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MATHEMATICS COMPETITIONS

JOURNAL OF THE

WORLD FEDERATION OF NATIONAL MATHEMATICS COMPETITIONS

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The aims of the Federation are:

1. ***to promote excellence in, and research associated with, mathematics education through the use of school mathematics competitions;***
2. ***to promote meetings and conferences where persons interested in mathematics contests can exchange and develop ideas for use in their countries;***
3. ***to provide opportunities for the exchanging of information for mathematics education through published material, notably through the Journal of the Federation;***
4. ***to recognize through the WFNMC Awards system persons who have made notable contributions to mathematics education through mathematical challenge around the world;***
5. ***to organize assistance provided by countries with developed systems for competitions for countries attempting to develop competitions;***
6. ***to promote mathematics and to encourage young mathematicians.***

World Federation of National Mathematics Competitions

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From the President

Dear Federalists,

I am happy to invite you to the 8th International Congress of the World Federation of National Mathematics Competitions (WFNMC). It will take place on July 18–24, 2018 near Graz, Austria, and the Chief Organizer is our Vice President Robert Geretschläger. You will be able to register for the Congress starting in late October 2017 on the Congress web site: `wfnmc-conference.uni-graz.at`.

The program of the Congress will include talks, workshops, mini-courses, problem posing session, plenary talks, and other events. Much of the Congress work will center on the following four themes:

a) Building Bridges between Problems of Mathematical Research and Competitions;

b) Creating Problems and Problem Solving;

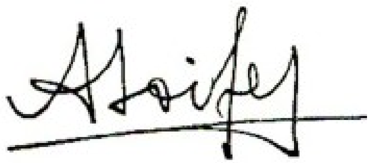
c) Competitions around the World;

d) Work with Students and Teachers.

You are invited to submit proposals for talks, workshops, mini-courses, and other means of entertainment of mathematical kind. Proposals should include titles, authors or organizers, and clear concise descriptions proposals. Please, submit your proposals to both the Chair of the WFNMC Program Committee Kiril Bankov (`kbankov@fmi.uni-sofia.bg`) and WFNMC President Alexander Soifer (`asoifer@uccs.edu`), by January 15, 2018.

At the Congress, we will hold the General Meeting of the Federation, which will include election of the officers. During the IMCI-13 Congress in Hamburg, I put together a group of 18 fine authors, many of whom are members of the Federation. The result is an impressive 400-page compendium *Competitions for Young Mathematicians: A Perspective from Five Continents,* Springer, Germany, July 2017. You can find it on Amazon near you.

I am eagerly looking forward to seeing you all in the Austrian Alps. Meanwhile please accept my best wishes on a Healthy and Prosperous New Year!



Alexander Soifer
President of WFNMC
October 2017

From the Editor

Welcome to *Mathematics Competitions* Vol. 30, No. 1.

First of all I would like to thank again the Australian Mathematics Trust for continued support, without which each issue of the journal could not be published, and in particular Heather Sommariva and Pavel Calábek for their assistance in the preparation of this issue.

Submission of articles:

The journal *Mathematics Competitions* is interested in receiving articles dealing with mathematics competitions, not only at national and international level, but also at regional and primary school level. There are many readers in different countries interested in these different levels of competitions.

- The journal traditionally contains many different kinds of articles, including reports, analyses of competition problems and the presentation of interesting mathematics arising from competition problems. Potential authors are encouraged to submit articles of all kinds.
- To maintain and improve the quality of the journal and its usefulness to those involved in mathematics competitions, all articles are subject to review and comment by one or more competent referees. The precise criteria used will depend on the type of article, but can be summarised by saying that an article accepted must be correct and appropriate, the content accurate and interesting, and, where the focus is mathematical, the mathematics fresh and well presented. This editorial and refereeing process is designed to help improve those articles which deserve to be published.

At the outset, the most important thing is that if you have anything to contribute on any aspect of mathematics competitions at any level, local, regional or national, we would welcome your contribution.

Articles should be submitted in English, with a black and white photograph and a short profile of the author. Alternatively, the article can be submitted on an IBM PC compatible disk or a Macintosh disk. We prefer LaTeX or TeX format of contributions, but any text file will be helpful.

Articles, and correspondence, can also be forwarded to the editor by mail to

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Jaroslav Švrček
October 2017

Call for nominations

Paul Erdös Award 2018

The Awards Committee of the World Federation of Mathematical Competitions hereby launches the process of selecting awardees of the Paul Erdős Prize 2018 by issuing a call for nominations. Up to three awards may be given.

Nominations must be received by the Chair of the Committee by January 31, 2018.

Requirements for Nominations for the Paul Erdős Award

The following documents and additional information must be submitted in English:

- A one- to two-page statement which includes achievements of the nominee and a description of the contributions by the nominee which reflects the objectives of WFNMC.
- Nominees present contact information, home or business address, telephone, e-mail addresses.
- The names and contact information of at least four people who have agreed to act as referees. Note that this is an international award, whereas all the work attributable to the candidate may be entirely within ones own country, the award is directed at how the nominee is viewed internationally and it is highly recommended that one or two of the referees be from countries other than the nominees own.

Nominating Authorities

The aspirant to the Award may be proposed through the following authorities:

- Chair of the Awards Committee, Mara de Losada `mariadelosada@gmail.com`
- President of the World Federation of National Mathematics Competitions, Alexander Soifer `asoifer@uccs.edu`

- Members of the World Federation of National Mathematics Competitions Executive Committee
- Regional Representatives of WFNMC

The Federation encourages the submission of such nominations from Directors or Presidents of Institutions and Organisations, from Chancellors or Presidents of Colleges and Universities, and from members of the international community working in mathematics competitions.

How Does Age Factor Matter a Case Study

Borislav Lazarov & Albena Vassileva

Borislav Lazarov is an associate professor in the Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, where he had received his PhD degree in 1998. As a student he participated the 19th IMO. He is the Head of the Educational and Research Program Chernorizec Hrabar *and the Chairman of* Chernorizec Hrabar Tournament *– the first multiple choice math competition in Bulgaria.*

Albena Vassileva works at the Institute of Mathematics and Informatics, Bulgarian Academy of Sciences. She has been a member of the team organizing the Chernorizec Hrabar Tournament *in Bulgaria almost from the beginning 25 years ago. She is also involved in the organization of several conference-like competitions for high school students with interests in mathematics, computer science and IT.*

1 Introduction

The Chernorizec Hrabar math tournament (ChH) started in 1992 adopting the format of the Australian Mathematics Competition (AMC) [6]. In the beginning there was just one competition paper for all participants who were advanced 9th–11th grade students [3]. The present format of ChH includes 5 divisions for students of two consecutive grades, starting with 3rd&4th and finishing with 11th&12th graders. Such organization is technically convenient for the organizers but causes some didactical challenges, especially in junior divisions. The statistics we get gives a hint to focus our attention on some age dependent specifics in order to clarify which problems contribute to statistical significant differences in students' performance. Below we are going to share our observations on the last two ChH competition papers for 5th-6th grade.

2 The Age Factor

We refer to the origin – AMC. The two graphs in Figure 1 show a typical difference in performance of students from different age-strata on a same competition paper [9].

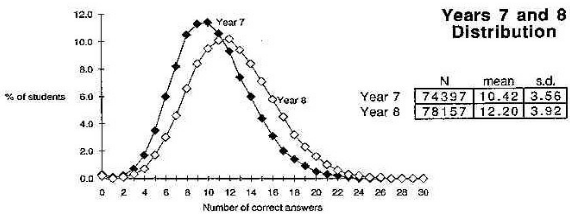


Figure 1: Distribution Rows of AMC 1990 Junior Division

There is a shift of the 8th graders graph on the right and slightly down which means that 8th graders perform better than 7th graders in average. Further we will consider the role of age in students' performance, calling it *age factor*. We are interested in the age factor mainly as statistical characteristic rather than as individual feature. Such standing point does not allow the conclusions made for an age stratum to be formally transferred to a particular individual.

Returning to Figure 1 we can say that AF causes more uniform coverage of the scoring range in the elder students' stratum. The magnitude of the shift of the graphs corresponds to the difference of the mean values. The same situation repeats in ChH in the early age divisions up to 7th-8th grade but then it disappears. Sometimes even 11th grade students perform better than 12th graders (which was the case in the 2015 issue of ChH). Similar is the case with AMC 1990 Intermediate Division where the 9th and 10th graders graphs almost coincides, but a significant shift of the graphs appears again in the Senior Division [9]. So the question *how and why the AF matters* is natural.

3 General information about the study

The largest groups among the participants at ChH are of 5th and 6th grade students. This is why we focused our attention on the last two competition papers for 5th&6th graders [4], [5]. The format of the competition papers was 20 multiple choice questions with one correct answer and 4 distractors each. The correct response is worth 7 points, the blank is 3 and a wrong response is 0. Such scoring provides guessing mathematical expectation of 28 points which guarantees rather poor result (take into account that a blank response sheet is worth 60 points). Hence the students are not encouraged to guess the answer and we can accept that the correct answer was actually obtained by solving the problem. At least this is our assumption for the upper part of the variation row. According to the score system, we consider the scores between 80 and 85 points as average, also a score of 100 points and up is a very good result.

The total number of participants in this age group in 2014 was 1762 and in 2015 was 1845. The range of scoring was 14-140 points. The shape of the variation rows (VR) is normal-distribution-like for the entire population and there is a shift of the VR shape of 6th grade with respect to the VR of 5th grade, which is similar to the one in Figure 1 [8].

The population considered in our study is restricted to the participants in one particular competition center which is typical for middle-size town in Bulgaria. There were: 5th grade students – 72 in 2014 and 56 in 2015; 6th graders – 44 in 2014 and 54 in 2015. Below we will study some particular examples and we will search an explanation of the existence or lack of age-dependent significant differences.

4 The age-factor index

We modify the usual formula for calculating the test-item characteristics in the following manner. Further the code of the test (*item*) will be (YYYY,N) which means the N-th test item in the YYYY-year competition paper. Denote by $T_n^{(item)}$ the difficulty of the corresponding test item for the n-th grade stratum ($n = 5,\ 6$). The values for $T_n^{(item)}$ are calculated as (100 minus the percentage of the number of correct responses), i.e. as the percentage of the incorrect and the blank responses altogether. The values for the discriminatory power $D_n^{(item)}$ are half of the difference between the difficulties for the fourth and the first quartile.

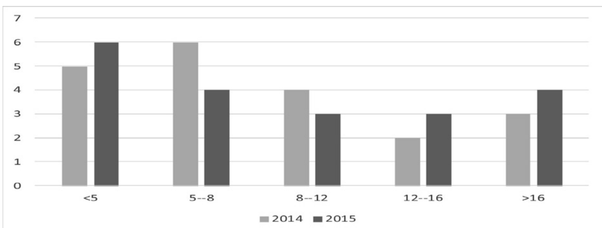


Figure 2: Distribution of Δ^{item} for 2014 and 2015 competition papers

We define the *age-factor index* (AFI) of the test item as the difference $\Delta^{(item)} = T_5^{(item)} - T_6^{(item)}$. The distribution of $\Delta^{(item)}$ for both competition papers under consideration is given in Figure 2.

The following example will show the didactical range of the variable $\Delta^{(item)}$.

(2015,10) Which is the arithmetic operation replaced by the symbol ∇ in the equation $(555 + 5 - 55)\nabla(5 - 5 : 5) - 5 = 2015$?

(A) addition (B) subtraction (C) division (D) multiplication
(E) none of the operations is appropriate

Solution. $505\nabla 4 - 5 = 2015$; $505\nabla 4 = 2020 \iff 505{\cdot}4 = 2020$.

Answer (D).

Parameters. $T_5^{(2015,10)} = 7.1$, $T_6^{(2015,10)} = 5.6$ and $\Delta^{(2015,10)} = 1.5$; $D_5^{(2015,10)} = 3.6$, $D_6^{(2015,10)} = 10.7$.

Comment. The unknown thing here is an arithmetic operation. There are no similar problems included in the school curriculum. So this is a pure competition problem. $\Delta^{(2015,10)}$ tells us only that this problem was equally easy for both age strata. Our interpretation of the values $D_{5,6}^{(2015,10)}$ is: the 6th grade students in the top quartile managed to solve the problem slightly better than the top 5th graders.

The above example shows that there are age dependent differences which are not recognized by the AFI. But we will see further how the AFI could be an useful additional instrument in our didactical analysis.

We expect AFI to be positive and a negative value could be considered as pathology. The next example illustrates this rare phenomenon.

(2015,4) In the summer of 2016 we will celebrate the 100th anniversary of Dimitar Spisarevski. On December 20, 1943, lieutenant Spisarevski died while destroying an American bomber attacking Sofia. How old was the lieutenant when he accomplished this feat?

(A) 43 (B) 16 (C) 27 (D) 20 (E) 36

Solution. Dimitar Spisarevski was born on July 19, 1916, and therefore he was $43 - 16 = 27$ years old.

Answer (C).

Parameters. $T_5^{(2015,4)} = 5.4$, $T_6^{(2015,4)} = 7.4$ and $\Delta^{(2015,4)} = -2$; $D_5^{(2015,3)} = 7.1$, $D_6^{(2015,4)} = 10.7$.

Comment. The values of $D_{5,6}^{4}$ advocate that the problem is easy but not bad at all. Perhaps it was underestimated by the elder students, but was not by the 5th graders.

5 Examples and comments

Further our didactical analysis is splited into two parts. First we will see how important is the school curriculum for the performance of students at ChH. In the second part we will observe the progress of the students' math competition competence from 5th to 6th grade.

The role of the school curriculum

We will illustrate how the preparation in school affects the students performance at ChH in the 5th and 6th grade age strata by four examples. The next two test items are of high difficulty, despite they are related to some topics included in the school curriculum.

(2014,9) What is the angle between the clock hands at 10:10 o'clock?

(A) $120°$ (B) $118°$ (C) $115°$ (D) $112°$ (E) none of these

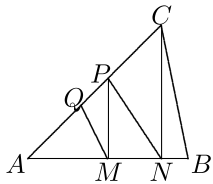
Solution. The minute hand is $60°$ clockwise after the 12 hour mark. In 60 minutes the hour hand travels $30°$. In 10 minutes the hour hand travels $\frac{1}{6}$ of the arc between 10 and 11 marks, i.e. $5°$. Thus, the angle between the hour hand and the 12 hour mark is $55°$.

