VOLUME 28 NUMBER 1 2015

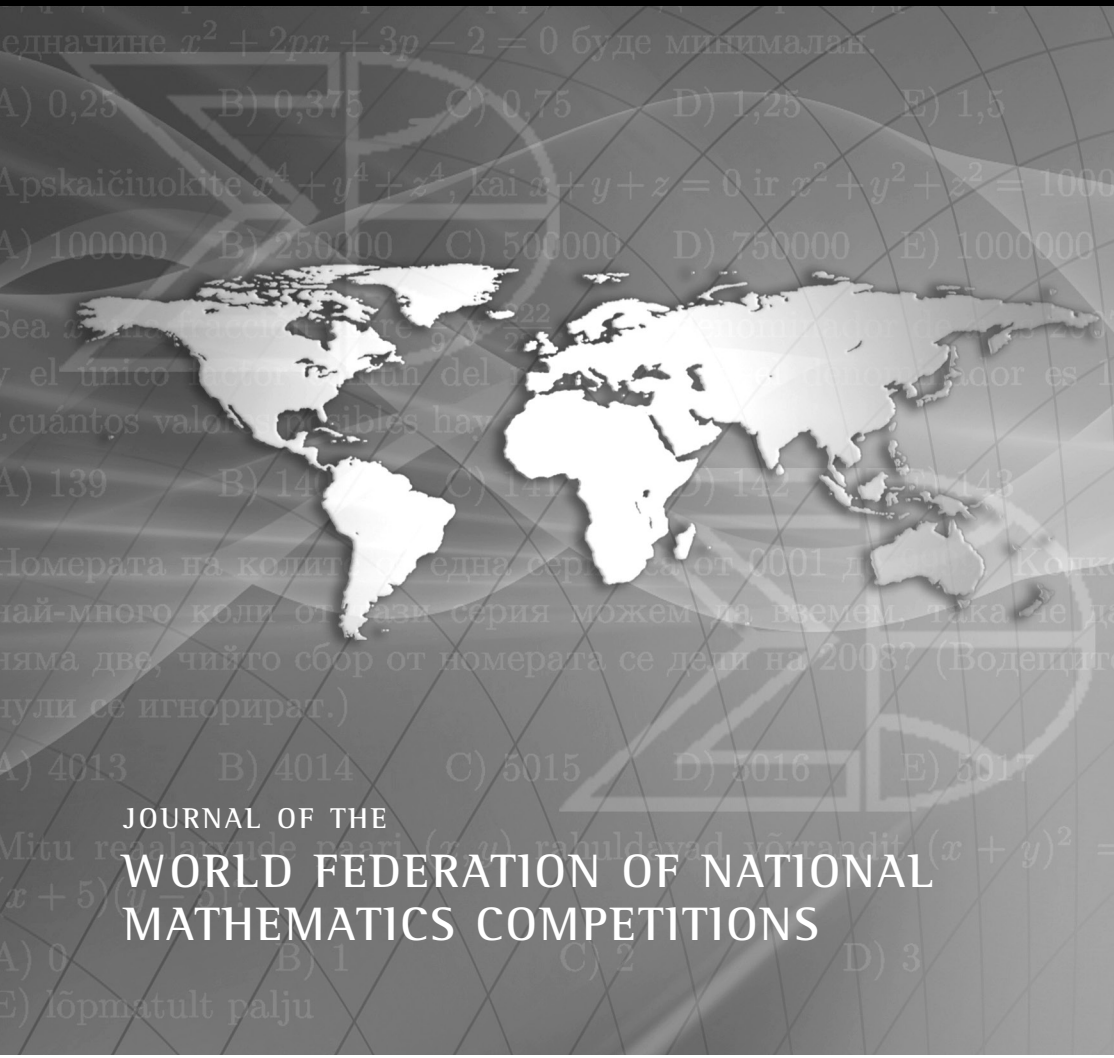
MATHEMATICS COMPETITIONS

JOURNAL OF THE
WORLD FEDERATION OF NATIONAL MATHEMATICS COMPETITIONS

AMT PUBLISHING

VOLUME 28 NUMBER 1 2015

MATHEMATICS COMPETITIONS



JOURNAL OF THE

WORLD FEDERATION OF NATIONAL MATHEMATICS COMPETITIONS

AMT PUBLISHING

MATHEMATICS COMPETITIONS
JOURNAL OF THE WORLD FEDERATION OF NATIONAL MATHEMATICS COMPETITIONS

(ISSN 1031 – 7503)
Published biannually by

AMT PUBLISHING
AUSTRALIAN MATHEMATICS TRUST
UNIVERSITY OF CANBERRA
LOCKED BAG 1
CANBERRA GPO ACT 2601
AUSTRALIA

With significant support from the UK Mathematics Trust.

Articles (in English) are welcome.

Please send articles to:

The Editor
Mathematics Competitions
World Federation of National Mathematics Competitions
University of Canberra Locked Bag 1
Canberra GPO ACT 2601
Australia

Fax:+61-2-6201-5052

or

Dr Jaroslav Švrček
Dept. of Algebra and Geometry
Faculty of Science
Palacký University
Tr. 17. Listopadu 12
Olomouc
772 02
Czech Republic

Email: svrcek@inf.upol.cz

MATHEMATICS COMPETITIONS VOLUME 28 NUMBER 1 2015

CONTENTS PAGE

World Federation of National Mathematics Competitions

Executive

North America: Professor Ian VanderBurgh
University of Waterloo
Waterloo
CANADA

Oceania: Professor Derek Holton
605/228 The Avenue
Parkville Vic 3052
AUSTRALIA

South America: Professor Radmila Bulajich
Facultad de Ciencias, UAEM
Av. Universidad 1001, Col. Chamilpa
Cuernavaca, Morelos 62209
MEXICO

The aims of the Federation are:

1. ***to promote excellence in, and research associated with, mathematics education through the use of school mathematics competitions;***
2. ***to promote meetings and conferences where persons interested in mathematics contests can exchange and develop ideas for use in their countries;***
3. ***to provide opportunities for the exchanging of information for mathematics education through published material, notably through the Journal of the Federation;***
4. ***to recognize through the WFNMC Awards system persons who have made notable contributions to mathematics education through mathematical challenge around the world;***
5. ***to organize assistance provided by countries with developed systems for competitions for countries attempting to develop competitions;***
6. ***to promote mathematics and to encourage young mathematicians.***

From the President

Dear Fellow Federalists!

The 13th International Congress on Mathematical Education in Hamburg is approaching. The organizers have included the Topic Study Group, TSG 30: Mathematics Competitions, and has just released the Second Announcement:

http://icme13.org/announcements/second

Most of the TSG 30 organizers are well familiar to you:

Co-chairs:	María Falk de Losada (Colombia) mariadelosada@gmail.com
	Alexander Soifer (USA) asoifer@mail.uccs.edu
Team members:	Christian Reiher (Germany)
	Jaroslav Svrcek (Czech Republic)
	Peter Taylor (Australia)
IPC Liaison person:	Binyan Xu (China)

Our Federation will also have time slots allocated to us.

So, start your mental engines, and propose your contributions to the co-chairs. I eagerly look forward to seeing all of you in Hamburg!

Cordially,

Alexander Soifer
President of WFNMC
June 2015

From the Editor

Welcome to *Mathematics Competitions* Vol. 28, No. 1.

First of all I would like to thank again the Australian Mathematics Trust for continued support, without which each issue of the journal could not be published, and in particular Heather Sommariva, Bernadette Webster and Pavel Calábek for their assistance in the preparation of this issue.

Submission of articles:

The journal *Mathematics Competitions* is interested in receiving articles dealing with mathematics competitions, not only at national and international level, but also at regional and primary school level. There are many readers in different countries interested in these different levels of competitions.

- The journal traditionally contains many different kinds of articles, including reports, analyses of competition problems and the presentation of interesting mathematics arising from competition problems. Potential authors are encouraged to submit articles of all kinds.
- To maintain and improve the quality of the journal and its usefulness to those involved in mathematics competitions, all articles are subject to review and comment by one or more competent referees. The precise criteria used will depend on the type of article, but can be summarised by saying that an article accepted must be correct and appropriate, the content accurate and interesting, and, where the focus is mathematical, the mathematics fresh and well presented. This editorial and refereeing process is designed to help improve those articles which deserve to be published.

At the outset, the most important thing is that if you have anything to contribute on any aspect of mathematics competitions at any level, local, regional or national, we would welcome your contribution.

Articles should be submitted in English, with a black and white photograph and a short profile of the author. Alternatively, the article can be submitted on an IBM PC compatible disk or a Macintosh disk. We prefer LaTeX or TeX format of contributions, but any text file will be helpful.

Articles, and correspondence, can also be forwarded to the editor by mail to

The Editor, *Mathematics Competitions*
Australian Mathematics Trust
University of Canberra Locked Bag 1
Canberra GPO ACT 2601
AUSTRALIA

or to

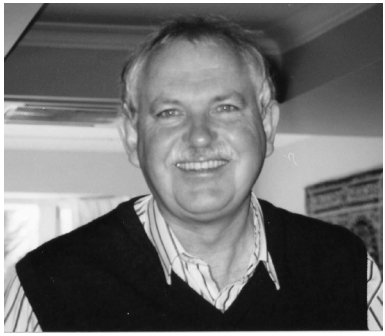
Dr Jaroslav Švrček
Dept. of Algebra and Geometry
Palacký University of Olomouc
17. listopadu 1192/12
771 46 OLOMOUC
CZECH REPUBLIC

jaroslav.svrcek@upol.cz

Jaroslav Švrček
June 2015

Classifying Methods of Problem Solving—and my Favourites

Peter J Taylor



Peter Taylor graduated with a Ph.D. in Applied Mathematics in 1972 at the University of Adelaide. He is Emeritus Professor of Mathematics at the University of Canberra, and was previously Executive Director of the Australian Mathematics Trust and Chairman of the Education Advisory Committee of the Australian Mathematical Sciences Institute. He is a Past President of WFNMC and was Co-chair of ICMI Study 16 Challenging Mathematics in and beyond the Classroom.

1 Diophantine equations

In the Australian Mathematics Competition we have always felt free to pose Diophantine problems, even though they are not explicitly part of a normal classroom experience. They are very intuitive, but it is nice to visit the subject and look for systematic ways of finding certain integer solutions of linear equations.

2 Pigeonhole principle

This elementary idea, thought to have been first articulated as such by Dirichlet and often known as Dirichlet's principle, is, as is well known, simple but powerful, helping to tighten up a solution once the principle is understood. The following is an example (taken from International Mathematics Tournament of Towns) of an accessible problem whose solution is best wrapped up using this idea.

Example 1 Ten friends send greeting cards to each other, each sending 5 cards to different people. Prove that at least two of them sent cards to each other.

Strategy. The words 'at least' are the ones which give the experienced student the clue that the pigeonhole principle will be useful here. The question is going to require a count of how many routes there are, and how many different routes are taken. Since routes occur in pairs (one for each direction) the objective of the proof will be to find that more than half the routes must be used, as then the pigeonhole principle will require that at least two are such a pair.

Solution. Since each of the ten friends can send to nine others, there are 90 available routes. However, each pair of friends is involved in 2 routes, so that there are 45 pairs. If more than 45 cards are sent, then by the pigeonhole principle, two of the transmissions must be on the same route in opposite directions. In this case since each student actually sends 5 cards, there are 10 times 5, or 50 (≥ 45) transmissions altogether and thus there are two friends who send cards to each other.

