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MATHEMATICS COMPETITIONS

JOURNAL OF THE

WORLD FEDERATION OF NATIONAL MATHEMATICS COMPETITIONS

AUSTRALIAN MATHS TRUST

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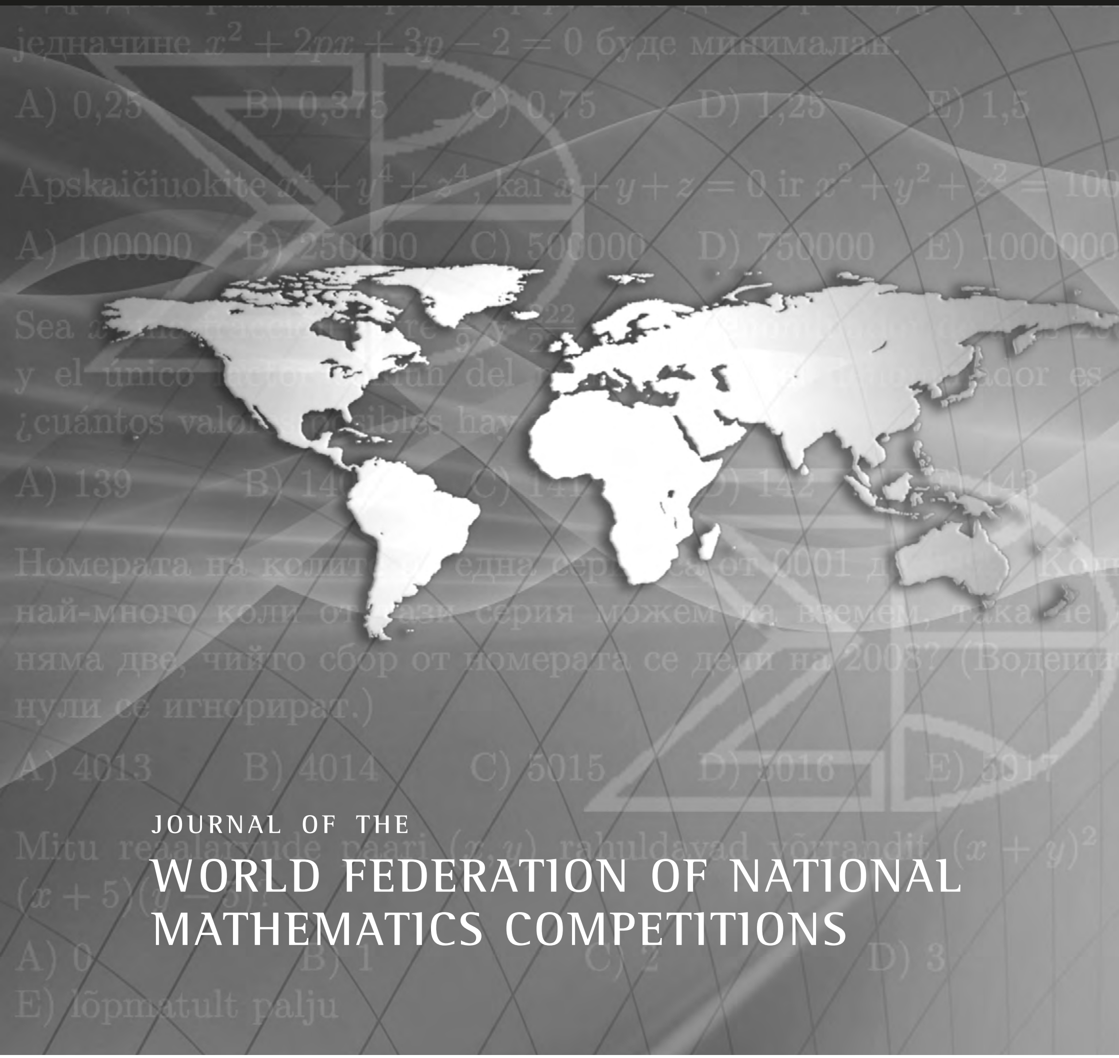

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World Federation of National Mathematics Competitions

Executive

The aims of the Federation are:

1. ***to promote excellence in, and research associated with, mathematics education through the use of school mathematics competitions;***
2. ***to promote meetings and conferences where persons interested in mathematics contests can exchange and develop ideas for use in their countries;***
3. ***to provide opportunities for the exchanging of information for mathematics education through published material, notably through the Journal of the Federation;***
4. ***to recognize through the WFNMC Awards system persons who have made notable contributions to mathematics education through mathematical challenge around the world;***
5. ***to organize assistance provided by countries with developed systems for competitions for countries attempting to develop competitions;***
6. ***to promote mathematics and to encourage young mathematicians.***

WORLD FEDERATION OF NATIONAL MATHEMATICS COMPETITIONS
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From the President

Dear Federalists,

It has been a high honor and responsibility to be your elected President for six years, 2012–2018, from ICME in South Korea to our Austrian Congress. We have had wonderful WFNMC mini-conferences in 2012 and 2016, our great Congresses in 2014 and 2018, organized by Maria Falk de Losada and Ricardo Losada and by Robert Geretschläger respectively. We produced an impressive book 400-page compendium *Competitions for Young Mathematicians: A Perspective from Five Continents*, Springer, Germany, July 2017. A new book of 2018—Graz proceedings is being compiled by Robert, and will be published in 2019. Finally, we were joined by new fine colleagues from around the world, and returned mathematicians from the United Kingdom and Poland to an active participation in the Federation.

Our gratitude goes to the outgoing long-term Editor of *Mathematics Competitions* (MC) Jaroslav Švrček. Starting with the next issue, I will serve as the MC Editor. My Assistant Editor will be Sergey Dorichenko of Russia. So, starting now, please send your articles for *Mathematics Competitions* to me at `asoifer@uccs.edu`. You can find the formatting instructions on the web pages of the Federation: `http://www.wfnmc.org/journalinvandformat.html`.

The future of this journal, founded by Peter O'Halloran in the 1980s, is in your hands—and minds. Make it a success by writing for MC well and often!

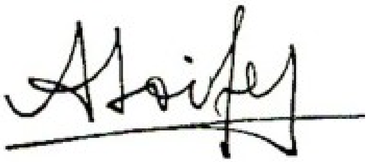
Beyond scholarly exchanges, our Congresses and Competitions Topic Groups and mini-conferences at ICME's have become reunions of old friends, and opportunities to make new ones. I hope to see you all during

ICME in Shanghai-2020 and at WFNMC Congress in Sofia, Bulgaria, in 2022.

I am passing the presidential baton—or is it an Olympic flaming torch?—to the new President Kiril Bankov. I have no doubt Kiril will continue a fine tradition of leadership and transparency established by the Federation's past presidents.

My best wishes to you and yours on a Healthy and Prosperous New Year!

Alexander Soifer
President of WFNMC
Editor of Mathematics Competitions
November 2018



From the Editor

Welcome to *Mathematics Competitions* Vol 31, No 2.

First of all I would like to thank again the Australian Mathematics Trust for continued support, without which each issue of the journal could not be published, and in particular Bernadette Webster, Alexandra Carvajal and Pavel Calábek for their assistance in the preparation of this issue.

Submission of articles:

The journal *Mathematics Competitions* is interested in receiving articles dealing with mathematics competitions, not only at national and international level, but also at regional and primary school level. There are many readers in different countries interested in these different levels of competitions.

- The journal traditionally contains many different kinds of articles, including reports, analyses of competition problems and the presentation of interesting mathematics arising from competition problems. Potential authors are encouraged to submit articles of all kinds.

- To maintain and improve the quality of the journal and its usefulness to those involved in mathematics competitions, all articles are subject to review and comment by one or more competent referees. The precise criteria used will depend on the type of article, but can be summarised by saying that an article accepted must be correct and appropriate, the content accurate and interesting, and, where the focus is mathematical, the mathematics fresh and well presented. This editorial and refereeing process is designed to help improve those articles which deserve to be published.

At the outset, the most important thing is that if you have anything to contribute on any aspect of mathematics competitions at any level, local, regional or national, we would welcome your contribution.

Articles should be submitted in English, with a black and white photograph and a short profile of the author. Alternatively, the article can be submitted on an IBM PC compatible disk or a Macintosh disk. We prefer LaTeX or TeX format of contributions, but any text file will be helpful.

Articles, and correspondence, can also be forwarded to the editor by mail to

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Jaroslav Švrček
February 2019

A History of the Federation through My Eyes

Alexander Soifer

Born and educated in Moscow, Alexander Soifer has for over 38 years been a Professor at the University of Colorado, teaching math, and art and film history. He has published over 300 articles, and a good number of books. In the past several years, 7 of his books have appeared in Springer: The Scholar and the State: In the search of Van der Waerden*;* The Mathematical Coloring Book: Mathematics of Coloring and the Colorful Life of Its Creators*;* Mathematics as Problem Solving*;* How Does One Cut a Triangle?*;* Geometric Etudes in Combinatorial Mathematics*;* Ramsey Theory Yesterday, Today, and Tomorrow*; and* Colorado Mathematical Olympiad and Further Explorations: From the Mountains of Colorado to the Peaks of Mathematics. *He has founded and for 32 years ran the Colorado Mathematical Olympiad. Soifer has also served on the Soviet Union Math Olympiad (1970–1973) and USA Math Olympiad (1996–2005). He has been Secretary of the World Federation of National Mathematics Competitions (WFNMC) (1996–2008), and Senior Vice President of the WFNMC (2008–2012); from 2012 he has been the president of the WFNMC. He is a recipient of the Federation's Paul Erdős Award (2006). Soifer's Erdős number is 1.*

Professor Peter O'Halloran of Australia, Executive Director of the Australian Mathematics Trust, envisioned the World Federation of National Mathematics Competitions.

O'Halloran's *singular vision*, energy and will made the Federation possible. It was born in 1984 in Adelaide, Australia, during the 5th International Congress on Mathematical Education (ICME-5).

O'Halloran led the Federation for 10 years, almost until his passing in 1994. Academician Blagovest H. Sendov of Bulgaria was handpicked

Figure 1: Peter O'Halloran receiving his David Hilbert Award from the Chairman of the AMT Board Professor Don Aitkin, at his home on 31 August 1994. Professor Bernard H. Neumann looks on.

by O'Halloran as his successor, but the successor was not into games of mathematical kind. He aspired to the Presidency of Bulgaria, and in 1995 became the second highest politician of his country, Chairman of Bulgaria's National Assembly. The steering wheel of the Federation stood unattended.

Then there came Ronald G. Dunkley of Canada. In 1996 he was asked to become the President of the Federation. A minute before the General Membership Meeting of the Federation during ICME-8 in Seville, Spain, Ron asked me whether I was willing to help. "Sure," I replied, not knowing what Ron had in mind. At the meeting, Ron commenced a democratisation of the Federation. He nominated 'critical' thinkers to the Executive Committee: Tony Gardiner of Great Britain to Vice-President, and me to Secretary and member of the Program Committee. Ron made the first Constitution of the Federation possible. Six years earlier, in 1990, Ron Dunkley and his Canadian colleagues Ron Scoins, Dr. Oakland, and others, invented and hosted the unforgettable First Congress of the Federation in their native Waterloo, Canada.

Figure 2: From left: Dimiter Dimitrov, Kiril Bankov, Alexander Soifer, Ronald Dunkley, Peter Taylor & Konstantin Tsiskartidze, 3rd Congress WFNMC, Zhongshan, China, 1998.

In 2000 in Japan, the Federation elected the new President, Peter J. Taylor of Australia. Peter continued the democratisation of the Federation by proposing a Constitutional Amendment limiting President's service to one term, which first applied to Peter himself. Peter also organized the lovely 4^{th} Congress in 2002 in Melbourne, Australia.

Figure 3: Three Presidents: from left, Peter Taylor, Alexander Soifer & Petar Kenderov, 4th Congress of WFNMC, Melbourne, 2002.

In 2004, academician Petar S. Kenderov of Bulgaria was elected President. For the first time the Federation earned its history, written by President Kenderov. Earlier, in 1994, Petar spearheaded the 2nd Congress of the Federation in Pravetz, Bulgaria. At about the same time, Bulgarian colleagues created the logo for the Federation.

The year 2008 saw the birth of the first Madam President, Maria Falk de Losada of Colombia. A few days earlier, Maria organized, funded, and ran the first Mini-Conference of the Federation a day before ICME-11 in Monterrey, Mexico. The dinner in the Ethnographic Museum, among the masterpieces of Pre-Columbian Art, was unforgettable. Maria continued this most important undertaking by organizing and running the Mini-Conference in 2012, before ICME-12 in Seoul, South Korea. In 2016, she kindly took upon herself the logistics and funding of the Mini-Conference in Hamburg, Germany, right before ICME-13. Maria also organized the Federation's 7th Congress in 2014 in Barranquilla, Colombia.

In 2012, at the General Meeting of the Federation in Seoul, during ICME-12, the membership elected me to the Presidency. Two years earlier, in 2010, I created the International Program of the Federation's 6th Congress in Riga, Latvia, on request of the person, who envisioned that Congress but fell ill, Professor Agnis Andžāns. Ever since joined the Federation 32 years ago, my main goal has been to bring Mathematical Olympiads closer to research mathematics, to build two-way bridges between the two. After all, according to Professor Boris Delone of Russia, the only difference between an Olympiad problem and a research problem is that the former takes 5 hours while the latter 5,000 hours to solve.

In 2018, during the 8th Congress of the Federation in Semriach, Austria, the membership elected Kiril Bankov of Bulgaria to serve as the next President. Robert Geretschläger became Senior Vice-President. Sergey Dorichenko of Russia and Krzysztof Ciesielski of Poland became Vice Presidents. David Crawford of the United Kingdom was elected Secretary. Peter Taylor will continue as Treasurer (and our web master). I will occupy two seats on the Executive, Immediate Past President and Publications Officer, but of course will have just one vote :). We elected fine members for our Standing Committees: Awards, Program, and Regional Representatives. You can find the Committees membership on our website: `http://www.wfnmc.org/about.html`.

Figure 4: Madam President Maria Falk de Losada and Senator Ricardo Losada, Canberra, 2000. Photo by A. Soifer

1 Publications of the Federation

The founding visionary Peter O'Halloran realized the importance of publications, and already in 1985 commenced the publication of the newsletter of WFNMC. In 1988 the newsletter was upgraded to a biannual journal *Mathematics Competitions.* For many years, until his retirement, Warren Atkins of Australia was the editor of *Mathematics Competitions.* Since 2004, the editor has been Jaroslav Švrček of Czech Republic. As an editor and publisher of a journal myself, I can appreciate the great job Warren and Jaroslav have done.

As the next editor of *Mathematics Competitions*, with Sergey Dorichenko serving as the Assistant Editor, responsible for having accepted by me articles typeset by the Moscow Center for Continuing Mathematics Education (MCCME).

2 Awards of the Federation

In 1991, Peter O'Halloran introduced two international awards of the Federation: the David Hilbert Award and the Paul Erdős Award, for mathematicians prominent on an international or national scale in mathematical enrichment activities. Both Hilbert and Erdős were mathematicians of genius, but I feared that having two awards would inspire an unintended interpretation of one award being higher than another. When in 1998 I expressed this concern, it did not create support of my fellow Executives. However, a few years later, the Executive reached a consensus on keeping just one award. We chose Paul Erdős, as the person of our time, whom some of us knew well personally. The Federation now presents up to four Paul Erdős awards at Congresses of the Federation and at ICME's.