Answer (C).

Parameters. $T_5^{(2014,9)} = 90.3$, $T_6^{(2014,9)} = 84.1$ and $\Delta^{(2014,9)} = 6.2$; $D_5^{(2014,9)} = -2.8$, $D_6^{(2014,9)} = 13.6$.

Comment. Such statistics means that the 5th graders and most of the 6th graders actually did not solve the problem. The situation in the top quartile of 6th grade is much better. The value of $\Delta^{(2014,9)}$ is below the average but in fact the age factor is more appreciable. The negative $D_5^{(2014,9)}$ indicates gambling instead of solving in the lowest quartile of 5th graders. The problem is close to the 5th grade school curriculum and no additional topics related to angles are included in it between 5th and 6th grade. The 6th graders who form the top quartile participate regularly in math competitions. So maybe their competition practice occurs significant for success.

(2014,17) Triangle ABC is divided into five smaller triangles with equal areas: AMQ, PMQ, MNP, CPN and BCN. If AB is 45 cm, how long (in cm) is MN?



(A) 12 (B) 15 (C) 16 (D) 18 (E) none of these

Solution. From $[ANC] = 4[BNC]$ follows that $AN = 4BN$, which yields $BN = 9$, $AN = 36$. From $[APM] = 2[MNP]$ follows that $AM = 2MN$, which yields $MN = 12$.

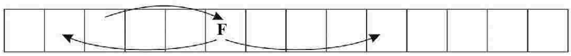
Answer (A).

Parameters. $T_5^{(2014,17)} = 98.6$, $T_6^{(2014,17)} = 86.4$ and $\Delta^{(2014,17)} = 12.2$; $D_5^{(2014,17)} = -2.8$, $D_6^{(2014,17)} = 4.5$.

Comment. The equality $\dfrac{[PKL]}{[PMN]} = \dfrac{KL}{MN}$ for collinear points K, L, M, N is implicitly included in the school curriculum – there are some examples in the 5th grade textbooks which refer to it. Similar configurations like this in (2014,17) are also rather familiar to a regular participant in math competitions. The negative $D_5^{(2014,17)}$ means that the correct answers given in the first quartile are a result of guessing. Thus the value of $\Delta^{(2014,17)}$ could be explained by age-dependent differences in approaching the problem. The age factor here is related to the students' experience – the elder students of the top quartile prefer to leave blank if they are not sure in their result.

The two test items that follow are typical for math competitions, particularly for ChH. There are no topics in school curriculum close to the procedural and combinatorial approach applied in them.

(2014,19) A box is 14 cm long and 1 cm wide and is divided into 14 square cells of 1×1 cm. The flea F is located initially in the center of the furthest left cell. The first jump of F is 1 cm long and it goes to the center of the adjacent cell. Each next jump is 1 cm longer than the previous one and the flea always jumps to the center of a cell in the box. What is the biggest possible number of jumps required for the flea to land into the furthest right cell?



(A) 6 (B) 10 (C) 12 (D) 14 (E) none of these

Solution. We will add the jumps to the left and subtract the jumps to the right from the cell number. The longest possible jump is 13 cm (otherwise the flea will land outside the box). The flea could make 13 jumps like this:

$$1 + 2 - 3 + 4 + 5 - 6 + 7 - 8 + 9 - 10 + 11 - 12 + 13.$$

Answer (E).

Parameters. $T_5^{(2014,19)} = 70.8$, $T_6^{(2014,19)} = 70.5$ and $\Delta^{(2014,19)} = 0.3$; $D_5^{(2014,19)} = 8.3$, $D_6^{(2014,19)} = 4.5$.

Comment. We doubt the largest fraction of the students who solved this problem proceeded like us in the solution above. More likely, they just tested one or another road of F – we expected this. But the surprise came with the AFI: practically same results in both age strata.

(2014,20) In how many ways can the number 5 be represented as a sum of smaller positive integers if the order of summands matters? For example, $3 = 1 + 2 = 2 + 1 = 1 + 1 + 1$, i.e. the number 3 is represented in three such ways.

(A) 5 (B) 10 (C) 15 (D) 30 (E) none of these

Solution. We think of the summands as sums of ones, for example, if $5 = 1 + 2 + 2$, we consider this representation: $5 = 1 + (1 + 1) + (1 + 1)$. It is equivalent to the following breaking of five ones with the divider *: 1*11*11. Continuing this line of thinking, we can easily establish that the number of ways we are looking for is equal to the number of ways to break down 11111 with at least one *. But this number is equal to the number of ways to put or not to put a divider * in 4 places with at least one divider present, i.e. $16 - 1$.

Answer (C).

Parameters. $T_5^{(2014,20)} = 59.7$, $T_6^{(2014,20)} = 50.0$ and $\Delta^{(2014,20)} = 9.7$; $D_5^{(2014,20)} = 19.4$, $D_6^{(2014,20)} = 0$.

Comment. The (2014,19) comment is applicable to this problem too. Relatively small numbers (e.g. 5) allow to list all possible representations. However, there is considerable difference between (2014,19) and (2014,20) AFIs. We think that here the age factor affects mainly the time elapsed in organizing a correct way of counting.

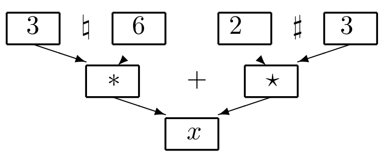
The role of the problem complexity

The next three examples will show another side of age dependent math abilities.

(2015,3) Let us define two new operations for the numbers a and b:

$$a\natural b = ab - a - b \quad \text{and} \quad a\sharp b = ab + a + b.$$

For example, $5\natural 2 = 5 \cdot 2 - 5 - 2 = 3$, $5\sharp 2 = 5 \cdot 2 + 5 + 2 = 17$. Which is the number x in the diagram?



(A) 20 (B) 12 (C) 22 (D) 18 (E) none of these

Solution. $3\natural 6 = 9,\ 2\sharp 3 = 11,\ x = 9 + 11.$

Answer (A).

Parameters. $T_5^{(2015,3)} = 32.1$, $T_6^{(2015,3)} = 11.1$ and $\Delta^{(2015,3)} = 9.7$; $D_5^{(2015,3)} = 21.0$, $D_6^{(2015,3)} = 7.1$.

Comment. The introduction of a new concept in a contest paper is always risky. This test-item combines the very well known technique of calculating expression value in a diagram with the also familiar funny-looking operations. Our previous research claims that the easiest problems about arithmetic expressions are of diagram type [2]. However, the large value of $\Delta^{(2015,3)}$ indicates that such combination of ideas is inside the 6th graders zone of proximal development (ZPD) but for the one third of the five graders it is beyond their ZPD [10].

(2015,12) Alex, Bobby and Charlie thought of the same two-digit number. Alex multiplied the number by 3, then added 3 to the result, then divided the new result to 3, subtracted 3 from the quotient and obtained the number A. Bobby added 3 to the initial number, multiplied the result to 3, subtracted 3 from the product, divided the result by 3 and obtained the number B. Charlie first subtracted 3 from the initial number, then multiplied the result by 3, added 3 to the product, divided the sum to 3 and obtained the number C. How are ordered the numbers A, B and C?

(A) A≥B≥C (B) C≥A≥B (C) A≥C≥B (D) B≥A≥C (E) C≥B≥A

Solution. Let X be the initial number. Performing the operations, we obtain

$$A = (X \cdot 3 + 3)/3 - 3 = X + 1 - 3 = X - 2,$$

$$B = ((X + 3) \cdot 3 - 3)/3 = (3X + 9 - 3)/3 = X + 2,$$

$$C = ((X - 3) \cdot 3 + 3)/3 = (3X - 9 + 3)/3 = X - 2.$$

The problem may be solved by using a specific value of $X \geq 10$, e.g. if $X = 12$, then $A = 10$, $B = 14$, $C = 10$.

Answer (D).

Parameters. $T_5^{(2015,12)} = 51.8$, $T_6^{(2015,12)} = 18.5$ and $\Delta^{(2015,12)} = 33.3$; $D_5^{(2015,12)} = 28.6$, $D_6^{(2015,12)} = 33$.

Comment. The test item has the largest AFI in the competition paper. Problems of this type appear in ChH competitions papers as easy ones, but usually they are written in schematic format, and then the difficulty is considerably smaller, e.g. $T_5^{(2014,10)} = 16.7$ [4]. The verbal format of the operation description appears to be insurmountable for half of the 5th graders but it causes no troubles in 6th grade.

(2015,15) How many natural numbers are there, whose sum of digits equals 7 and product of digits equals 8?

(A) 4 (B) 6 (C) 8 (D) 10 (E) 12

Solution. The digits of the desired numbers are factors of 8. A digit *8* makes the sum of the digits greater than 7. Thus only *1*, *2*, and *4* could be possible digits. There are two cases.
(1) *4* is among the digits; there cannot be another *4*, hence it must be exactly one *2* to get the product 8; to complete the sum exactly one *1* is needed; there are 6 numbers formed by the digits *4*, *2* and *1*.
(2) *4* is not among the digits; there must be 3 digits *2* to get the product 8; to complete the sum exactly one *1* is needed; there are 4 numbers formed by three *2* and one *1*.

Answer (D).

Parameters. $T_5^{(2015,15)} = 91.1$, $T_6^{(2015,15)} = 63.0$ and $\Delta^{(2015,15)} = 28.1$; $D_5^{(2015,15)} = 10.7$, $D_6^{(2015,15)} = 28.6$.

Comment. The problem is of complex type. The first step refers to divisibility and the second step is combinatorial. It is no surprise for us that the difficulty is high for both strata. The high value of AFI confirms the conjecture that combining math ideas is beyond ZPD of a considerable fraction of advanced students in 5th and 6th grade. However, the statistics gives reason to claim that for one third of the 6th graders such complex knowledge and skill could be achieved, i.e. complex problems are in their zone of actual development [10].

The next example points to the lack of direct correlation between problem complexity and difficulty.

(2015,6) The long-eared owl makes calls at equal intervals; the duration of a call is ignored. An owl needs 20 seconds to make 5 calls. How many seconds are needed for an owl to make 25 calls?

(A) 108   (B) 100   (C) 125   (D) 96   (E) 120

Solution. There are 4 intervals between the five calls, i.e. each interval is $20/4 = 5$ seconds long. There will be 24 intervals between 25 calls, therefore $24 \cdot 5 = 120$ seconds will be needed to make 25 calls.

Answer (E).

Parameters. $T_5^{(2015,6)} = 85.7$, $T_6^{(2015,6)} = 85.2$ and $\Delta^{(2015,6)} = 0.5$; $D_5^{(2015,6)} = 3.6$, $D_6^{(2015,6)} = 7.1$.

Comment. To our surprise, the statistical difficulty is too high for this easy one-step problem. Moreover, every issue of ChH includes such kind of problems, even in the competition papers for 3-4 grades. The small Δ^6 value says that it is an equally slippery problem for both age strata. On the other hand, the small $D^6_{5,6}$ means that the problem appears tricky for both advanced and less able students. The most frequent students response was (B) which corresponds to the "solution": 5 times more calls need 5 times more time.

6 Concluding remarks

In section 5.1 we have observed that the AFI of some problems which concern extracurricular topics is less than the AFI of test items closely related to the school curriculum. It was 25 years ago when W. Atkins wrote that the poor performance on such problems could be ... *not due to the inappropriateness of the skill for the particular age or grade level, but rather that the appropriate problem solving skills are not emphasized and developed in the typical school curriculum.* [1] Nowadays we can confirm this conclusion. This leads to the conjecture that the math knowledge and skills formed in extracurricular activities are more sustainable than the ones built in the frame of school curriculum.

The math knowledge and skills built in school at this age are strongly context dependent: out of the classroom environment, e.g. during math competitions, the knowledge and skills are applicable only by a small fraction of students. Also, the knowledge and skills created especially for competitions works successfully in the same context, i.e. during the math competitions. The ability to apply knowledge beyond the context of its formation (we call it *decontextualization* of the knowledge) is part of students competence. So we can conclude that math competence in 5th and 6th grade is (partially) developed just in a tiny part of the students.

The ability to synthesize elements of knowledge is the next step in the evolution of students' competence. The examples in section 5.2 clearly show that elements of *synthetic competence* are approachable in 6th grade, but not in 5th grade (in general). We believe that these age specifics should be taken into account by decision-makers in education.

The average value Δ of AFIs for an entire competition paper is also in our area of interest. So $\Delta_{2014} = 9.2$ could be interpreted as: 2014 competition paper is 9.2 *abstract difficulty units* harder to 5th grade participants than to 6th grade ones. For 2015 competition paper we have $\Delta_{2015} = 10.8$. The value of Δ is also coherent with the magnitude of the shift to the right of the graph of the 6th grade VR with respect to the 5th grade VR. But we are still not confident which value of Δ is acceptable for ChH competition papers in 5th and 6th grade.

Autors' contributions and acknowledgments

Borislav Lazarov: competition papers, conceptual frame, research design and analysis; Albena Vassileva: data processing.

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A Fraction Problem

Arthur Holshouser & Harold Reiter

Arthur Holshouser has had a lifelong passion for doing mathematics. He works on problems at least 5 hours each day and often much more. His favorites are plane geometry, polynomial algebra, combinatorial games, and number theory. His current passion is Brocard geometry and Poincelet polygons. His partnership with Harold Reiter has produced more than 50 articles.

As Harold Reiter completes his 54st year of teaching, his passion for lighting mathematical fires has continued to grow. In the past few years, he has enjoyed working with Indian children in Bangalore, Indonesian children and teachers, Navajo children, Saudi teachers and very young gifted students at Epsilon Camp in the United States. He spends time creating challenges including puzzles like KenKen (with modulo 6 or 7 arithmetic and with prime number entries). He continues to work with longtime coauthor Arthur Holshouser, on papers in number theory, geometry, and polynomial algebra.