Note. Discussion on such challenges, working out what might be the pigeons, and what might be the pigeonhole, can be very rewarding in mathematical circles.

3 Discrete optimisation

Discrete optimisation is quite a different skill than that found in calculus, where the variables are real (not integers), and situations are continuous. Here the variables are integer, and the standard method involves two steps.

1. Show optimality, that is, give an argument to show that the proposed solution cannot be exceeded.
2. Show that this demonstrated optimum exists.

The first part is usually a challenging mathematical argument, while the second requires no more than production of an optimum. Other

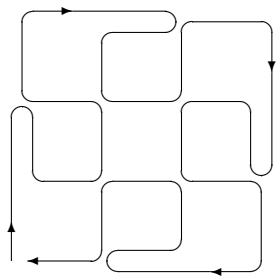
than the fact that the variables are integer, the method is usually given away by the openness of a statement requiring an optimum (maximum or minimum).

The following example, from the International Mathematics Tournament of Towns, is one in which there is also a nice use of Eulerian graph theory, which is such a useful tool in networking problems.

Example 2 A village is constructed in the form of a square, consisting of 9 blocks, each of side length l, in a 3×3 formation. Each block is bounded by a bitumen road. If we commence at a corner of the village, what is the smallest distance we must travel along bitumen roads, if we are to pass along each section of bitumen road at least once and finish at the same corner?

Strategy. This problem is also an excellent interactive classroom problem. Students can try for some time to improve their first results until everyone is convinced they have a result which cannot be beaten.

Solution (due to Andy Liu). The diagram shows a closed tour of length 28 and we claim this to be a minimum.



Each of the four corners is incident with two roads and requires at least one visit. Each of the remaining twelve intersections is incident with three or four roads and requires at least two visits. Hence the minimum is at least $4 + 12 \times 2 = 28$.

4 Proof by cases

Quite often experimentation with a situation leads to a conclusion that a result can only be established after an exhaustive consideration of a mutually exclusive set of cases, leading to the method usually known as proving by cases. The challenge is two-fold. First, one needs to identify the cases that might apply and to describe them in a way that is clear, efficient and non-overlapping. Secondly, one needs to ensure that the cases are exhaustive, that nothing is left out.

The method can be illustrated by the following problem, one of the more challenging problems from the Australian Mathematics Competition.

Example 3 The sum of n positive integers is 19. What is the maximum possible product of these n numbers?

Strategy. This problem is also excellent for classroom interaction. Students can try to obtain maximum products with various selections but soon discover that high numbers adding to 19 don't seem to help, while at the other extreme the number one is also useless. Normally discussions converge on the full solution.

Solution. We are looking for a maximum product

$$n_1 \cdot n_2 \cdot n_3 \cdot \cdots \cdot n_k$$

where $n_1 + n_2 + n_3 + \cdots + n_k = 19$.

- If any factor n_i is ≥ 5, it can be replaced with $2(n_i - 2)$ for a larger product, since $2(x-2) > x$ for $x \geq 5$; so every factor is ≤ 4.
- If any factor n_i is equal to 4, it can be replaced by 2×2 with no change to the product, so we shall do this and then every factor is less or equal 3.
- If any factor is 1, it can be combined with another factor, replacing $1 \cdot n_i$ by $n_i + 1$ which increases the product, so now all factors are 2 or 3.
- If there are three or more 2s, $2 \cdot 2 \cdot 2$ can be replaced by $3 \cdot 3$ to increase the product. So there are at most two 2s.

There is only one way that 19 can be written as such a sum: there are five 3s and two 2s. So the maximum product is $3^5 \cdot 2^2 = 972$.

Note. An extension could be to ask the student what happens if the number 19 is replaced by any other.

A further challenge for a more senior student studying calculus is to decide whether there is a continuous version of the problem and formulate it exactly.

5 Proof by contradiction

Some of the most famous proofs in mathematics are constructed by contradiction and are accessible from school mathematics, even if not in a formal syllabus.

The following problem, taken from the International Mathematics Tournament of Towns, is most easily solved by contradiction.

Example 4 There are 2000 apples, contained in several baskets. One can remove baskets and/or remove any number of apples from any number of baskets. Prove that it is possible to have an equal number of apples in each of the remaining baskets, with the total number of apples being at least 100.

Strategy. This is hardly in the form that a student might encounter at school, and unlike other examples in this paper the initial challenge is to figure out exactly what is being asked. The indeterminacy of the situation and the variety of possibilities for removal of apples and baskets boggles the mind. An efficient way to control the situation is to suppose that the result is false. As the reader will see in the solution, it is not so difficult in this direction to find a contradiction.

Solution. Assume the opposite. Then the total number of baskets remaining is not more than 99 (otherwise we could leave 1 apple in each of 100 baskets and remove the rest). Furthermore, the total number of baskets with at least two apples is not more than 49, the total number of baskets with at least three apples is not more than 33, etc. So the

total number of apples is not more than

$$\begin{aligned} 99 + 49 + 33 + 24 + \cdots &< 100\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{99}\right) \\ &< 100\left(1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{64}\right) < 100 \cdot 7 = 700 < 2000. \end{aligned}$$

We thus have a contradiction.

6 Counting by exhaustion

Combinatorial problems are popular in challenges because they can be less dependent on classroom knowledge and therefore be fair ways of identifying potential problem solvers, where they can use intuitive methods. Counting, or enumeration is a popular source of such problems. Counting problems, properly set, can be solved in the time allocated and they have the advantage of challenging the student later to try to generalise, to enable similar problems to be solved from an algorithm.

The following problem, composed by Australian National University mathematician Bob Bryce for the Australian Mathematics Competition, is an excellent example.

Example 5 I have four pairs of socks to be hung out side by side on a straight clothes line. The socks in each pair are identical but the pairs themselves have different colours. How many different colour patterns can be made if no sock is allowed to be next to its mate?

Discussion. The following solution lists and counts all the cases.

Solution. Call the socks aa, bb, cc, dd. There are $4 \times 3 \times 2 = 24$ ways of selecting the first three as abc (i.e. the first three different). These can be arranged as shown:

abc	a	$bdcd$ $cdbd$ $dbcd$ $dbdc$ $dcbd$ $dcdb$	6	
	b	symmetric with a	6	
	da	bcd bdc cbd cdb dbc dcb	6	
	db	symmetric with da	6	
	dc	symmetric with da	6	30

So there are $24 \times 30 = 720$ patterns commencing with abc. There are $4 \times 3 = 12$ ways of selecting the first three as aba. These can be arranged as shown:

aba	b	$cdcd$ $dcdc$	2	
	c	$bdcd$ $dbcd$ $dbdc$ $dcbd$ $dcdb$	5	
	d	symmetric with c	5	12

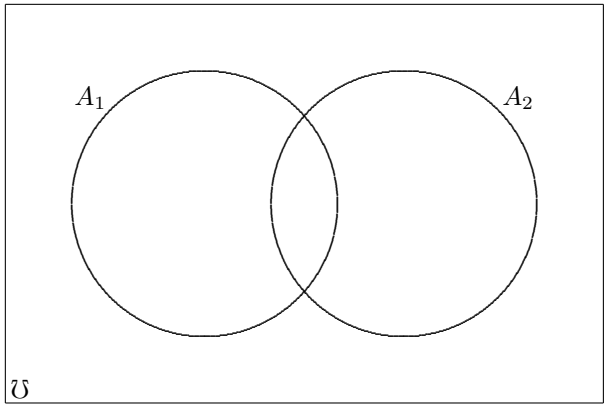
So there are $12 \times 12 = 144$ patterns commencing with aba. Finally, $720 + 144 = 864$.

7 Counting systematically

Problems such as the socks question motivate one to generalise to other problems, e.g. 5, 6 or more pairs of socks. Exhaustive counting quickly

becomes just that, exhausting. The following alternative solution, due to ANU mathematician Mike Newman, using the the inclusion-exclusion principle, achieves generalisation.

Alternative Solution of Example 5. Consider the problem for 2 pairs of socks, as illustrated in the Venn diagram.



The set A_1, for instance, is the set of arrangements in which pair 1 is together. We wish to compute

$$N(\overline{A_1 \cup A_2}) = N(\mathcal{U}) - \sum_{i=1}^{2} N(A_i) + N(A_1 \cap A_2).$$

In this problem

$$N(\mathcal{U}) = \frac{4!}{2^2}.$$

(We divide by 2^2 because the socks within each pair can be changed without changing the arrangement.) Now

$$N(A_1) = N(A_2) = \frac{3!}{2^1}$$

and

$$N(A_1 \cap A_2) = \frac{2!}{2^0},$$

so the solution is

$$\frac{4!}{4} - 2\frac{3!}{2} + 2 = 6 - 6 + 2 = 2.$$

The inclusion-exclusion principle generalises to give, in the $n-$pair case

$$\begin{aligned} N\left(\overline{\bigcup_{i=1}^{n} A_i}\right) &= N(\mho) - \sum_{i=1}^{n} N(A_i) + \sum_{i\neq j} N(A_i \cap A_j) \\ &\quad - \sum_{i\neq j\neq k} N(A_i \cap A_j \cap A_k) \\ &\quad + \cdots + (-1)^n N\left(\bigcap_{i=1}^{n} A_i\right). \end{aligned}$$

In the case of 3 pairs, this gives

$$\frac{6!}{2^3} - 3\frac{5!}{2^2} + 3\frac{4!}{2^1} - 3! = 90 - 90 + 36 - 6 = 30.$$

In the case of 4 pairs, this gives

$$\begin{aligned} \frac{8!}{2^4} - 4\frac{7!}{2^3} + 6\frac{6!}{2^2} - 4\frac{5!}{2} + 4! &= 2520 - 2520 + 1080 - 240 + 24 \\ &= 864. \end{aligned}$$

In the case of n pairs this gives

$$\binom{n}{0}\frac{(2n)!}{2^n} - \binom{n}{1}\frac{(2n-1)!}{2^{n-1}} + \binom{n}{2}\frac{(2n-2)!}{2^{n-2}} + \cdots + (-1)^n\binom{n}{n}\frac{n!}{2^n}.$$

Note that the first two terms always cancel each other in this particular problem.

The Mathematical Content section of my AMT History site (`https://sites.google.com/site/pjt154/home`) generalisations to other types of counting methods, including derangements, partial derangements and the Polya Necklace Method.

8 Inverse thinking

Sometimes there can be useful challenges involved by thinking in the inverse direction (the method below can also be described as "working backwards"). Here is a problem from the Mathematics Challenge for Young Australians.

Example 6 Let us define a Fibonacci sequence as one in which each term is the sum of the two preceding terms. The first two terms can be any positive integers. An example of a Fibonacci sequence is $15, 11, 26, 37, 63, 100, 163, \ldots$.

1. Find a Fibonacci sequence which has 2000 as its fifth term.
2. Find a Fibonacci sequence which has 2000 as its eighth term.
3. Find the greatest value of n such that 2000 is the nth term of a Fibonacci sequence.

Discussion. Generally one thinks of a Fibonacci sequence in the forward direction. Here, as is common in an inverse thinking scenario, instead of being given the data and then finding the results, we are given the results and are asked to find the data. It is a challenge for students to think this way.