The great Paul Erdős attended the 2nd Congress of the Federation in 1994 in Pravetz, Bulgaria. The organizer Petar Kenderov asked me to invite Paul, which is what I did.

Figure 5: Doing Math on Excursion: Jaroslav Švrček (left), Paul Erdős, and Josef Molnár, 2nd WFNMC Congress, Pravetz, Bulgaria, 1994

The 1994 winners of the Paul Erdős Award had the distinct honor of receiving it from Paul Erdős. They included Qiu Zonghu, China:

Figure 6: Even dinner cannot stop our doing Math: Paul Erdős, Alexander Soifer, and Kiril Bankov, 2nd WFNMC Congress, Pravetz, Bulgaria, 1994

Urgengtserengiin Sanjmyatav, Mongolia: Jordan Tabov, Bulgaria: and Peter Taylor, Australia.

3 Congresses of the Federation

Starting in 1990, the Federation has been holding its International Congresses every four years. I have already mentioned several of them above. Let me list all of them here, so that you can see their amazing geography:

- WFNMC-1: Waterloo, Canada (1990)
- WFNMC-2: Pravets, Bulgaria (1994)
- WFNMC-3: Zhongshan, China (1998)
- WFNMC-4: Melbourne, Australia (2002)
- WFNMC-5: Cambridge, United Kingdom (2006)
- WFNMC-6: Riga, Latvia (2010)
- WFNMC-7: Barranquilla, Colombia (2014)
- WFNMC-8: Semriach, Austria (2018)

– WFNMC-9: to be held in Sofia, Bulgaria (2022)

Permit me to add some information about the last three Congresses of the Federation.

6th WFNMC Congress, Riga, Latvia, 2010

The 6th Congress of the Federation took place in the historic Riga, Latvia. Local Organizing Committee was headed by Dr. Dace Bonka and included Dr. Liiga Ramaana and other dedicated Latvian Mathematicians. I created an elaborate program that included talks, workshops, problem hour, etc. We were treated to fine concerts of classical and Latvian folk music, some performed by professional musicians and others by students majoring in mathematics. Bach and Vivaldi performed on Europe's largest organ in the Riga Dom were unforgettable. The following plenary talks were presented:

– *Some Olympiad Problems in Combinatorics and their Generalizations*, by Andris Ambainis (Latvia)
– *New Bridges between Research and Olympiad Problems*, by Alexander Soifer (USA)
– *Eyewitnessing the IMO—Decades of Stability and Change*, by Matti Lehtinen (Finland)
– *The Application of Mathematics as the Source of New Ideas*, by Andris Buiķis (Latvia)

7th WFNMC Congress, Barranquilla, Colombia, 2014

The 7th Congress of the Federation was magnificently organized by Maria Falk de Losada in Barranquilla, Colombia, in July-2014. All delegates lived together in a resort-hotel that allowed for much interaction. The Congress featured the Opening Keynote Address *Computers and Mathematics: Problems and Prospects* by the world-renowned mathematician and inspiring speaker Ronald L. Graham. The following Plenary Talks were given:

– *The organization of the International Mathematical Olympiad—IMO with particular emphasis on IMO 2013* by Maria Falk de Losada (Colombia);
– *A Few Thoughts on the Putnam* by Mark Krusemeyer (USA);

Figure 7: Delegates of the 6th Congress of the Federation, July 2010, Riga, Latvia

- *Math Olympiads for the Public Schools in Brazil II* by Michel Spira (Brazil);
- *Is there an impact of mathematical competitions on the development of mathematical research? The Romanian experience* by Radu Gologan (Romania);
- *A Paper about Pentagon Problems* by Robert Geretschläger (Austria).

The Closing Address *Predicting the Future: Four Classic Conjectures of Mathematics* was presented by Alexander Soifer. The Congress included many other talks, workshops, and exhibits. A tour of colorful Cartagena served as the icing on this delicious Congress. The following few photos may enliven your perception of the event.

At the General Meeting, the delegates decided to hold Federation's election during our own Congresses, which have been much better attended than our meetings during massive ICME Congresses. For that purpose, the terms of the Federation officers have been extended by two years.

Figure 8: The Russians are coming: Sergei Dorichenko (left) and Nikolai Konstantinov, Barranquilla, 2014

Figure 9: Colorful Cartagena welcomes you!

Figure 10: Opening and Closing Speakers: Ronald L. Graham (right) & Alexander Soifer, Barranquilla, 2014

8th Congress, Semriach (Graz), July 2018

All local arrangements, registration, lodging, meals, excursions, etc. were beautifully organized by our Vice-President Robert Geretschläger. Zita Hauptmann Geretschläger created a lovely ambiance at the congress. The Program was created by Senior Vice-President and Chair of the Program Committee Kiril Bankov and me. There will be a book assembling the talks and the workshop of the Congress, edited by Robert Geretschläger, and published by World Scientific. It will also include a chapter on competition problems proposed by the participants of the Congress.

During the Congress three Constitutional Amendments were approved, limiting service on Standing Committees and most members of the Executive to two consecutive 4-year terms, and allowing past presidents to remain active voting members of the Executive. The following plenary talks were presented:

– *From the Lifting-the-Exponent-Lemma to Elliptic curves with iso-*

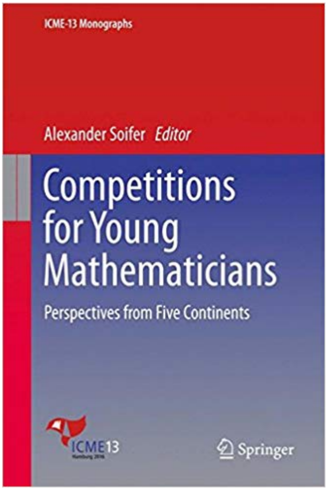
Figure 11: Robert and Zita Geretschläger in Riga, 2010

morphic groups of points—how Olympiad Mathematics influences Mathematical Research, by Clemens Heuberger (Austria);

- *Beyond the Rainbow—Thoughts on the Potential of Mathematics Competition Problems in the Classroom*, Opening the section "Work with Students and Teachers", by Robert Geretschlaeger, (Austria);
- *The Impact of Mathematical Olympiads on the Mathematics Community of Colombia*, Opening the section "Competitions around the World", by Maria Falk de Losada (Colombia);
- *Building Bridges Between Olympiads & Mathematics: Three Long-Distance Trains of Thought*, Opening the section "Building Bridges between Problems of Mathematical Research and Competitions", by Alexander Soifer (USA);
- *Problems: The Treasure of The Mathematics Competitions*, Opening the section "Creating Problems and Problem Solving", by Kiril Bankov (Bulgaria).

4 Our book *Competitions for Young Mathematicians: Perspectives from Five Continents*

Springer International Publishing, 2017, was "dedicated to all those people around the world, who are passing baton to next generations of mathematicians." Permit me to list its contents:



The Convenor and Chair of International Program Committee of ICME-13, wrote a most complimentary foreword for our book. Here is a quotation from her text:

> Mathematical competitions are a chance for mathematically talented young scholars to experience mathematics as a research-oriented discipline. These competitions offer the chance to get insight into the beauty of mathematical structures at a high level, which many of these young mathematicians usually will not experience at home. Furthermore, these competitions allow to meet other talented young mathematicians, exchange their ideas with them and experience that they are not singular and isolated youngsters, but part of an important community.
>
> Despite this high importance of mathematical competitions, either as mathematical Olympiad or as mathematical tournament of towns or other kinds of mathematical competitions, there exists hardly any scientific research about mathematical competitions. This is surprising, because these mathematical competitions have a long tradition and a high influence on the promotion of young talented mathematicians.
>
> At the occasion of the 13th International Congress on Mathematical Education (ICME-13) a Topic Study Group on Mathematics Competitions took place, at which famous researchers working in this field met and exchanged about the state-of-the-art in this field. This intensive work together with papers from related groups forms the basis of this book. The book provides an excellent overview about the current discussion, topical themes and experiences with mathematical competitions. It starts with reflections on goals of mathematics education, problems coming from geometry or combinatorics being used in mathematical competitions. The next parts reflect on the role of competitions in the classroom, this theme is hardly researched so far. Then two examples of mathematical competitions are analyzed. The last two parts focus on the present state of mathematical competitions and its future and a bridge between competitions and 'real' mathematics.
>
> To summarize, this book is more than overdue and reflects from an academic perspective on the potential of mathematical competitions for mathematics education in general.

> I wish to congratulate the editor—Alexander Soifer—and the contributors to this timely and excellent book.

Hamburg, Germany
Gabriele Kaiser
Convenor of the 13th International Congress
on Mathematical Education, University of Hamburg

5 The State of the Federation: Summing Up

In this historical essay, I listed many talks to show what we do. In essence, we are what we do!

Through the 34 years, the Federation has expanded its interests, publications, and activities. It was the 4th organization to become an Affiliate of the International Commission of Mathematical Instruction (ICMI) of the International Mathematics Union (IMU). Our members Peter Taylor and Ed Barbeau led the way in an ICMI Study, resulting in a book. Our Riga 2010 Congress produced two books. The Springer "Perspective from Five Continents" book was primarily authored by our members. Robert Geretschläger is assembling a book of Graz-2018 Congress.

However, compared to the earlier years, the number of national IMO leaders, participating in the Federation activities, seems to have declined. I hope their number will grow, and in part for that we are scheduling our 9th Congress of the Federation in 2022 in Europe to immediately follow the International Mathematical Olympiad (IMO) that will be held in Norway.

The so-called "Giftedness" Group, attracted away some of our regular participants, such as our long term member Emilia Velikova and local organizers of our Riga-2010 Congress. It is not surprising considering that their circle of interests is broader than ours.

I encourage all my colleagues to become more active in our Congresses, our topic study groups at ICME Congresses, and in writing for our journal *Mathematics Competitions* (send your submissions to the Editor, i.e., to me at asoifer@uccs.edu). We define ourselves: we are what we do.

The Federation is 34 years old. I hope it will reach the Golden Anniversary in 2084 and continue beyond! I am grateful for the trust and support of the membership and the opportunity to be your President from 2012 through 2018.

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The Cuban Mathematics Olympiad: a fragmentary journey

Frank Gamboa de la Paz & Jorge Marchena Menéndez

Frank Gamboa is the student of Mathematics Sciences in University of Havana. He was member of the National Mathematics Preselection in Cuba during three courses, winner of bronze and gold medals in the Iberoamerican Collegiate Mathematics Olympiad in 2016 and 2017. Since 2016, he became trainer for the Cuban teams to the IMO, the Iberoamerican Mathematics Olympiad and the Centro American Mathematics Olympiad. Olympiad problems building and inequalities are his passions.

Jorge Marchena is the student of Mathematics Sciences in University of Havana. He was member of the National Mathematics Preselection and the National Programming Preselection in Cuba. He also was winner of bronze medals in the Iberoamerican Collegiate Mathematics Olympiad in 2017. Since 2016, he is competitor of the ACM-ICP developing a big interest in topics such as Computational Geometry and Combinatorics. He was Problem Coordinator of the XX Centroamerican and Caribbean Mathematics Olympiad in 2018, Havana, Cuba.

1 A quick overview to the Selection Process in Cuba

The whole country was witness to the Cuban Mathematics Olympiad for high school students in the second half of the February in 2018. This time 149 contestants from the three grades took part and finally, 28 of them were chosen to participate in the National Mathematics Preselection, in a preparative process that started in March 26th and will end in September along with last event of the Olympic year, the Iberoamerican Mathematics Olympiad. During this period was realized in Cuba from the 15th to 23rd of June the Centro-American Mathematics Olympiad, and also took place in the International Mathematics Olympiad at Cluj-Napoca, Romania, in July.

The academic course in Cuba starts in September first week and the selective process has its beginnings in the contests to a base level in the educational centers. The next phase is composed by the municipal contests (municipal phase) to which those students with high notable results in the first phase are invited. Finally in January, the Cuban Mathematics Olympiad (OMCC by its Spanish acronym) takes place in every single one of the 16 provinces.

As expected, the difficulty of the contest gets necessarily higher from one phase to another, and has as main aim the National Olympiad. Once the international Olympiad training is started, near to the end of March or the beginning of April, competitors of the National Preselection are subjected to the team selection tests with a high difficulty and very diverse problems.

That way, competitors with the four highest scores of 10th grade are invited to be part of the National Team to OMCC (the Centro American teams to OMCC can be confirmed up to four members since 2018 according to the international judge members). In addition, the four highest scorers of 11th and 12th grade do also enter into the national team. The highest scorer (Cub 1) gets to represent Cuba in the IMO in the meantime the other four students represent Cuba on the Iberoamerican Mathematics Olympiad. Occasionally only the best four positions are classified in case that student (Cub 1) could also be allowed to participate in the Iberoamerican Mathematics Olympiad. This fact is given due to since 11 years ago, the Cuban teams to IMO contain a single member, excluding Mexico 2005, in which, a team of four participated and the other five places had been in vacancy caused by economic problems that have stopped Cuba full participation in these competitions since then. Between all the Cuban incursions to these Olympiads till 2017, we have reached and achieved 24 mentions, 37 bronze medals, 7 silver medals and a unique gold medal (IMO 2001), according to statistics from [1].

2 Cuban Mathematics Olympiad 2017: most representative problems

In the Cuban Mathematics Olympiad, celebrated in 2017, on the second day can be found a problem for 12th grade as question #3:

Problem Let p be a prime number. Show that the equation

$$m^2 + n^2 = (m, n, p) \cdot [m, n, p],$$

has positive integer solutions if and only if p can be expressed as sum of two perfect squares.

Solution 1: We will first prove that if $p = a^2 + b^2$ for some integers a, b there exists a positive integer solution for the above equation. Notice that if $d = (a, b)$, then $d^2 \mid p$, by which $d = 1$, i.e. a and b are relative primes. Furthermore, $p \nmid a$ and $p \nmid b$, due to in a contrary way $(a, b) \geq p > 1$. Let $m = a^2 b$ and $n = ab^2$. We already know that $(m, n, p) = 1$ and $[m, n, p] = a^2 b^2 (a^2 + b^2)$, and finally

$$m^2 + n^2 = a^2 b^2 (a^2 + b^2) = (m, n, p) \cdot [m, n, p].$$

Let's see the other implication. We will assume that the equation has positive integer solutions. Let (m, n) be one of them. It is evident that $(m, n, p) \mid p$, by which $(m, n, p) = 1$ or $(m, n, p) = p$. If $(m, n, p) = p$, it follows that $p \mid m$ and $p \mid n$, there exist integers a, b, such that $m = ap$ and $n = ap$. Note that $[m, n, p] = p \cdot [a, b]$. Substituting in the equation and simplifying we obtain

$$a^2 + b^2 = [a, b].$$

Using the Arithmetic and Geometry Means inequality (AM$-$GM inequality), we conclude that

$$a^2 + b^2 \geq 2ab > ab \geq [a, b].$$

Therefore, we reach a contradiction. Then $(m, n, p) = 1$, which implies $p \nmid (m, n)$. We can assume without the loss of generality that $p \nmid m$. Note that $p \mid [m, n, p]$, by which $p \mid m^2 + n^2$, this directly implies that $p \nmid n$, because $(m, n, p) = 1$. Let $d = (m, n)$ and $m_1 n_1$ be coprime positive integers such that $m = dm_1$ and $n = dn_1$. Substituting in the equation and simplifying we obtain the equality

$$d(m_1^2 + n_1^2) = m_1 n_1 p,$$

and given that $1 = (m, n, p) = (d, p)$, it follows that $d \mid m_1 n_1$. On the other hand, it's easy to prove that $(m_1^2 + n_1^2, m_1 n_1) = 1$, so $m_1 n_1 \mid d$. This implies that $d = m_1 n_1$. Finally

$$p = m_1^2 + n_1^2 = \left(\frac{m}{d}\right)^2 + \left(\frac{n}{d}\right)^2.$$

This problem, solved by a small number of students in the Olympiad, though it does not offer a very high difficulty, goes through a fact that stayed out the light in some solutions: the searched primes are of the form $4k + 1$ (except 2).