1 Abstract

In recent years, mathematics educators have begun to realize that understanding fractions and fractional arithmetic is the gateway to advanced high school mathematics. Published in the current issue of the peer-reviewed journal Psychological Science, the study [7] found that

understanding fractions and division at age 10 predicted success in algebra and overall math achievement in high school, even after statistically controlling for a wide range of factors including parents' education and income, and children's age, gender, I.Q., reading comprehension, working memory, and knowledge of whole number addition, subtraction and multiplication.

Yet, U.S. students continue to do poorly when ranked internationally on fractional arithmetic. This essay is intended to help mathematicians interested in working with ambitious students who want to develop a deeper understanding of some fundamental ideas. We do this by posing and solving the following problem: If e/f is a positive rational number reduced to lowest terms, we call $e+f$ the *size* of e/f. If $0 \leq a/b < c/d$ are given rational numbers (reduced to lowest terms), what is the fraction e/f of smallest size between a/b and c/d. We give two methods including a continued fraction method for finding e/f. At the end of the paper, we use the ideas developed here to generalize this problem in two different ways. Also, at the end, we pose several more problems. These problems are not intended to be used in a classroom with young students. On the other hand, they can all be used to deepen teachers' understanding of fractions and fractional arithmetic. Key Words: Fraction, mediant, rational number, unit fraction, continued fraction, floor function, ceiling function.

2 Introduction

The Common Core State Standards proposes a model for fractions that enables both conceptual understanding and computational facility. The suggested model begins with *unit* fractions. A unit fraction has numerator 1 and denominator a positive integer. Unit fractions play in the area of fractions more or less the same role that place value numbers play in building decimal representations and computation. See [6]. A fraction is then defined as an integer multiple of a unit fraction. See [3] and [8]. There are two important types of problems related to fractions. One is fractional arithmetic and the other is comparison of fractions. Both these types are addressed here. In this paper, we give two solutions (including a continued fraction solution) to the following equivalent Problems 1,1$'$. In Section 3, we first prove that the Problems 1,1$'$ are equivalent problems. In Section 3, we also give an elementary solution to Problem 1. We also prove Theorem 1 which states that the solutions to problems 1, and 1′ are the same. Theorem 1 is the foundation of this paper. At the end of the paper, we generalize Problems 1, 1′ and Theorem 1 in two different ways by using the ideas given in Section 4.

Problem 1 Suppose $0 \le \frac{a}{b} < \frac{c}{d}$ where $\frac{a}{b}, \frac{c}{d}$ are fractions that are reduced to lowest terms and $0 = \frac{0}{1}$. Find a fraction $\frac{e}{f}$ (reduced to lowest terms) such that $0 \le \frac{a}{b} < \frac{e}{f} < \frac{c}{d}$ and such that f is a positive integer that has the smallest possible value. For this smallest possible positive integer f, compute the smallest possible positive integer e so that $0 \le \frac{a}{b} < \frac{e}{f} < \frac{c}{d}$.

Problem 1′ Suppose $0 \le \frac{a}{b} < \frac{c}{d}$ where $\frac{a}{b}, \frac{c}{d}$ are fractions that are reduced to lowest terms and $0 = \frac{0}{1}$. Find a fraction $\frac{\overline{e}}{\overline{f}}$ (reduced to lowest terms) such that $0 \le \frac{a}{b} < \frac{\overline{e}}{\overline{f}} < \frac{c}{d}$ and such that $\overline{e}$ is a positive integer that has the smallest possible value. For this smallest possible positive integer $\overline{e}$, let us compute the smallest possible positive integer $\overline{f}$ that can go with $\overline{e}$ so that $0 \le \frac{a}{b} < \frac{\overline{e}}{\overline{f}} < \frac{c}{d}$.

Theorem 1 Suppose $\frac{a}{b}, \frac{c}{d}$ have the same values in both Problems 1, 1′. Then the solutions $\left(e, f, \frac{e}{f}\right)$ and $\left(\overline{e}, \overline{f}, \frac{\overline{e}}{\overline{f}}\right)$ in Problems 1, 1′ are the same. That is, $\left(e, f, \frac{e}{f}\right) = \left(\overline{e}, \overline{f}, \frac{\overline{e}}{\overline{f}}\right)$.

3 An Elementary Solution to Problems 1, 1′ and a Proof of Theorem 1

We first show that Problems 1, 1′ are equivalent problems. First, we note that the solutions to $0 = \frac{a}{b} < \frac{e}{f} < \frac{c}{d}$ and $0 = \frac{a}{b} < \frac{\overline{e}}{\overline{f}} < \frac{c}{d}$ are $e = \overline{e} = 1$ and $f = \overline{f}$ where $f = \overline{f}$ is the smallest positive integer that satisfies $\frac{d}{c} < f = \overline{f}$. Next, note that $0 < \frac{a}{b} < \frac{\overline{e}}{\overline{f}} < \frac{c}{d}$ is true if and only if $0 < \frac{d}{c} < \frac{\overline{f}}{\overline{e}} < \frac{b}{a}$ which is the same form as $0 < \frac{a}{b} < \frac{e}{f} < \frac{c}{d}$. Note that e and f are essentially interchanged. Therefore, Problems 1, $1'$ are equivalent problems.

Solution to Problem 1. Note that $0 \leq \frac{a}{b} < \frac{e}{f} < \frac{c}{d}$ if and only if $0 \leq \frac{af}{b} < e < \frac{cf}{d}$ where e is an integer lying strictly between $\frac{af}{b}$ and $\frac{cf}{d}$. Note that if $0 \leq \frac{af}{b} < e < \frac{cf}{d}$ then $0 \leq \frac{a}{b} < \frac{e}{f} < \frac{c}{d}$. Also, the length of the integral $\left(\frac{af}{b}, \frac{cf}{d}\right)$ gets arbitrarily big as f gets big. The idea that the interval length $f(c/d - a/b)$ gets arbitrarily large for large values of f is called the Archimedean property of real numbers.

Thus, the solution to Problem 1 is to find the smallest positive integer f such that there exists a positive integer e that lies strictly between $\frac{af}{b}$ and $\frac{cf}{d}$. Also, for this smallest positive integer f, we choose e as the smallest positive integer lying in $\left(\frac{af}{b}, \frac{cf}{d}\right)$. Thus, e is the smallest positive integer greater than $\frac{af}{b}$. Thus, if $\frac{af}{b}$ is not an integer then $\left\lceil \frac{af}{b} \right\rceil = e$ and if $\frac{af}{b}$ is an integer then $\left\lceil \frac{af}{b} \right\rceil + 1 = \frac{af}{b} + 1 = e$. Note that $\frac{e}{f}$ is automatically reduced to lowest terms in the above solution.

A calculator can be used to carry out the above solution.

Proof of Theorem 1. Suppose $\frac{e}{f}$ is the solution to Problem 1 and $\frac{\overline{e}}{\overline{f}}$ is the solution to Problem $1'$ where $\frac{a}{b}, \frac{c}{d}$ have the same values in both problems. We must show that $\left(e, f, \frac{e}{f}\right) = \left(\overline{e}, \overline{f}, \frac{\overline{e}}{\overline{f}}\right)$. Now $\frac{e}{f}$ solves $0 \leq \frac{a}{b} < \frac{e}{f} < \frac{c}{d}$.

By Problem $1'$, $\overline{e}$ is the smallest positive integer such that a solution $0 \leq \frac{a}{b} < \frac{\overline{e}}{\overline{f}} < \frac{c}{d}$ exists. Therefore, $(*)\, \overline{e} < e$ or $\overline{e} = e$. Now $\frac{\overline{e}}{\overline{f}}$ solves $0 \leq \frac{a}{b} < \frac{\overline{e}}{\overline{f}} < \frac{c}{d}$. By the definition of f in Problem 1, we have $f \leq \overline{f}$. Also, $\overline{f} = f$ is impossible by the definition of f and e in Problem 1. Therefore, $f < \overline{f}$.

Now $0 \leq \frac{a}{b} < \frac{\overline{e}}{\overline{f}} < \frac{c}{d}$ implies $0 \leq \frac{a\overline{f}}{b} < \overline{e} < \frac{c\overline{f}}{d}$.

Also, $0 \leq \frac{a}{b} < \frac{e}{f} < \frac{c}{d}$ implies $0 \leq \frac{af}{b} < e < \frac{cf}{d}$. From the solution to Problem 1, we know that e is the smallest integer greater than $\frac{af}{b}$. Now $f < \overline{f}$ implies $\frac{af}{b} < \frac{a\overline{f}}{b}$ if $\frac{a}{b} \neq 0$ or $\frac{af}{b} = \frac{a\overline{f}}{b}$ if $\frac{a}{b} = 0$.

Thus, $\frac{af}{b} \leq \frac{a\overline{f}}{b} < \overline{e}$.

We are supposing $\overline{e} < e$.

Now e is the smallest integer greater than $\frac{af}{b}$ and $\frac{af}{b} \leq \frac{a\overline{f}}{b} < \overline{e}$ implies $e \leq \overline{e}$. Therefore, $\overline{e} < e$ is impossible. Therefore, from $(*)$ we have $e = \overline{e}$. By definition of f, we have $f \leq \overline{f}$. Since $e = \overline{e}$, the smallest integer $\overline{f}$ such that $e = \overline{e}$ satisfies $0 \leq \frac{a}{b} < \frac{\overline{e}}{\overline{f}} < \frac{c}{d}$ is $\overline{f} = f$. Therefore, $\left(e, f, \frac{e}{f}\right) = \left(\overline{e}, \overline{f}, \frac{\overline{e}}{\overline{f}}\right)$.

If $0 < \frac{x}{y}$ is a positive fraction, reduced to lowest terms, define the size function $s\colon \mathcal{Q}^{+} \cup \{/\} \to \mathcal{Z}^{+} \cup \{/\}$ by $s(\frac{x}{y}) = x + y$.

From Theorem 1, we know that the solution $0 \leq \frac{a}{b} < \frac{e}{f} < \frac{c}{d}$ that we gave to Problem 1 will also be the fraction $\frac{e}{f}$ (where $\frac{e}{f}$ is reduced to lowest terms) such that $\frac{e}{f}$ is the fraction of the smallest possible size $e + f$ that lies strictly between $\frac{a}{b}, \frac{c}{d}$.

Thus, if $0 \leq \frac{a}{b} < \frac{e}{f} < \frac{c}{d}$, from Theorem 1, $s(\frac{e}{f})$ has the smallest possible value if and only if both e and f have the smallest possible values.

In Section 10, we use this to generalize Problems 1, $1'$.

Example 1 Find a fraction $\frac{e}{f}$ (reduced to lowest terms) such that $0 < \frac{7}{10} < \frac{e}{f} < \frac{11}{15}$ and such that f has the smallest possible value.

For this smallest possible value of f, choose the smallest possible value of e so that $0 < \frac{7}{10} < \frac{e}{f} < \frac{11}{15}$. From Theorem 1, these values $\left(e, f, \frac{e}{f}\right)$ will also compute the fraction $\frac{e}{f}$ (reduced to lowest terms) such that $\frac{e}{f}$ lies in $0 < \frac{7}{10} < \frac{e}{f} < \frac{11}{15}$ and also $\frac{e}{f}$ has the smallest possible size.

Solution. We easily carry out the elementary solution with a calculator.

$1 \cdot \frac{7}{10} = .7, \quad 1 \cdot \frac{11}{15} = .7333333.$
$2 \cdot \frac{7}{10} = 1.4, \quad 2 \cdot \frac{11}{15} = 1.466666.$
$3 \cdot \frac{7}{10} = 2.1, \quad 3 \cdot \frac{11}{15} = 2.199999.$
$4 \cdot \frac{7}{10} = 2.8, \quad 4 \cdot \frac{11}{15} = 2.93333.$
$5 \cdot \frac{7}{10} = 3.5, \quad 5 \cdot \frac{11}{15} = 3.666666.$
$6 \cdot \frac{7}{10} = 4.2, \quad 6 \cdot \frac{11}{15} = 4.399999.$
$7 \cdot \frac{7}{10} = 4.9, \quad 7 \cdot \frac{11}{15} = 5.133333.$

Thus, $f = 7$ is smallest positive integer such that $7 \cdot \frac{7}{10} = 4.9 < e < 7 \cdot \frac{11}{15} = 5.133333.$

Also, $e = 5$ is the smallest e that can go with $f = 7$ and the answer is $\frac{e}{f} = \frac{5}{7}$.

In Section 8, we solve this example in two more (equivalent) ways. One way uses continued fractions.

4 Two Transformations, Translation and Inversion

As we explain in detail later, to solve Problem 1 in a more advanced way, we use the following Transformations 1 and 2 to keep simplifying Problem 1 until it can be easily solved. Then we reverse ourselves and work back to the original solution. Continued fractions can be used to "keep the books" for us.

Transformation 1, translation by an integer For fixed integer $n \in \{1, 2, 3, \dots\}$, consider the transformation $t \to t + n$. For $0 \le x < y$ and x, y are rational, this transformation maps the interval $[x, y]$ to $[x + n, y + n]$.

Suppose, $\frac{e}{f} \in (x, y)$ and $\frac{\overline{e}}{\overline{f}} \in (x, y)$ where $0 \le x < y$ and x, y are rational and $\frac{e}{f}, \frac{\overline{e}}{\overline{f}}$ are positive fractions reduced to lowest terms. Also, $\frac{e}{f} \ne \frac{\overline{e}}{\overline{f}}$.

Suppose $\frac{e}{f}$ is also the fraction in (x, y) having the smallest possible size. Thus, $f \le \overline{f}, e \le \overline{e}$ and at least one of $f < \overline{f}, e < \overline{e}$ is true. Consider the translation $t \to t+n, [x,y] \to [x+n, y+n]$. Thus, $\frac{e}{f} \to n + \frac{e}{f} = \frac{nf+e}{f}$ and $\frac{\overline{e}}{\overline{f}} \to n + \frac{\overline{e}}{\overline{f}} = \frac{n\overline{f}+\overline{e}}{\overline{f}}$. Of course, both $\frac{nf+e}{f}$ and $\frac{n\overline{f}+\overline{e}}{\overline{f}}$ are reduced to lowest terms since $\frac{e}{f}$ and $\frac{\overline{e}}{\overline{f}}$ are reduced to lowest terms.

