$$

Note that $R > 0$, so $m > n$, and $x_2 < 0$. Therefore $x = x_1$.

The condition $R - y = p$ is written as $v(m^2 - mn - n^2) = p$. If $v = 1$, then $n \mid m^2 - 2n^2$, and hence $n \mid m^2$. This is impossible because m, n are relatively prime. Therefore $m^2 - mn - n^2 = 1$ and $v = p$. Since $n \mid vm(m^2 - 2n^2)$, $\gcd(m, n) = 1$, and $\gcd(n, m^2 - 2n^2) = \gcd(n, m^2) = 1$, then $n \mid v$. Therefore $n = 1$ or $n = p$. If $n = 1$, then $m = 2$ which generates the solution $(x, y, p) = (4p, 2p, p)$. However, if $x = 4p$ and $y = 2p$, then $R = 3p$. But by triangle inequality $R + R > x + y$, so $6p > 6p$, which is a contradiction. Therefore $n = 1$ does not yield a solution and hence $n = p$.

If $n = v = p$ we have

$$\begin{aligned} x &= m(m^2 - 2p^2) = m((m^2 - p^2) - p^2) = m(mp + 1 - p^2), \\ y &= p^2 m, \\ R &= p(m^2 - p^2) = p(pm + 1) = p^2 m + p. \end{aligned}$$

Since $m^2-mn-n^2=1$ and $n=p$, we have $m^2-pm-(p^2+1)=0$. This can be viewed as a quadratic in m with discriminant $\Delta_2 = 5p^2+4$. Therefore we need to find p such that $5p^2+4=w^2$, which implies $5p^2=w^2-4=(w-2)(w+2)$. Then w must be of the form $5k+2$ or $5k+3$. If $w=5k+2$, then $p^2=k(5k+4)$. Since $5k+4>k$, then $k=1$ and $5k+4=p^2$, so $p=3$. If $w=5k+3$, then $p^2=(k+1)(5k+1)$. But $5k+1>k+1$, so $k+1=1$, and $k=0$. Then $w=3$, so $p=1$. But 1 is not prime. Therefore the solution is $p=3$ with $x=35$ and $y=45$. So that $(x,y,p)=(35,45,3)$.

Problem 3.

In triangle ABC, let ℓ_a and ℓ_b be the angle bisectors from the vertices A and B respectively. Consider the extensions of ℓ_a and ℓ_b to points A' and B' respectively, such that $AA'=2\ell_a$, and $BB'=2\ell_b$. Show that there are infinitely many integer sided non-equilateral triangles ABC where the points A',C,B' are collinear and also the perimeter is divisible by 6.

Solution:

Denote by a, b, c the sides opposite to the vertices A, B, C respectively. We proceed by barycentric coordinates to find an equivalent condition depending on a, b, c for the collinearity of the points A', C, B'. Let

$$A(1,0,0), \quad B(0,1,0), \quad C(0,0,1).$$

Denote by X the intersection point of ℓ_a and BC. By the Angle Bisector Theorem we know that $\frac{BX}{XC} = \frac{c}{b}$. Thus X has coordinates $\left(0, \frac{b}{b+c}, \frac{c}{b+c}\right)$. This is the midpoint of the segment AA', hence

$$A'\left(-1, \frac{2b}{b+c}, \frac{2c}{b+c}\right).$$

In a similar manner we obtain

$$B'\left(\frac{2a}{a+c}, -1, \frac{2c}{a+c}\right).$$

So, the points A', C, B' are collinear if and only if

$$\begin{vmatrix} -1 & \frac{2b}{b+c} & \frac{2c}{b+c} \\ 0 & 0 & 1 \\ \frac{2a}{a+c} & -1 & \frac{2c}{a+c} \end{vmatrix} = 0.$$