The student can do this by searching through various second-last terms and working back. In doing so, depending on which term they choose, they can work back uniquely but some choices will not go back far. If the second last term is less than 1000, the third last term is greater than 1000 and that is as far as we can go, as the next term would be negative. We do not do much better if the second last term is too high.

The student can eventually focus in on a small range of values for which the sequence can be traced back a few terms, and then finally the one which goes back optimally.

Solution.

1. Note that the fifth term in the standard Fibonacci series is 5, a factor of 2000. So multiplying the first five terms by 400 yields 400, 400, 800, 1200, 2000. Many other sequences are possible and can easily be found by trial and error. Alternative Method: Systematic trialling of possible numbers for the fourth term eventually shows that if the fourth term is 1333, the previous three terms are 667, 666 and 1, giving 1, 666, 667, 1333, 2000. Note that if the fourth term is 1334 or more, the third term is 666 or less, the second term is 668 or more, and the first term must be negative, giving an invalid sequence. Similarly, if the fourth term is 1001, the previous terms are 999, 2 and 997. Note that if the fourth term is 1000 or less, then the third term is 1000 or more, the second term is 0 or negative, giving an invalid sequence. It follows that the selection for the fourth term of any integer between 1001 and 1333 inclusive will lead to a valid sequence with 2000 as the fifth term. Selection for the fourth term of any integer outside this range will lead to an invalid sequence. There are 333 valid sequences.

2. Systematic trialing of possible numbers for the seventh term shows that the selection of any integer between 1231 and 1249 inclusive will lead to a valid sequence with 2000 as the eighth term. Note that if the seventh term is 1230 or less, then the sixth term is 770 or more, the fifth term is 460 or less, the fourth term is 310 or more, the third term is 150 or less, the second term is 160 or more and the first term is negative, giving an invalid sequence. Similarly, if the seventh term is 1250 or more, the sixth term is 750 or less, the fifth term is 500 or more, the fourth term is 250 or less, the third term is 250 or more and the second term is 0 or negative, giving an invalid sequence. Selection for the seventh term of any integer outside the range 1231 and 1249 inclusive will, as shown, not lead to a valid sequence. There are 19 valid sequences. For example, a seventh term of 1231 yields 3, 152, 155, 307, 462, 769, 1231, 2000.

3. With similar reasoning to that in **2.**, systematic trialing of possible numbers for the term preceding 2000 shows that the selection of any integer between 1236 and 1238 inclusive will lead to a valid sequence with 2000 as the tenth term. Writing each of these

sequences in reverse for ten terms yields:
2000, 1236, 764, 472, 292, 180, 112, 68, 44, 24.
2000, 1237, 763, 474, 289, 185, 104, 81, 23, 58.
2000, 1238, 762, 476, 286, 190, 96, 94, 2, 92.
Note that the latter two sequences cannot be extended further, since an extra term will be negative. However the first sequence can be extended to three more terms: 20, 4, 16. It follows that there is exactly one sequence of 13 terms:
2000, 1236, 764, 472, 292, 180, 112, 68, 44, 24, 20, 4, 16. This 13-term sequence is the one of maximum length.

Further Discussion. The Golden Ratio can be used in extended thinking of this problem.

As is well known Fibonacci sequences are generated via what are known as recurrence relations of the form $x_{n+2} = x_{n+1} + x_n$, and the ratio x_{n+1}/x_n of successive terms gets closer and closer to what is known as the Golden Ratio, whose value is $(1 + \sqrt{5})/2$, which equals 1.61803398875... (A student should try checking this by calculating the ratio of some successive ratios for higher n.)

So if we are looking for a lengthy sequence ending in 2000 we might expect the second last term to be approximately 2000/1.61803398875..., which is 1236.0679775379..., closest to 1236, which was the actual second-last term in the longest sequence, as we discovered.

9 Invariance

Discovering an invariant in a problem can lead to a simple resolution of an otherwise intractable problem. The method of invariance applies in a situation where a system changes from state to state according to various rules, and some property which is important to the statement of the problem remains unchanged in each transition. The property which doesn't change is known as the invariant.

Looking out for invariants can be rewarding. Question 3 of the 2000 IMO involved an if and only if situation which was very difficult in one of the directions. An Australian student became one of only 14 students to

solve the problem by spotting an invariant and bypassing the difficulty, a situation not anticipated by the jury. This helped him win a Silver Medal.

This method is very well illustrated by the following famous problem from the International Mathematics Tournament of Towns, not just for its mathematical properties, but for other various associated aesthetic features.

Example 7 On the island of Camelot live 13 grey, 15 brown and 17 crimson chameleons. If two chameleons of different colours meet, they both simultaneously change colour to the third color (e.g. if a grey and brown chameleon meet they both become crimson). Is it possible they will all eventually be the same colour?

Strategy. At first sight this problem looks very difficult, and the imagining of two chameleons with dull colours touching noses becoming a bright colour is also a distraction. With problems like this one should either try to solve a simpler version first, or play around with it by going through a few phases, looking at what happens in various phases and looking for a successful solution. Doing this can cause frustration after a while as the answer is no and one can go for some time without being able to find a successful pathway.

This is also a case where I found working backwards to help. Since there are 45 chameleons altogether, a successful answer would be 45 of one colour, none of the other two. Working backwards in one step gives a configuration 1, 1, 43 (it doesn't matter which are which colours), then maybe 2, 2, 41, and others might be 3, 3, 39 or 1, 4, 40. Eventually one frequently often recognises a case in which all are multiples of 3. After further exploration, working backwards, one eventually notices that all configurations working backwards have all three the same value modulo 3, whereas in our case our starting configuration has the three different values modulo 3. One then notices that during one nose-touch one colour increases by two (the new colour of the two chameleons touching) whereas each of their former colours has decreased by one, and that whatever happens, in one nose-touch the population of each colour will

always involve each of the three numbers modulo 3, so the outcome with all three being zero modulo 3 is not possible.

Solution. The starting configuration has populations 0, 1, and 2 modulo 3. This situation remains invariant no matter which two chameleons touch noses. So the desired configuration, which has all three equal to zero modulo 3 is not possible.

10 Colouring

There is a beautiful class of problems which involve colourings. They can be illustrated by the following very nice problem from the International Mathematics Tournament of Towns.

Example 8 A 7 by 7 square is made up of sixteen 1 by 3 tiles and one 1 by 1 tile. Prove that the 1 by 1 tile lies either at the centre of the square or adjoins one of its boundaries.

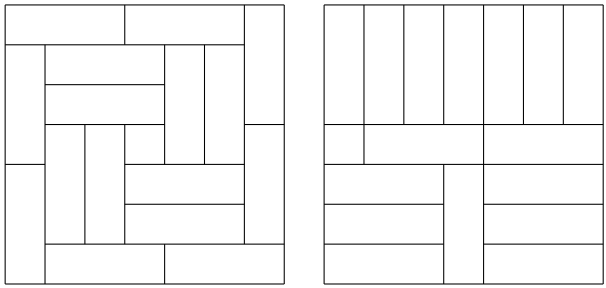
Discussion. This problem has a rather surprising result and at first sight, with all the combinations possible, seems almost impossible to prove. But colouring with 3 colours and looking at the resultant way in which a 3 by 1 domino, or should I say triomino, might cover squares of the board makes the problem accessible. The solution was devised by my colleague John Campbell.

Solution. Label each of the squares as follows.

A	B	C	A	B	C	A
B	C	A	B	C	A	B
C	A	B	C	A	B	C
A	B	C	A	B	C	A
B	C	A	B	C	A	B
C	A	B	C	A	B	C
A	B	C	A	B	C	A

Each 1×3 tile must cover squares labeled A, B and C, because they would be filling adjoining squares. Since there are 17 As and only 16 Bs and 16 Cs, the 1×1 square must therefore occupy a square marked A. However, because the orientation of the square is not relevant, we should eliminate all those occurrences of As which do not remain as As on rotation by, say multiples of $90°$. This leaves only those in the corners, the four along the midpoints of each outer edge, and the one in the centre. This completes the proof.

Note. The following diagram show two such tilings, one with the 1×1 square in the centre, and the other one with it on the edge.



In fact, by rotating the bottom-left 4×4 square in the diagram on the right, all three possible positions of the monomino can be achieved (centre side, centre and corner).

11 Geometry

I will not add much on geometry, the oldest of the formal mathematics disciplines. It is still the strongest of the IMO disciplines, where normally two of the six problems are generally geometric, with Euclidean proof normally required (although if a coordinate proof is complete it can attract 7 rather than otherwise 0 points). I do not recall three dimensional Euclidean geometry in the IMO in recent years.

Geometry has suffered massive cuts in the syllabi of various countries since the 1980s. This has happened to the extent that the Australian Mathematics Competition is no longer able to set circle geometry problems for years 9 and 10 because it has ceased to be taught at these levels.

Teachers who supported geometry's downgrade said the reason was simple—people do not use geometry in later life. This overlooks the fact that geometry, with its theorems, logic and structure was the main branch of mathematics for developing logical reasoning, a vital skill in later life.

I think geometry has also suffered because the textbooks have been very dry: old fashioned, theorem, proof, theorem proof, exercise, etc. but with no motivating discussion. It doesn't have to be like that. We set an Australian Mathematics Competition problem in 1983 (to be found on my website) where we asked for the most northerly point from which one can see the Southern Cross. It was well known that the Southern Cross can be seen in some northern latitudes, as French Aviation pioneers used it for navigation when crossing the Sahara in establishing routes to South America. The solution involved a nice use of circle geometry with a cyclic quadrilateral.

12 Logic

It was a standard procedure, particularly in the early days, for us to put in the Australian Mathematics Competition paper each year what we would call a logic problem. The most famous, and quite typical, was the following, from the 1984 paper.

Example 9 Albert, Bernard, Charles, Daniel and Ellie play a game in which each is a frog or kangaroo. Frogs' statements are always false while kangaroos' statements are always true.

Albert says that Bernard is a kangaroo.
Charles says that Daniel is a frog.
Ellie says that Albert is not a frog.
Bernard says that Charles is not a kangaroo.
Daniel says that Ellie and Albert are different kinds of animals.

How many frogs are there?

Solution. Let a single-headed arrow represent '... says ... is a kangaroo' and a double-headed arrow represent '... says ... is a frog'. Then we have

$$\begin{tikzcd}
E \arrow[r] & A \arrow[dr] & \\
 & & B \arrow[dl] \\
D & C \arrow[l] &
\end{tikzcd}$$

Assume Ellie is a kangaroo, and hence that his statement is true.
Thus: Albert is a kangaroo,
Bernard is a kangaroo,
Charles is a frog and
Daniel is a kangaroo. But this is not possible since Ellie and Albert are then both kangaroos, contrary to Daniel's statement. This proves that Ellie is not a kangaroo, but a frog instead. Being a frog, Ellie's statement is false.
Thus: Albert is a frog,
Bernard is a frog,
Charles is a kangaroo and
Daniel is a frog. There are 4 frogs.

Comment. We had some fun with our French colleagues and combined with them to arrange for the kangaroos to tell lies and frogs to tell the truth in the French translation used in New Caledonia and French Polynesia.