Now $s(\left(\frac{nf+e}{f}\right) = (n+1)\,f + e < (n+1)\,\overline{f} + \overline{e} = s\left(\frac{n\overline{f}+\overline{e}}{\overline{f}}\right)$. Thus, in the translation $t \to t+n, [x,y] \to [x+n, y+n]$, we see that the fraction $\frac{e}{f}$ of the smallest possible size in (x,y) maps into the fraction $n + \frac{e}{f} = \frac{nf+e}{f}$ in $(x+n, y+n)$ that also has the smallest possible size.

Of course, when we transform back again $t \to t-n, [x+n, y+n] \to [x,y]$ we see that the fraction $\frac{e'}{f'}$ in $(x+n, y+n)$ having the smallest possible size is mapped into the fraction $\frac{e}{f} = \frac{e'}{f'} - n$ in (x,y) that also has the smallest possible size. Thus, the smallest size property is an invariant in the translations $t \to t+n, [x,y] \to [x+n, y+n]\,, t \to t-n, [x+n, y+n] \to [x,y]\,.$

We now call both $t \to t+n, t \to t-n$ translation by an integer. This very important invariant helps to simplify and solve Problem 1. We show later that $t \to t-n, [x+n, y+n] \to [x,y]$ simplifies Problem 1.

Transformation 2, inversion Consider the transformation $t \to \frac{1}{t}$. For $0 < x < y$ and x, y rational, inversion maps the interval $[x,y]$ to $\left[\frac{1}{y}, \frac{1}{x}\right]$ and maps the interval $\left[\frac{1}{y}, \frac{1}{x}\right]$ to $[x,y]\,.$

Now if $0 < x < \frac{e}{f} < y$ where x, y are rational and $\frac{e}{f}$ is a fraction reduced to lowest terms, we see that $t \to \frac{1}{t}$ maps $\frac{e}{f} \to \frac{f}{e}$. Of course, the size of $\frac{e}{f}$ is $e+f$ and the size of $\frac{f}{e}$ is $e+f$. Thus, the size of fractions (reduced to lowest terms) is invariant under $t \to \frac{1}{t}$.

Also, the smallest size fraction $\frac{e}{f}$ in (x,y) maps into the smallest size fraction $\frac{f}{e}$ in $\left(\frac{1}{y}, \frac{1}{x}\right)$ and vice-versa.

Observation 1 The elementary solution to Problem 1 is often much easier to carry out with a calculator if we first use inversion. We then solve Problem 1 for this new interval and then we transform back again.

5 Easy Standard Solutions to Problem 1

In Problem 1, the following cases 1–5 for the intervals $\left[\frac{a}{b}, \frac{c}{d}\right], 0 \le \frac{a}{b} < \frac{c}{d}$, with $0 \le \frac{a}{b} < \frac{e}{f} < \frac{c}{d}$, are straightforward when computing the smallest size $\frac{e}{f} \in \left(\frac{a}{b}, \frac{c}{d}\right)$.

In all cases 1–5, $n, m \in \{0, 1, 2, 3, \dots\}$ and $0 < \epsilon < 1, 0 < \overline{\epsilon} < 1$ and $\epsilon, \overline{\epsilon}$ are rational. Also in each case, we list other restrictions on $n, m, \epsilon, \overline{\epsilon}$ when more restrictions are needed. We give the solution for the smallest size $\frac{e}{f}$ in each case 1–5. The proof of Case 3 is given in Section 9.

Case 1. $\frac{e}{f} \in (n, m), 0 \le n < m.$

If $n + 2 \le m$ we have $\frac{e}{f} = n + 1$.

If $m = n + 1$, we have $\frac{e}{f} = n + \frac{1}{2} = \frac{2n+1}{2}$.

Case 2. $\frac{e}{f} \in (n, m + \overline{\epsilon}), 0 \le n < m, 0 < \overline{\epsilon} < 1.$

$\frac{e}{f} = n + 1.$

Case 3. $\frac{e}{f} \in (n + \epsilon, m), 0 \le n < m, 0 < \epsilon < 1.$

If $n + 2 \le m$, we have $\frac{e}{f} = n + 1$.

If $n + 1 = m$, we have $\frac{e}{f} = n + \frac{\overline{f}-1}{\overline{f}}$ where $\overline{f}$ is the smallest positive integer that satisfies $\overline{f} > \frac{1}{1-\epsilon}$.

We prove this Case 3 in Section 9.

Case 4. $\frac{e}{f} \in (n, n + \overline{\epsilon}), 0 \le n, 0 < \overline{\epsilon} < 1$. We have $\frac{e}{f} = n + \frac{1}{\overline{f}}$ where $\overline{f}$ is the smallest positive integer that satisfies $\overline{f} > \frac{1}{\overline{\epsilon}}$. To see this, note that $\frac{e}{f} = n + \frac{\overline{e}}{\overline{f}} < n + \overline{\epsilon}$ is equivalent to $\frac{\overline{e}}{\overline{\epsilon}} < \overline{f}$. We use Theorem 1 in the following.

Note that $\overline{f}$ has the smallest possible value if and only if $\overline{e} = 1$ and when $\overline{e} = 1, \overline{f}$ is the smallest positive integer such that $\overline{f} > \frac{1}{\overline{\epsilon}}$.

Case 5. $\frac{e}{f} \in (n + \epsilon, m + \overline{\epsilon})\,, 0 \le n < m, 0 < \epsilon < 1, 0 < \overline{\epsilon} < 1$. We have $\frac{e}{f} = n + 1$.

6 Two Hard Cases

Cases 6, 7 are the two hard cases when solving Problem 1.

Case 6. $\frac{e}{f} \in (n + \epsilon, n + \overline{\epsilon})\,, n \in \{1, 2, 3, \dots\}\,, 0 < \epsilon < \overline{\epsilon} < 1,\ \epsilon, \overline{\epsilon}$ are rational.

Case 7. $\frac{e}{f} \in (\epsilon, \overline{\epsilon})\,, 0 < \epsilon < \overline{\epsilon} < 1, \epsilon, \overline{\epsilon}$ are rational.

We solve Cases 6, 7 as follows.

In case 6, we use translation: $t \to t - n$ to transform $[n + \epsilon, n + \overline{\epsilon}]$ $\to [\epsilon, \overline{\epsilon}]$, which is case 7. Translation transforms the smallest size $\frac{e}{f} \in (n + \epsilon, n + \overline{\epsilon})$ into the smallest size $\frac{e}{f} \to \frac{\overline{e}}{\overline{f}} = \frac{e}{f} - n = \frac{e - nf}{f} \in (\epsilon, \overline{\epsilon})\,.$

Also, note that the size of $\frac{e}{f}$ is reduced in the translation $\frac{e}{f} \to \frac{e}{f} - n = \frac{e - fn}{f}$ since $e - fn + f < e + f$. It is of interest that the sizes of all rational numbers $\frac{x}{y} \in (n + \epsilon, n + \overline{\epsilon})$ are reduced in the translation $t \to t - n$ including the reduction of the size of $n + \epsilon \to \epsilon, n + \overline{\epsilon} \to \overline{\epsilon}$. However, it is the reduction in the size of the smallest size $\frac{e}{f} \in (n + \epsilon, n + \overline{\epsilon}) \to \frac{\overline{e}}{\overline{f}} = \frac{e}{f} - n \in (\epsilon, \overline{\epsilon})$ that is the most important. Of course, this reduction in the size of $\frac{e}{f} \to \frac{\overline{e}}{\overline{f}} = \frac{e}{f} - n$ is simplifying the Problem 1.

In case 7, we use inversion $t \to \frac{1}{t}$ to transform $[\epsilon, \overline{\epsilon}] \to \left[\frac{1}{\overline{\epsilon}}, \frac{1}{\epsilon}\right]$. Note that $1 < \frac{1}{\overline{\epsilon}} < \frac{1}{\epsilon}$ since $0 < \epsilon < \overline{\epsilon} < 1$. Of course, inversion transforms the smallest size $\frac{e}{f} \in (\epsilon, \overline{\epsilon})$ into the smallest size $\frac{f}{e} \in \left(\frac{1}{\overline{\epsilon}}, \frac{1}{\epsilon}\right)$. The sizes of all fractions $\frac{x}{y} \in (\epsilon, \overline{\epsilon})$ are invariant under inversion $\frac{x}{y} \to \frac{y}{x} \in \left(\frac{1}{\overline{\epsilon}}, \frac{1}{\epsilon}\right)$ since $x + y = y + x$.

Now $\frac{f}{e} \in \left(\frac{1}{\overline{\epsilon}}, \frac{1}{\epsilon}\right)$ must come under one of the Cases 1-6. We finish this analysis in Section 7.

7 Solving Problem 1

To solve Problem 1, we note that we can immediately solve the easy standard Cases 1–5. To handle Cases 6, 7 we use translation and inversion as we explained in Section 6. In Case 6, we use translation $t \to t - n$ and in Case 7 we use inversion $t \to \frac{1}{t}$. When we use translation $t \to t - n, [n + \epsilon, n + \overline{\epsilon}] \to [\epsilon, \overline{\epsilon}]$, the sizes of all fractions $\frac{x}{y} \in [n + \epsilon, n + \overline{\epsilon}]\,, \frac{x}{y} \to \frac{x}{y} - n$, are reduced including a reduction in size of $n + \epsilon \to \epsilon, n + \overline{\epsilon} \to \overline{\epsilon}$, and a reduction in size of the smallest size $\frac{e}{f} \in (n + \epsilon, n + \overline{\epsilon})$ where $\frac{e}{f} \to \frac{e}{f} - n$ and $\frac{e}{f} - n \in (\epsilon, \overline{\epsilon})$. This reduction of the smallest size $\frac{e}{f}$ is by far the most important.

Now Case 6 transformed into Case 7. However, Case 7 is transformed into one of the six cases 1-6. Now inversion $\frac{e}{f} \to \frac{f}{e}$ leaves the sizes of the smallest sizes $\frac{e}{f}$ and $\frac{f}{e}$ invariant, while translation $\frac{e}{f} \to \frac{e}{f} - n$ reduces the size of the smallest size $\frac{e}{f}$. If we keep transforming Cases 6, 7 back and forth to each other, we would eventually run into an impossibility since we cannot keep reducing the size of the smallest size $\frac{e}{f}$ forever. Thus, as we transform Cases 6, 7 back and forth, eventually we must use inversion $t \to \frac{1}{t}$ to transform Case 7 into one of the easy standard Cases 1-5.

When we reach one of the easy standard Cases 1-5, we solve this easy case for the smallest size $\frac{\overline{e}}{f}$ that lies in the interval of that case. Then we reverse ourselves step by step and go back until we compute the smallest size $\frac{e}{f} \in \left(\frac{a}{b}, \frac{c}{d}\right)$ where $0 \leq \frac{a}{b} < \frac{c}{d}$ is given in Problem 1.

If we study continued fractions, we observe that mentioned translation $[n + \epsilon, n + \overline{\epsilon}] \to [\epsilon, \overline{\epsilon}]$ and inversion $[\epsilon, \overline{\epsilon}] \to \left[\frac{1}{\overline{\epsilon}}, \frac{1}{\epsilon}\right]$, are just computing the continued fraction expansions for the two end points $n+\epsilon \to \epsilon, \epsilon \to \frac{1}{\epsilon}$ and $n + \overline{\epsilon} \to \overline{\epsilon}, \overline{\epsilon} \to \frac{1}{\overline{\epsilon}}$. Thus, the continued fraction expansions of the two end points of our intervals provide a convenient bookkeeping scheme for keeping track of what we are computing. We can use continued fractions to compute Cases 6, 7 back and forth until we transform Case 7 into one of the easy standard Cases 1-5. We then put the answer to the easy Case 1-5 into the continued fractions and use the continued fraction to compute the final answer $\frac{e}{f}$ to the original Problem 1 where $0 \leq \frac{a}{b} < \frac{e}{f} < \frac{c}{d}$.

All of this will become clear in Section 8 when we work through four examples. In the first example of Section 8, we first work the example directly without using continued fractions. Then we work through the entire example using continued fractions. This will show the reader step by step exactly how the continued fraction method is carried out. We emphasize that the continued fraction method is just a convenient bookkeeping scheme for keeping track of our calculations.

8 Four Examples for Solving Problem 1

We now go through four examples of finding the smallest size $\frac{e}{f} \in \left(\frac{a}{b}, \frac{c}{d}\right), 0 \le \frac{a}{b} < \frac{c}{d}$. In the first example, we first solve the Problem 1 directly by using transformation 1, 2. We transform Cases 6, 7 back and forth until we run into one of the easy Cases 1-5. Then we solve this easy case. Then we use the answer to this easy case to work backwards to compute the original answer $\frac{e}{f}$ to our Problem 1.

Then we show how these same direct calculations can also be carried out with continued fractions. This parallel work will show the reader exactly what is going on. As stated before, transforming Cases 6, 7 back and forth by using translation and inversion creates a continued fraction of $\frac{a}{b}, \frac{c}{d}$. When we run into one of the easy Cases 1-5, we substitute the answer to this easy case directly into the continued fraction of either $\frac{a}{b}$ or $\frac{c}{d}$. Then we use the continued fractions to compute the answer $\frac{e}{f}$ to our original Problem 1. The examples will make this clear to the reader.

Example 1 Find the smallest size $\frac{e}{f}$ that lies on $0 < \frac{7}{10} < \frac{e}{f} < \frac{11}{15}$.

Note that Example 1 was also solved in Section 3.

Solution 1. We transform Cases 7, 6 back and forth until we run into one of the easy Cases 1-5. Then we solve the Problem 1 for Cases 1-5 and then we work backwards to compute $0 < \frac{7}{10} < \frac{e}{f} < \frac{11}{15}$. $t \to \frac{1}{t} : \left[\frac{7}{10}, \frac{e}{f}, \frac{11}{15}\right] \to \left[\frac{15}{11}, \frac{f}{e}, \frac{10}{7}\right] = \left[1 + \frac{4}{11}, \frac{f}{e}, 1 + \frac{3}{7}\right]$. Note that $\frac{e}{f}$ maps into $\frac{f}{e}$ and $\frac{f}{e}$ is the smallest size fraction in $\left(1 + \frac{4}{11}, 1 + \frac{3}{7}\right)$.