From the second row this determinant is $(a+c)(b+c) = 4ab$. This is equivalent to $c(a+b+c) = 3ab$. We will find positive integer solutions to the equation

$$c^2 + (a+b)c - 3ab = 0. \tag{5}$$

The discriminant of this quadratic equation should be a perfect square, obtaining

$$a^2 + 14ab + b^2 = x^2. \tag{6}$$

This Diophantine equation has been studied extensively, [2],

$$a = 14n^2 - 2mn, \quad b = m^2 - n^2$$

satisfies (6) since

$$\begin{aligned}&(14n^2 - 2mn)^2 + 14(14n^2 - 2mn)(m^2 - n^2) + (m^2 - n^2)^2\\&= n^4 - 28mn^3 + 198m^2n^2 - 28m^3n + m^4,\\&= (m^2 - 14mn + n^2)^2.\end{aligned}$$

Thus,

$$c = 8mn - 7n^2 - m^2.$$

It is simple to verify that the triple (a, b, c) satisfies equation (5). It remains to find conditions for the positive integer parameters m, n such that a, b, c are the sides of triangle ABC. From a, b, c being positive numbers and also from the Triangle Inequality we obtain that $\frac{11}{5} < \frac{m}{n} < 4$. Clearly, the perimeter $a + b + c$ is $6n^2 + 6mn$, divisible by 6. This completes the proof.

A few such triangles are

$$(36, 21, 27)$$
$$(28, 45, 35)$$
$$(84, 40, 56)$$
$$(78, 55, 65)$$
$$(152, 65, 95)$$
$$(66, 91, 77)$$
$$(144, 84, 108).$$

Comment: There are many triangles with the given property and perimeter **not** divisible by 6, for example, $(7, 12, 9)$ and $(10, 21, 14)$.

3 New solutions to $c^2 + (a+b)c - 3ab = 0$

Our intention in this section is to show new solutions to the Diophantine equation

$$c^2 + (a+b)c - 3ab = 0, \tag{7}$$

with the property $6 \mid a+b+c$. Unfortunately, the resulting a, b, c are not necessarily sides of a triangle. The problem of finding *all* positive solutions remains open for us, maybe the reader will be able to solve it.

The discriminant of this quadratic equation should be a perfect square, obtaining

$$a^2 + 14ab + b^2 = x^2. \tag{8}$$

This new equation can be rewritten as

$$x^2 + 3(4b)^2 = (a + 7b)^2,$$

and denoting $y = 4b$, and $z = a + 7b$, this leads to solving the Diophantine equation

$$x^2 + 3y^2 = z^2. \tag{9}$$

Since we are only interested in finding infinitely many solutions, and not necessarily all of them, we divide by z^2, and we denote $u = \frac{x}{z}, v = \frac{y}{z}$. Now, the equation becomes

$$u^2 + 3v^2 = 1,$$

and we are looking for rational solutions, i.e. rational points on the ellipse. With this aim, we consider a line with rational slope passing through the point $(1, 0)$. That is to say, $v = \alpha u - \alpha$, where α is a rational number. The result of intersecting this line with the conic is

$$u = \frac{3\alpha^2 - 1}{3\alpha^2 + 1}, \quad v = \frac{-2\alpha}{3\alpha^2 + 1}.$$

Replacing α by $-\alpha$, the following triple (x, y, z) is solution to equation (9).

$$\begin{aligned} x &= 3\alpha^2 - 1, \\ y &= 2\alpha, \\ z &= 3\alpha^2 + 1. \end{aligned}$$

Setting $\alpha = 4k$, the triple can be rewriten as

$$\begin{aligned} x &= 48k^2 - 1, \\ y &= 8k, \\ z &= 48k^2 + 1. \end{aligned}$$

Now, working back, we obtain infinitely many solutions a, b, c to equation (7). Namely,

$$\begin{aligned} a &= 48k^2 - 14k + 1, \\ b &= 2k, \\ c &= 6k - 1. \end{aligned}$$

Where k is a positive integer. Clearly $a + b + c = 48k^2 - 6k = 6k(8k - 1)$.

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Remembering Ronald Lewis Graham

October 31, 1935 – July 6, 2020

From the editor...

WFNMC has been truly fortunate in having highly distinguished mathematicians as keynote speakers at their federation congresses, among them Paul Erdős, John Conway and Ron Graham. We were privileged to listen to Ron speak at our VII Congress in Barranquilla, Colombia in 2014.

According to the American Mathematical Society (AMS), which he served as president, Ron Graham was "one of the principal architects of the rapid development worldwide of discrete mathematics in recent years."