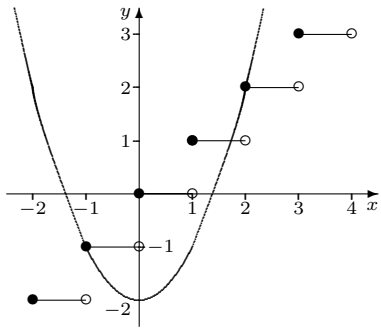
13 Graphical methods

A very useful method if life gets difficult is to attempt graphical thinking. This can particularly help when a student confronts a formula or equation in which there are structures such as modulus signs, fractional powers, rational functions, greatest integer functions and similar. The following example illustrates a standard low-level approach useful in the early stages of problem solving, but could be systematised more in teaching.

Example 10 Find all the solutions of $x^2 - [x] - 2 = 0$.

Solution and comment. In questions like this one can go straight into a graphical approach. But one needs to be careful. A common method is to try to develop the left-hand side function in stages. In this case the left-hand side becomes a little messy and the author's experience is that not all people find all the solutions to be obvious. It is easy to draw qualitatively the function $x^2 - 2$ for example, but then subtracting $[x]$ is messy.

A better strategy in this case is to note that the equation is equivalent to $x^2 - 2 = [x]$ and to draw the two simpler curves of each side, as below.



This leads to a clear discovery that there are three solutions, two of which are integer, $(-1, -1)$ and (2,2), and the one where $y = 1$ and it turns out using symbolic calculation that this root is at $(\sqrt{3}, 1)$.

14 Probability

Probability is not a topic as such in the International Mathematical Olympiad, nor major international competitions generally. In the Australian Mathematics Competition we did try to set however an interesting probability problem each year, mainly because we found probability the most interesting branch of statistics, and also because, like combinatorics, which we embraced, we felt we could set questions which could be solved without formal knowledge of the theory. We even felt we could work round simple Bayesian problems in such a way. Statistics did become part of the Australian syllabus from about the time the Australian

Mathematics Competition started, but its syllabus focus would be on aspects such as data analysis. We did frequently set questions on aspects of mean, in addition to probability.

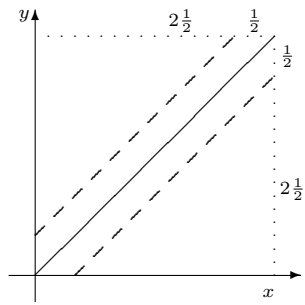
It is not so interesting for me to illustrate with a standard problem, but having just discussed graphical methods, the following former Australian Mathematics Competition problem I find very interesting.

Example 11 Two forgetful friends agree to meet in a coffee shop one afternoon but each has forgotten the agreed time. Each remembers that the time was somewhere between 2 pm and 5 pm. Each decides to go to the coffee shop at a random time between 2 pm and 5 pm, wait half an hour, and leave if the other doesn't arrive. What is the probability that they meet?

Solution. This is a very difficult problem to solve, even for an experienced solver, if trying by any sort of symbolic method. In grappling with it one soon understands the situation much better if looking for a graphical-based solution, as discussed above.

We can do this as follows:

Let one friend's arrival time be x and the other be y. They can both take random values, say, between 0 and 3. They will meet if $|x-y| \leq \frac{1}{2}$.



The probability of their meeting is then

$$\frac{3^2 - (2\frac{1}{2})^2}{3^2} = \frac{9 - 6\frac{1}{4}}{9} = \frac{2\frac{3}{4}}{9} = \frac{11}{36}.$$

15 Problem solving by reviving past ideas and evolving new ones

Problem solving is an exciting part of mathematics, sometimes at the cutting edge of knowledge, and I have been on problem-setting committees in which we have discovered new results and published refereed papers.

It can be quite exciting to discover forgotten techniques which can be used to solve modern problems. On my website I cite the rediscovery (in the West anyway) of barycentric coordinates as a useful technique, and record that a recent member of the UK IMO team, probably to the chagrin of his leader, who had to mark his work, solved many geometry problems using them. They are particularly useful in certain types of problem, for example, proving collinearity of three lines.

Then there is the evolution of new techniques. The website also shows how the method of the moving parallel evolved through papers in the West and discussion in the Russian student journal *Kvant*, enabling the solution of some nice polygon dissection problems.

16 Conclusion

I have attempted a classification of what I would call immediate beyond-school mathematics problems, and in doing so have used the opportunity to discuss some of my favourite problems, on occasions alluding to pedagogy and describing how some can be used in enrichment classes. All of the above methods are ones I have found to work well in the mathematics circle I used to work with in Canberra.

References

[1] Barbeau E. J. and Taylor P. J. (eds.), *Challenging Mathematics In and Beyond the Classroom, The 16th ICMI Study*, Springer 2009.

[2] Tabov J. B. and Taylor P. J., *Methods of Problem Solving, Book 1*, AMT Publishing, Canberra, 1996.

[3] Tabov J. B. and Taylor P. J., *Methods of Problem Solving, Book 2*, AMT Publishing, Canberra, 2002.

[4] Tabov J. B., Kolev E. M. and Taylor P. J., *Methods of Problem Solving, Book 3*, AMT Publishing, Canberra, 2012.

Peter J Taylor
Canberra
AUSTRALIA
email: pjt013@gmail.com

About a Paradoxical Drawing

Pavel Tlustý

Pavel Tlustý, a professor of Mathematics at the University of South Bohemia in České Budějovice, graduated with a Ph.D. in Probability and Statistics in 1991 at the Charles University of Prague. He is interested in the didactics of mathematics focusing mainly on probability and stochastic methods. He has been involved in the preparation of students for Olympiads since 1990, and is a co-author of mathematics textbooks. Some of them have been awarded the Silver Medal for the Best European schoolbook at the Book Fair 2009 in Frankfurt am Main.

Mathematics describes and examines different situations of the real world. It is natural that we sometimes try to solve a case which seems to be, without a careful investigation, paradoxical. Especially, the theory of probability is rich in paradoxes. This paper deals with one such a paradox.

We start with a simple example.

Problem 1 In a bag, there are w white and b black balls. What is the probability that a randomly drawn ball is black? What is the probability that the ball drawn in the second draw is black, if we put the first ball back into the bag?

It is quite simple to observe that both questions deals with the same situation. It is, therefore, clear that the probability in both cases equals $b/(b+w)$.

Now, we change the situation.

Problem 2 In a bag, there are w white and b black balls. What is the probability that a randomly drawn ball in the second draw is black if we do not put the first ball back into the bag?

In this case, we can see that the ball situation in the bag before the second draw depends on the result of the first draw. A different situation happens, if we draw a white ball in the first draw, to if we draw a black ball in the first draw. Usually in these situations, we introduce the term "conditional probability" and for solving Problem 2 we use the formula of total probability (see [1], Theorem 1B). Using the formula, we obtain the following solution:

Solution. We denote $P(B_n)$ the probability of drawing a black ball in the nth draw ($n \in \mathbb{N}$), $P(W_1)$ the probability of drawing a white ball in the first draw, $P(B_2|W_1)$ the probability of drawing a black ball in the second draw, if we draw a white ball in the first draw, $P(B_2|B_1)$ the probability of drawing a black ball in the second draw, if we draw a black ball in the first draw.

According to the total probability formula, we obtain the equation

$$P(B_2) = P(B_1) \cdot P(B_2|B_1) + P(W_1) \cdot P(B_2|W_1).$$

We know that

$$P(B_1) = \frac{b}{b+w}, \quad P(W_1) = \frac{w}{b+w}.$$

Similarly, we calculate that

$$P(B_2|B_1) = \frac{b-1}{b-1+w}, \quad P(B_2|W_1) = \frac{b}{b+w-1}.$$

After the substitution, we get

$$P(B_2) = \frac{b}{b+w} \cdot \frac{b-1}{b-1+w} + \frac{w}{b+w} \cdot \frac{b}{b+w-1} = \frac{b}{b+w}.$$

The result shows us that the probability of drawing of a black ball in the second draw does not depend on the fact of whether we put the ball drawn into the first draw back to the bag or not. This situation seems to be paradoxical.

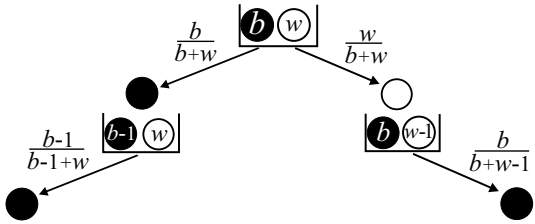


Fig. 1

If we choose a different approach using so-called "stochastic tree", we may avoid the concept of the conditional probability, as is illustrated by the following solution:

Another solution. A random trial consists of two draws (steps). In Fig. 1, there is a stochastic tree of such a trial, the numbers assigned to particular tree branches represent corresponding probabilities. Using the stochastic tree, we can see that we can draw a black ball in the second draw in two different ways (the tree has two branches). According to the rule of sum (see, for example, [2]), the resulting probability equals the sum of probabilities of the particular branches. According to the rule of product (see, for example, [2]), the probability of each of the branches is given by the product of the particular drawings. It gives us

$$P(B_2) = \frac{b}{b+w} \cdot \frac{b-1}{b-1+w} + \frac{w}{b+w} \cdot \frac{b}{b+w-1} = \frac{b}{b+w}.$$

Basic combinatorics principles are also used in the following solution.

Another solution. Because of the fact that all possible drawings of two balls from the bag are equally probable, the stochastic model of the trial is a the classic probability space. In the classic probability space, we calculate the probability of an event as a quotient of the number of favourable events to the total number of the possible events.

- The total number of possible drawing results:
  Before the first draw, there are $b+w$ balls in the bag and we can have $b+w$ different drawing results. Before the second draw, there are $b+w-1$ balls and we can, therefore, have $b+w-1$ different

drawing results. According to the rule of product, there are, in total, $(b+w)\cdot(b+w-1)$ different drawing results of two balls.

- The number of drawings in which we draw a black ball in the second draw:
  There are b black balls in the bag and so it is possible to draw a black ball in the first draw in b different ways. In the second draw, we draw any remaining ball, it may be done in $b+w-1$ different ways. According to the rule of product, there are, in total, $b\cdot(b+w-1)$ different results.

We obtain

$$P(B_2)=\frac{b\cdot(b+w-1)}{(b+w)\cdot(b+w-1)}=\frac{b}{b+w}.$$

This universal algorithm, that is, which is also possible to use in the situation when we are interested in the probability that the ball drawn in the nth draw is black $n=1,2,\ldots,b+w$.

Then we get

$$P(B_n)=\frac{b\cdot(b+w-1)\cdot\cdots\cdot(b+w-(n-1))}{(b+w)\cdot(b+w-1)\cdot\cdots\cdot(b+w-(n-1))}=\frac{b}{b+w}.$$

We can see a surprising fact that when drawing balls (one by one), the probability of drawing a black ball is the same in all the draws!!!

Solution 4. We put w white and b black balls in a bottle. We shake the bottle and turn it upside down. The balls start to fall out of the bottle (Fig. 2). We are interested in the order in which the balls fall out.

Because of the symmetry, it is obvious that each of the balls can fall out with the same probability at any place from $b+w$ positions and therefore

$$P(B_n)=\frac{b}{b+w}$$

for $n\in\{1,2,\ldots,b+w\}$.

The situation may be generalised.