Next, $t \to t-1$: $\left[1+\frac{4}{11}, \frac{f}{e}, 1+\frac{3}{7}\right] = \left[\frac{4}{11}, \frac{e'}{f'}, \frac{3}{7}\right]$ where $\frac{e'}{f'} = \frac{f}{e} - 1$. Note that $\frac{f}{e}$ maps into $\frac{e'}{f'} = \frac{f}{e} - 1$ where $\frac{e'}{f'}$ is the smallest size fraction in $\left(\frac{4}{11}, \frac{3}{7}\right)$. $t \to \frac{1}{t}$: $\left[\frac{4}{11}, \frac{e'}{f'}, \frac{3}{7}\right] \to \left[\frac{7}{3}, \frac{f'}{e'}, \frac{11}{4}\right] = \left[2+\frac{1}{3}, \frac{f'}{e'}, 2+\frac{3}{4}\right]$. Note that $\frac{e'}{f'}$ maps into $\frac{f'}{e'}$ and $\frac{f'}{e'}$ is the smallest size fraction in $\left(2+\frac{1}{3}, 2+\frac{3}{4}\right)$. $t : t-2$: $\left[2+\frac{1}{3}, \frac{f'}{e'}, 2+\frac{3}{4}\right] \to \left[\frac{1}{3}, \frac{\overline{e}}{\overline{f}}, \frac{3}{4}\right]$ where $\frac{\overline{e}}{\overline{f}} = \frac{f'}{e'} - 2$. Note that $\frac{f'}{e'}$ maps into $\frac{\overline{e}}{\overline{f}} = \frac{f'}{e'} - 2$. $t \to \frac{1}{t}$: $\left[\frac{1}{3}, \frac{\overline{e}}{\overline{f}}, \frac{3}{4}\right] \to \left[\frac{4}{3}, \frac{\overline{f}}{\overline{e}}, 3\right] = \left[1+\frac{1}{3}, \frac{\overline{f}}{\overline{e}}, 3\right]$. This is Case 3. Of course, $\frac{\overline{f}}{\overline{e}}$ is the smallest size fraction in $\left(1+\frac{1}{3}, 3\right)$. From Case 3, $\frac{\overline{f}}{\overline{e}} = 2$. Working backwards, we have the following.

$$\frac{\overline{f}}{\overline{e}} = 2,\ \frac{\overline{e}}{\overline{f}} = \frac{1}{2},\ \frac{f'}{e'} = \frac{\overline{e}}{\overline{f}} + 2 = \frac{5}{2},$$
$$\frac{e'}{f'} = \frac{2}{5},\ \frac{f}{e} = \frac{e'}{f'} + 1 = \frac{7}{5},\ \frac{e}{f} = \frac{5}{7}.$$

$\frac{e}{f} = \frac{5}{7}$ is the same answer that we computed in Section 3.

We note that the calculations that we just made are fairly hard to keep track of. We now use continued fractions to go through the same calculations in an easier and more compact way.

Solution 2 (Continued Fractions).

$$\frac{e}{f} \in \left(\frac{7}{10}, \frac{11}{15}\right).$$
$$\frac{7}{10} \to \frac{1}{1+\frac{3}{7}} \to \frac{1}{1+\frac{1}{2+\frac{1}{3}}}.$$
$$\frac{11}{15} \to \frac{1}{1+\frac{4}{11}} \to \frac{1}{1+\frac{1}{2+\frac{3}{4}}} \to \frac{1}{1+\frac{1}{2+\frac{1}{1+\frac{1}{3}}}}.$$

Note that the numbers in these continued fractions are the same numbers that we computed in Solution 1. Especially note the end $\left(1+\frac{1}{3}, 3\right)$.

We now deal with $\frac{\overline{e}}{\overline{f}} \in \left(1+\frac{1}{3}, 3\right)$ which is Case 3. $\frac{\overline{e}}{\overline{f}} = 2$ is the smallest size fraction such that $\frac{\overline{e}}{\overline{f}} \in \left(1+\frac{1}{3}, 3\right)$. Substituting $\frac{\overline{e}}{\overline{f}} = 2$ for either

3 or $1+\frac{1}{3}$, we compute $\frac{e}{f} = \frac{1}{1+\frac{1}{2+\frac{1}{2}}} = \frac{5}{7}$ which is the same answer that we computed in the first direct solution. The continued fractions contain exactly the same information as the direct solution. However, the information in the continued fractions is much more compact and easy to understand.

Example 2 $0 < \frac{4}{77} < \frac{e}{f} < \frac{3}{56}$.

$$\frac{4}{77} = \frac{1}{19+\frac{1}{4}}, \quad \frac{3}{56} = \frac{1}{18+\frac{2}{3}}.$$

We now deal with $\frac{\overline{e}}{\overline{f}} \in \left(18+\frac{2}{3}, 19+\frac{1}{4}\right)$ which is Case 5. The smallest size $\frac{\overline{e}}{\overline{f}} \in \left(18+\frac{2}{3}, 19+\frac{1}{4}\right)$ is $\frac{\overline{e}}{\overline{f}} = 19$. We now substitute $\frac{\overline{e}}{\overline{f}} = 19$ for $19+\frac{1}{4}$ or $18+\frac{2}{3}$. This give us $\frac{e}{f} = \frac{1}{19}$ which is the known answer.

Example 3 $0 < \frac{8}{75} < \frac{e}{f} < \frac{7}{65}$.

$$\frac{8}{75} \to \frac{1}{9+\frac{3}{8}} \to \frac{1}{9+\frac{1}{2+\frac{2}{3}}}.$$
$$\frac{7}{65} \to \frac{1}{9+\frac{2}{7}} \to \frac{1}{9+\frac{1}{3+\frac{1}{2}}}.$$

We now deal with $\frac{\overline{e}}{\overline{f}} \in \left(2+\frac{2}{3}, 3+\frac{1}{2}\right)$ which is Case 5. The smallest size $\frac{\overline{e}}{\overline{f}} \in \left(2+\frac{2}{3}, 3+\frac{1}{2}\right)$ is $\frac{\overline{e}}{\overline{f}} = 3$. We now substitute 3 for $2+\frac{2}{3}$ or $3+\frac{1}{2}$. This gives us $\frac{1}{9+\frac{1}{3}} = \frac{3}{28}$ which is the known answer.

Example 4 $0 < \frac{10}{259} < \frac{e}{f} < \frac{3}{77}$.

$$\frac{10}{259} \to \frac{1}{25+\frac{9}{10}} \to \frac{1}{25+\frac{1}{1+\frac{1}{9}}}.$$
$$\frac{3}{77} \to \frac{1}{25+\frac{2}{3}} = \frac{1}{25+\frac{1}{1+\frac{1}{2}}}.$$

We now deal with $\frac{\overline{e}}{\overline{f}} \in (2,9)$ which is Case 1. The smallest size $\frac{\overline{e}}{\overline{f}} \in (2,9)$ is $\frac{\overline{e}}{\overline{f}} = 3$. We now substitute 3 for 2, or 9. This gives us

$$\frac{e}{f} = \frac{1}{25+\frac{1}{1+\frac{1}{3}}} = \frac{1}{25+\frac{3}{4}} = \frac{4}{103}$$

which is the known answer.

9 Solving Case 3 of Section 4

We need to find the smallest size $\frac{e}{f} \in (n+\epsilon, n+1)$, where numbers $n \in \{0,1,2,\dots\}$, $0 < \epsilon < 1$ and $\frac{e}{f}$ is reduced to lowest terms. Using Theorem 1, we can let $\frac{e}{f} = n + \frac{\overline{e}}{\overline{f}}$ where $1 \leq \overline{e} < \overline{f}$ and $\frac{\overline{e}}{\overline{f}}$ is reduced to lowest terms and $\frac{\overline{e}}{\overline{f}} \in (\epsilon, 1)$ and $\frac{\overline{e}}{\overline{f}}$ has the smallest possible size which is true if and only if both $\overline{e}, \overline{f}$ have the smallest possible values. Now $n+\frac{\overline{e}}{\overline{f}} \in (n+\epsilon, n+1)$ if and only if $(n+1)-\left(n+\frac{\overline{e}}{\overline{f}}\right) < (n+1)-(n+\epsilon)$.

That is, $\frac{\overline{f}-\overline{e}}{\overline{f}} < 1-\epsilon$.

Now $\frac{\overline{f}-\overline{e}}{1-\epsilon} < \overline{f}$. If we observe that $\frac{2}{1-\epsilon} > \frac{1}{1-\epsilon} + 1$, we see that $\overline{f}$ has the smallest possible value if and only if we let $\overline{e} = \overline{f} - 1$ and $\overline{f}$ is defined as the smallest positive integer satisfying $\overline{f} > \frac{1}{1-\epsilon}$.

By Theorem 1, $\frac{e}{f} = n + \frac{\overline{f}-1}{\overline{f}}$ must be the solution to Case 3.

10 Generalizations

The methods in this paper can be slightly modified to solve the same Problem 1 for the case where $0 \leq \frac{a}{b} < \frac{e}{f} < \frac{c}{d}$ and $\frac{a}{b}, \frac{c}{d}$ are allowed to be irrational numbers.

Next, suppose $x > 0, y > 0$ are fixed. The solution in this paper is also computing the fraction $\frac{e}{f}$ (reduced to lowest terms) such that $0 \leq \frac{a}{b} < \frac{e}{f} < \frac{c}{d}$ and such that $xe + yf$ is minimized. This follows from Theorem 1. This can be generalized as follows.

A function $F(x,y): (0,\infty) \times (0,\infty) \to R$ is said to be strictly increasing if $x \leq \overline{x}, y \leq \overline{y}$ and at least one of $x < \overline{x}, y < \overline{y}$ implies $F(x,y) < F(\overline{x},\overline{y})$.

The solution in this paper is also computing the fraction $\frac{e}{f}$ (reduced to lowest terms) such that $0 \leq \frac{a}{b} < \frac{e}{f} < \frac{c}{d}$ and such that $F(e,f)$ is minimized.

Examples of $F(e,f)$ are $F(e,f) = e \cdot f, F(e,f) = e^2 + 3f, F(e,f) = ef + f^2 + 4e, F(e,f) = e + f$.

Problems 1,$1'$ and Theorem 1 can also be generalized as follows.

Suppose $f(x): (0,\infty) \to (0,\infty)$ and $g(x): (0,\infty) \to (0,\infty)$ are strictly increasing. Also, suppose $F(x,y): (0,\infty) \times (0,\infty) \to R$ is strictly increasing as defined above. We generalize Problems 1, $1'$ and Theorem 1 as follows. We use the above $f(x), g(x), F(x,y)$ in Problem 1, $1'$ and in Theorem 1.

Problem 1 Suppose $0 < \frac{a}{b} < \frac{c}{d}$ where $\frac{a}{b}, \frac{c}{d}$ are fractions that are reduced to lowest terms.

Find an ordered pair (e,f), where $e,f \in \{1,2,3,4,\dots\}$, such that $0 < \frac{a}{b} < \frac{g(e)}{f(f)} < \frac{c}{d}$ and such that $f \in \{1,2,3,\dots\}$ has the smallest possible value. For this smallest possible value of f, let us compute the smallest possible $e \in \{1,2,3,\dots\}$ that can go with f so that $\frac{a}{b} < \frac{g(e)}{f(f)} < \frac{c}{d}$. In Problem 1, we assume that there exists $e', f' \in \{1,2,3,\dots\}$ such that $\frac{a}{b} < \frac{g(e')}{f(f')} < \frac{c}{d}$.

Problem $1'$ Suppose $0 < \frac{a}{b} < \frac{c}{d}$ where $\frac{a}{b}, \frac{c}{d}$ are fractions that are reduced to lowest terms. Find an ordered pair $(\overline{e},\overline{f})$, where $\overline{e},\overline{f} \in \{1,2,3,4,\dots\}$, such that $\frac{a}{b} < \frac{g(\overline{e})}{f(\overline{f})} < \frac{c}{d}$ and such that $\overline{e} \in \{1,2,3,\dots\}$

has the smallest possible value. For this smallest possible value of $\overline{e}$, let us compute the smallest possible $\overline{f} \in \{1, 2, 3, \dots\}$ that can go with $\overline{e}$ so that $0 < \frac{a}{b} < \frac{g(\overline{e})}{f(\overline{f})} < \frac{c}{d}$. In Problem $1'$, we assume that there exists $e', f' \in \{1, 2, 3, \dots\}$ so that $0 < \frac{a}{b} < \frac{g(e')}{f(f')} < \frac{c}{d}$.

Theorem 1 Suppose $\frac{a}{b}, \frac{c}{d}$ have the same values in both Problems $1, 1'$. Then the solutions (e, f) and $(\overline{e}, \overline{f})$ in Problems $1, 1'$ are the same. That is $(e, f) = (\overline{e}, \overline{f})$.

The proof that Problems $1, 1'$ are equivalent is trivial. We simply write Problem $1'$ as $0 < \frac{d}{c} < \frac{f(\overline{f})}{g(\overline{e})} < \frac{b}{a}$.

The elementary solutions to the new Problems $1, 1'$ are exactly the same as the elementary solutions to Problems $1, 1'$ that are given in Section 3.

Also, the proof of the new Theorem 1 is almost exactly the same as the proof of Theorem 1 that is given in Section 3.

From Theorem 1, we know that the solution $(e, f) = (\overline{e}, \overline{f})$ that we computed in Problems $1, 1'$ is also computing the ordered pairs $\{(e, f) : e, f \in \{1, 2, 3, \dots\}\}$ such that $F(e, f)$ is minimized where $F(x, y) : (0, \infty) \times (0, \infty) \to R$ is a strictly increasing function and such that $0 < \frac{a}{b} < \frac{g(e)}{f(f)} < \frac{c}{d}$. This follows because, by theorem 1, $f = \overline{f}$ and $e = \overline{e}$ are minimized.