His friendship with Paul Erdős gave the mathematics world almost 30 coauthored papers. It also led to Ron's conception in 1979 of the Erdős number, or number of degrees a mathematician was distanced from Erdős as co-author. As a direct co-author, Ron's Erdős number, of course, was 1. Ron was also known for the Graham number, given in 1977 as an approximate solution to a problem of Ramsey Theory, and noted for a long time in the Guinness Book of Records.

Ron Graham was a close friend of Alexander Soifer, past president of WFNMC, and they jointly attempted to get Paul Erdős to publish a book of his open problems. When asked to help us remember Ron Graham, Alexander Soifer wrote:

"Ron was certainly among the most influential mathematicians of the U.S. He served as President of the American Mathematical Society, President of the Mathematical Association of America, member of the American Academy of Arts and Sciences, member of the Hungarian Academy of Sciences, Chief Scientist of Bell Labs, Endowed Chair at the University of California San Diego, etc., etc. Ron was not only a member of the National Academy of Sciences – he served two terms as its Treasurer. He showed me once his impressive Treasurer's office at the Academy, across a narrow street from the U.S. Department of State. All these accolades did not spoil Ron's personality. He has always been open, friendly, curious, generous to a fault, sprinkling our conversations with lovely humor.

I remember him serving as the Chair of the Jury of the 2001 International Mathematical Olympiad in Washington, D.C.; for representatives of about 90 countries Ron was an easy going, humorous, yet principled shepherd.

Paul Erdős "preached" (his term) to professionals, young and old. His open problems inspired generations. But he was not a natural lecturer, whereas Ron certainly was. His lectures were well composed – as pieces of music – elegant, deep, inspiring, yet lightened by humor. Ron excited every audience, from professional mathematicians to high school students who came to MIT to compete in the USA Mathematical Olympiad.

In 2014, Ron gave a brilliant keynote address at the Congress of World Federation of National Mathematics Competitions (WFNMC) in Barranquilla, Colombia. All his audiences got their share of open problems and conjectures, some with cash prizes for first solutions.

WFNMC Keynote speakers Ronald Graham[1]
and Alexander Soifer, Barranquilla, July 22, 2014

Ron's interests were not limited to mathematics and computer science. He was a fine gymnast, ping pong player," unicyclist and professional-level juggler with a passion for sports.

Peter Taylor also recalled Ron Graham as Chairman of the IMO Jury in Washington in 2001, and spoke of the outstanding job he did, amid difficulty. Recalling the help Ron as president of the MAA had given him, Peter mentioned that Ron would, in his view, be the "most impressive mathematics manager" as well a great mathematician.

Ronald Graham was one of the leading mathematicians in the field of Ramsey Theory, yet he was never far from the goal that WFNMC has upheld of encouraging and aiding young mathematicians to develop their talent. He will be sorely missed by people intrigued with problem solving worldwide.

(See another photo of Ron and learn more about his support of solvers of open problems in Alexander Soifer's article in this issue of *Mathematics Competitions*.)

International Mathematics Tournament of Towns

Andy Liu

Andy Liu is a Canadian mathematician. He is a professor emeritus in the Department of Mathematical and Statistical Sciences at the University of Alberta. Liu attended New Method College in Hong Kong.He then did his undergraduate studies in mathematics at McGill University, and earned his Ph.D. in 1976 from the University of Alberta, under the supervision of Harvey Abbott, with a dissertation about hypergraphs. He was the leader of the Canadian team at the International Mathematical Olympiad in 2000 (South Korea) and 2003 (Japan) and acts as vice-president of the Tournament of Towns.

Selected Problems from the Fall 2019 Papers

1 An illusionist lays the 52 cards of a standard deck in a row. In each step, the audience chooses an integer k not greater than the length of the row, and the illusionist removes either the kth card from the left or the k card from the right. The illusionist announces in advance that the Three of Clubs should be the last card which remains. For which initial positions of the Three of Clubs can the illusionist guarantee the success of the trick?