It is well known that an odd prime p can be represented as sum of two perfect squares if and only if $p = 2$ or $p = 4k + 1$, for some positive integer k: this result is known as Fermat's Theorem about the sum of two perfect squares or the Fermat's Christmas Theorem [2]. Another well-known fact within the Elementary Number Theory is that if $p = 4k + 3$ is a prime and $p \mid x^2 + y^2$ with $x, y \in \mathbb{Z}$, then $p \mid x$ and $p \mid y$. So that way, the following solution to the same problem is a little bit shorter and but also more complicated to figure out.

Solution 2: We will only prove that if the equation has positive integer solutions, then p is as desired. The other side of the proof is the same as the above solution. In fact, let (m, n) be a solution. Let's assume that $p = 4k + 3$ for certain positive integer k. Because $p \mid [m, n, p]$, then

$p \mid m^2 + n^2$, by which $p \mid m$ and $p \mid n$, so $(m, n, p) = p$, hence

$$m^2 + n^2 = p \cdot [m, n, p] = p^2 \cdot \left[\frac{m}{p}, \frac{n}{p}\right] \leq p^2 \cdot \frac{m}{p} \cdot \frac{n}{p} = mn < 2mn,$$

which is absurd. By this, $p = 2$ or $p = 4k + 1$, and the conclusion is given by Fermat's Christmas Theorem.

The Cuban Mathematics Olympiad is divided into two exams within two days: the three grades common test (day #1) and the individual tests for each grade (day #2).

The next problem, also attached to Olympiad of 2017, is the problem #3 of the first day. The statistics gave prove that in the Olympiad, the average scores for this exercise is smaller than 1, in a total of 7 points; this fact makes it a high difficulty problem. Before showing you the statement, it is important to emphasize that Mathland (the country of the Mathematics) is a fictitious place used to give place to combinatorial problems, with an elegant and familiar touch to the students in the Cuban MO.

Problem In Mathland there exist provinces as positive integers. From province p it is only possible to travel to provinces $2p$ and $p+1$. Find the least amount of required moves to get from province 2017 to province 2^{1997}.

Solution 1: It could seem intuitive to start to multiply by 2, up to the maximal province p, where if we multiply again by 2 we will get to a province $k > 2^{1997}$. Once in p, it would be enough to add 1 several times up to get to 2^{1997}. Apparently, this does not seem absurd, but note that $2^{11} = 2048 > 2017$ is the nearest power of two to 2017, by which the amount of times that we should multiply by 2, according to the above strategy is 1986, because $2017 \cdot 2^{1986} < 2^{1997}$ and $2017 \cdot 2^{1987} > 2^{1997}$. But the amount of times that we should then add 1, would be in order of 2^{1986}, which is a huge number. A most reasonable strategy would be adding 1 up to the first power of 2, which is obtained in $2048 - 2017 = 31$ steps, and then multiply 1986 times by 2 up to 2^{1997}, with a total of 2017 moves. There is no doubt that, by this way it is possible to make the required journey with 2017 moves. We will prove then, that this cannot be possible in less than 2017 moves.

Consider the function $G\colon A \to \mathbb{Z}$ with $A = \{(a,b) \in \mathbb{Z}_{>0} \times \mathbb{Z}_{>0} : b \geq a\}$, defined by $G(a,b) = 0$ if $a = b$; if $b \geq 2a$ with b even, then

$$G(a,b) = 1 + G\left(a, \frac{b}{2}\right),$$

and $G(a,b) = 1 + G(a,b-1)$ in any other case. Let's see the following claims.

Claim 1 For all pairs (a,b), holds $G(a,b+1) \leq 1 + G(a,b)$.

Proof: We proceed by strong induction on b, taking a as fixed. Notice that $G(a,a+1) = 1$ and $G(a,a+2) \leq 2$; that way $G(a,a+2) \leq 1 + G(a,a+1)$. This proves the claim for $b = a+1$. We assume that for all j with $a < j \leq k$ it holds that $G(a,j+1) \leq 1 + G(a,j)$. If k is even, then $k = 2t$ for certain integer t; it follows that $k+1 = 2t+1$, and

$$G(a,k+1) = G(a,2t+2) = G(a,2t) + 1 = G(a,k) + 1$$

and in this case, the claim holds. Assume then that k is odd, meaning that $k = 2t+1$ for a certain integer t; that way $k+1 = 2t+2$. If $t+1 > a$, we would have

$$\begin{aligned} G(a,k+1) &= G(a,2t+2) = 1 + G(a,t+1) \leq 2 + G(a,t) \\ &= 1 + G(a,2t) = G(a,2t+1) \\ &= G(a,k) < G(a,k) + 1 \end{aligned}$$

where the inequality is strict; now, if $t+1 = a$, then

$$G(a,k+1) = G(a,2t+2) = 1 + G(a,t+1) = 1 \leq G(a,k) + 1.$$

The final case is that $t+1 = a$, but this situation is evident due to $G(a,k+1) = G(a,k) + 1$ and the claim have been proved.

We will prove that function G represents algorithmically the sequence to follow steps to get from one province a to another b using the allowed travels.

Claim 2 We denote by $M(a,b)$, the least amount of moves to get from a to b using the maps $x \mapsto 2x$ and $x \mapsto x+1$. Then $M(a,b) = G(a,b)$ for all $(a,b) \in A$.

Proof: Let's use strong induction one more time on b. Note that $M(a,a+1) = 1 = G(a,a+1)$ and the base case holds. Assume that for all j with $a < j \leq k$, it holds that $M(a,j) = G(a,j)$. With a reproductive way, we split the analysis in two cases. If $k = 2t$, then $k+1 = 2t+1$, hence $M(a,k+1) = 1+M(a,k)$, because to have $k+1$ odd, the last map added was 1; that way, as $M(a,k) = G(a,k)$ by hypothesis of induction, the conclusion is direct to be $G(a,k+1) = 1+G(a,k)$. Then, we assume that $k = 2t+1$. If $t+1 > a$, we should analyze two possible cases because (remember the definition of M)

$$M(a,k+1) = \min\left\{1 + M(a,k), 1 + M\left(a, \frac{k+1}{2}\right)\right\}.$$

We analyze that $M(a,k+1) = 1 + M(a,k)$. In that case

$$M(a,k+1) = 1+M(a,2t+1) = 1+G(a,2t+1) = 2+G(a,2t) = 3+G(a,t)$$

The other possibility implies that, according to Claim 1

$$M(a,k+1) = 1+M(a,t+1) = 1+G(a,t+1) \leq 2+G(a,t) < 3+G(a,t).$$

We can conclude that

$$M(a,k+1) = 1 + M\left(a, \frac{k+1}{2}\right) = 1 + G\left(a, \frac{k+1}{2}\right) = G(a,k+1).$$

If $t+1 = a$, then $M(a,k+1) = 1$, because by simply multiply a per 2 and $G(a,k+1) = 1 + G(a,t+1) = 1$. Finally, if $t+1 < a$, then $k+1 < 2a$, and by this reason, we can only get to $k+1$ with the maps $x+1$ in this case; hence $M(a,k+1) = k-a+1$ and similar $G(a,k+1) = k-a+1$, in virtue of definitions of G.

To finish this problem it is enough to compute $M\left(2017, 2^{1997}\right)$

$$\begin{aligned} M\left(2017, 2^{1997}\right) &= G\left(2017, 2^{1997}\right) \\ &= 1986 + G\left(2017, 2^{11}\right) = 1986 + 31 = 2017 \end{aligned}$$

and the problem is solved.

The previous solution was the official one proposed to the Olympiad with the above formulation of the problem; however, we could generalize and make it lose its nice structure with the following statement.

Problem Let D be a digraph with N vertices $v_1, v_2, ..., v_N$ such that, for every vertex v_k there exist exactly two edges going from v_k: one that connects it with v_{k+1} and another with v_{2k}. Find the least amount of edges that we need for move up to the vertex v_B from v_A, where A, B are positive integers such that $1 \leq A \leq B \leq N$.

We are not going to represent another explicit solution on this problem, however we will comment on some ways to attack it. The first one would be purely computational, by modeling the problem; in fact, we could create a counting function of the amount of passed edges, and optimize this function about the restrictions of the graph in the previous formulation. We present here a possible model for the problem:

$$\begin{array}{rcll} \min & \sum\limits_{i=A}^{B} \sum\limits_{j=i}^{B} c_{i,j} & & \\ \text{s.t. } c_{k,2k} + c_{k,k+1} & = & 1, & A < k < B, \\ c_{1,2} & = & 1, & \\ c_{i,j} & = & 0, & i = [A, B], \quad j = [B+1, ...2B] \\ c_{i,j} & = & \{0,1\} & i, j \in [A, B] \end{array} .$$

Here c_{ij}, represent the decision of choosing or not the edge that joins the i, j vertices to define the path. This model can be solved through the usage of an adequately implicit enumeration algorithm, i.e. DFS (Depth First Search).

An alternative method of solving the problem would be a refined way exposed in the website `www.mathlinks.ro`, which is a well-known online mathematics forum. If we write the number 2^{1997} in binary system, we will obtain the number $a = 100...00$ with 1997 zeros exactly, and 2017 in binary system is $b = 11111100001$. Then, the statement is equivalent to find the lowest amount of mappings of type $x \mapsto x + 1$ and $x \mapsto 2x$, to obtain a coming from b.

The mappings of type $x \mapsto 2x$ applied to a number N adds one 0 at the end of the binary representation of N. The ones of type $x \mapsto x+1$, if the last digit is a 0 then it is changed into a 1, and in the contrary case it is changed into a 0 and all previous digits are updated following the same idea. Then, each $N = 2^k - 1$ with $k \geq 1$, this map adds one more digit to the binary representation of N. Then, a non-refined conclusion would be to use the previous maps to go from an 11-digit number to a 1998-digit one; if we exclusively use the previous mappings of type $x \mapsto 2x$, we would need $1998 - 11 = 1987$ applications up to the 1998-digit number, however, this number will be formed by several digits equal to 1. Now, during the process, we could apply $x \mapsto x + 1$ at any moment, but it is obvious that if we do this at the beginning, we will obtain the desired minimum. The reason for this affirmation is that we should add 1s until we find to the next power of 2; if we do maps of type $x \mapsto 2x$ before doing the ones of type $x \mapsto x + 1$, then, we will need more applications to find the next power of 2.

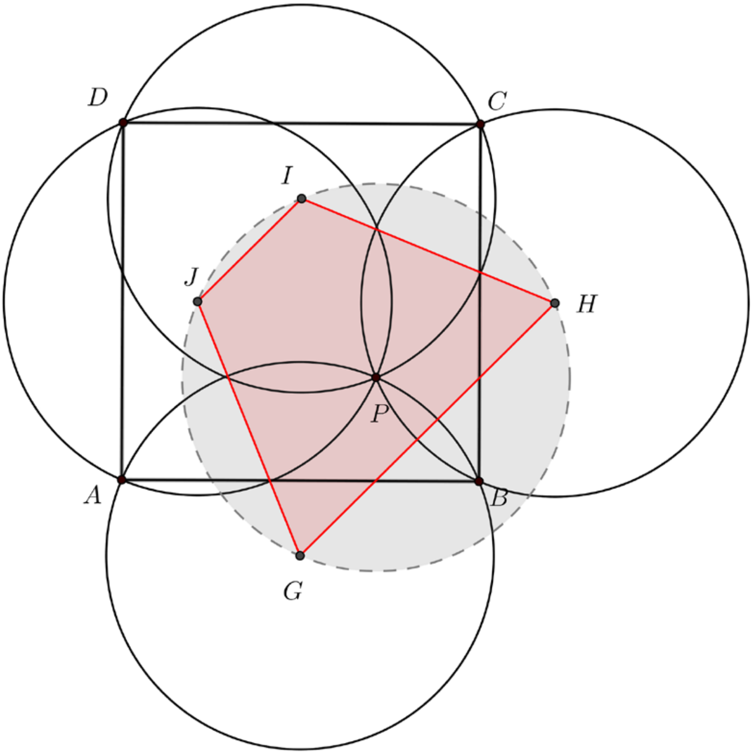
The nearest power of 2 to 2017 is $2^{11} = 2048$, and so we would need $2048 - 2017 = 31$ maps of type $x \mapsto x + 1$ and $1997 - 11 = 1986$ maps of type $x \mapsto 2x$, for a total of $31 + 1986 = 2017$ mappings. The problem is that this solution consists in supposing that the optimal strategy to make movements from A to B is to get always to the power of 2 and further than A, which does not seems evident.

3 Cuban Mathematics Olympiad 2018: three problems with beautiful solutions

In the last part of this article, we will debate three problems of the Olympiad 2018 with their official and alternative solutions; some of these where offered by students in the contest itself. The first one of these problems corresponds to the #3 problem of day #1, that is, the common exam, where, fortunately, more than one perfect score was reached. As is usual with geometrical problems this exercise came out spontaneously in the assistant Geogebra, meanwhile one of the authors played to create circumferences over a square. The solution 1 was taken from the official solution to the problem in the Olympiad, and it's a contribution of Jorge Estrada, professor of Mathematics at the University of Havana.

Problem 1 Let $ABCD$ be a square. Determine the geometrical place of all points P inside the square such that the circumcenters of triangles ABP, BCP, CDP and DAP are concyclic.

Solution 1: Let's prove that the square diagonals are the geometrical place. In effect, if P is over the diagonals it is easy to check that the given circumcenters form an isosceles trapezium, which is clearly cyclic. Later, the diagonals accomplish the required property. Then, let us assume P is such that all circumcenters are concyclic. If P is the center of $ABCD$,



the circumcenters make up a square. If it is not the case, let us assume without loss of generality that $P \notin AC$, and let's prove that $P \in BD$.

Let G, H, I and J be the circumcenters of triangles APB, BPC, CPD and DPA, respectively. Let P' be the image of translate P through the vector $\overrightarrow{AB}$. You shall notice that $AP \perp JG$, $BP \perp GH$, $CP \perp HI$ and $DP \perp IJ$, this implies $\angle APB = \pi - \angle JGH$, and $\angle CPD = \pi - \angle JIH$.

By hypothesis, $GHIJ$ is a cyclical quadrilateral, then $\angle JGH + \angle JIH = \pi$, and hence $\angle APB + \angle CPD = \pi$. By translation, $\angle APB = \angle CP'D$, and then $CP'DP$ is cyclical. It follows that $\angle P'DC = \angle P'PC$. Due to $\angle P'DC = \angle PAB$ by translation properties and $\angle P'PC = \angle PCB$, since they are alternate angles, we have $\angle PAB = \angle PCB$. Let's look now at the next lemma.

Lemma Consider the points X, Y, Z such that $|XY| = |YZ|$. Within $\angle XYZ$ let's take one point $T \notin ZX$, such that the oriented angles $\angle YXT = \angle TZY$. Then, T lies on the bisector of $\angle XYZ$.