11 More Sample Problems

1. Suppose $0 < \frac{a}{b} < \frac{c}{d}$ where $\frac{a}{b}, \frac{c}{d}$ are fractions that are reduced to lowest terms. Find an ordered pair (e, f), where $e, f \in \{1, 2, 3, \dots\}$ such that $0 < \frac{a}{b} < \frac{e^2}{f} < \frac{c}{d}$ and such that $e + f$ is minimized.

   Solution. Note that
   $$0 < \frac{d}{c} < \frac{f}{e^2} < \frac{b}{a}.$$
   We choose e so that e is the smallest positive integer for which there is a positive integer f satisfying $0 < \frac{d}{c}e^2 < f < \frac{b}{a}e^2$. For

this smallest positive integer e we choose f to be the smallest positive integer such that $0 < \frac{d}{c}e^2 < f < \frac{b}{a}e^2$. Note that for any $n \in \{1, 2, \ldots\}$, we can solve $0 < \frac{a}{b} < \frac{e^n}{f} < \frac{c}{d}$ and $0 < \frac{a}{b} < \frac{e}{f^n} < \frac{c}{d}$ the same way. We can also solve $0 < \frac{a}{b} < \frac{e^n}{f^n} < \frac{c}{d}$, $n \in \{1, 2, \ldots\}$ by solving $0 < (\frac{a}{b})^{1/n} < \frac{e}{f} < (\frac{c}{d})^{1/n}$. Similarly, we can solve $0 < \frac{a}{b} < \frac{e^n}{f^m} < \frac{c}{d}$ by solving $0 < (\frac{a}{b})^{1/n} < \frac{e}{f^{m/n}} < (\frac{c}{d})^{1/n}$.

2. Suppose $0 < \frac{e}{f}$ is a fraction reduced to lowest terms. Find the smallest x satisfying $0 \le x < \frac{e}{f}$ and the largest y satisfying $\frac{e}{f} < y$ such that (x, y) has the property that $\frac{e}{f}$ is the smallest size fraction in (x, y).

Solution. There are only a finite number of fractions m/n (reduced to lowest terms) such that both $m/n < e/f$ and $m + n \le e + f$, where $m + n$ is the size of m/n and $e + f$ is the size of e/f. Also, there are only a finite number of fractions $\overline{m}/\overline{n}$ (reduced to lowest terms) such that both $e/f < \overline{m}/\overline{n}$ and $\overline{m}+\overline{n} \le e+f$, where $\overline{m}+\overline{n}$ is the size of $\overline{m}/\overline{n}$. The solution to the problem is to choose x as the largest m/n and y as the smallest $\overline{m}/\overline{n}$.

12 More Problems for the Reader

3. For positive integers m and n, the decimal representation for the fraction m/n begins 0.711 followed by other digits. Find the least possible value for n. Hint: Find the smallest size m/n such that $0 < 0.711 < \frac{m}{n} < 0.712$. Any of the three methods in this paper can be used.

Solution. $n = 45$. Repeatedly add copies of 0.711 to itself until you get a number that is not quite an integer. This happens several times, but the smallest n for which there is an integer between 0.711 and 0.712 is $n = 45$. Since $45 \cdot 0.711 < 32 < 45 \cdot 0.712$, it follows that the fraction we seek is $32/45$.

4. Notice that $\frac{6}{12} = \frac{3}{6} = \frac{63}{126}$ and $\frac{2}{3} = \frac{18}{27} = \frac{218}{327}$. With this in mind, for two positive integers a and b, we call the number we get from writing the digits of a followed by the digits of b the *concatenation* of a and b. We write the concatenation of a and b as $a \oplus b$. For example $13 \oplus 5 = 135$ and $123 \oplus 456 = 123456$. Explore the conjecture If $\frac{a}{b} = \frac{c}{d}$, then $\frac{a}{b} = \frac{a \oplus c}{b \oplus d}$.

Solution. The conjecture is not true: $\frac{1}{2} = \frac{5}{10}$, but $\frac{1}{2} \neq \frac{15}{210}$. On the other hand, suppose c and d have the same number k of digits. Then $a \oplus c = 10^k a + c$ and $b \oplus d = 10^k b + d$. Also, suppose $\frac{a}{b} = \frac{c}{d} = u$. The $a = bu, c = du$ and we have

$$\frac{a \oplus c}{b \oplus d} = \frac{10^k a + c}{10^k b + d} = \frac{10^k bu + du}{10^k b + d} = u.$$

5. Bart and Lisa shoot free throws in two practice sessions to see who gets to start in tonight's game. Bart makes 5 out of 11 in the first session while Lisa makes 3 out of 7. Who has the better percentage? Is it possible that Bart shoots the better percentage again in the second session, yet overall Lisa has a higher percentage of made free throws? The answer is yes! This phenomenon is called Simpson's Paradox. Also see [2] for a geometric explanation of the paradox.

6. Notice that

$$\frac{19}{95} = \frac{1\cancel{9}}{\cancel{9}5} = \frac{1}{5}.$$

Can you find more pairs of two-digit numbers, with the smaller one on top, so that cancellation of this type works? Do you have them all?

Solution. Let us first build an equation using place value notation. Note that the equation can be written

$$\frac{10a + b}{10b + c} = \frac{a}{c},$$

where a, b, c are digits and $a < c$. This leads to $c(10a + b) = a(10b + c)$, which we message to get $10ac - ac = 10ab - bc$. This, in turn leads to $9ac + bc = c(9a + b) = 10ab$. From this it follows that either $c = 5$ or $9a + b$ is a multiple of 5.

Case 1. $c = 5$. Then

$$\frac{10a + b}{10b + 5} = \frac{a}{5}.$$

Next, letting $a = 1, \ldots a = 4$, etc. we find that

$a = 1:$ $5b + 50 = 10b + 5$ from which it follows that $b = 9$.

$a = 2:$ $100 + 5b = 20b + 10$ from which it follows that $90 = 15b$ and $b = 6$.

$a = 3:$ $\frac{30+b}{10b+5} = \frac{3}{5}$ gives $150 + 5b = 30b + 15$ which has not integer solutions.

$a = 4:$ $\frac{40+b}{10b+5} = \frac{4}{5}$ gives rise to $200 + 5b = 40b + 20$ which also has not solutions.

Why do we need not to check any higher values of a?

Case 2. $9a + b$ is a multiple of 5. Again we consider $a = 1, a = 2$, etc.

$a = 1:$ $9 + b = 10$ or $9 + b = 15$.If $b = 9$ we get $c = 5$ a case we already considered. If $b = 6$ we get $c = 4$, a new solution. $ab = 16, bc = 64$, and $a/c = 1/4$.

$a = 2:$ $18 + b$ is either 20 or 25. One leads to $b = 2$ and other $b = 7$ and neither of these works.

$a = 3:$ This leads to $b = 8$ which does not produce an integer value for c.

$a = 4$: $9 \cdot 4 + b = 45$, so $b = 9$. This leads to $c = 8$.

The four solutions are

$$\frac{19}{95} = \frac{1\cancel{9}}{\cancel{9}5} = \frac{1}{5}.$$

$$\frac{26}{65} = \frac{2\cancel{6}}{\cancel{6}5} = \frac{2}{5}.$$

$$\frac{16}{64} = \frac{1\cancel{6}}{\cancel{6}4} = \frac{1}{4}.$$

$$\frac{49}{98} = \frac{4\cancel{9}}{\cancel{9}8} = \frac{4}{8} = \frac{1}{2}.$$

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Scanning Cats

Yen-Kang Fu

Yen-Kang Fu is now a 10th grader in the class of science, in Taipei Municipal Jianguo High School. He has participated in math competitions since 2012. His most remarkable awards include a silver medal in BIMC 2013, a bronze medal in ITMO 2015, and a gold medal in TIMC 2016. Since his acquaintance with Professor Wen-Hsien Sun, the Chairman of Chiu-Chang mathematics Education Foundation and Professor Andy Liu of Canada, Fu has found out that mathematics is not just calculation, but the combination of creation, imagination, innovation, logic, and truth.

The m cool cats of the Catatonic Choir put on a show at the Catacoustic Centrium. Decorated in catmints and catnips, they emerged from backstage through a catwalk, to a round of enthusiastic applause. It went downhill from there. At first, the audience began to take catnaps. Then catcalls were heard. Their confidence shaken, the members allowed the final chorus to degenerate into a caterwaul. Various categories of objects, some launched from catapults, rained on the stage. It was a catastrophic performance. The local paper, *Cat of Nine Tales*, whipped the choir with a cataract of catacaustic comments.

The conductor came to the conclusion that there were copy cats in the choir, masquerading as cool cats. The only way to tell them apart was to use the catscans from the Catoptrics Consortium. The cheapest model could tell the exact number of copy cats among those scanned, but without identifying them individually. As a promotion, the company offered one free use of a large catscan which could take in any number of cats at a time. A group scan revealed that there were indeed n copy cats among them, where $0 < n < m$. A small catscan was then rented. It took in exactly k cats at a time, $1 < k < m$. Since the fee per use was extremely high, and the choir's budget was smaller than a budgie, it was essential that the catscan should be used as few times as possible.

Denote by $f(m, n, k)$ the minimum number of times the catscan must be used. The special case $f(20, 4, 3)$ was featured in an Omniheurists' Contest when *The Puzzling Adventures of Dr. Ecco* was first published by W. H. Freeman. The best submitted answer, according to author Dennis Shasha of the Courant Institute, was $f(20, 4, 3) \leq 10$, embodied as a 71-page computer printout with no explanations. When the book was republished by Dover later, Dennis asked that the reference to the contest be removed. A over-zealous editor deleted the contest problems as well.

In this paper, we present a readable proof that $f(20, 4, 3) \leq 11$. Number the cats from 1 to 20. Scan the first six of the following groups, namely, (1,2,3), (4,5,6), (7,8,9), (10,11,12), (13,14,15), (16,17,18) and (19,20). There are four cases.

Case 1 The distribution of the copy cats is 3-1.
We may assume that 1, 2 and 3 are copy cats.

Subcase 1(a) The other copy cat is one of 19 and 20.
The seventh scan (1,2,19) yields two or three copy cats. This allows us to determine whether 19 or 20 is the other copy cat. So seven scans suffice here.

Subcase 1(b) The other copy cat is one of 4, 5 and 6.
The seventh scan is (4,19,20) and the eighth scan is (5,19,20). If either yields one copy cat, then it is 4 in the former case and 5 in the latter case. If neither yields any copy cats, then 6 is a copy cat. So eight scans suffice here.

Case 2 The distribution of the copy cats is 2-2.
We may assume that two of 1, 2 and 3 are copy cats.

Subcase 2(a) The other copy cats are 19 and 20.
The seventh scan is (1,19,20). If it yields two copy cats, then 2 and 3 are copy cats. Otherwise, 1 is a copy cat. The eighth scan (2,19,20) also yields two or three copy cats. This allow us to tell whether 2 or 3 is the remaining copy cat. So eight scans suffice here.

Subcase 2(b) The other copy cats are two of 4, 5 and 6.
This is the complement of Subcase 3(a) below.

Case 3 The distribution of the copy cats is 2–1–1.

Subcase 3(a) The copy cats are 19, 20, one of 1, 2 and 3 and one of 4, 5 and 6.
The seventh scan is (1,4,19) and the eighth scan is (2,5,20). Each yields one, two or three copy cats, as summarized in the following chart.

		Seventh		
		1	2	3
Eighth	1	(3,6) are copy cats	(1,6) or (3,4)	(1,4) are copy cats
	2	(2,6) or (3,5)	(1,5) or (2,4)	Not Possible
	3	(2,5) are copy cats	Not Possible	Not Possible

A ninth scan, chosen from (1,19,20), (2,19,20) and (3,19,20), will settle the remaining uncertainties.

Subcase 3(b) The copy cats are two of 1, 2 and 3, one of 4, 5 and 6 and one of 19 and 20.
This can be handled as in Subcase 3(c) below, if we pretend that we do not know 18 is a cool cat.

Subcase 3(c) The copy cats are two of 1, 2 and 3, one of 4, 5 and 6 and one of 7, 8 and 9.
The seventh scan is (1,4,7) and the eighth scan is (2,4,7). If they yield the same number of copy cats, then 1 and 2 are copy cats. Otherwise, 3 is a copy cat, and we can also tell which of 1 and 2 is a copy cat. Moreover, we know how many copy cats are among 4 and 7. It may happen that both are copy cats. If neither is a copy cat, then two more scans, namely (5,19,20) and (8,19,20), will identify the remaining copy cats. Suppose exactly one of 4 and 7 is a copy cat. The ninth scan is (4,6,8) and the tenth scan is (5,7,8). Each yields zero, one or two copy cats, as summarized in the following chart.

		Ninth		
		0	1	2
Tenth	0	Not Possible	(4,9) are copy cats	Not Possible
	1	Not Possible	(6,7) are copy cats	(4,8) are copy cats
	2	(5,7) are copy cats	Not Possible	Not Possible

Case 4 The distribution of the copy cats is 1-1-1-1.
We may assume that one of 1, 2 and 3, one of 4, 5 and 6 and one of 7, 8 and 9 are copy cats.

Subcase 4(a) The other copy cat is one of 19 and 20.
This can be handled as in Subcase 4(b) below, as though we do not know that 18 is a cool cat. Interchange the labels (10,11,12) and (18,19,20).

Subcase 4(b) The other copy cat is one of 10, 11 and 12.
The seventh scan is (1,7,10), the eighth scan is (2,7,10), the ninth scan is (4,8,11) and the tenth scan is (5,8,11). As in Subcase 3(c), we know which of 1, 2 and 3 is a copy cat, and which of 4, 5 and 6 is a copy cat. Moreover, we know the number of copy cats among 7 and 10, and among 8 and 11. Each is zero, one or two, as summarized in the following chart.

		7 and 10		
		0	1	2
8 and 11	0	(9,12) are copy cats	(7,12) or (9,10)	(7,10) are copy cats
	1	(8,12) or (9,11)	(7,11) or (8,10)	Not Possible
	2	(8,11) are copy cats	Not Possible	Not Possible

An eleventh scan, chosen from (7,19,20), (8,19,20) and (9,19,20), will settle the remaining uncertainties.