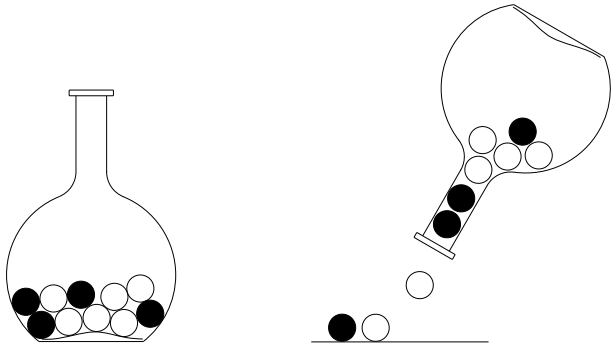


Fig. 2

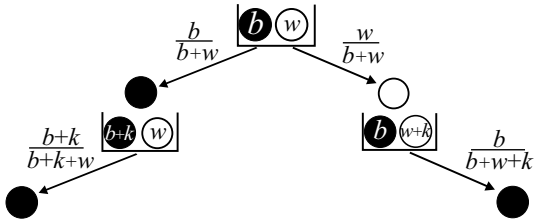


Fig. 3

Problem 3 In a bag, there are w white and b black balls. What is the probability that a randomly drawn ball in the second draw is black, if we put the ball drawn from the first draw back into the bag and

- add k balls of the same colour,
- take out k balls of the same colour, where $k \leq \min(b, w)$?

Solution. We denote by k a whole number meeting the assignment of the problem. The corresponding stochastic tree is shown in Fig. 3. We can see, from Fig. 3, that

$$P(B_2) = \frac{b}{b+w} \cdot \frac{b+k}{b+w+k} + \frac{w}{b+w} \cdot \frac{b}{b+w+k} = \frac{b}{b+w}.$$

The three problems show a paradoxical situation. The probability of drawing a ball

- does not depend on the way of drawing, that is, if we put the drawn balls back into the bag or not,
- is the same for all draws, that is,

$$P(B_1) = P(B_2) = \ldots = P(B_{b+w}),$$

- does not depend on adding or taking out balls (according to fixed rules).

References

[1] Grimmett G. and Welsh D., *Probability: An Introduction, Oxford Science Publications*, 1986, Theorem 1B.

[2] Rosen Kenneth H., ed., *Handbook of discrete and combinatorial mathematics*, CRC press, 1999.

Pavel Tlustý
Faculty of Economics
University of South Bohemia
České Budějovice
CZECH REPUBLIC

The 25th *Baltic Way* Team Contest 6–10 November 2014, Vilnius, Lithuania

Romualdas Kašuba

Romualdas Kašuba teaches mathematics, communication skills and ethics at Vilnius University, Lithuania. He obtained his Ph.D. at the University of Greifswald in Germany. Since 1979 he has been a jury member of the Lithuanian Mathematical Olympiad. Since 1996 he has been the Deputy Leader of the Lithuanian IMO team and since 1995 also the leader of Lithuanian team at the Baltic Way *team contest. In 1999 he initiated the Lithuanian Olympiad for youngsters. He is the author of the book* What to do when you do not know what to do?*, Part I and II, Riga 2006-2007 and also of issues* Once upon a time I saw the puzzle*, Part I, II and III, Riga 2007–2008 (all in English). He also represents Lithuania at ICMI.*

1 Historical background and preliminary remarks

Mathematics Olympiads in Lithuania started in 1951, at the same time as in Latvia and Estonia. Historically, the first serious mathematical competition in Lithuania that happened in that year was the city championship in Vilnius, the capital city of Lithuania. That competition was guided by a young mathematician, who later on became the rector of Vilnius University, namely Jonas Kubilius. In the following year that competition was extended and was conducted throughout Lithuania.

Within all the three Baltic countries, which, in those days, as the consequence of the Molotov–Ribbentrop Pact (signed in secret in the year

1939, just before World War II), had become incorporated into the former Soviet Union, this mathematical competition was carried out in three levels—school, regional and the final level.

Taking into account that the final level usually happened in March or April, the Olympiad process as the whole, at least in Lithuania, was essentially restricted to the first half of the year, leaving the second half of the year without any remarkable mathematical events.

The idea to do something in order to fill that remarkable gap was practically realized by the young Lithuanian mathematician Algirdas Zabulionis, who acted under the practical support of the rector of Vilnius University who had already become a well-known mathematician—Jonas Kubilius. In the year 1986, the first edition of that event took place in the Mathematics department of Vilnius University.

The event always happened in autumn and was strictly connected with the previously mentioned main Lithuanian mathematical Olympiad. Still, at the same time it was not an individual, but strictly a team competition, which tended to lead the students not only to solve but also to collaborate. The idea of collaboration when solving is a highly logical idea—it would be enough to mention and remember that the whole process of creation of any science has a clear highly collective nature. That is practically the same as to remind that each coming generation of science is dealing with the problems that are left and which were being developed by all the previous generations of scientists.

In that team contest, during a four-hour period, a team consisting of 5 persons, has to deal with 20 problems selected by the jury strictly for that competition. Each participating team is supposed to and expected to deliver one solution to each proposed problem.

The link with the main competition, that is, with that individual Olympiad, was also always underlined by the principle of selection of teams to that team contest. The main idea was (and remains till the present) that the region is invited to participate if it did remarkably well on the final stage of the individual contest throughout the country.

In general, about 20 teams participated in that annual competition, which usually took place on the last Saturday in September. The papers

were checked within a few hours of completion of the competition, mainly by the leading students and scientists of Vilnius University, many of whom were participants of that contest in the previous years.

After the well-known events, when Lithuania and the other Baltic States regained their independence, that team contest gained international importance at once and under the famous name *Baltic Way*. At that stage of internationalization of that mathematical event, the influence and support of the famous Latvian mathematician and scientific organizer Agnis Andžans, should never be forgotten.

After a few years, Finland and, following it, other states in the region around the Baltic Sea joined that team competition.

The formula and format of that team competition have remained practically the same as it was first in Lithuania, with the exception that the time for solutions was fixed to be exactly 4.5 hours.

Nowadays, the teams from all the states surrounding the Baltic Sea are participate, so that the event has its regular roster of participants. There is only one exception in that stable participation and that exception is Iceland, which is regarded to be a "Super-Baltic" State, in order to honor that essential and highly remarkable fact that Iceland was the first country that confirmed the independence of all three Baltic States, that is, of Lithuania, Latvia and Estonia.

Below we present the set of problems selected together with the results.

2 Problems

Problem 1 Show that

$$\cos(56^\circ) \cdot \cos(2 \cdot 56^\circ) \cdot \cos(2^2 \cdot 56^\circ) \cdot \cdots \cdot \cos(2^{23} \cdot 56^\circ) = \frac{1}{2^{24}}.$$

Problem 2 Let $a_0, a_1, \ldots, a_N$ be real numbers satisfying $a_0 = a_N = 0$ and

$$a_{i+1} - 2a_i + a_{i-1} = a_i^2$$

for $i = 1, 2, \ldots, N-1$. Prove that $a_i \leq 0$ for $i = 1, 2, \ldots, N-1$.

Problem 3 Positive real numbers a, b, c satisfy $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=3$. Prove that the inequality

$$\frac{1}{\sqrt{a^3+b}}+\frac{1}{\sqrt{b^3+c}}+\frac{1}{\sqrt{c^3+a}}\leq\frac{3}{\sqrt{2}}$$

holds.

Problem 4 Find all functions f defined on all real numbers and taking real values such that

$$f(f(y))+f(x-y)=f(xf(y)-x)$$

for all real numbers x, y.

Problem 5 Given positive real integers a, b, c, d that satisfy equalities

$$a^2+d^2-ad=b^2+c^2+bc \qquad \text{and} \qquad a^2+b^2=c^2+d^2,$$

find all possible values of the term $\dfrac{ab+cd}{ad+bc}$.

Problem 6 In how many ways can we paint 16 seats in a row, each red or green, in such a way that the number of consecutive seats painted in the same color is always odd?

Problem 7 Let $p_1, p_2, \ldots, p_{30}$ be a permutation of positive integers $1, 2, \ldots, 30$. For how many permutations does the equality

$$\sum_{k=1}^{30}|p_k-k|=450$$

hold?

Problem 8 Albert and Betty are playing the following game. There are 100 blue balls in a red bowl and 100 red balls in the blue bowl. In each turn a player must take one of the following moves:

a) Take two red balls from the blue bowl and put them in the red bowl.

b) Take two blue balls from the red bowl and put them in the blue bowl.

c) Take two balls of different colors from one bowl and throw the balls away.

They take alternative turns and Albert starts. The player who first takes the last red ball from the blue bowl or the last blue ball from the red bowl wins. Determine who has the winning strategy.

Problem 9 What is the least possible number of cells that can be marked on a $n \times n$ board such that for each $m > \frac{n}{2}$ both diagonals of any $m \times m$ sub-board contain a marked cell.

Problem 10 In a country there are 100 airports. Super-Air operate direct flights between some pairs of airports (in both directions). The *traffic* of an airport is the number of airports with which it has a direct Super-Air connection. A new company, Concur-Air, establishes a direct flight between two airports if and only if the sum of their traffics is at least 100. It turns out that there exists a round trip of Concur-Air flights that lands at every airport exactly once. Show that then there also exists a round trip of Super-Air flights that lands at every airport exactly once.

Problem 11 Let Γ be the circumcircle of an acute triangle ABC. The perpendicular to AB from C meets AB at D and Γ again at E. The bisector of angle C meets AB at F and Γ again at G. The line GD meets Γ again at H and the line HF meets Γ again at I. Prove that $AI = EB$.

Problem 12 Triangle ABC is given. Let M be the midpoint of the segment AB and T be the midpoint of the arc BC not containing A of the circumference of ABC. The point K inside the triangle ABC is such that $MATK$ is an isosceles trapezoid with $AT \parallel MK$. Show that $AK = KC$.

Problem 13 Let $ABCD$ be a square inscribed in a circle ω and let P be a point on a shorter arc AB of ω. Let $CP \cap BD = R$ and $DP \cap AC = S$. Show that triangles ARB and DSR have equal areas.

Problem 14 Let $ABCD$ be a convex quadrilateral such that the line BD bisects the angle ABC. The circumcircle of triangle ABC intersects the sides AD and CD in the points P and Q, respectively. The line through D and parallel to AC intersects the line BC and BA at the points R and S, respectively. Prove that the points P, Q, R and S lie on a common circle.

Problem 15 The sum of the angles A and C of a convex quadrilateral $ABCD$ is less than $180°$. Prove that $AB{\cdot}CD+AD{\cdot}BC < AC(AB+AD)$.

Problem 16 Determine whether $712! + 1$ is a prime number.

Problem 17 Do there exist pairwise distinct rational numbers x, y and z such that

$$\frac{1}{(x-y)^2} + \frac{1}{(y-z)^2} + \frac{1}{(z-x)^2} = 2014?$$

Problem 18 Let p be a prime number and n be a positive integer. Find the number of quadruples (a_1, a_2, a_3, a_4) with $a_i \in \{0, 1, \ldots, p^n - 1\}$ for $i = 1, 2, 3, 4$ such that

$$p^n \mid (a_1a_2 + a_3a_4 + 1).$$

Problem 19 Let m and n be relatively prime positive numbers. Determine all possible values of $\gcd(2^m - 2^n, 2^{m^2+mn+n^2} - 1)$.