Proof: Let M be the intersection point of XT and YZ; N the intersection point of ZT and XY; F and G projections of T on XY and YZ. We have that the triangles YXM and YZN are equal. Later, $|YM| = |YN|$, and since $|XY| = |YZ|$ we also have $|XN| = |ZM|$. Moreover, $\angle YNT = \angle YMT$, then $\angle TNX = \angle TMZ$. We conclude that XNT and TMZ are equal triangles. It follows that $|TF| = |TG|$ are equal altitudes within equal triangles, i.e. T is on the bisector of $\angle XYZ$.

Using the previous lemma, we conclude that P is on the bisector of $\angle ABC$. Following the fact that $ABCD$ is a square, that bisector is BD, i.e. $P \in BD$.

The key of all known solutions is to prove that $\angle PAB = \angle PCB$, or similar relations with points P, A, B, C, D. From that point, there exist other ways to conclude without using the previous lemma. The one we will present now was offered by the 11th grade student Sofía Albizu Campos, golden medalist in the XIX OMCC 2017 in El Salvador with a perfect score and golden medalist with the best score in the Cuban Mathematics Olympiad 2018. The conclusion uses the Law of Sines.

Solution 2: Consider that Q, R, S and T are the cirumcenters of the triangles PAB, PBC, PCD and PDA respectively. It is not difficult to check that $SR \perp PC$ and $QT \perp PA$. Since SQ and RT intersects perpendicularly, we can deduce that $SQ \perp CD$, and following it we have $\angle RSQ = \angle PCD$. Analogously $\angle PAD = \angle QTR$. Since the quadrilateral is cyclical, it follows that $\angle PAD = \angle PCD$, i.e. $\angle PAB = \angle PCB$. Applying the Law of Sines for the triangles PAB and PCB respectively we have:

$$\frac{AB}{\sin\angle APB} = \frac{PB}{\sin\angle PAB}, \quad \frac{BC}{\sin\angle BPC} = \frac{PB}{\sin\angle PCB}$$

Following the previous conclusions and as $AB = BC$, it follows that $\sin\angle APB = \sin\angle BPC$. At this point, we have two possible situations: $\angle APB = \angle BPC$ or $\angle APB + \angle BPC = \pi$. In the first case, we have $P \in BD$. In the second one $P \in AC$, then P belongs to the diagonals of $ABCD$. On the other side, is clear that if P belongs to the diagonals, the quadrilateral $QRST$ is an isosceles trapezium, and hence is cyclical. In conclusion, the searched locus are the square diagonals.

It is not difficult to prove that if we don't restrain P at the interior of $ABCD$, and we just demand that P doesn't belong to the straight line which determines the square sides (in that case, triangles shall be degenerated), then the locus are the diagonals of $ABCD$. However, for avoiding the unnecessary analysis of some cases it was preferred to state the problem in the way shown. Many students just proved that the diagonals of $ABCD$ accomplished the required property, and hence, that they were a subset of the requested locus.

Along with this geometrical problem, two others problems were evaluated in the first day of the Cuban Mathemathics Olympiad. The problem # 1, was without any doubts the easier and the one with higher accumulated scores between the two exams, accomplishing the Problem Selection Committee expectations. However, when this easy Number Theory exercise was elaborated the very first time it wasn't *easy* at all. The original statement said:

Problem 2 For certain pairs of integer numbers (m, n), we say that a rational number r is (m, n)-representable if there exist integers a and b such that

$$k = \frac{a^3 - m}{b^3 + n}.$$

If we can guarantee also that $a, b > 0$, then r is said (m, n)-megarepresentable,

a) Prove that 2018 is not $(3, 23)$-representable.

b) Prove that 2 is not $(1, 1)$-megarepresentable.

c) Prove that for all rational numbers $r > 0$ there exist positive integers m, n for which r is $(-m^3, n^3)$-megarepresentable.

In the end it was decided to just ask for case *a*) with the following formulation, which minimized the problem difficulty almost in its totality.

Problem 2a Prove that there do not exist integer numbers x, y that verify the equation

$$\frac{x^3 - 3}{y^3 + 23} = 2018.$$

Solution 1: To prove it let's use modular congruence choosing adequately n that allows us to arrive to a contradiction. Let's analyze $n = 7$. Let's prove that the number $2018y^3 + 23 \cdot 2018 + 3$ is not a perfect cube modulo 7. By simple inspection we can notice that $y^3 \equiv 0, \pm 1 \pmod 7$, (also we can note that if p is an odd number, then $a^{\frac{p-1}{2}}$ is congruent with ± 1 modulo p always that $p \nmid a$), and hence

$$x^3 \equiv 2018y^3 + 23 \cdot 2018 + 3 \equiv 2y^3 + 4 + 3 \equiv 2y^3 \pmod 7.$$

Necessarily $x \equiv y \equiv 0 \pmod 7$, then $7^3 \mid (x^3 - 2018y^3) = 23 \cdot 2018 + 3 = 46417$, but this is a contradiction.

What is more curious about this exercise is that just a few given answers used modulo 7. The majority of students preferred modulo 9.

Solution 2: Let's analyze modulo 9. In effect, we have that $y^3 \equiv 0, \pm 1$ (congruence are taken modulo 9), since $x^3 \equiv 2018y^3 + 23 \cdot 2018 + 3 \equiv 2y^3 + 4$. If $y^3 \equiv 0$, then $x^3 \equiv 4$; if $y^3 \equiv 1$, then $x^3 \equiv 6$; if $y^3 \equiv -1$, then $x^3 \equiv 2$. For any of these three cases we get to a contradiction, from which we conclude that there exist not integer solutions.

It's worthy to comment about the other two exercises presented in the original statement of the problem. Case b) is equivalent to proving that there are no positive integer solutions for the equation $2m^3 + 3 = n^3$ (even more, the only pair (m, n) of integers that actually accomplish this is $(-1, 1)$, info gathered with Wolfram Alpha). This problem was posted on the website `www.mathlinks.ro` looking for an elemental solution without the use of Ring Theory. The unique one arrived after many tries uses the famous Liouville Theorem for Diophantine approximations, which states in one of its versions:

Liouville Theorem If α is a rational algebraic number with n degree over the rational numbers, then there exists $\widetilde{\alpha} > 0$ such that

$$\left|\alpha - \frac{p}{q}\right| > \frac{\widetilde{\alpha}}{q^n}.$$

The another version of the original question is exactly the problem N2 of the Short List of IMO 1999, and it can be found in [3]. An excellent generalization of the problem stated in the case a) would be to describe the class M of all integers k that can be represented in the form

$$k = \frac{x^3 - 3}{y^3 + 23}$$

for some pair (x, y) of integer numbers. In particular, the Olympiad problem asks proof that $2018 \notin M$.

The last problem we want to introduce to you was the problem # 2 of the individual exam for 12th grade students of the Cuban Mathematics Olympiad 2018, another about geometry. It was proposed by Robert Bosch to the Problem Selection Committee, known Cuban problem solver and a person who gained our admiration due to his outstanding contributions to the Olympiad world. Robert told us when he showed us this problem that he was unable to reach a purely synthetic solution, any of his solutions strongly required the use of trigonometry, or at least of metrical calculations. It was also considered the possibility of giving a special prize in the Olympiad to those students who were capable to find a synthetic solution. And regarding the Number 6 problem of IMO

1988, the capabilities of students for reaching elegant solutions can not be underestimated.

Problem 3 Consider the circumferences Γ_1, Γ_2 and Γ_3 with their centers on A, B and C and with radius lengths a, b and c respectively. Let's suppose that Γ_2 is tangent to Γ_1 in P and Γ_3 is tangent to Γ_1 in Q. Let be r the exterior tangent segment to Γ_2 and Γ_3. It is known that the segments r and PQ are located in different half-planes with respect to the straight line BC. Prove that

$$|PQ|^2 = \frac{a^2|r|^2}{(a+b)(a+c)}.$$

Solution 1: Let's denote the angle $\angle PAQ = \alpha$ and $d = |r|$. Let us consider a few auxiliary constructions: the segments AB, AC, BC, the radius lines of C_2 and C_3 that are perpendicular to r, and lastly we trace a segment starting from the point B and that is perpendicular to the C_3 radius line. It's clear that this new segment has length d. Look at the next figure. The solution uses the Law of Cosines and Pythagoras' Theorem. The following relations are true:

$$\begin{aligned} PQ^2 &= 2a^2(1-\cos\alpha), && (1)\\ BC^2 &= (a+b)^2 + (a+c)^2 - 2(a+b)(a+c)\cos\alpha, && (2) \end{aligned}$$

applying the Law of Cosines in the triangles PAQ and BAC respectively. Note that we shall prove the following equality

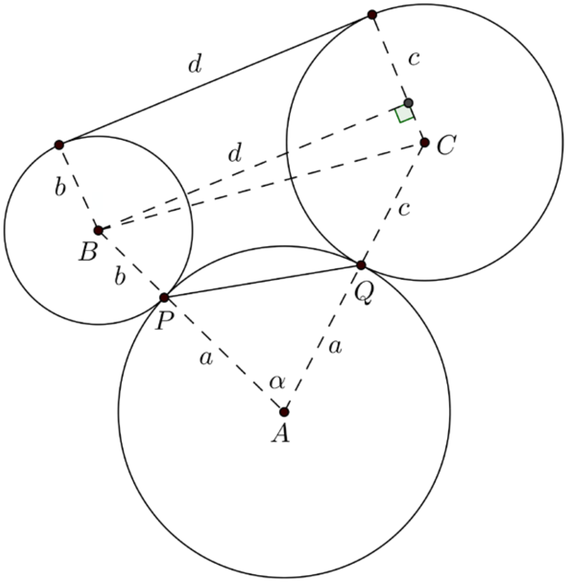
$$d^2 = 2(1-\cos\alpha)(a+b)(a+c).$$

Now from Pythagoras' Theorem we have

$$BC^2 = d^2 + (c-b)^2,$$

and if we equate with equation (2) it results

$$\begin{aligned} d^2 &= (a+b)^2 + (a+c)^2 - (c-b)^2 - 2(a+b)(a+c)\cos\alpha,\\ &= 2a^2 + 2ab + 2bc + 2ca - 2(a+b)(a+c)\cos\alpha,\\ &= 2(a+b)(a+c) - 2(a+b)(a+c)\cos\alpha,\\ &= 2(1-\cos\alpha)(a+b)(a+c). \end{aligned}$$



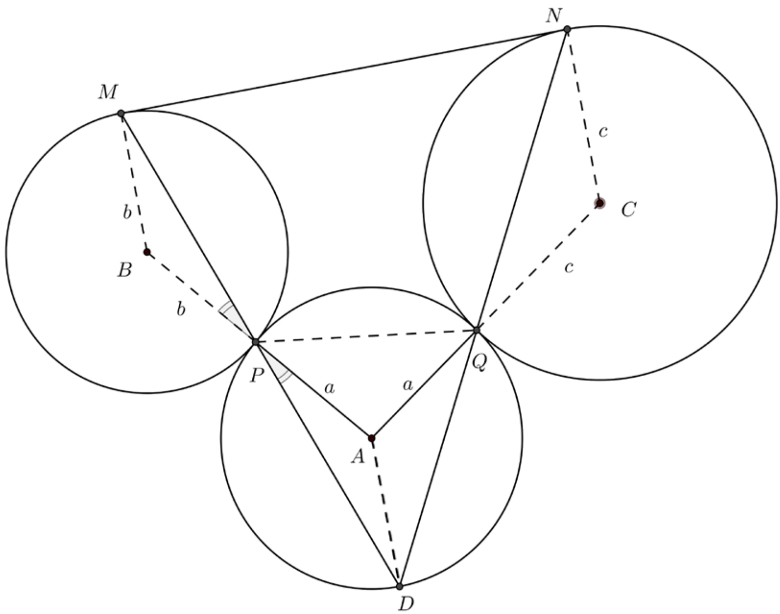
It was not a complicated problem in the Olympiad, but beautiful enough for captivating the attention of the tribunal members when it came out the first synthetic solution, offered by a girl student of 12th grade, member of the National Mathematics Preselection 2018, and gold medalist in this edition of the Olympiad.

Solution 2: Let M and N be tangential points of r with Γ_2 and Γ_3 respectively. Let be $\alpha = \angle PAQ$, $\beta = \angle QCB$, $\gamma = \angle BCN$, $\delta = \angle CBM$ and $\epsilon = \angle PBC$. The angle $\angle QNM$ is semi-inscribed on the arc NQ, hence $2\angle QNM = \angle QCN = \beta + \gamma$. Analogously $2\angle PMN = \delta + \epsilon$. On the other hand

$$\alpha = 180^\circ - 2\angle AQP,$$

$$\angle AQP + \angle PQN + \angle NQC = \angle AQP + \angle PQN + 90^\circ - \angle QNM.$$

It implies that $2\angle NQP = \alpha+\beta+\gamma$. In the same way, $2\angle QPM = \alpha+\delta+\epsilon$. Since $PMNQ$ is a quadrilateral, we know that $\angle NQP + \angle QPM +$



$\angle PMN + \angle QNM = 360^\circ$ and hence

$$\alpha + \beta + \gamma + \delta + \epsilon = 360^\circ.$$

So, noticing the opposite angles sum, we realize that this is actually a cyclical quadrilateral, then, it follows that

$$\frac{PQ}{|r|} = \frac{DP}{DN} = \frac{DQ}{DM}.$$

If we consider the sum of the triangle DMN within angles is clear that we obtain $2\angle MDN = \angle PAQ$, then $D \in \Gamma_1$, and hence A is the triangle DPQ circumcenter, later $AD = AP = AQ = a$. Angles $\angle APD$ and $\angle BPM$ are equal. Since $AP = AD = a$ and $PB = BM = b$, moreover the triangles APD and MPB are similar. Analogously, it is obtained that the triangles DAQ and CQN are similar. This way we know that

$$\frac{MP}{DP} = \frac{b}{a}, \quad \frac{NQ}{DQ} = \frac{c}{a} \Longrightarrow \frac{DM}{DP} = \frac{a+b}{a}, \quad \frac{DN}{DQ} = \frac{a+c}{a}.$$

Then, we can conclude

$$\frac{|r|^2}{|PQ|^2} = \frac{DN \cdot DM}{DP \cdot DQ} = \frac{(a+b)(a+c)}{a^2}.$$

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Problems: The Treasure of Mathematics Competitions

Kiril Bankov

Kiril Bankov prepares future mathematics teachers as a professor of mathematics education at the University of Sofia and the Bulgarian Academy of Sciences in Bulgaria. He graduated and received his PhD in mathematics at the same University. Prof. Bankov has been involved in mathematics competitions in Bulgaria for more than 20 years as an author of contest problems and as a member of juries. Kiril Bankov was the Secretary of World Federation of National Mathematics Competitions (WFNMC) from 2008 till 2012. In 2012 he was elected as the Senior Vice President of WFNMC and in July 2018 he became the President of the Federation.

1 Introduction

There are hundreds of mathematics competitions around the world. Every year thousands of students participate and win awards. Their names remain in the lists and build the history of the competitions. No doubt this is an impressive number of people, many of whom continue their career as professional mathematicians. This is one of the goals of mathematics competitions: to stimulate the development of mathematical talents.