Solution:

We call the first position from either end an outside position, and every other position an inside position. Suppose the Three of Clubs starts in an outside position. If the audience chooses an inside position, the illusionist removes the corresponding card counting from either end. If the audience chooses an outside position, the illusionist removes the card from the end opposite to the Three of Clubs. After each step, the Three of Clubs remains in an ouside position. Eventually it will be the only card which remains. Suppose the Three of Clubs starts in an inside position and the audience keeps choosing inside positions. The two cards which start in the outside positions will remain there. So any card which starts in an inside position, including the Three of Club, will not be the last card which remains.

2. In each step, we may multiply a positive integer by 3 and then add 1 to the product. If the positive integer that results is even, we may divide it by 2. If the positive integer that results is odd, we may subtract 1 from it and then divide the difference by 2. Prove that starting with 1, we can obtain any positive integer in a finite number of steps.

Solution:

In the following table, the first row consists of the positive integers in order. The second row consists of the positive integers congruent modulo 3 to 1, in increasing order starting from 4. The third row consists of the positive integers alternatingly congruent modulo 3 to 2 and 0, in increasing order starting from 2 and 3 respectively. The table extends indefinitely to the right.

1	2	3	4	5	6	7	8	9	⋯
4	7	10	13	16	19	22	25	28	⋯
2	3	5	6	8	9	11	12	14	⋯

Note that each positive integer in the second row fills a gap in the third row which occurs to its right. In each column, the number in the second row is generated by the number in the first row, and the number in the third row is generated by the number in the second row. Since we start with 1, we can obtain all three numbers in the first column, in particular, the 2 in the third row. This allows us to obtain all three numbers in the second column, in particular, the 3 in the third row. This allows us to obtain all three numbers in the third column. The number in the third row is 5, skipping over the gap 4. However, as we have pointed out, the number 4 has already appeared. In this manner, we can obtain all three numbers in every column. Since the first row consists of all the positive integers, we have the desired result.

3. *ABC* is an acute triangle with area *S*. *K* is a point inside *ABC* while *L* and *M* are points on *BC* such that *KLM* is also an acute triangle, with area S'. Prove that $\frac{S}{AB+AC} > \frac{S'}{KL+KM}$.

Solution:

Reflect A and K to D and N respectively across the line BC. Since ABC is acute, the kite $ABDC$ has an incircle, with radius r. Similarly, the kite $KLNM$ also has an incircle, with radius $r' < r$ since it is inside $ABDC$. The area of $ABDC$ is $2S = r(AB + AC)$ and the area of $KLNM$ is $2S' = r'(KL + KM)$.

Hence $\frac{S}{AB+AC} = \frac{r}{2} > \frac{r'}{2} = \frac{S'}{KL+KM}$.

4. Counters numbered 1 to 100 are arranged in order in a row. It costs 1 dollar to interchange two adjacent counters, but nothing to interchange two counters with exactly k other counters between them. What is the minimum cost for rearranging the 100 counters in reverse order if

(a) $k = 3$;

(b) $k = 4$?

Solution:

(a) Paint the 100 positions red, yellow, blue, green, red, yellow, blue, green, and so on. Then two counters on positions of the same colour can be interchanged for free, while two counters in positions of adjacent colours can be interchanged for 1 dollar. If two counters on positions of adjacent colours are not adjacent themselves, we can bring them next to each other using only free moves. All counters in red positions must go to green positions, and vice versa. All counters in yellow positions must go to blue positions, and vice versa. Since red and green are adjacent colours, we can spend 25 dollars to interchange the 50 counters on them. Similarly, the 50 counters on yellow and blue positions can be interchanged for another 25 dollars, for a total cost of 50 dollars. We now use free moves to put every counter in its correct position. Since 100 counters must change the colours of their positions, and 1 dollar can only pay for 2 such changes, 50 dollars is the minimum cost.