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The 58th International Mathematical Olympiad, Rio de Janeiro, Brazil, 2017

The 58th International Mathematical Olympiad (IMO) was held 12–23 July 2017 in Rio de Janeiro Brazil. This was the largest IMO in history with a record number of 615 high school students from 111 countries participating. Of these, 62 were girls.

Each participating country may send a team of up to six students, a Team Leader and a Deputy Team Leader. At the IMO the Team Leaders, as an international collective, form what is called the *Jury*. This Jury was ably chaired by Nicolau Saldanha.

The first major task facing the Jury is to set the two competition papers. During this period the Leaders and their observers are trusted to keep all information about the contest problems completely confidential. The local Problem Selection Committee had already shortlisted 32 problems from 150 problem proposals submitted by 51 of the participating countries from around the world. During the Jury meetings three of the shortlisted problems had to be discarded from consideration due to being too similar to material already in the public domain. Eventually, the Jury finalised the exam questions and then made translations into the 57 languages required by the contestants.

The six questions that ultimately appeared on the IMO contest are described as follows.

1. An easy number theoretic sequence problem proposed by South Africa.
2. A medium to difficult functional equation proposed by Albania.
3. A difficult game theory problem with incomplete information proposed by Austria.
4. A relatively easy classical geometry problem proposed by Luxembourg.
5. A medium to difficult combinatorics problem reminiscent of the Erdős-Szekeres theorem. It was proposed by Russia.

6. A difficult problem, somewhat reminiscent of Lagrange interpolation, combining number theory and polynomials. It was proposed by the United States of America.

These six questions were posed in two exam papers held on Tuesday 18 July and Wednesday 19 July. Each paper had three problems. The contestants worked individually. They were allowed four and a half hours per paper to write their attempted proofs. Each problem was scored out of a maximum of seven points.

For many years now there has been an opening ceremony prior to the first day of competition. Following the formal speeches there was the parade of the teams and the 2017 IMO was declared open.

After the exams the Leaders and their Deputies spent about two days assessing the work of the students from their own countries, guided by marking schemes, which had been agreed to earlier. A local team of markers called *Coordinators* also assessed the papers. They too were guided by the marking schemes but are allowed some flexibility if, for example, a Leader brought something to their attention in a contestant's exam script that is not covered by the marking scheme. The Team Leader and Coordinators have to agree on scores for each student of the Leader's country in order to finalise scores. Any disagreements that cannot be resolved in this way are ultimately referred to the Jury.

Problem 1 turned out to be the easiest problem in the IMO for many years[1] with an average score of 5.94. In contrast, problem 3 ended up being the most difficult problem ever in the IMO's 58 year history. It averaged only 0.04. Just two contestants, Linus Cooper from Australia and Mikhail Ivanov from Russia, managed to score full marks on it, while 608 of the 615 contestants were unable to score even a single point.

The medal cuts were set at 25 for gold, 19 for silver and 16 for bronze. Consequently, there were 291 (=47.3%) medals awarded. The medal distributions[2] were 44 (=7.8%) gold, 90 (=14.6%) silver and 153 (=24.9%)

[1] We have to go back to the IMO in 1981 to find problems with higher average scores.

[2] The total number of medals must be approved by the Jury and should not normally exceed half the total number of contestants. The numbers of gold, silver, and bronze medals should be approximately in the ratio 1:2:3.

bronze. These awards were presented at the closing ceremony. Of those who did not get a medal, a further 222 contestants received an honourable mention for solving at least one question perfectly.

No contestant was able to achieve a perfect score of 42. The top score was 35 which was obtained by the following three students.

Amirmojtaba Sabour	Iran
Yuta Takaya	Japan
Hữu Quốc Huy Hoàng	Vietnam

In an effort to encourage female participation at the IMO, five girls were each given a special award at the closing ceremony.

The 2017 IMO was organised by Brazil's National Institute of Pure and Applied Mathematics (IMPA), and the Brazilian Mathematical Society.

The 2018 IMO is scheduled to be held July 3–14 in Cluj-Napoca, Romania. Venues for future IMOs have been secured up to 2022 as follows.

2019	United Kingdom
2020	Russia
2021	United States
2022	Norway

Much of the statistical information found in this report can also be found at the official website of the IMO.

www.imo-official.org

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First Day

Tuesday, July 18, 2017

Problem 1. For each integer $a_0 > 1$, define the sequence a_0, a_1, a_2, ... by:

$$a_{n+1} = \begin{cases} \sqrt{a_n} & \text{if } \sqrt{a_n} \text{ is an integer,} \\ a_n + 3 & \text{otherwise,} \end{cases} \qquad \text{for each } n \geqslant 0.$$

Determine all values of a_0 for which there is a number A such that $a_n = A$ for infinitely many values of n.

Problem 2. Let $\mathbb{R}$ be the set of real numbers. Determine all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that, for all real numbers x and y,

$$f\left(f(x)f(y)\right) + f(x+y) = f(xy).$$

Problem 3. A hunter and an invisible rabbit play a game in the Euclidean plane. The rabbit's starting point, A_0, and the hunter's starting point, B_0, are the same. After $n-1$ rounds of the game, the rabbit is at point A_{n-1} and the hunter is at point B_{n-1}. In the n^{th} round of the game, three things occur in order.

(i) The rabbit moves invisibly to a point A_n such that the distance between A_{n-1} and A_n is exactly 1.

(ii) A tracking device reports a point P_n to the hunter. The only guarantee provided by the tracking device to the hunter is that the distance between P_n and A_n is at most 1.

(iii) The hunter moves visibly to a point B_n such that the distance between B_{n-1} and B_n is exactly 1.

Is it always possible, no matter how the rabbit moves, and no matter what points are reported by the tracking device, for the hunter to choose her moves so that after 10^9 rounds she can ensure that the distance between her and the rabbit is at most 100?

Language: English

Time: 4 hours and 30 minutes
Each problem is worth 7 points

Second Day

Wednesday, July 19, 2017

Problem 4. Let R and S be different points on a circle Ω such that RS is not a diameter. Let ℓ be the tangent line to Ω at R. Point T is such that S is the midpoint of the line segment RT. Point J is chosen on the shorter arc RS of Ω so that the circumcircle Γ of triangle JST intersects ℓ at two distinct points. Let A be the common point of Γ and ℓ that is closer to R. Line AJ meets Ω again at K. Prove that the line KT is tangent to Γ.

Problem 5. An integer $N \geqslant 2$ is given. A collection of $N(N+1)$ soccer players, no two of whom are of the same height, stand in a row. Sir Alex wants to remove $N(N-1)$ players from this row leaving a new row of $2N$ players in which the following N conditions hold:

(1) no one stands between the two tallest players,

(2) no one stands between the third and fourth tallest players,

$\vdots$

(N) no one stands between the two shortest players.

Show that this is always possible.

Problem 6. An ordered pair (x, y) of integers is a *primitive point* if the greatest common divisor of x and y is 1. Given a finite set S of primitive points, prove that there exist a positive integer n and integers $a_0, a_1, \ldots, a_n$ such that, for each (x, y) in S, we have:

$$a_0x^n + a_1x^{n-1}y + a_2x^{n-2}y^2 + \cdots + a_{n-1}xy^{n-1} + a_ny^n = 1.$$

Language: English

Time: 4 hours and 30 minutes
Each problem is worth 7 points

Mark Distribution by Question

Mark	Q1	Q2	Q3	Q4	Q5	Q6
0	40	183	608	47	451	557
1	16	110	3	93	46	24
2	17	26	0	42	47	9
3	5	138	0	14	9	5
4	12	79	1	15	0	4
5	54	10	1	4	2	2
6	25	8	0	6	1	0
7	446	61	2	394	59	14
Total	615	615	615	615	615	615
Mean	5.94	2.30	0.04	5.03	0.97	0.29

The medal cuts were set at 25 for gold, 19 for silver and 16 for bronze.

Some Country Totals

Rank	Country	Total
1	South Korea	170
2	China	159
3	Vietnam	155
4	United States of America	148
5	Iran	142
6	Japan	134
7	Singapore	131
7	Thailand	131
9	Taiwan	130
9	United Kingdom	130
11	Russia	128
12	Georgia	127
12	Greece	127
14	Belarus	122
14	Czech Republic	122
14	Ukraine	122
17	Philippines	120
18	Bulgaria	116
18	Italy	116
18	Netherlands	116
18	Serbia	116

Rank	Country	Total
22	Hungary	115
22	Poland	115
22	Romania	115
25	Kazakhstan	113
26	Argentina	111
26	Bangladesh	111
26	Hong Kong	111
29	Canada	110
30	Peru	109
31	Indonesia	108
32	Israel	107
33	Germany	106
34	Australia	103
35	Croatia	102
35	Turkey	102
37	Brazil	101
37	Malaysia	101
39	France	100
39	Saudi Arabia	100

Distribution of Awards at the 2017 IMO

Country	Total	Gold	Silver	Bronze	HM
Albania	67	0	0	1	5
Algeria	70	0	0	1	4
Argentina	111	1	2	1	2
Armenia	99	0	2	2	1
Australia	103	0	3	2	1
Austria	74	0	2	0	2
Azerbaijan	98	0	0	4	2
Bangladesh	111	0	2	2	2
Belarus	122	1	1	4	0
Belgium	80	0	1	2	2
Bolivia	41	0	0	0	4
Bosnia and Herzegovina	95	0	0	4	2
Botswana	19	0	0	0	1
Brazil	101	0	2	1	3

Country	Total	Gold	Silver	Bronze	HM
Bulgaria	116	0	4	2	0
Cambodia	11	0	0	0	1
Canada	110	1	2	2	1
Chile	67	0	0	1	4
China	159	5	1	0	0
Colombia	81	0	0	1	5
Costa Rica	58	0	0	0	5
Croatia	102	0	2	3	1
Cuba	13	0	0	0	1
Cyprus	93	0	0	5	1
Czech Republic	122	1	2	2	1
Denmark	77	0	0	1	5
Ecuador	66	0	0	1	4
Egypt	3	0	0	0	0
El Salvador	57	0	0	1	3
Estonia	72	0	1	0	4
Finland	56	0	0	0	6
France	100	0	2	2	2
Georgia	127	1	2	3	0
Germany	106	0	1	3	2
Ghana	6	0	0	0	0
Greece	127	1	4	1	0
Guatemala	20	0	0	0	1
Honduras	12	0	0	0	0
Hong Kong	111	1	1	3	1
Hungary	115	2	1	1	1
Iceland	45	0	0	0	3
India	90	0	0	3	3
Indonesia	108	0	2	3	1
Iran	142	2	3	1	0
Iraq	13	0	0	0	1
Ireland	80	0	0	2	4
Israel	107	0	3	2	0
Italy	116	2	1	1	2
Ivory Coast	11	0	0	0	0
Japan	134	2	2	2	0

Country	Total	Gold	Silver	Bronze	HM
Kazakhstan	113	1	2	1	1
Kenya	8	0	0	0	0
Kosovo	55	0	0	1	2
Kyrgyzstan	75	0	0	2	3
Latvia	84	0	0	3	2
Liechtenstein	22	0	0	0	2
Lithuania	69	0	0	2	3
Luxembourg	45	0	0	1	1
Macau	94	1	0	0	5
Macedonia (FYR)	77	0	0	1	4
Malaysia	101	0	2	2	2
Mexico	96	0	1	2	3
Moldova	83	0	1	0	4
Mongolia	93	0	1	2	3
Montenegro	42	0	0	1	2
Morocco	75	0	0	1	4
Myanmar	15	0	0	0	1
Nepal	3	0	0	0	0
Netherlands	116	1	2	1	1
New Zealand	94	0	0	3	3
Nicaragua	44	0	0	1	2
Nigeria	51	0	0	0	4
Norway	71	0	0	2	3
Pakistan	58	0	0	1	3
Panama	15	0	0	0	1
Paraguay	48	0	0	0	2
Peru	109	0	2	3	1
Philippines	120	0	3	3	0
Poland	115	1	0	5	0
Portugal	89	0	0	2	2
Puerto Rico	55	0	0	0	4
Romania	115	0	3	2	1
Russia	128	1	3	2	0
Saudi Arabia	100	0	2	2	1
Serbia	116	0	4	2	0
Singapore	131	2	1	2	1

Country	**Total**	**Gold**	**Silver**	**Bronze**	**HM**
Slovakia	75	0	0	1	5
Slovenia	90	0	0	2	4
South Africa	81	0	0	2	4
South Korea	170	6	0	0	0
Spain	86	0	0	3	2
Sri Lanka	80	0	0	3	3
Sweden	91	0	1	2	3
Switzerland	83	0	0	1	5
Syria	85	0	1	0	5
Taiwan	130	1	4	1	0
Tajikistan	95	0	0	3	3
Tanzania	5	0	0	0	0
Thailand	131	3	0	2	1
Trinidad and Tobago	15	0	0	0	1
Tunisia	59	0	0	1	3
Turkey	102	0	1	3	2
Turkmenistan	93	0	0	2	4
Uganda	22	0	0	0	1
Ukraine	122	1	2	2	1
United Kingdom	130	3	0	2	1
United States of America	148	3	3	0	0
Uruguay	43	0	0	0	3
Uzbekistan	69	0	1	0	4
Venezuela	59	0	0	2	2
Vietnam	155	4	1	1	0
Total (111 teams, 615 contestants)		**48**	**90**	**153**	**222**

N.B. Not all countries sent a full team of six students.

International Mathematics Tournament of Towns Selected Problems from the Spring 2017 Tournament

Andy Liu

Problems

1. Prove that the graph of any monic quadratic polynomial with a repeated root passes through a point (p, q) such that $x^2 + px + q$ also has a repeated root.

2. A circle is drawn with each side of an n-gon as diameter. Is it possible that all these circles pass through a point which is not a vertex of the n-gon, where

   (a) $n = 10$;

   (b) $n = 11$?