It is not only the participants that build the history of competitions. Actually, for any competition there are two important sets: the participants, and the problems. The problems are the other side of the mathematics competitions. They are the intellectual product that remains in the history. There are hundreds of thousands of ideas that are implemented in the problems presented on mathematics competitions. It is impossible to review or consider all of them. Two types of "unforgettable" problems need special attention:

(i) Problems that do not need particular knowledge from mathematics school curriculum to be understood but their solutions require deep thinking, mathematical reasoning, experience, and a lot of intuition. Sometimes the solutions are quite unexpected. These problems are one of the best examples of the beauty of mathematics.

(ii) Problems that lead to interesting generalizations. Sometimes they are sources of unsolved problems.

This chapter presents examples of unsolved problems that are inspired by problems presented on mathematics competitions.

2 Problem for Exploring of a Period

The base of the problem is the following:

Situation Let $n > 2$ cells be arranged into a circle. Each cell can be occupied by 1 or 0. The following operation is admissible: draw another n cells—one between any two of the existing cells; in these new cells write 0 if the numbers in the neighboring existing cells are equal, and 1 if these numbers are different; then delete the existing cells.

The situation was used in a problem presented on a mathematics competition in 1975 in former Yugoslavia:

Problem 2.1 In the given situation, let $n = 9$ and four of the cells be occupied by 1, the other five be occupied by 0. Is it possible to obtain 0s in all nine cells in a finite number of admissible steps?

Solution. The answer is "No" and here is the argument. Assume that in a finite number of admissible steps all nine cells contain 0s. Then in the second to the last arrangement all nine cells contain 1s. Therefore, in the previous arrangement any two neighboring cells contain different numbers, which is impossible, having nine cells.

The problem gives rise to variety of generalizations. Different variations of the initial arrangements can be considered, depending on the number n of the cells, and on the number and the positions of the initial 1s. An interesting generalization is presented in the following problem.

Problem 2.2 In the given situation, initially there is a 1 in one cell and 0s elsewhere. For which values of n is it possible to obtain 0s in all cells in a finite number of admissible steps?

Let n be such a number that it is possible to obtain 0s in all cells in a finite number of admissible steps. Because of the arguments presented in the solution of Problem 2.1, n must be an even number. This is why only even values of n are interesting.

I have presented this as an open problem in my talk at TSG-30 in ICME-13, Hamburg, 2016. At that time I did not know the answer. Soon after the end of ICME-13, I had to prepare my talk as a chapter for a book. I tried to find a solution to the problem and to include it in the chapter. My desk was full of sheet of papers with circles and numbers. I used a computer to check for different values of n. The time was passing... Just before the timeline for submission of the chapter, I realized that the Sierpinski's triangle may help and I proved that the answer to Problem 2.2 is that the required values of n are only all powers of 2 (Bankov, 2017).

After Problem 2.2 is solved, the question is what happens for the other values of n.

Obviously, if n is not a power of 2 the execution of the admissible operation will never end. On the other hand, there are a finite number of arrangements of 0 and 1 in the cells of the circle. This means that after a certain number of steps the arrangements of the numbers will cyclically repeat. The least number of the steps in this repetition is called a period. Here is an open problem.

Open problem Explore how the period depends on n.

3 Large Subset of Disjoint Figures

This part presents an open problem as a generalization of a problem given on two famous mathematics competitions. It is about a set of disjoint figures on the plane. Before presenting the problems, let us consider the one dimensional case.

Problem 3.1 Let M be a finite set of segments on the line, the length of which union is equal to L. Then there is a disjoint subset of M, the sum of the length of whose segments is not less then $\frac{L}{2}$.

A solution can be found in Schkljarskij at al., 1974. More interesting is that this statement presents the best possible result, i.e. for every $\delta > 0$ there is a covering of a given segment of length L by a finite set M of segments, so that the sum of the length of the segments of any disjoint subset is less than $\frac{L}{2} + \delta$.

A possible two dimensional case presents the following problem that was set on the Moscow Mathematical Olympiad 1979, and on the Austrian-Polish Mathematics Competition 1983.

Problem 3.2 Let M be a finite set of circles in the plane, the area of which union is equal to A. Prove that there is a disjoint subset of M, the sum of the area of whose circles is not less than $\frac{A}{9}$.

Solution. The proof uses the Principle of Mathematical Induction on the number of the circles of M. The statement is obvious if M contains 1 or 2 circles. Let n be a natural number, $n \geq 3$. Assume that the statement is true if M contains k circles for every $k < n$. We will prove that the statement is true if M contains n circles, $M = \{K_1, K_2, \ldots, K_n\}$. Let K be the circle of M having the largest radius R among all circles of M. Denote by $A(K)$ the area of K. If $A(K) \geq \frac{A}{9}$, the required subset consists of one circle, which is K. Otherwise, denote by $3K$ the circle concentric with K with radius $3R$. If a circle C of M has a common point with K, then $C \subset 3K$, because R is the largest radius of the circles of M, (Figure 3.1). Because $A(3K) = 9A(K) < A$, there are circles of M that do not have a common point with K. Denote the set of these circles with N. Obviously, the area of the union of the circles of N is not less than $A - 9A(K)$. According to the inductive assumption, there is a disjoint subset P of N the sum of the area of which circles is not less than $\frac{A-9A(K)}{9} = \frac{A}{9} - A(K)$. Then the set $P \cup \{K\}$ is a disjoint subset of M, the area of whose circles is not less than $\frac{A}{9}$.

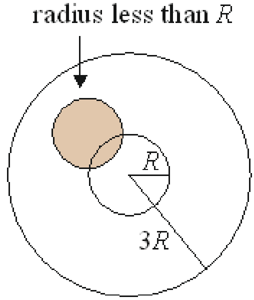

Figure 1: Illustration of the solution of Problem 3.2

I have presented this solution to Problem 3.2 because the same method can be used for the proof of a generalization of this problem. The generalization considers not only circles, but any set of bounded figures. First, we need to get familiar with the notion of a neighborhood.

Denote by $d(X, Y)$ the distance between the points X and Y in the plane. Let K be a bounded figure in the plane. The number $d(K) = \sup_{X,Y}\{d(X, Y); X \in K, Y \in K\}$ is called a *diameter* of K. Let Z be a point in the same plane. We call a distance between the point Z and the figure K the number $d(Z, K) = \inf_{Y}\{d(Z, Y); Y \in K\}$. For any $\varepsilon > 0$, the *neighborhood* $O_\varepsilon(K)$ of K with radius equal to ε, is the set of all points X in the plane with distance not greater than ε apart from K, i.e. $O_\varepsilon(K) = \{X; d(X, K) \leq \varepsilon\}$.

To visualize the notion of a neighborhood, think of the following experiment. Throw figure K in a water and look how the waves are spreading. Their shape has the form of the neighborhood of K.

For example, if K is a circle with radius R, the neighborhood of a circle with radius ε is a concentric circle with radius $R + \varepsilon$. The neighborhood of a square with side a is shown in Figure 2. Its area is equal to

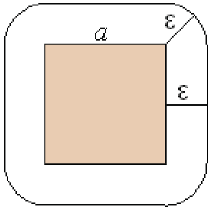

Figure 2: Neighborhood of a square

$a^2 + 4a\varepsilon + \pi\varepsilon^2$. The neighborhood of an equilateral triangle with side a is shown in Figure 3. Its area is equal to $\frac{a\sqrt{3}}{4} + 3a\varepsilon + \pi\varepsilon^2$.

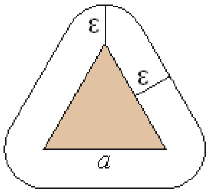

Figure 3: Neighborhood of an equilateral triangle

The next theorem is a generalization of Problem 3.2.

Theorem Let $M = \{K_1, K_2, \ldots, K_m\}$. be a finite set of bounded figures on the plane, the area of which union is equal to A. For every $i = 1, 2, \ldots, m$ let us denote by d_i the diameter of K_i, $\lambda_i = \frac{A(K_i)}{A(O_{d_i}\left(K_i\right))}$, and $\lambda = \min\{\lambda_1, \lambda_2, \ldots, \lambda_m\}$. Then there is a disjoint subset of M, the sum of the area of whose figures is not less than λA.

The proof of the theorem (Bankov, 1996) uses the method of the solution of Problem 3.2.

Namely, let K be the figure of M having the largest diameter among all figures of M. If $A(K) \geq \lambda A$, we are done. If not, consider the set N of the figures of M that do not have a common point with K. Prove that

N is not empty. Use the Principle of Mathematical Induction to select a disjoint subset of N with "large" area, and add K to this subset.

Corollary Let M be a finite set of squares in the plane, the area of which union is equal to A. Then there is a disjoint subset of M, the sum of the area of whose squares is greater than $\frac{A}{13}$.

The statement follows from the fact that in the case of squares.

$$\lambda = \lambda_1 = \lambda_2 = \cdots = \lambda_m = \frac{1}{1 + 4\sqrt{2} + 2\pi} > \frac{1}{13}.$$

State a similar statement for a finite set of equilateral triangles in the plane.

The question now is whether the numbers $\frac{1}{9}$ (in Problem 3.2), λ (in the Theorem), $\frac{1}{13}$ (in the corollary) give the best possible results. This means, whether it is true that if we increase any of these numbers (for example, λ) by a "small" number $\delta > 0$, there is a set of bounded figures that cover the area A but any disjoint subset covers the area less than $\lambda + \delta$. The answer is negative. For example, in Shkljarski at al., 1974, the following problem can be found.

Problem 3.3 A square K of area A is covered by a finite set M of squares, whose sides are parallel to the sides of K. Prove that there is a disjoint subset of M, the area sum of which squares is not less than $\frac{A}{9}$.

Certainly, Problem 3.3 gives a better result that can be obtained by the Theorem.

Here is what is known about the best possible result.

Let Ω be a set of bounded figures in the plane. For example, Ω could be the set of all squares with parallel sides, or the set of all squares, or the set all circles, or the set of all regular polygons, etc. Let $M = \{K_1, K_2, \ldots, K_m; K_i \in \Omega, i = 1, 2, \ldots, m\}$ be a finite set of figures on the plane, each of which belongs to Ω. Denote by A the area of the union of the figures of M, and $D(M) = \{K_{i_1}, K_{i_2}, \ldots, K_{i_n}\}$ any disjoint subset of M, and $A\big(D(M)\big) = A(K_{i_1}) + A(K_{i_2}) + \cdots + A(K_{i_n})$. The main problem is to find the number

$$\mu = \inf_{M \in \Omega} \sup_{D(M)} A\big(D(M)\big).$$

The notation $M \in \Omega$ means that M is any finite set of figures of Ω.

Roughly speaking, to determine μ, consider all disjoint subsets of M; take the one that has the maximum area; then consider all finite sets of figures of Ω and take the minimum across these maximum areas.

Certainly, the number μ depends on the set Ω. The determination of μ is not an easy task. Here are some results.

Let Ω be the set of all *squares with parallel sides.* The Hungarian mathematician T. Rado, 1928, conjectured that $\mu = \frac{A}{4}$. The German mathematician R. Rado, 1950, proved that $\mu > \frac{A}{8.75}$. Ten years later, the Russian mathematician Zalgaller, 1960, proved that $\mu > \frac{A}{8.6}$. Until 1973 the conjecture of T. Rado seems to be true. But the Hungarian mathematician Ajtai, 1973, constructed a set of squares with parallel sides which disproves it. This made the problem much more attractive.

The next open problems focus on concrete sets Ω.

Open Problem 3.4 Find μ if Ω is the set of all squares with parallel sides.

Open Problem 3.5 Find μ if Ω is the set of all circles.

Open Problem 3.6 Find μ if Ω is the set of all squares.

Open Problem 3.7 Find μ if Ω is the set of all equilateral triangles.

Open Problem 3.8 Find μ if Ω is the set of all regular polygons.

Certainly, some other similar open problems arise, which the reader may create themselves.

4 Problems on a Square Grid

This part is inspired by the "Chess 7×7" problem, presented by A. Soifer on the 21st Colorado Mathematical Olympiad, April 16, 2004 (Soifer, 2016). Even though the problem is presented as a play between two chess teams, it is equivalent to the following:

Problem 4.1

a) 22 squares are colored in a square grid 7×7. Prove that there are 4 colored squares that form a rectangle.

b) Prove that the statement in a) is not true if 21 squares are colored.

This problem gives the best possible results, i.e. it determines the minimum number of squares that must be colored in a 7×7 square grid such that there are 4 colored squares that form a rectangle.

Denote by $C(n)$ the minimum number of squares that have to be colored in a square grid $n \times n$, such that there are 4 colored squares that form a rectangle. Problem 4.1 shows that $C(7) = 22$.

Open Problem 4.2 Find $C(n)$ for any n. (It is interesting to consider values of n that are greater than 7.)

I am finishing this paper with one more problem for coloring squares on a square grid that also gives rise to an interesting open problem.

Problem 4.3

a) 6 squares are colored in a square grid 4×4. Prove that there are two rows and two columns that contain all colored squares.

b) Prove that the statement in a) is not true if 7 squares are colored.

Solution.

a) Because of the Pigeonhole Principle, there is a row that contains at least two colored squares. If this row contains exactly two colored squares, there is one more row with at least two colored squares; the remaining two colored squares may be chosen in two columns. If this row contains more than two colored squares, we choose one more row with a colored square; the remaining two colored squares may be chosen in two columns.

b) An example is presented on Figure 4.

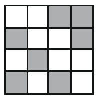

Figure 4: Example for part b)

Denote by $D(n;k)$ $1 < k < n$, the maximum number of squares that have to be colored in a square grid $n \times n$, such that there are k rows and k columns that contain all colored squares. Problem 4.3 shows that $D(4;2) = 6$.

Open Problem 4.4 Find $D(n;k)$ for any $n > 4$ and $1 < k < n$.

This is a version of a plenary talk for the section "Creating Problems and Problem Solving" of the 8th Congress of WFNMC, Graz, July 2018.

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The Julia Robinson Mathematics Festival A Complement and Alternative to Competitions

Mark Saul

Mark Saul is the Executive Director of the Julia Robinson Mathematics Festival. Saul grew up in the Bronx, got his BA from Columbia University and his Ph.D. from New York University. He then spent 35 years in and around New York, teaching mathematics in classrooms from grades 3 through 12. He initiated a student exchange program between Russian and American students, as well as an "Intel/Westinghouse" style competition for students of mathematics in China. He has given talks and led workshops in more than 20 countries, including Russia, Bulgaria, China, Saudi Arabia, South Africa, and India. He served as President of the American Regions Mathematics League, mathematics field editor of Quantum (the English-language version of the Russian journal Kvant).

1 An Autobiographical Introduction

Like many students with an interest and ability in mathematics, I rarely learned anything new in my standard math classes, until college. I always read ahead in the book, always had fun with the new concepts I learned, always got high grades on my tests—and rarely did much in class but anticipate the teacher in my mind while politely waiting for a 'stretch' question, or looking for a new way to derive the result being discussed.

This changed when I got onto the math team. There we did hard problems. There I learned the joy of struggling with mathematics, and made friends doing it. The only interesting mathematics I encountered before college was in contests and contest preparation.

This was half a century ago, and things have certainly changed. Interested students now find a landscape of advanced coursework, after school activities, summer programs, and web resources. These resources provide access to exciting and challenging mathematics that students can work on at their own pace. And students of more modest ability, or lower motivation, also find activities that stretch their idea of mathematics and the role it can play in their lives.

The point of my autobiographical sketch is that competition is a vehicle. We use it to stimulate challenging mathematical thought and for motivating the study of advanced mathematics.