(b) Paint the 100 positions red, yellow, brown, blue, green, red, yellow, brown, blue, green, and so on. Then two counters in positions of the same colour can be interchanged for free, while two counters in positions of adjacent colours can be interchanged for 1 dollar. If two counters on positions of adjacent colours are not adjacent themselves, we can bring them next to each other using only free moves. All counters in red positions must go to green positions, and vice versa. All counters in yellow positions must go to blue positions, and vice versa. All counters on brown positions must stay in brown positions. Since red and green are adjacent colours, we can spend 20 dollars to interchange the 40 counters on them. Yellow and brown are adjacent colours, as are blue and brown. We can spend 40 dollars to interchange the following pairs of counters: (2,3), (4,2), (7,4), (9,7), (12,9), (14,12), (17,14), (19,17), …, (92,89), (94,92), (97,94) and (99,97). Now every counter is in a position of the correct colour, except that counter 3 is in a yellow position while counter 99 is in a brown position. Since yellow and brown are adjacent colours, these two counters can be interchanged for 1 dollar, bringing the total cost to 61 dollars. We now use free moves to put every counter in its correct position. The 40 counters in red and green positions must change the colours of their positions. Since 1 dollar can only pay for 2 such changes, 20 dollars are required. The 40 counters in yellow and blue positions must also change the colour of their positions. Since yellow and blue are not adjacent colours, 40 dollars are required. However, these changes cannot be made without involving at least one counter in a brown position. It follows that 61 dollars is the minimum cost.

5. The weight of each of 100 coins is unknown, but is one of 1 gram, 2 grams or 3 grams, and there is at least one of each kind. Show how the weight of each coin can be determined using at most 101 weighings on a balance.

Solution:

More generally, we prove that $n+1$ weighings are sufficient for $n\geq 3$ coins. The key step is identifying one coin of weight 2 grams or two coins with total weight 2 grams. The weight of each untested coin can then be determined in one weighing. We use induction on

n. For $n = 3$, we have one coin of each weight. Comparing them pairwise takes only 3 weighings. Suppose the result holds for some $n \geq 3$. Consider now $n+1$ coins. In the preliminary stage, compare coin A with the others one at a time, until equilibrium is not achieved for the first time. We may assume that this happens on the first weighing, with coin B, as otherwise we can apply the inductive hypothesis. By symmetry, we may assume that $A<B$. In the second weighing, we compare B with coin C. If there is equilibrium, the inductive hypothesis applies again. If $B<C$, then A is of weight 1 gram, C is of weight 3 grams, and we have identified B as a coin of weight 2 grams. Henceforth, we assume that $B>C$. In the third weighing, we compare A with C. If equilibrium is not achieved, B is of weight 3 grams, the lighter of A and C is of weight 1 grams, and we have identified the heavier one as a coin of weight 2 grams. Suppose $A=C$. In the fourth weighing, we compare $A+C$ with B. If $A+C=B$, then each of A and C is of weight 1 gram, and we have identified B as a coin of weight 2 grams. If $A+C<B$, then B is of weight 3 grams and we have identified A and C as two coins with total weight 2 grams. If $A+C>B$, then B is of weight 1 gram, and we have identified each of A and C as a coin of weight 2 grams. Since we have used 4 weighings to determine the weights of A, B and C, we have sufficient weighings left to determine the weights of the remaining coins.

6. Prove that for each positive integer m, there exists at least one integer $n>m$ such that both mn and $(m+1)(n+1)$ are squares of integers.

Solution:

Note that n must be the product of m and the square of an integer. We choose this integer to be a linear function of m, namely $pm+q$ for some positive integers p and q. Then

$$n+1 = m(pm+q)^2 +1 = p^2m^3+2pqm^2+q^2m+1.$$

This must be divisible by $m+1$, and the quotient must be the square of an integer. We perform the following long division.

$$
\begin{array}{rrrrr}
 & & p^2m^2 & +p(2q-p)m & +(q^2-2pq+p^2) \\
\cline{2-5}
m+1 &)p^2m^3 & +2pqm^2 & +q^2m & +1 \\
 & p^2m^3 & +p^2m^2 & & \\
\cline{2-4}
 & & p(2q-p)m^2 & +q^2m & \\
 & & p(2q-p)m^2 & +p(2q-p)m & \\
\cline{3-5}
 & & & (q^2-2pq+p^2)m & +1 \\
 & & & (q^2-2pq+q^2)m & +(q^2-2pq+p^2) \\
\cline{4-5}
\end{array}
$$

It follows that $q^2-2pq+p^2= 1$, so that $|p-q|=1$. Now the quotient $p^2m^2+p(2q-p)m+1$ must be the square of $pm+1$. Hence $2q-p=2$. Combined with $q-p=1$, we have $q=1$ and $p=0$, which is not acceptable. Combined with $p-q=1$, we have $q=3$ and $p=4$.