Problem 20 Consider a sequence of positive integers $a_1, a_2, a_3, \ldots$ such that for $k \geq 2$ we have

$$a_{k+1} = \frac{a_k + a_{k-1}}{2015^i},$$

where 2015^i is the maximal power of 2015 that divides $a_k + a_{k-1}$. Prove that if the sequence is periodic then its period is divisible by 3.

3 Results

Baltic Way

8 November 2014, Vilnius

Results

	Algebra					Combinatorics					Geometry					Number theory					Total	Ranking
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20		
St. Petersburg	5	5	5	5	5	4	5	5	5	0	5	5	5	2	5	5	5	0	5	5	86	I
Germany	5	5	5	4	5	0	5	4	5	0	5	5	5	1	5	5	5	1	5	0	75	II
Poland	5	5	5	5	5	4	5	0	5	0	5	5	5	0	0	5	5	5	5	0	74	III
Lithuania	5	5	5	5	5	5	3	5	0	0	5	5	5	5	0	5	5	0	5	0	73	
Sweden	5	5	5	5	2	5	5	4	4	0	5	5	5	0	0	5	4	1	5	0	70	
Belarus	5	5	5	5	5	4	5	1	5	0	5	5	5	0	0	1	4	2	2	0	64	
Iceland	5	1	5	1	0	5	4	0	3	0	5	5	5	0	0	5	1	0	5	0	50	
Latvia	0	5	0	0	5	5	5	5	5	0	0	5	5	0	0	1	1	0	5	0	47	
Norway	5	5	0	5	0	5	5	4	0	0	0	0	5	0	0	5	2	0	5	0	46	
Estonia	1	5	0	0	1	5	4	3	5	0	0	5	1	0	0	0	5	0	5	0	40	
Finland	0	5	0	0	0	5	0	5	3	0	1	5	5	0	0	0	0	0	0	0	29	
Denmark	0	5	0	1	0	4	5	5	0	0	5	0	0	0	0	0	0	0	0	0	25	
	3.4	4.7	2.9	3	2.8	4.3	4.3	3.4	3.3	0	3.4	4.2	4.3	0.7	0.8	3.1	3.1	0.8	3.9	0.4	56.6	

Jury chairman:

Belarus Denmark Estonia Finland Germany Iceland Latvia Lithuania Norway Poland St. Petersburg Sweden

Place	Country	Score (from 100 points)
1	St. Petersburg (Russia)	86
2	Germany	75
3	Poland	74
4	Lithuania	73
5	Sweden	70
6	Belarus	64
7	Iceland	50
8	Latvia	47
9	Norway	46
10	Estonia	40
11	Finland	29
12	Denmark	25

The next issue of the team competition *Baltic Way* will take place in Sweden in November 2015.

Romualdas Kašuba
Lithuanian Baltic Way *leader*
Department for Methodic of Mathematics and Informatics
Vilnius University
REPUBLIC OF LITHUANIA
email: romualdas.kasuba@mif.vu.lt

Further remarks on a research agenda for WFNMC: Research into the nature and characterization of geometric thinking based on students' solutions of competition problems

María Falk de Losada

María Falk de Losada is a world famous authority in the work with mathematically gifted pupils. She was, first of all, one of the founders of the Mathematical Olympiad in Colombia. She has worked as a professor at the Universidad Antonio Narino in Bogotá for many years. In 2008 she was elected at the WFNMC mini-conference in Monterrey as the president of WFNMC for the period 2008–2012.

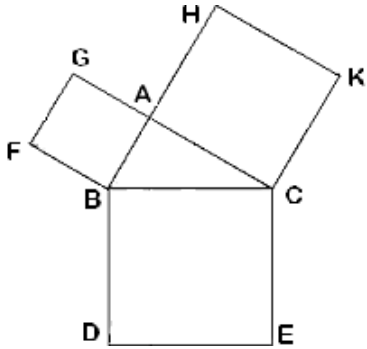
Recently I found myself reading an article written by Guershon Harel [2010, p. 87–94] in which he stresses the fact that mathematics education should focus on the development of the mathematical reasoning of the student, reasoning which in turn he breaks up into two components: ways of understanding and ways of thinking—an analysis that has both piqued my interest and stimulated much thought. Both understanding and thinking contribute to the construction of meaning for mathematical concepts, leading to strategic knowledge, strategic in that it is in turn used to foster further understanding and to empower new ways of thinking. Furthermore, Harel maintains that the field of mathematics education has focused almost exclusively on the component of mathematical understanding.

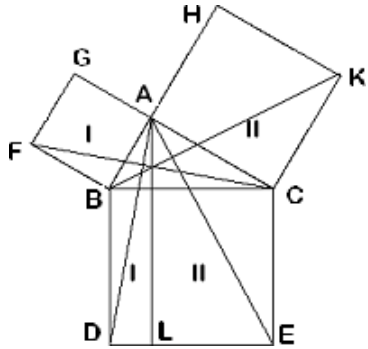
I have been working somewhat sporadically on a project aimed at characterizing geometric thinking, and have found that the literature contains some advances on the theoretical level and others on the pedagogical—didactic level, altogether falling short when it comes to describing the nature and characterization of geometric thinking. The first year of the project was aimed at gleaning clues to mathematical thinking in historical contexts, not only from writings but also from artifacts, remnants of architecture, tools and ornaments that reveal the geometric knowledge, ideas and fancy of their makers.

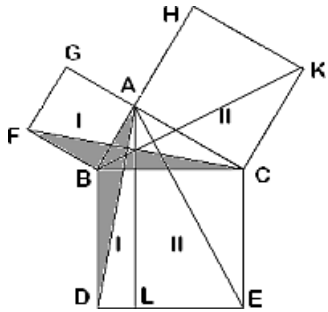
I have spoken on previous occasions on the Greek concept of area, especially as revealed in Euclid. Euclid does not include the concept of area among his definitions. It is clear that the theoretical basis of the concept is the area of the square, hence the problems of quadrature. It is also clear that, although area is not defined, the guiding principle of its study is the concept of figures of equal area. (Somewhat similar to Frege's definition of (cardinal) number that springs from the notion of similarly-numbered sets.) But Euclid's proofs reveal three additional components of the concept of area he eses:

1. Congruent figures have equal areas (not explicitly stated as a definition or postulate, but employed in proving propositions).
2. Decomposition and recomposition is used to obtain figures equal in area to figures of known area.
3. Comparison of the areas of figures according to such decomposition and recomposition yields the desired result. He builds on these components to prove his propositions on area, such as parallelograms on equal bases and between the same parallels lines have equal area.

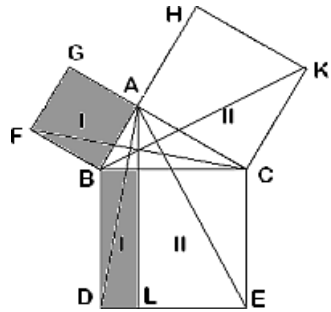
This is shown in multiple situations, one of the most prominent being Euclid's proof of the Pythagorean Theorem which we show pictorially as follows.



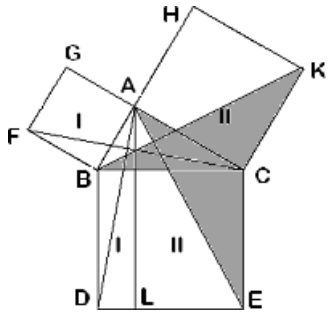




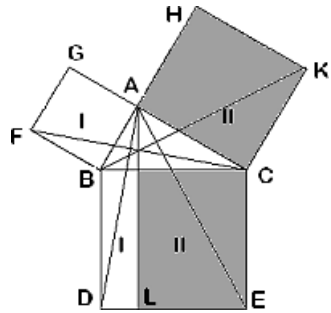
First pair of congruent triangles.



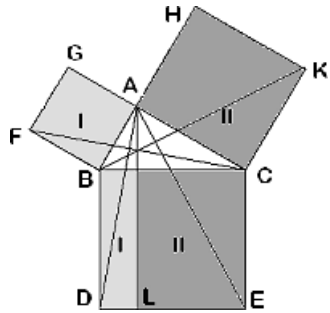
Parallelograms with area equal to twice that of the congruent triangles.



Second pair of congruent triangles.



Parallelograms with area equal to twice that of the congruent triangles.



The (area of the) square on the hypotenuse of a right triangle is equal to the sum of the (areas of the) squares on the two legs.

Note that argument is based on decomposing the square on the hypotenuse into two rectangles and proving that each of these rectangles has area equal to the area of the square on one of the legs.

In 2011, together with my student Diana Pérez, I looked into the construction of meaning for the concept of area in sixth grade students, starting with geometric puzzles (manipulatives), decomposition of figures into smaller congruent components on paper using the imagination, building (composing) other figures with the same components (though not necessarily the same number of such components), comparison of the figures according to such decomposition and recomposition, definition of area, development of the area formulas given the formula for the area of a rectangle and deriving the formulae for parallelograms, triangles, trapezoids and others showing precisely their relations with one another through decomposition and recomposition. The project showed that students, most of whom had not met the topic of area previously, had constructed robust meaning for the concept as measured by their solutions to non-routine problems concerning area taken from mathematics competitions (AMC (Australia), AMC (USA)).

Our approach to studying and characterizing geometric thinking proceeds along these lines:

1. examination of historical evidence because it reveals the path taken by the mathematical community in generating and refining concepts

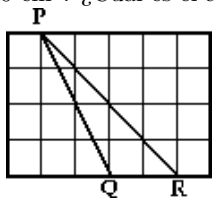
as well as the thinking strategies used to prove the relevant statements and solve the pertinent problems;

2. design of activities based on the solution of interesting, non-routine and challenging problems that invite the student to construct meaning for the concepts under consideration; and
3. analysis of the thinking of children in such problem-solving situations as revealed in their solutions and the justifications or proofs that they provide.

Furthermore, we are conscious that mathematics competitions provide an excellent resource for research on the mathematical thinking of students in the context of just such an examination of their solutions, justifications and proofs. There are some clear drawbacks, given that there is no systematic posing of problems to thoroughly study a topic, nor can there be, as one of the most important ingredients of competitions is the unexpected nature of the problems they contain. We look at correct solutions or ones with strong advances towards a correct solution, under the premise that these are the results aimed at when providing opportunities for children to construct meaning and use it for solving and proving. We will not mention incorrect solutions in this paper, but errors have been noted and can be used for another of our projects, the positive use of errors.

In October, 2013 we posed the following problem on the final round of the Colombian Math Olympiad for Primary Schools (grades three to five).

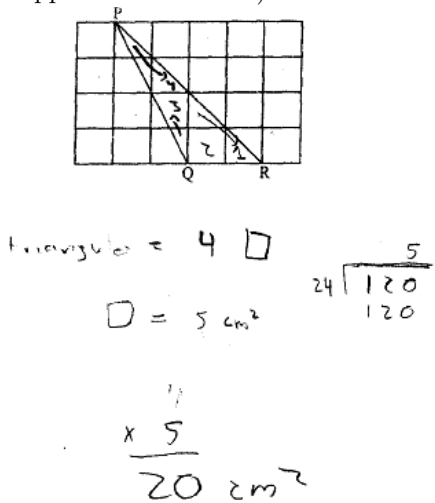
1. El área del rectángulo cuadriculado (compuesto por cuadrados) que se muestra es de 120 cm^2. ¿Cuál es el área del triángulo PQR?