My most recent work is on another vehicle that takes us to the same place: the Julia Robinson Mathematics Festival. We have developed, test driven, and brought it to scale. In many ways, it builds on and extends our experience with competitions, and offers mathematics to students for whom competition may be less attractive than we might wish.

2 What is a Julia Robinson Mathematics Festival?

A Julia Robinson Mathematics Festival (JRMF) is a non-competitive mathematical event that occurs outside of the formal classroom. A local host secures a space with a capacity of about ten tables, with room to move around between the tables. Each table is dedicated to a game, puzzle, problem set, or other mathematical activity, and each table has a 'facilitator': a mathematician, professional, or older student, who manages the activity.

The local host invites about 100 students (roughly 10 per table) into the space for 2–3 hours. Students visit the tables, find an activity that attracts them, and work on it for as long as they choose. They can take the problems home for further study. The facilitator at the table provides hints and materials, manages the social environment, and finds ways to prevent the students from getting frustrated with the sometimes difficult problems. The facilitator does not teach or explain. Rather, the learning is guided by the students themselves and their interactions with the activity.

There is usually no schedule to be followed, nor is there any evaluation of the students' achievement (except to correct errors that students may not see themselves). Students go as far as they want, using as much time as they want. We, however, evaluate the quality of our activities by collecting data on how long students work on each activity, rather than how far they get in solving a problem. Students typically form impromptu groups and make new friends as they work on the activities. Their motivation comes from their natural sense of intrigue, and from the social situation.

While the local host provides a venue, tables, and facilitators, the national organization, a not-for-profit program within the American Institute of Mathematics, lends support in various ways. First and foremost, the national organization maintains a data bank of more than 100 activities from which local hosts can select. We are also developing 'facilitators' guides' for many of these activities. We are constantly adding to and improving our activities, largely based on feedback from the field. The chief criterion for selecting activities is their holding power: we collect information on how long students are engaged in each activity at a given age.

The national JRMF organization can also provide facilitator training. Sometimes we arrange for a visit by an experienced Festival host, who will work for an hour or so before the Festival to prepare facilitators for the experience. Often this is done by pairing facilitators. First one facilitator acts as a student, exploring the activity of the other facilitator (which he or she has not yet seen). Then the two switch roles. This process typically takes an hour or so. Since the facilitators will be doing in the training exactly what they do with the students in the Festival, they can segue directly into the Festival following the training.

The national organization also offers additional materials (at cost) to local hosts—tee shirts, table cloths, banners, posters, and press releases—that can help with local organization. Often, we can put local hosts in touch with local volunteer facilitators, groups of student participants, or even local funders.

3 The Audience

A JRMF can reach a wide variety of students, including groups not often served by competitions. Among the almost 200 Festivals we have held so far, many have been for students as advanced as high school and others for children as young as age ten. By including their parents in the activity, we have held Festivals serving even younger students. Festivals have been held in which most of the students have had experience in high-level national competitions, and also in schools and communities whose students have not had access to competitions.

Notably, our Festivals attract and retain girls in equal numbers to boys, continuing all the way through high school age. This is in sharp contrast to the experience of the competition community. (Some of the best data on this phenomenon can be found in Glenn Ellison and Ashley Swanson, "The Gender Gap in Secondary School Mathematics at High Achievement Levels: Evidence from the American Mathematics Competitions", *Journal of Economic Perspectives* Volume 24, Number 2, Spring 2010, Pages 109–128 (see `https://economics.mit.edu/files/7598` (accessed June 2018); see also `https://theatlantic.com/education/archive/2016/04/girls-math-international-competiton/478533/` (accessed June 2018)). In every Festival we have observed, girls are close to 50 % of the population, working side by side with boys and in the same roles. The exception is Festivals which are organized solely for girls, which have been popular and successful.

Of particular interest is the success of Festivals in working class and high-poverty areas. Schools in these areas tend not to participate in competitions. Why would they want to be compared to schools whose students have many more resources, and have had a tradition of training and success in competition? But in a Julia Robinson Mathematics Festival, no schools are being compared. So a Festival is a useful way to get high-level mathematics into these schools and communities.

4 The Content

Without the long history of mathematical competitions, it would be difficult to find content suitable for a JRMF. Indeed, the founder and originator of the idea, Nancy Blachman, began by working mathematics problems with her father, problems that were meant to select students for an on-site contest at St. Mary's College in Moraga, California. She enjoyed solving these qualifying problems with her father much more than solving contest problems by herself.

So Blachman 'detached' the mathematics from the competition, using social engagement rather than competition as motivation. She found ways to provide opportunities for many students to work on problems together with a facilitator, someone who asked more questions than they answer—just as her father had done. The first Julia Robinson Mathematics Festival was hosted by Google in their cafeteria on April 22, 2007.

The development of activities for a JRMF involves many sources, and not just competitions. Researchers have contributed 'slices' of their work. We have reshaped old puzzles and bits of mathematical folklore to fit the genre. Some examples appear below.

Example 1 (Square puzzles) These are courtesy of Gordon Hamilton of Math Pickle (`http://mathpickle.com/`). These puzzles may seem simple, but for elementary school students, they require a certain amount of insight. For middle school students, an algebraic version is a useful extension. Older students can be invited to construct such puzzles, a task that can be challenging. The example shown below involves arithmetic, but there are also puzzles involving algebraic expressions for the sides of the squares.

Note that the large rectangle containing the squares may not itself be a square.

Example 2 Digit sums and graphs. This set of puzzles was written by Josh Zucker, based on an idea of Erich Friedman (`https://www2.stetson.edu/~efriedma/mathmagic/1208.html` accessed June 2018).

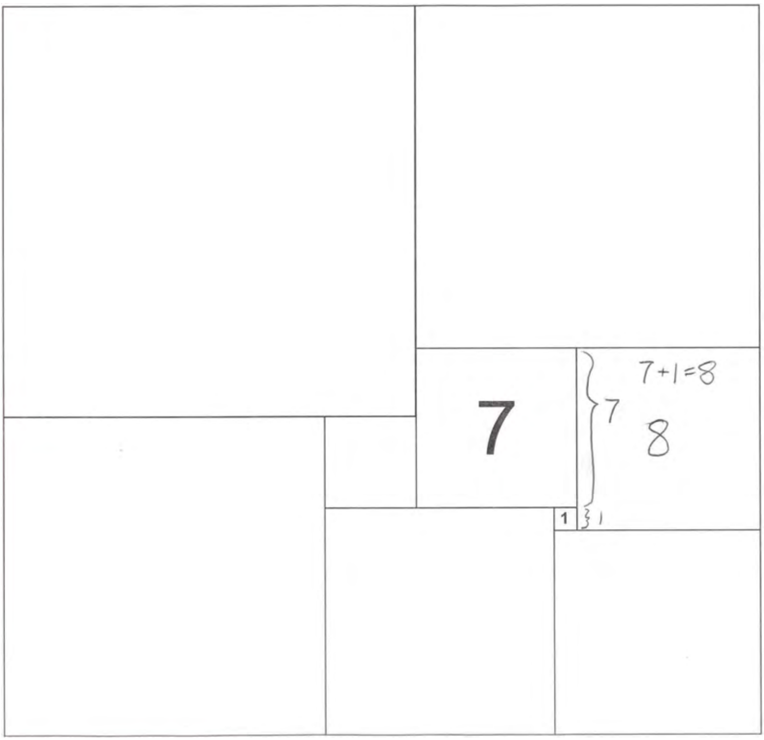
Julia Robinson Mathematics Festival

Squaring Puzzles

by Gord Hamilton, Math Pickle

These abstract squaring puzzles give students addition and subtraction practice with numbers usually below 100. They also link these numerical activities to geometry. What a beautiful way to practice subtraction! *—Gord Hamilton, Founder of Math Pickle.*

The number in each square represents the length of a side of that square. Determine the length of a side of all the squares in this rectangle and the lengths of the sides of the rectangle.



Find more square and subtracting puzzles here:
mathpickle.com/project/squaring-the-square/.

Simply understanding what is required can be difficult even for high school students. The construction of such a puzzle can require significant analysis.

Example 3 Color Triangles. This example illustrates three types of problems:

1. Part of the attraction is the manipulative used to state the problem. Younger students are attracted by the bright colors, and stay to work on the problem.
2. The problem has a low threshold and a high ceiling. Anyone can understand it and explore the situation. But a full solution is quite difficult.
3. The problem is not one which JRMF participants are expected to solve—and certainly not in the time allotted to a Festival. Exploring the problem is productive, whether or not a solution is achieved.

The activity: A row of five discs is laid out, each disc one of three colors (say red, yellow, blue). A row of 4 discs is constructed below it, using the following rules:

- If the two discs above the new disc are the same color, the new disc is this color.
- If the two discs above the new disc are of different colors, the new disc is the third color.

A new row of 3 discs is constructed below the four discs, then a row of two discs, then a 'row' consisting of a single disc.

The problem is to determine the color of the last disc, given the colors of the first five discs.

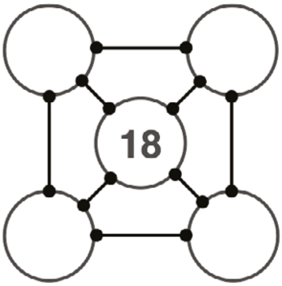
A general solution to the problem can be found at `https://ijpam.eu/contents/2013-85-1/6/6.pdf` (accessed June 2018). Students typically go through several stages of deliberation in working with the problem. The first stage is simply understanding the construction of each new row. Sometimes the students don't see that the color of the final disc is determined, and think they are being asked to guess it, as if it were a game of chance. For some students, comprehending the deterministic nature of the situation is itself a breakthrough.

Julia Robinson Mathematics Festival®

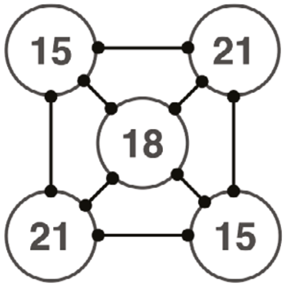
Digit Sums & Graphs

In each diagram, fill in the circle with positive whole numbers in such a way that each circle's number is the sum of the digits of all the numbers connected to it. Thanks to Erich Friedman for this idea!

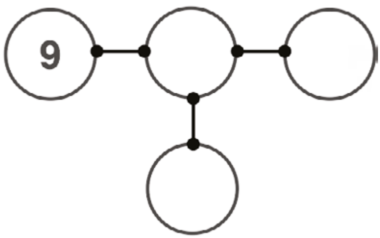
EXAMPLE

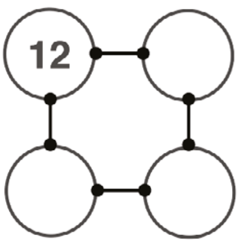


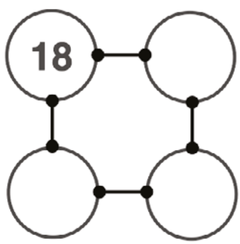
SOLUTION

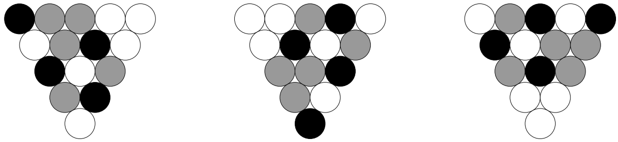


The solution works because
15 = (2+1) + (1+8) + (2+1) for the two corners
21 = (1+5) + (1+8) + (1+5) for the other two corners
18 = (1+5) + (2+1) + (1+5) + (2+1) in the center









Students later are able to notice "local" patterns. For example, if a row consists of discs of all one color, then all the discs beneath that row are that color, and the last disc is also that color. The process of forming and testing hypotheses is motivated by the complexity of the patterns and by the social situation.

One pattern that students notice, which may not be particularly helpful, is that we can 'read' the completed triangle from different directions: if we rotate a completed triangle by 120 degrees (or simply consider a different one of its sides as having generated the triangle), the new triangle obeys the same rules as the original.

Another insight students have is that the initial row need not be of length 5. It can be of any length. Typically, this starts students on a path of case-by-case solution, which is too complicated to succeed. However, they learn a lot by experimenting.

A full solution rests on the observation that if we start a triangle with four discs (rather than five), then the final disc can be predicted from just the first and fourth disc. The middle two discs do not affect the outcome. The same is true for a row of ten discs, or of 28, or of $3^n + 1$ discs. Students often have trouble seeing this, and the pedagogical problem of how to support them in the discovery is so far unsolved.

The Color Triangles is a clear example of a problem from which students can learn without actually having achieved a solution.

For more on this problem, see: `https://wordplay.blogs.nytimes.com/2013/05/13/triangle-mysteries/?mtrref=www.google.com\&gwh=E2DE0F4FD0E6999B0357F29804B6DDC5\&gwt=pay`, or `https://static01.nyt.com/images/blogs/wordplay/posts/Triangle_Mysteries_Behrends_Humble.pdf` (Both accessed July 2018.) There has been extensive online discussion of this problem, including a number of connections to deeper mathematical topics.

Example 5 Nim. This is of course an entire set of games, some very difficult and some unsolved.

We usually start with "one row nim": A row of N counters are set up and players take turns. In one turn, they can remove $\{1, 2, 3, \ldots, M\}$ counters, where M and N can vary. (A typical initial game is $M = 5$, $N = 12$). The winner is the person who takes the last counter. (Or, the loser is the person who cannot move.)

The general solution appears after much experimentation, in the form of game playing. Festival attendees play against each other or against the facilitator. The facilitator will not advise the students on strategy, except to guide them on what to observe. And the facilitator does not always arrange to win.

For most values of N, the first player has a winning strategy: a row of counters which is a multiple of $M + 1$ is a losing position. From any other position, a player can put her opponent in a losing position. So unless the initial row is a multiple of $M + 1$, the first player wins.

A more detailed description of typical student experiences can be found at `https://cims.nyu.edu/cmt/assets/pdfs/AA\_Problems/OneRowNim.pdf` (accessed June 2018). (This reference discusses the *misère* form, in which the player who takes the last counter loses, but the analysis and description of the learning can be easily adapted to the present problem.)

Another nim game often used is two-row nim, in which players can take as many counters as they want from either of two rows. See `https://cims.nyu.edu/cmt/assets/pdfs/AA\_Problems/OnePieceChess.pdf`, accessed June 2018.

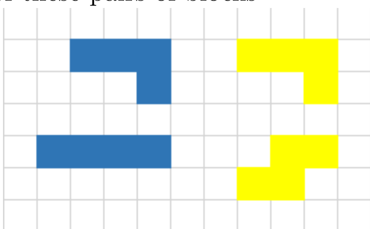
The best known version of nim involves three rows, and has a solution using binary arithmetic. It is a difficult pedagogical problem to go from the two-row version to the three-row version without simply teaching the students the computations involved in the binary arithmetic solution. This should not be done, and certainly not at a JRMF: the students will not learn much more than how to win this particular game.

Another difficult, but interesting, variant of nim starts with N counters, but the players can take only 2 or 3 counters. A charming account of this game is given in А. Шень, Игриы и стратегии с точки зрения математики, издательство МЦНМО, москва 2007, available at http://math.ru/lib/files/pdf/shen/Shen-games.pdf (accessed June 2018.)