For any positive integer m, we can choose $n = m(4m + 3)^2$.

Then $mn = (m(4m+3))^2$ and $n+1 = 16m^3+24m^2+9m+1 = (m+1)(4m+1)^2$. Hence $(m+1)(n+1) = ((m+1)(4m+1))^2$.

7. Peter has an $n \times n$ stamp, n>10, such that 102 of the unit squares are coated with black ink. He presses this stamp 100 times on a 101×101 grid, each time leaving a black imprint on 102 unit squares of the grid. Is it possible that the grid is black except for one unit square at a corner?

Solution:

Remove the square at the intersection of the first row and the first column. Shade the rest of the first column but leave the rest of the first row unshaded. Divide the remaining part of the grid into four 50×50 quadrants and shade the second and the fourth ones. Then the shaded regions can be mapped into the unshaded regions by a 90° rotation. Peter's stamp is 101×101, the same size as the grid. The inked squares consist of the same row of squares in the two shaded quadrants, along with the shaded squares in the same rows in the first column. By shifting the stamp up and down, Peter can make all shaded squares black. Then he can make the unshaded squares black by rotating the stamp 90°. The diagram below illustrates with a 9 × 9 grid.

	8	7	6	5	8	7	6	5
1	1	1	1	1	8	7	6	5
2	2	2	2	2	8	7	6	5
3	3	3	3	3	8	7	6	5
4	4	4	4	4	8	7	6	5
1	8	7	6	5	1	1	1	1
2	8	7	6	5	2	2	2	2
3	8	7	6	5	3	3	3	3
4	8	7	6	5	4	4	4	4

8. A cube consisting of $(2n)^3$ unit cubes is pierced by several needles parallel to the edges of the cube, each piercing exactly $2n$ unit cubes. Each unit cube is pierced by at least one needle. A subset of these needles is regular if there are no two needles in the subset that pierce the same unit cube.

(a) Prove that there exists a regular subset consisting of $2n^2$ needles such that all of them have either the same direction or two different directions.

(b) What is the maximum size of a regular subset that is guaranteed to exist?

Solution:

(a) Call the needles x-needles, y-needles and z-needles according to their directions. Take the larger of the numbers of x-needles and y-needles in each $2n \times 2n$ xy-layer, the larger of the numbers of y-needles and z-needles in each $2n \times 2n$ yz-layer, and the larger of the numbers of z-needles and x-needles in each $2n \times 2n$ zx-layer. Let k be the minimum of all these $6n$ maxima. Consider the layer where the maximum is k. We may assume that it is an xy-layer. It contains $2n-k$ rows and $2n-k$ columns free of x-needles and y-needles. The $(2n-k)^2$ unit cubes at their intersection must pierced with z-needles. Paint these z-needles red. There are exactly k yz-layers which do not contain red needles. In each such layer, we can choose at least k y-needles, and the total from these k layers is at least k^2 y-needles. Add them to the red needles, and we have a regular subset of size

$$k^2+(2n-k)^2=k^2+4n^2-4nk+2k^2=2n^2+2(n-k)^2\geq 2n^2.$$

(b) By (a), a regular subset of $2n^2$ needles is guaranteed to exist. We now construct an example in which the maximum regular subset consists of exactly $2n^2$ needles, so that this is the desired value. Divide the $2n \times 2n \times 2n$ cube into eight $n \times n \times n$ cubes. Pierce with needles in each of the three directions all n^3 unit cubes in the southwest $n \times n \times n$ cube in the bottom layer, as well as all n^3 unit cubes in the northeast $n \times n \times n$ cube in the top layer. Then every unit cube in the $2n \times 2n \times 2n$ cube is pierced by at least one of the $6n^2$ needles. To obtain a regular subset, we must remove at least 2 needles from each of the $2n^3$ cubes that are being pierced in all three directions. This means the removal of at least $4n^2$ needles, leaving behind the desired regular subset of size $2n^2$.

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