Roughly one-tenth of the 500 students taking part in the final round solved the problem, and we found essentially three types of solution, each revealing the meaning the students had constructed for the concept

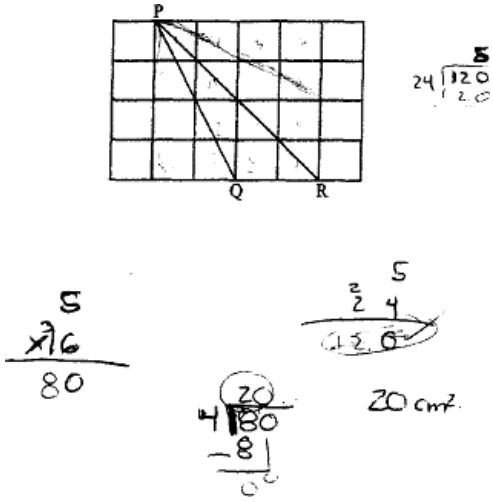
of area and the ways in which they were capable of using that meaning to generate a strategy and solve the problem.

The great majority of the students used the concept of area of a figure (or region of the plane) as the number of unit squares contained in the region. The figure is drawn on a grid, the area of each square unit of the grid is calculated and the figure is decomposed in pieces that are recomposed into unit squares and counted. (This is always possible for figures drawn on a grid and whose vertices are reticular points of the grid, as the slopes of the lines involved guarantee that the recompositions indeed fit together as the students affirm they do.) The students do not refer to lengths nor formulas, as was anticipated when posing the problem since the side lengths of the unit squares are equal to $\sqrt{5}$. (It is quite probable that the way the problem is presented has influence over the way students approach a solution.)

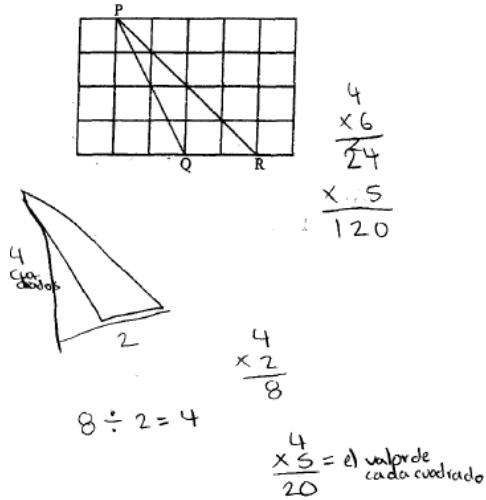


Other students calculated the area of a parallelogram which is composed of two copies of the given triangle or a square composed of four triangles of equal area among them the given triangle, this time using a product to calculate the number of unit squares it contains and the area of each unit square, and finally taking the appropriate fraction of this area. And there was at least one student who considered that the area of the triangle was one-half of the area of a certain right triangle. These students used

a combination of decomposition – recomposition, and comparison. Their thinking remains Euclidean.

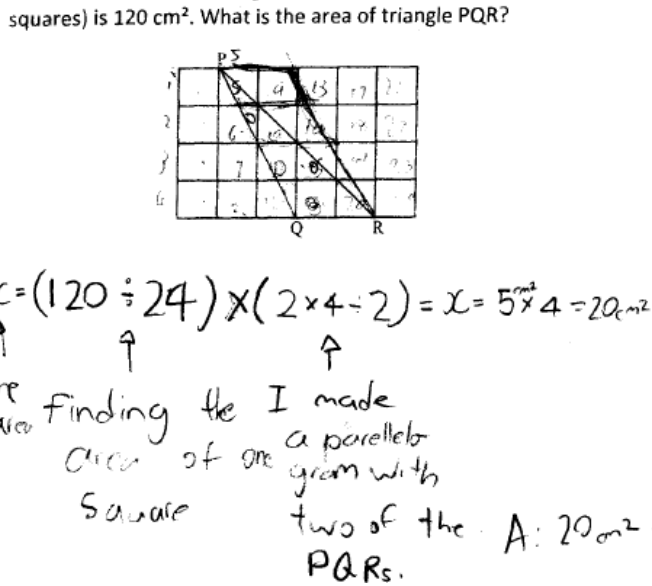


Only one student used the formula for the area of a triangle (almost) directly.



One student solved the problem constructing a parallelogram with two

of the given triangles, using the formula for the area of a parallelogram, dividing by 2 and calculating the product of base and height proportionally according to the product of the base and height of the given rectangular grid.



These are the only two students who have moved beyond Euclidean thinking, using directly and correctly an arithmetic–algebraic formula and employing proportional reasoning to reach the result.

The students reveal the meaning they have constructed for the concept of area, meaning that gives them a basis for understanding the problem at hand and how to solve it. They then use this understanding to think through the problem, generate strategies and put them to work. Here they are thinking mathematically. They justify their reasoning according to the meaning they have constructed for the concept of area. (Number of square units covered by the figure, one-half of base times height, or even, perhaps its Euclidean antecedent, triangles on equal bases and between the same parallels have equal area.)

On the subject of the characterization of geometric thinking, there is lit-

tle to be found in the literature. Most articles focus on activities designed to stimulate geometric thinking, defined implicitly as what students do when they solve the geometric problems they encounter. As such, success is measured by engagement, motivation and progress toward a viable solution, but little or no attention is paid to the description of what it means to think geometrically.

One notable exception is David Tall who in 2013 published a new book titled *How Humans Learn to Think Mathematically.* Tall's book reveals many years of research and reflection and as such deserves our respect, and yet I believe he is far off the mark when it comes to characterizing geometric thinking and I would like to analyze today with you the reasons behind this claim.

Tall starts with a Piagetian point of view with regard to learning, zeroing in on reflective abstraction and claiming that geometry starts with observation of objects and abstracts properties from the objects themselves, whereas arithmetic starts with actions upon objects and abstracts from the operations the learner performs.

Tall tells us that "mathematical thinking emerges in two different forms: space and shape focusing on the properties of objects through structural abstraction and number focusing on actions such as sorting and counting that lead on to the concept of number through operational abstraction." [Tall, 2012, locations 1003–7, kindle version].

He states further that "Geometrical meaning is a long journey for a child. It begins with initial perceptions of figures and talking about them to realize that the same figure may look different in different orientations. It becomes more precise through using language to describe properties of the figures by verbal definitions and the use of these definitions to infer that certain properties imply others, leading to Euclidean geometry and beyond.

At each stage, the focus is on the properties of objects, which involves a structural abstraction of the properties." [Tall, 2012, Kindle version, locations 819-829]

At first glance, one is struck by the thinking behind of singling out

geometric thinking in this way. Some questions that immediately arise are:

- What can be said about geometric models for algebraic and arithmetic ideas and processes?
- Is transformational geometry about objects or actions?
- Do predominantly relational (geometry), functional (algebra) and operational (arithmetic) ideas, which share a unified mathematical basis, require separate and different explanations of the mathematical thinking involved?

On the other hand, Tall sets measurement in the realm of abstracting from actions, explaining that the thinking related to measurement starts with actions such as measuring, differentiating between geometric thinking and the thinking involved with measurement in these words: "As soon as geometry involves measurement, mathematical thinking requires the blending of embodiment and symbolism." His example is proving the formula for the area of a spherical triangle, and his proof involves decomposition and comparison. But he argues that there is an element of symbolism because it is necessary to introduce the value of the surface area of the entire sphere which is $4\pi r^2$. Tall calls this the "symbolic surface area".

Many comments come to mind. To me this is unacceptable. The point that he raises is that measurement entails algebraic-like formulae. But the root concept is that of comparison, comparing the area of the spherical triangle with the surface area of the sphere. It can be done in Euclidean terms, where there is no reference to formulae, only comparison of the areas (or lengths or volumes) of figures, or it can be done in quasi-algebraic terms, where certain measurements are taken as the "unit" of comparison. But the geometric thinking requires operations and the abstraction rests on analysis of the results of the actions.

When speaking of measurement, Tall makes the claim that the learner begins with actions, such as measuring, and abstracts the concept from actions, making measurement akin to arithmetic and the concepts abstracted from operations performed on the objects and not from the objects themselves. But he clearly does not have Euclidean measurement in mind, and it is confusing to think that children construct meaning for

the concept of area, for example, solely on the basis of actions such as measuring. Does Tall also have in mind operations such as decomposition and recomposition? If he does, he does not mention them. Does he have the idea of comparison of two different figures based on the results of the decomposition into congruent parts?

According to Tall geometric thought begins with abstraction from objects. However, none of the geometric figures is given without the intervention of a human creator (although the full moon does resemble a circular disk). Tall compares the process with the categorization behind the use of language, words like "dog" or "bird". And it is true that geometrically shaped objects (somewhat imprecise) do inhabit the world of the child and stimulate his or her thought, but they were not there before a human being put them there.

Geometric thinking can be seen as lead by a community, the thinking is embedded in the objects available to the learner. And by asking the right questions and proposing the right activities and problems, the learner can reconstruct the actions that lead to understanding, assimilating and solving the problem at hand.

I would argue that the difference is not on the level of mathematical thought but on the level of educational goals; educational goals regarding geometry show "abject poverty". It is the teacher and the school that mimic a geometry of objects and their recognition as goals for the development of geometrical thinking. Whereas, to cite an example, problems of construction, as in Euclid, immediately take the student into a world of actions that are the basis for the meaning the student constructs for the geometric concepts involved. Tall treats some of this but does not see it.

Note our example, only two students solved the problem acting on the symbolic (arithmetic-algebraic) formula that represents the area of the triangle. One student is able to see how the formula for the area of a triangle can be used (almost) directly. The second student manipulates the symbols $b \times h$ that represent the area of the rectangular grid in relation to a parallelogram he draws in order to be able to see its height directly, expressing with them the area of the triangle, comparing directly the numeric ratio. But other students also act, using the rela-

tionship between the triangle and other figures drawn by them between the same parallel lines, act drawing other figures that embed the given triangle in a parallelogram, a right triangle or a square, act analyzing the relationship between areas geometrically and only then interpreting these results in terms of their numerical values in order to arrive finally at an answer to the question posed.

The geometric thinking done in the context of solving a well-posed problem involving measurement requires constructing, decomposing, recomposing, comparing, all actions of the subject, transformation of the figure and analysis of the results, not only abstraction of properties from the figure.

Our task becomes that of analyzing geometric thinking that is unrelated to measurement. We would argue that geometric thinking is much richer than Tall acknowledges, and that it is not limited to the analysis of the properties of objects but rather constitutes a natural extension of the human act of creating the figures themselves, once the fruit of a startling creative act and now the given of visual experience, transforming them with a firm purpose in mind.

Where does transformational geometry stand in his analysis of geometric thinking? Or where combinatorial geometry? Is visual thinking condemned to abstract from objects or is visual thinking best seen as analysis of the results of our actions (not gleaned from actions but leading us to perform actions such as drawing an auxiliary line, circumscribing a circle that is not given about a figure, representing algebraic entities figurally and analyzing the new circumstances)?