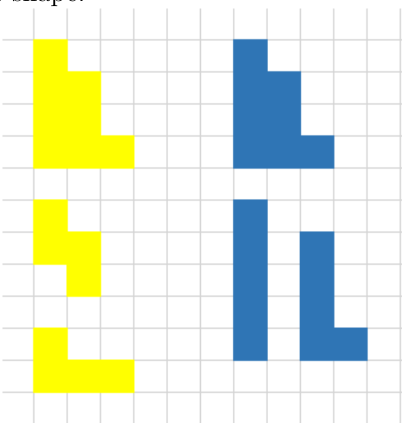
Example 6 Toys: We often use commercially available activities. One of our favorites is Shapeometry, developed and marketed by ThinkFun (`https://www.thinkfun.com/products/shapeometry/`, accessed June 2018).

Shapeometry uses a manipulative consisting of plastic tetrominos and pentaminos. The player is give two sets of pairs of pieces, and is asked to make the same shape with each pair. The target shape is not given.

For example, each of these pairs of blocks



can form the same shape:



But it is not easy to see what that shape might be or how to arrange the pieces.

There are about 60 problems, arranged by level of difficulty. Shapeometry offers an intuitive introduction to geometric transformations and an intuitive ideas of area. Because the study of geometry in schools has recently been suffering—worldwide—we are particularly eager to include geometric activities.

5 A comparison of JRMF and Competition Materials

Here is an extended example of a problem situation that can be adapted to various levels of competition as well as to a festival. We give some examples of how the same mathematical content can appear in these different educational situations.

Broken Calculators I've got a collection of calculators someone left me. Unfortunately, all of them are broken, each in a different way. Can I use them to do calculations?

Every calculator displays 0 at the beginning.

Calculator 1

There are only two buttons that do anything on this calculator. Button A adds 3 and button B adds 7.

Problem 1 (format for an elementary short answer contest): What will I get if I press $ABAB$?

(That is, first I press button A, then button B, then button A again, then button B again.)

Problem 1 (format for a JRMF): Show that I can make the calculator show 20 by hitting $ABAB$. What do I get when I hit $ABBA$?

Discussion: Notice that the student can draw his or her own conclusions in each case, but that the JRMF version invites the student to generalize. The insight here is that the order in which we press the buttons will not matter.

Problem 2 (format for a middle school contest): What is the largest whole number that I can NOT make by pressing keys on this calculator?

Problem 2 (format for a JRMF or an elementary school Olympiad): It's pretty easy to see that I can't get the calculator to show 8. What other numbers are impossible? What is the largest positive integer you can't display on calculator 1?

Solution: First two general notes:

a) The order in which we press the buttons doesn't make a difference, since we are just adding (and addition is commutative).

b) Every time we press another button we increase the display on this calculator. We can never make the display smaller.

From (b), it's pretty clear that we cannot get 1 or 2.

We can get 3 simply by pressing button A.

Suppose we could get 4. We can never have pressed B, or we would overshoot our goal right away. If we had 4, then the button press before that must have shown a 1, and we have already seen that this is impossible. Thus we cannot get 4.

Similar reasoning shows that we cannot get 5.

We can get 6 by pressing AA.

We can get 7 by pressing B.

We can show that we cannot get 8 by direct computation: we cannot have more than 2 presses of A, and not more than 1 press of B. No combination works.

Or, we could reason as before: The order of our buttons doesn't matter for this calculator. If we got 8 as a sequence of button presses, we can arrange their order to have all the A's first, then all the B's. The last press cannot be an A, since then there would be no B's, and we would get a multiple of 8. And if the last press is a B, then the calculator must have shown 1 before the B button was pressed, and we have already seen that this is not possible.

$$9 = AAA, \qquad 10 = AB.$$

We can show that because we cannot get 8, we also cannot get 11. Suppose 11 were showing. The last button pressed was either A or B.

If the last press was A, then before this the number shown was $11-3 = 8$, and we already know this is impossible. If the last button pressed was B, then before this the number shown was $11 - 7 = 4$, and we already know this is impossible. So we cannot get 11.

$$12 = AAAA, \qquad 13 = BAA, \qquad 14 = BB.$$

And now we note that if we can get a number N, then we can get $N+3$. So from 12 we can get 15, 18, 21,...; from 13 we can get 16, 19, 22,...; and from 14 we can get 17, 20, 23,... Every number greater than 11 belongs to one of these sequences.

So the largest impossible number is 11. (Note that we used no algebra in writing this proof.)

Calculator 2

On calculator 2, button A adds 1, and button B multiplies by 3.

Problem 3 (format for a middle school short answer contest): When I press $ABBAAAA$, I will get 13. But there is a shorter sequence of button presses that will get me 13. Can you find it?

Solution: We work backwards, and try to get 1 from 13. We cannot have pressed B to get 13 because 13 is not divisible by 3. So the last button pressed must have been A, and the last number shown on the display must have been 12.

How can we get 12? Well, if we had pressed B to get 12, then the display must have held 4. And it is not hard to see that we can get 4 by pressing ABA. The complete sequence is $ABABA$.

(Extra credit. Or a discussion after the contest): Let us try to form all sequences that give us 13. As in the solution above, the first and last button pressed must be A.

$$Axx\ldots xxA.$$

If the next to last button is B, we will get $ABABA$, by the reasoning above. (Well, not quite. We never showed that there is any other such sequence. But it's not hard to add to the reasoning to show this.)

If the next to last button is A, then either we have all A's (length 13, the maximum possible), or we must keep subtracting 1 until we have another multiple of 3. We then have $Axxx\ldots xAAAA$. Before the four A's are pressed, we would have had 9. There are only a few ways to get 9.

How can we get 9? We continue to work backwards. If the last button pressed was B, then we must have had 3, which forces us to have ABB. This gives us the string in the problem. And if the last button pressed was A, we need to get to a multiple of 3, and we have $Ax\ldots xAAA$. Repeating the reasoning above, we must have had 6 before the last 3 As.

How can we get 6? Well, we certainly can use $AAAAAA$. This again gives us thirteen A's to get 13.

But can we get 6 with B? Yes: AAB. This gives us $AABAAAAAAA$, which exhausts the possibilities.

In summary, $13 = AAA\ AAA\ AAA\,AAA\ A = AAB\ AAA\ AAA\ A = ABB\ AAA\ A = ABABA$.

A case by case analysis, which we will not give here, can show that four presses cannot give us 13.

In a Festival, this question provides a flexible environment. Younger students can explore case-by-case, as shown above. Older or more advanced students can reason more succinctly.

Problem 4 (In an Olympiad environment): Determine with proof the smallest number of button presses it takes to get to 102. What about 511?

In a JRMF environment, we can simply ask:

What is the smallest number of button presses it takes to get to 102? What about 511?

Then the facilitator can see how formal the discussion should be.

Here is a "naïve" solution, which you can expect from a good middle school student:

The number 102 is 3×34. So if we can get 34, one more press of button B will give us 102. In turn, 34 is $3 \times 11 + 1$. So if we can get 11, two more presses gives us 34. We can get 11 by pressing $ABBAA$. The complete string that gives 102 is then $ABBAABAB$.

For 511: $510 = 3 \times 170$, so if we can get 170, one more press of button B will give us 510, then pressing A gives 511. So we need to get 170, after which BA gives 511. $170 - 2 = 168 = 3 \times 56$. So we need to get to 56, after which BAA gives 170. $56 - 2 = 54 = 3 \times 3 \times 6$, so if we get 6, then $BBAA$ will give us 56. To get six we press AAB. The complete sequence is $AAB(6)BBAA(56)BAA(170)BA(511)$.

The 'backwards' form of the algorithm seems to be: go back to the lowest multiple of 3, then divide.

But to show how these problems have a 'high ceiling', we give a much more sophisticated solution, due to Behzad Mehrdad:

The operations that this calculator performs can be described in base 3 as A: adding 1 to the numeral, and B: shift the digits over to the left one place and filling the rightmost place with a 0. Note that the calculator doesn't actually show base 3 computations. It's something we have in our heads to explain the workings of the calculator.

For example, if we want the display to show 49, we convert the number we want to base 3: $49 = 1\times27+2\times9+1\times3+1 = 1211_3$, and this numeral actually gives us instructions to generate 49 on our broken calculator:

We press A to get a 1. This will end up as the leftmost digit 1. Then we press B to shift this digit over. We get 1×3 (without simplifying!). Then we press A twice to add 2. We get $1\times3+2 = 5_{10} = 12_3$. Then we press B to shift these digits over. We get $3(1\times3+2) = 1\times9+2\times3 = 15_{10} = 120_3$. Then we press A to add 1. We get $1\times9+2\times3+1 = 16_{10} = 121_3$. Then we press B to shift digits over. We get $1\times27+2\times9+1\times3 = 48_{10} = 1210_3$. Finally, we press A to add one, and get our result.

The complete string is then $ABAABABA$.

This algorithm gives us some bonuses. First, it shows that we don't need to press A more than twice in a row (because there is no need for digits larger than 2 in ternary notation). Perhaps more importantly, it shows that any shorter string of A's and B's would generate a smaller number, so this must be the shortest length string.

In just the same way, the rest of the following JRMF problem set can be adapted to furnish contest problems of various formats.

The complete JRMF problem set for the Broken Calculator Problem appears in an appendix to this article. The reader is invited to solve the problems, and also to think about how they might be re-formatted for a competition.

6 Conclusion

Mathematics has lent itself to competition since ancient times (Archimedes' "Cattle Problem"). And competition has sometimes stimulated significant results (the solution of the general cubic and quartic equations by the mathematical competitors of the Italian renaissance, Stokes' Theorem).

But mathematics is flexible, and some competitions do not attract some very important groups of students. A Julia Robinson Mathematics Festival offers deep mathematics to a broad variety of students. Other places students can learn informal mathematics include recreational journals, math circles, and web pages. We are fortunate to be living in a time when all these vehicles for mathematical activity are evolving rapidly.

7 Acknowledgments

The following people and organizations have given me invaluable help in writing this article: Nancy Blachman, Dan Finkel, Gordon Hamilton, Scott Kim, Alice Peters, Bill Ritchie, Thinkfun, Inc., Josh Zucker. Activities due to others are used with permission.

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APPENDIX: The ‘Broken Calculator’ Activity
as used in many Julia Robinson Mathematics Festivals

(Original problem by Josh Zucker. This version by M. Saul and B. Mehrdad, December 2016)

Broken Calculators

I’ve got a collection of calculators someone left me. Unfortunately, all of them are broken, each in a different way. Can I use them to do calculations? Every calculator displays 0 at the beginning, except calculator (5).

Calculator 1

There are only two buttons that do anything on this calculator. Button A adds 3 and button B adds 7.

Problem 1: Show that I can make the calculator show 20 by hitting ABAB. What do I get when I hit ABBA?

Problem 2: It's pretty easy to see that I can't get the calculator to get 8. What other numbers are impossible? What is the largest positive integer you can’t display on calculator 1?

Calculator 2

On calculator 2, button A adds 1, and button B multiplies by 3.

Problem 3. When I press ABBAAAA, I will get 13. But there is a shorter sequence of button presses that will get me 13. Can you find it?

Problem 4: What is the smallest number of button presses it takes to get to 102? What about 511?

Calculator 3

On calculator 3, button A adds 6, button B divides by 2, and button C divides by 3.

Problem 5: Show how you can get Calculator 3 to display any positive integer.

Problem 6: What is the smallest number of button presses it takes to display 99 on calculator 3?

Calculator 4

On calculator 4, button A adds 5, button B adds 7, and button C takes the square root.

Problem 7. Prove that it is possible to display any positive integer greater than 1.

Calculator 5

Calculator 5 has two memory slots. Wherever you are in your sequence of button presses, there is a number displayed on the calculator. At that point, memory slot Alpha holds the last number you've generated, and memory slot Beta holds the number you generated before that. Both memory slots start with 0.

On this calculator, button A adds the contents of Alpha to the current number, and button B adds the contents of Alpha plus the contents of Beta to the current number.

This calculator always displays 1 at the start, with 0 in each of the memory-slots.

Problem 8: What do you get if you hit button A once?

Problem 9: What do you get if you hit button A for a second time?

Problem 10: ….and again?

Problem 11: What happens if you keep hitting button A? Write out the first ten numbers in the sequence of numbers displayed.

Problem 12: What sequence is generated if you keep hitting button B? Write out the first ten terms.

Problem 13 (Difficult): Can this calculator display any positive integer?

An Exploratory Problem

Ryan Fang, Caleb Ji & Evan Liang

We recently attended an Exploratory Problem Session, during which the participants made individual contributions to the collective effort in solving the problems. We present one of them here.

Problem For which positive integers k is there a configuration of 100 arcs on a circle such that each arc intersects exactly k others?

It was immediately observed that $k = 1$ is possible. We could just have 50 disjoint stacks of 2. This idea yielded the solutions for the first group of values $k = 3$, 4, 9, 19, 24, 49 and 99. We simply took 25, 20, 10, 5, 4, 2 and 1 disjoint stacks of 4, 5, 10, 20, 25, 50 and 100, respectively.

We worked on $k = 2$ for a while until someone came up with the construction of 100 arcs "holding hands" around the circle. This led to the solutions for the second group of values $k = 5$, 11, 14, 29, 59 and 74. We simply took 50, 25, 20, 10, 5 and 4 stacks of 2, 4, 5, 10, 20 and 25, respectively, "holding hands" around the circle.

We had some difficulty with the next case $k = 6$, since 100 arcs could not be divided into equal stacks of 3 or 7. So we re-examined our construction for second group. Here we marked on the circle a number of evenly spaced points, and each arc started and ended at these points. We called this a canonical representation. The representations in the first group were also canonical, since each arc started and ended at the same point.

The question was raised whether any valid construction could be represented canonically. We claimed that the answer was affirmative. What was needed was the trimming off of redundant parts of the arcs. This we would achieve in two stages.

Suppose one arc was entirely contained in another without being identical to it. Since both arcs intersected exactly k other arcs, we could shrink the longer arc so that it coincided with the shorter one without disturbing the intersection pattern.

We now assigned the clockwise direction to the arcs and marked all the starting points, making them evenly spaced. Suppose there was an arc whose end point was unmarked. We could retract this end point counterclockwise until it reached the first marked point. As we had seen, this might result in a degenerate arc of length 0. Again the intersection pattern was not disturbed. Thus the claim was justified.

We returned to tackle the case $k = 6$. We marked the 100 starting points $P_1, P_2, \ldots, P_{100}$. We considered the arc which started at P_m for some integer m. If it were to intersect exactly 6 other arcs, they should be the 3 which started at P_{m-3}, P_{m-2} and P_{m-1} along with the 3 which started at P_{m+1}, P_{m+2} and p_{m+3}. This would be the case if the arc ran from P_m to P_{m+3}, so that its length is 3. This new construction allowed us to solve all the cases where $k = 2t$ is even. We just made t the common arc length.

Someone made the observation that in all of the solved cases, the length of the arcs was constant. Would this always be true? After some discussion, we came up with a simple proof.

We would divide the circle into n equal parts by the marked points $P_1, P_2, \ldots, P_n$. An arc starting from P_2 must be at least as long as an arc starting from P_1, since it was not contained in the latter. Similarly, an arc starting from P_3 must be at least as long as an arc starting from P_2, and so on. Going around the circle, an arc starting from P_1 must be at least as long as an arc starting from P_n. It follows that all arcs have a common length.