3. A set of positive numbers has sum a_1. For any integer $k \geq 2$, a_k is the sum of the k-th powers of the numbers in the set. Is it possible that

   (a) $a_1 > a_2 > a_3 > a_4 > a_5$ and $a_5 < a_6 < a_7 < \cdots$;

   (b) $a_1 < a_2 < a_3 < a_4 < a_5$ and $a_5 > a_6 > a_7 > \cdots$?

4. Each token weighs a non-integer number of grams. Any object of integer weight from 1 gram to 40 grams can be balanced by placing some tokens on the same pan. What is the smallest possible number of tokens?

5. On a $1 \times n$ board, a grasshopper can jump to the 8th, the 9th or the 10th square in either direction. Find an intneger $n \geq 50$ such that starting from some square of the board, the grasshopper can visit every square exactly once.

6. On the plane are ten lines and one triangle. Every line is equidistant from two of the triangle's vertices. Prove that either at least two of these lines are parallel or at least three of them pass are concurrent.

7. Is it possible to dissect a cube into two pieces which can be reassembled into a convex polyhedron with only triangular and hexagonal faces?

8. In triangle ABC, $\angle A = 45°$. When the median from A is reflected across the altitudes from B and from C, the two images intersect at a point X. Prove that $AX = BC$.

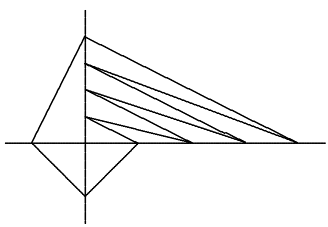
9. Find all positive integers n which has a multiple with digit sum k for any integer $k \geq n$.

10. Each of 36 gangsters belongs to several gangs. There are no two gangs with the same membership. Two gangsters are enemies if they do not both belong to the same gang. Each gangster has at least one enemy in every gang to which he does not belong. What is the largest possible number of gangs?

Solutions

1. Let the polynomial be $y = (x - r)^2$. Since it passes through (p, q), $q = (p - r)^2$. Since $x^2 + px + q$ also has repeated roots, $p^2 - 4q = 0$. Hence $p^2 = 4q = 4(p - r)^2$. If $p = 2p - 2r$, then $p = 2r$ and $q = r^2$. This is the original polynomial. If $p = 2r - 2p$, then $p = \frac{2}{3}r$ and $q = \frac{1}{9}r^2$, so that $x^2 + px + q = (x + \frac{1}{3}r)^2$.

2. The vertices of the polygon must lie alternately on two perpendicular lines passing through the common point of the circles.



(a) The diagram above shows an example for $n = 10$.

(b) We cannot have $n = 11$ or any other odd values as the polygonal line will not close.

3. *Solution by Ryan Morrill.*

(a) It can happen. Take the given positive integers to be $\frac{1}{2}$ and $1 + \frac{1}{64}$. Since there are only two numbers the sequence can only change between increasing and decreasing at most once, so it suffices to check $a_4 > a_5$ and $a_5 < a_6$. For the first inequality, notice that $(1 + \frac{1}{2^6})^4 < 2$. Then

$$\begin{aligned} a_4 - a_5 &= \left(\tfrac{1}{2}\right)^5 + \left(1 + \tfrac{1}{2^6}\right)^5 - \left(\tfrac{1}{2}\right)^4 - \left(1 + \tfrac{1}{2^6}\right)^4 \\ &= \tfrac{1}{2^4}\left(\tfrac{1}{2} - 1\right) + \tfrac{1}{2^6}\left(1 + \tfrac{1}{2^6}\right)^4 \\ &= \tfrac{1}{2^6}\left(1 + \tfrac{1}{2^6}\right)^4 - \tfrac{1}{2^5} \\ &< 0. \end{aligned}$$

For the second inequality, notice that $(1 + \frac{1}{2^6})^5 > 1$. Then, by a calculation similar to the above, we have $a_6 - a_5 = \frac{1}{2^6}(1 + \frac{1}{2^6})^5 - \frac{1}{2^6} > 0$.

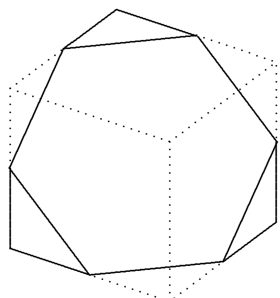
(b) It cannot happen. Out of the given positive integers, at least one must be larger than 1 as otherwise the sequence will be decreasing for all terms. The nth power of such a number gets larger without bound as n increases. Hence for large enough n, a_n must always grow without bound, so it is impossible to get an eventually decreasing sequence.

4. *Solution by Ryan Morrill.*
Seven tokens are sufficient. Let their weights be 0.5, $0.5 \times 2 - 0.5 = 0.5$, $0.5 \times 2 + 0.5 = 1.5$, $1.5 \times 2 - 0.5 = 2.5$, $2.5 \times 2 + 0.5 = 5.5$, $5.5 \times 2 - 0.5 = 10.5$ and $10.5 \times 2 + 0.5 = 21.5$ grams respectively. It is routine to verify that every object with integer weight up to 41 grams can be balanced. Suppose we only have six tokens. Put one of them aside. Then there are $2^5 = 32$ sets formed from the other five tokens. We associate each set with a companion set formed by adding the token set aside. Since this token has a non-integer weight, at most one set from each associated pair can balance an object with integer weight. Since $32 < 40$, six tokens are not sufficient.

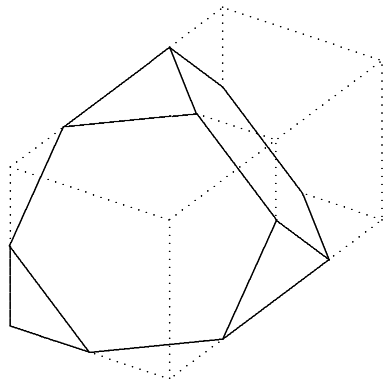
5. *Solution by Ryan Morrill.*
If there are 63 squares, it is impossible for the grasshopper to complete its task. Number the squares 1 to 63 and divide them into seven blocks of 9 squares. Colour the blocks alternately red and blue. The grasshopper can jump from a red square to a red square at most 7 times, namely between the pairs (1,9), (9,19), (19,27), (27,37), (37,45), (45,55) and (55,62). There are 27 blue squares. Even if the grasshopper never jumps from a blue square to another blue square, it can only jump from a blue square to a red square 27 times. The starting square may be red, but this accounts for only 27+7+1=35 red squares. We have a contradiction since there are 36 red squares.

6. A line equidistant from two vertices of a triangle is either parallel to the side they determine, or passing through the midpoint of that side. If there are at least four lines of the first type, two of them must be parallel since each of them is parallel to one of three sides. If there are at most three lines of the first type, then there are at least seven line of the second type. Three of them must be concurrent since each of them passes through one of three midpoints.

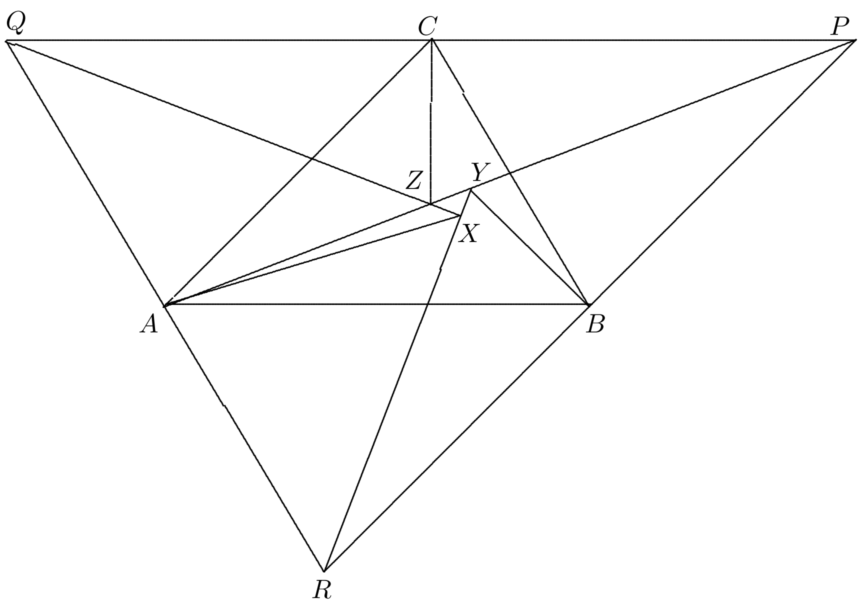
7. A cube can be dissected into two congruent pieces as shown in the diagram below. Each consists of one hexagonal face, three pentagonal faces (hidden) and three triangular faces.



When the two pieces are reassembled by abutting one pentagonal face from each piece, two pairs of pentagonal faces merge into two hexagonal faces (hidden) and two pairs of triangular faces merge into two triangular faces. The resulting convex polyhedron, as shown in the diagram below, has four hexagonal faces and four triangular faces.



8. *Solution by Central Jury.*
Let PQR be the triangle such that A, B and C are the respective midpoints of QR, RP and PQ. Then the median from A



passes through P. The altitude from B is perpendicular to CA and therefore to RP. Hence the reflection of the median AP at the point Y on this altitude passes through R. Similarly, the reflection of the median AP at the point Z on the altitude from C passes through Q. Let the extension of QZ intersect RY at X. Now $\angle XZY = \angle QZA = 2(90^\circ - \angle QZC) = 2\angle CPY$. Similarly, $\angle XYZ = 2\angle BPY$. It follows that $\angle ZXY = 180^\circ - \angle XYZ - \angle XZY = 180^\circ - 2\angle BPC = 180^\circ - 2\angle CAB = 90^\circ$. Hence A is the circumcentre of the right triangle XQR, so that $AX = \frac{1}{2}QR = BC$.

9. Solution by Steven Chow.
Note first that n cannot be a multiple of 3. Otherwise, all multiples of n will have digit-sums divisible by 3. For any $k \geq n$ which is not a multiple of 3, no multiple of n can have digit-sum k. Let n be any positive integer relatively prime to 30. Then $10^{\phi(n)} \equiv 1 \pmod{n}$ by Euler's Theorem. Let $k \geq n$ be any integer. The number $M = \sum_{i=0}^{k-1} 10^{i\phi(n)}$ is congrrent to k modulo n. It has digit-sum k since it consists of k copies of 1 separated by copies of 0. If k is a multiple of n, then M is the desired multiple. Otherwise, there exists a positive integer $d < n$ such that $k + 9d$ is a multiple of n. Let each of the last d copies of 1 in M trade places with the copy of 0 to the left. In other words, we replace $10^{i\phi(n)} \equiv 1 \pmod{n}$ by $10^{i\phi(n)+1} \equiv 10 \pmod{n}$ for $0 \leq i \leq d-1$. The new number still has digit-sum k, but is now congruent to $k + 9d \equiv 0 \pmod{n}$. Finally, let n be any positive integer relatively prime to 3. Let m be the number obtained from n by removing the 2s and the 5s from its prime factorization, and let M be the desired multiple for m as constructed above. By adding a sufficient number of 0s at the end, this will be a multiple of n which still has digit-sum k. In the special case $m = 1$, take M to be the number consisting of k copies of 1.

10. *Solution by Central Jury.*
More generally, let there be $n \geq 2$ gangsters and let $g(n)$ be the

largest possible number of gangs. Define

$$f(n) = \begin{cases} 3^k & n = 3k, \\ 4(3^{k-1}) & n = 3k+1, \\ 2(3^k) & n = 3k+2. \end{cases}$$

Note that if n is partitioned into any number of positive integers, the largest product of these integers is $f(n)$. If we take $f(0) = f(1) = 1$, then $f(k)f(n-k) \le f(n)$ for $0 \le k \le n$. We claim that $g(n) = f(n)$ for all $n \ge 2$. In particular, $g(36) = 3^{12}$. We first prove that $g(n) \ge f(n)$ for all $n \ge 2$. Partition the n gangsters into groups of 3. If there is one left over, add him to an existing group. If there are two left over, start a new group. Now form all possible gangs with one member from each group. The number of gangs is $f(n)$, and these gangs satsify all conditions of the problem. Hence $g(n) \ge f(n)$. We now use mathematical induction on n to prove that $g(n) \le f(n)$ for all $n \ge 2$. We may take $g(0) = 1 = f(0)$ and $g(1) = 1 = f(1)$. It is routine to verify that $g(n) = n = f(n)$ for $2 \le n \le 4$. For $n \ge 5$, assume that $g(k) = f(k)$ for $0 \le k \le n-1$. We shall prove that $g(n) \le f(n)$. Construct a graph with n vertices representing the n gangsters. Two vertices are joined by an edge if and only if the gangsters they represent are enemies. If the graph is not connected, let there be k vertices in one component, $1 \le k \le n-1$. Then each gang is a union of a subgang consisting of some gangsters represented by vertices in this component and a subgang consisting of some of the remaining gangsters. There are at most $g(k)$ subgangs of the first type, and at most $g(n-k)$ subgangs of the second type. Hence $g(n) \le g(k)g(n-k)$ since no two gangs can have identical membership. By the induction hypothesis, $g(k)g(n-k) \le f(k)f(n-k) \le f(n)$. Henceforth, assume that the graph is connected. Suppose it is in fact a cycle or a path. Then there are three consecutive vertices each of degree 2. Let them represent the gangsters B, C and D, with B also hostile to A and D to E. If a gang does not contain any of B, C and D, then C has no enemies in it, which is a contradiction. Consider the gangs which contain B. Then it can contain neither A nor C. Thus there are at most $g(n-3)$ of them. Similarly, there are at most $g(n-3)$ angs which contain C, and $g(n-3)$ gangs which contain D. It is possible that a gang may contain both B and D, but this will only reduce the total number of gangs. By the induction hypothesis, $g(n) \leq 3g(n-3) \leq 3f(n-3) \leq f(n)$. Finally, suppose there is a vertex of degree at least 3. Let it represent a gangster W who has at least three enemies X, Y and Z. The number of gangs not containing W is at most $g(n-1)$. If a gang contains W, it cannot contain any of X, Y and Z. Hence the number of gangs containing W is at most $g(n-4)$. By the induction hypothesis, $g(n) \leq g(n-1) + g(n-4) \leq f(n-1) + f(n-4) \leq f(n)$.

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