We were now ready for the final assault. We let a_m be the number of arcs starting from P_m, $1 \le m \le n$. Then the number of arcs intersecting an arc which started from P_m would be

$$a_{m-t} + \cdots + a_{m-1} + (a_m - 1) + a_{m+1} + \cdots + a_{m+t} = k,$$

t being the common arc length. Summing from $m = 1$ to n, we obtained $100(2t + 1) = n(k + 1)$ since $a_1 + a_2 + \cdots + a_n = 100$. We now observed that 8 could not divide $100(2t + 1)$. It followed that we could not have

$k \equiv 7 \pmod 8$. All other values of k under 100 were possible. The following chart showed values of t and n for possible odd values of k.

k	1	3	5	9	11	13	17	19	21	25	27	29	33
t	0	0	1	0	1	3	4	0	5	6	3	1	8
n	50	25	50	10	25	50	50	5	50	50	25	10	50
k	35	37	41	43	45	49	51	53	57	59	61	65	67
t	4	9	10	5	11	0	6	13	14	1	15	16	8
n	25	50	50	25	50	2	25	50	50	5	50	50	25
k	69	73	75	77	81	83	85	89	91	93	97	99	
t	3	18	9	19	20	10	21	4	11	23	24	0	
n	10	50	25	50	50	25	50	10	25	50	50	1	

It was pointed out that 100 could be replaced by any positive integer. However, it was not expected that such a generalization would require any new ideas.

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International Mathematics Tournament of Towns Selected Problems from the Fall 2017

Andy Liu

1. Among 100 coins in a row are 26 fake ones which form a consecutive block. The other 74 coins are real, and they have the same weight. All fake coins are lighter than real ones, but their weights are not necessarily equal. Find at least one fake coin using a standard two-pan balance only once.

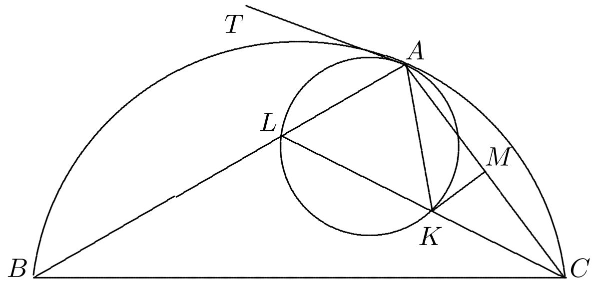
Solution. Weigh #1 to #25 against #76 to #100. If there is equilibrium, all these 50 coins are real. It follows that #50 and #51 are both fake. If there is no equilibrium, we may assume by symmetry that #76 to #100 are heavier. Then these 25 coins are real, whereas at least one of #1 to #25 is fake. It follows that #25 and #26 are both fake.

2. Let M be the midpoint of the side AC of triangle ABC. Let L be the point on AB such that CL bisects $\angle BCA$. The line through M perpendicular to AC intersects CL at K. Prove that the circumcircles of triangles ABC and AKL are tangent.

Solution. Construct the tangent the circumcircle of ABC at A, as shown in the diagram below. Then $\angle TAB = \angle ACB$. Since K lies on the perpendicular bisector of AC, $\angle KCA = \angle KAC$. Since CL is the bisector of $\angle BCA$, $\angle KCA = \angle KCB$. Hence $\angle KAC = \angle KCB$. Now

$$\angle AKL = \angle KAC + \angle KCA = \angle KCB + \angle KCA = \angle ACB = \angle TAL.$$

It follows that TA is also tangent to the circumcircle of AKL at A, so that the two circumcircles are tangent to each other.



3. We have a faulty two-pan balance with which equilibrium may only be obtained if the ratio of the total weights in the left pan and in the right pan is 3:4. We have a token of weight 6 kg, a sufficient supply of sugar and bags of negligible weight to hold them. In each weighing, you may put the token or any bags of sugar of known weight on the balance, and add a bag of sugar so that equilibrium is obtained. Is it possible to obtain a bag of sugar of weight 1 kg?

Solution by Ryan Morrill. Put the token in the left pan and a bag of sugar in the right pan to obtain equilibrium. The weight of the bag is 8 kg. Replace the token by a bag of sugar to obtain equilibrium. The weight of the bag is 6 kg. Put the token and the 6-kg bag in the right pan. Put the 8-kg bag in the left pan and add a bag of sugar to the left pan to obtain equilibrium. The weight of this bag is 1 kg.

4. One hundred doors and one hundred keys are numbered 1 to 100 respectively. Each door is opened by a unique key whose number differs from the number of the door by at most one. Is it possible to match the keys with the doors in n attempts, where

(a) $n = 99$;

(b) $n = 75$;

(c) $n = 74$?

Solution by Ryan Morrill.

(a) For $1 \leq k \leq 99$, try door k with key k. If all attempts are successful, we know everything. Suppose the kth attempt fails for some k. Then key k must open door $k + 1$ while key $k + 1$ must open door k. This actually saves us one question.

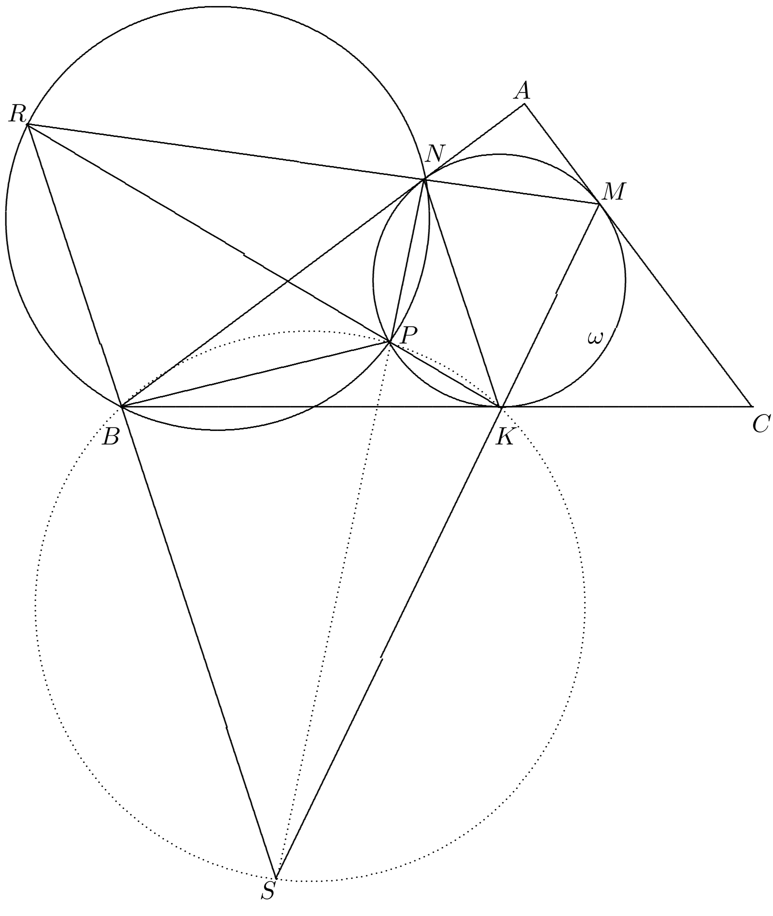
(b) We claim that 3 attempts can settle 4 doors with 4 keys. Try door 3 with key 3. If the attempt is successful, one more attempt will settle doors and keys 1 and 2. Even if the doors and keys go beyond 4, one more attempt will settle door 4 and key 4. Suppose the attempt fails. Try door 2 with key 3. If the attempt is successful, then key 1 must open door 1 and key 2 must open door 3. One more attempt will settle door 4 and key 4. Suppose the attempt fails. Then key 3 must open door 4 and key 4 must open door 3. One more attempt will settle doors and keys 1 and 2. Iterating this process, we can accomplish the task in 75 attempts.

(c) Suppose that key k opens door k for $1\ ek \leq 100$ but we do not know that. We claim that we need at least 75 attempts. For doors and keys 1 to 4, there are at least five possible scenarios.
(1) Key k opens door k for $1 \leq k \leq 4$.
(2) There is a switch between 1 and 2.
(3) There is a switch between 2 and 3.
(4) There is a switch between 3 and 4.
(5) There is a double switch between 1 and 2 as well as between 3 and 4.
Since two attempts can distinguish among at most four scenarios, we need at least three attempts to settle doors and keys 1 to 4. The same argument may be applied to each successive block of four doors and keys, justifying our claim.

5. The digits of two integers greater than 1 are in reverse order of each other. Is it possible that every digit of their product is 1?

Solution by Central Jury. Suppose such a pair of positive integers exist. Let the digits of one of them be a_1, a_2, $\ldots$, a_{k-1}, a_k in that order. Since the product of the two numbers ends in a 1, we must have $(a_1, a_k) = (1, 1)$, (9,9), (3,7) or (7,3). Since the product also starts with a 1, only the first case is possible. Now, the product is less than $(2 \times 10^k - 1)(2 \times 10^k - 1) = 4 \times 10^{2k} - 2$, so that it is a $2k - 1$ digits. In the second column from either side when the multiplication is performed, we must have $a_k a_2 + a_{k-1} a_1 + a_2 + a_{k-1} = 1$ since any carrying over will make the leftmost digit of the product greater than 1. Using an analogous argument on successive pairs of columns towards the middle, we arrive at the conclusion that middle digit must be $a_1^2 + a_2^2 + \cdots + a_k^2 \geq 2 \neq 1$. This is a contradiction.

6. The incircle of triangle ABC is tangent to BC, CA and AB at K, M and N respectively. The extensions of MN and MK intersect the exterior bisector of $\angle ABC$ at R and S respectively. Prove that RK and S intersect on the incircle of ABC.



Solution. Let RK cut the incirle ω at P. Let $\angle NKM = \alpha$, $\angle KMN = \beta$ and $\angle MNK = \gamma$. Then $\alpha + \beta + \gamma = 180°$. Now $\angle RPN = \beta$ since it is an exterior angle of $KMNP$. We also have $\angle BNK = \beta$ since BN is tangent to ω. Finally, since NK and RS are both perpendicular to the bisector of $\angle ABC$, $\angle RBN = \beta = \angle RPN$, so that $BPNR$ is cyclic. Note that $\angle MSR = \alpha$ and $\angle MRS = \gamma$. Hence $\angle NPK = 180° - \alpha$ and $\angle NPB = 180° - \gamma$. It follows that $\angle BPK = 180° - \alpha$, so that $BPKS$ is also cyclic. We have $\angle KPS = \angle KBS = \beta = \angle RPN$. Since P lies on RK, it also lies on NS.

7. Let a positive number be given. A piece of cheese is cut so that a new part is created with each cut. Moreover, after every cut, the ratio of the weight of any piece to the weight to any other one must be greater than r.

(a) Prove that if $r = 0.5$, we can cut the cheese so that the process will never stop.

(b) Prove that if $r > 0.5$, then at some point we will have to stop cutting.

(c) What is the greatest number of parts we can obtain if $r = 0.6$?

Solution by Howard Halim.

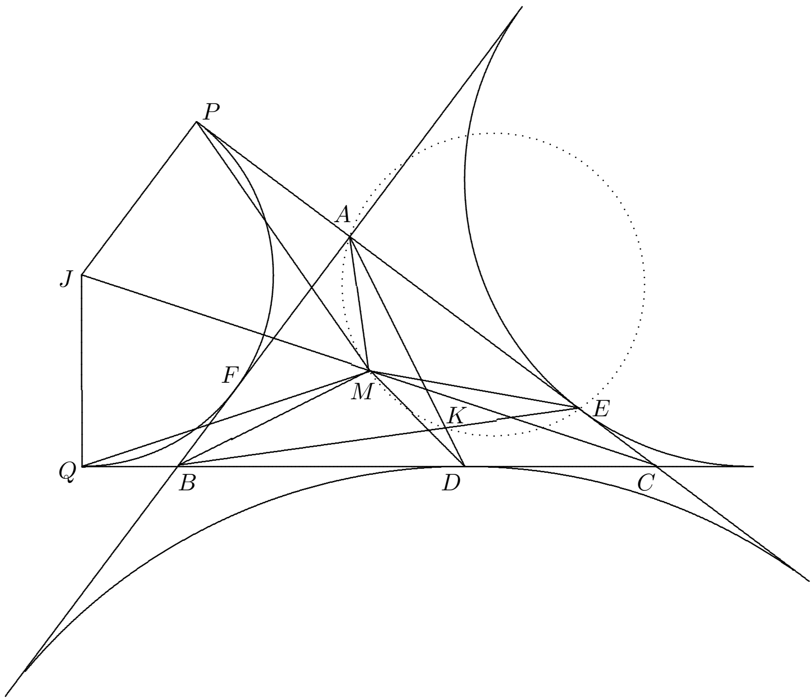
(a) Let the initial piece be of weight 1. We can cut it into two pieces of weight $\frac{1}{2}$. We then cut each into two pieces of weight $\frac{1}{4}$. Continuing this way, we will always have pieces of only two different weights, one being half of the other.

(b) Let $r > 0.5$. Suppose to the contrary that cutting can continue forever. Consider any two pieces A and B with respective weights a and b where $a \geq b$. Then A must be cut before B, as otherwise one of the pieces resulting from cutting B has weight at most $\frac{b}{2}$ and hence less than ra. Suppose the pieces resulting from cutting A have weights c and d with $c \geq d$ and $c + d = a$. Then $d \leq \frac{a}{2}$, so that $r < \frac{d}{b} < \frac{a}{2b}$. It follows that $\frac{a}{b} > 2r > 1$. Suppose at some stage we have n pieces with respective weights $a_1 > a_2 > \cdots > a_n$. Then $a_1 > 2ra_2 > (2r)^2 a_3 > \cdots > (2r)^{n-1} a_n$. Hence $(\frac{1}{2r})^{n-1} > \frac{a_n}{a_1} > r$. This cannot hold forever as $\frac{1}{2r} < 1$.

(c) From (b), n pieces may be obtained for $r = 0.6$ if $(\frac{1}{2\times 0.6})^{n-1} > 0.6$. This holds for $n-1=3$ but not for $n-1=4$. It follows that at most 4 pieces may be obtained. We now show that 4 pieces may in fact be obtained. We may take the weight of the initial piece of cheese to be 32. We first cut it into two pieces with respective weights 18 and 14, observing that $\frac{14}{18} > 0.6$. Next, we cut the heavier piece into two, each with weight 9, and observe that $\frac{9}{14} > 0.6$. Finally, we cut the piece with weight 14 into two, each with weight 7.

8. The excircles of triangle ABC opposite A and B are tangent to BC and CA at D and E, respectively. K is the point of intersection of AD and BE. Prove that the circumcircle of triangle AKE passes through the midpoint of JC, where J is the excentre of ABC opposite C.

Solution by Central Jury. Let the excircle with centre J touch the extension of CA at P, the extension of CB at Q, and the side AB at F. Then $CA + AF = CA + AP = CP = CQ = CB + BQ = CB + BF$. Hence CF bisects the perimeter of ABC. Thus AD and BE also bisect the perimeter of ABC, so that $AE = BD$. Similarly, we can prove that $BQ = CE$ and $DC = AP$. Let M be the midpoint of CJ. Then CM bisects $\angle BCA$. Since M is the circumcentre of the cyclic quadrilateral $CPJQ$, $MP = MC = MQ$ so that $\angle MPC = \angle MCP = \angle MCQ = \angle MQC$. Perform a rotation about M equal to $\angle CMP = \angle CMQ$. Then the line BC lands on the line CA, with Q landing on C, B on E, D on A and C on P. It follows that $MA = MD$, $MB = ME$, and both $\angle AMD$ and $\angle BME$ are equal to the angle of rotation. Hence AMD and BME are similar isosceles triangles. Thus $\angle DAM = \angle BEM$, so that $AEKM$ is a cyclic quadrilateral.



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