Euclid himself addressed two kinds of propositions in which a proof was required, on the one hand those that we would call theorems, and on the other those that we would call construction problems. His reasoning is somewhat different in the two cases. On the one hand he proceeds creatively to draw certain straight lines or circles, and uses what he has already proved to show that indeed what has resulted is the construction sought. On the other hand, he uses varying methods, including auxiliary constructions, some of them no less than brilliant, to establish general relationships. Indeed he does define the basic objects of study, and his arguments proceed from his definitions, but he does not begin with

objects, but with figures constructed by human action. His is a process of "formalization" of a body of knowledge.

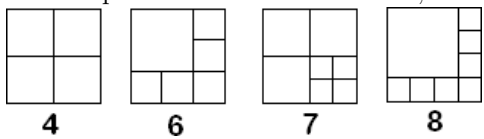
It appears that Tall is not analyzing geometric thinking as it developed historically and as it continues to develop, nor is he analyzing the thinking used in solving non-routine geometry problems. Rather he is addressing the usual expectations that schools have when touching the shallow geometry most frequently attempted in education. His analysis is curriculum driven, based on a curriculum with low expectations for geometry at that. He is not looking at geometry and the range of ideas, concepts, methods, interests that geometric thinking encompasses. He is characterizing geometric thinking in terms of curricular goals.

Going back to Harel's comment, the ways of understanding that the student brings to the problem, as in our example, the meaning (s)he has constructed for the concept of area, shape the strategy for solving the problem, while the ways of thinking involved move the strategy forward, achieving the relations necessary for making the strategy successful. In certain circumstances success yields a new understanding, hones the meaning the student has constructed for the concept adding to the understanding (yielding the construction of more robust meaning for the concepts involved) and thus providing a more complete basis for new strategies when facing a new problem. It is difficult to discern if Tall's treatment even gets as far as aspiring to describe students' geometric understanding, let alone their geometric thinking.

When characterizing geometric thinking, he does not mention problem solving which as we have tried to illustrate is a most appropriate context for rounding out understanding and developing thinking, and when he mentions problem solving his treatment immediately changes. He begins by referencing Pölya and then goes on to reference the writing of John Mason, with Leone Burton and Kaye Stacey, and their book *Thinking mathematically*. He centers on one of the problems in that book with a geometric flavor and does not seem to realize that the discussion he reports contradicts his claims as to the nature of geometric thinking.

The problem is to determine for which numbers n is it possible to subdivide a given square in n (non-overlapping) squares. A classic problem found in competitions, the solution on the geometric level requires al-

most exclusively ways of thinking while the generalization is based on ways of understanding. We all know the solution, the strategy based on understanding is that since each square can easily be divided into four squares (and trivially we must know the definition of square), if we find a way to subdivide the given square in n squares then we can subdivide it into $n+3$ squares. So let's consider small values of n. Considering small values of n, it so happens that we do not find a way for $n = 2, 3, 5$ suggesting that it is not possible in these cases but 6, 7 and 8 are possible.



And by increasing by 3, we conclude that it is possible for 4 and for all numbers greater than 5. Our understanding leads us to a strategy, we probably proceed from experience for 4, then think creatively in a way that guides our actions for 6 and 7 (8 is simply a repeat for the decomposition for 6 (and really for 4)), and then thinking and strategy (if it is possible for n then it is possible for $n+3$) direct our actions that combine to conquer the problem. Furthermore the problem tells us that geometric thinking and arithmetic-algebraic thinking interact and reinforce one another in ways that are fruitful and that would seem to contradict their separation by Tall into different categories. (Our proof is not complete, for we have failed to rule out definitely the cases $n = 2, 3$ and 5.)

Why does Tall not see this as something to be taken into consideration in his account of geometric thinking? I believe that this is an example that shows clearly that he links geometric thinking, and indeed his account of thinking in general, too closely to the objectives of a poor, an almost destitute, yet widespread, curriculum. Harel's comment here is crucial. Mathematics education as it is practiced places emphasis almost entirely on understanding as the main objective of the curriculum, a type of understanding that apparently comes about making the transition from process to concept (as explained by Tall, a precept). But understanding in fact is an important component of strategy, whereas mathematical thinking developed and practiced in the context of problem solving takes

understanding as a starting point, develops new ways of using it and contributes to sharpening that understanding.

In closing, then, I would claim that research into mathematical thinking not only is largely absent from the corpus of mathematics education research, as Harel says, but also that it must be studied in the context of solving genuine problems and that competitions are the main vehicle for such study. Thus I propose two things, or perhaps three. First, that research into mathematical thinking be done by those who see it every day in their work, that is, those who are involved in problem-solving competitions. And second, giving echoing my talk at WFNMC-6 in Riga, that those of us involved in problem-solving competitions work tirelessly to generate proposals for a more challenging mathematics curriculum for all students, one that will truly address the development of their mathematical thinking. And finally, and quite tentatively, the conduct of research that explores the link between understanding and thinking in the context of problem solving, does I believe gives reign to, or is driven by, creativity.

References

[1] Harel, Guershon. DNR-based Instruction in Mathematics as a Conceptual Framework in *Theories in Mathematics Education: Seeking New Frontiers*. Bharath Sriraman & Lyn English (eds). Berlin, Springer, 2010.

[2] Heath, Thomas. *Euclid: The Thirteen Books of the Elements, Vol. 1*. New York, Dover, 1956.

[3] Olimpiada para la Escuela Primaria, Ronda final, Octubre, 2013.

[4] Tall, David. *How Humans Learn to Think Mathematically: Exploring the three worlds of mathematics*. Cambridge University Press, 2013.

María Falk de Losada
Universidad Antonio Nariño
Olimpiadas Colombianas de Matemáticas
email: mariadelosada@gmail.com

Rook Paths

Yu-Cheng Daniel Chiu & Wei Ting Kelvin Shih

In this article we will consider some problems on Rook paths.

Problem 1.
There is a Rook on the square at the bottom-left corner of a $2 \times n$ chessboard. It may pass through each square at most once, and may not pass through all four squares which form a 2×2 subboard. What is the number of different Rook paths which end on any square of the rightmost column?

By symmetry, we may assume that the destination is the square at the top-right corner, while the starting square is

(a) at the bottom-left corner;
(b) at the top-left corner.

Call the numbers of such paths a_n and b_n respectively. Below are some initial data.

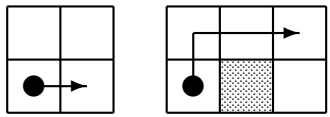
n	0	1	2	3	4	5	6	7	8	9	10
a_n	0	1	2	3	4	6	10	17	28	45	72
b_n	1	1	1	2	4	7	11	17	27	44	72

We take $a_0 = 0$ and $b_0 = 1$ since on an empty board, there is no room to move from the bottom row to the top row, but no movement is needed if we are already on the top row.

Some patterns are easy to spot.
Observation A. If $n \equiv 1$ or 4 (mod 6), then $a_n = b_n$.
Observation B. If $n \equiv 2$ or 3 (mod 6), then $a_n = b_n + 1$.
Observation C. If $n \equiv 5$ or 0 (mod 6), then $a_n = b_n - 1$.
We shall justify these results using a recursive approach.

Suppose the Rook is on the square at the bottom-left corner. Its first move may be to the right, and the number of ways of continuing from

there is a_{n-1}. If the first move is upward, then the next two moves must be to the right. The number of ways of continuing from there is b_{n-2}.



We have

$$a_n = a_{n-1} + b_{n-2}, \tag{1}$$
$$b_n = b_{n-1} + a_{n-2}. \tag{2}$$

The second recurrence relation can be proved in an analogous manner.

From (1), we have $b_{n-2} = a_n - a_{n-1}$. Substituting into (2), we have $a_{n+2} - a_{n+1} = a_{n+1} - a_n + a_{n-2}$. This is equivalent to

$$a_n = 2a_{n-1} - a_{n-2} + a_{n-4}.$$

By symmetry, we also have

$$b_n = 2b_{n-1} - b_{n-2} + b_{n-4}.$$

We can now justify our earlier observations by simultaneous induction. The basis can easily be verified from the initial data. Suppose the results hold up to some positive integer n. Let $n \equiv 1 \pmod 6$. Then $n \equiv 0$, $n-1 \equiv 5$ and $n-3 \equiv 3 \pmod 6$. Hence we have

$$\begin{aligned} a_{n+1} &= 2a_n - a_{n-1} + a_{n-3} \\ &= 2(b_n - 1) - (b_{n-1} - 1) + (b_{n-3} + 1) \\ &= 2b_n - b_{n-1} + b_{n-3} \\ &= b_{n+1}. \end{aligned}$$

The other five cases can be handled in an analogous manner.

A more striking pattern emerges when we consider $t_n = a_n + b_n$.

n	0	1	2	3	4	5	6	7	8	9	10
a_n	0	1	2	3	4	6	10	17	28	45	72
b_n	1	1	1	2	4	7	11	17	27	44	72
t_n	1	2	3	5	8	13	21	34	55	89	144

Note that $t_n = a_n + b_n = (a_{n-1} + b_{n-2}) + (b_{n-1} + a_{n-2}) = t_{n-1} + t_{n-2}$. Hence $\{t_n\}$ satisfies the same recurrence relation as that of the famous Fibonacci sequence.

To explain this rather surprising phenomenon, we use an alternative approach to the problem. Note that the Rook can never move to the left, and vertical moves cannot take place in adjacent columns. Moreover, the path is entirely determined by the vertical moves. The number of vertical moves in any path counted in $\{a_n\}$ must be odd, and the number of vertical moves in any path counted in $\{b_n\}$ must be even.

Suppose we wish to choose k of n vertical columns without choosing two adjacent columns. Denote the chosen columns by 1s and the remaining columns by 0s. This yields a binary sequence of length n with k 1s such that no two are adjacent. If we remove one 0 between every pair of 1s, we obtain a binary sequence of length $n-k+1$ with k 1s. Since the process is reversible, we have a one-to-one correspondence between the two classes of binary sequences. The number of binary sequences of either kind is $\binom{n-k+1}{k}$. It follows that

$$\begin{aligned} a_n &= \binom{n}{1} + \binom{n-2}{3} + \binom{n-4}{5} + \cdots, \\ b_n &= \binom{n+1}{0} + \binom{n-1}{2} + \binom{n-3}{4} + \cdots \\ t_n &= \binom{n+1}{0} + \binom{n}{1} + \binom{n-1}{2} + \cdots. \end{aligned}$$

The diagram below shows that each t_n is the sum of all entries of some diagonal in Pascal's triangle.

$$\begin{array}{cccccccl}
 & & & & & 1 & & \\
 & & & & 1 & \mathbf{1} & & \\
 & & & 1 & \mathbf{2} & 1 & & t_0 = 1 \\
 & & 1 & \mathbf{3} & 3 & \mathbf{1} & & t_1 = 1 + 1 = 2 \\
 & 1 & \mathbf{4} & 6 & \mathbf{4} & 1 & & t_3 = 2 + 1 = 3 \\
1 & \mathbf{5} & 10 & \mathbf{10} & 5 & \mathbf{1} & & t_4 = 1 + 3 + 1 = 5 \\
 & & & & & & & t_5 = 3 + 4 + 1 = 8
\end{array}$$

Now t_n is the number of binary sequences of length n, with no two 1s adjacent. First we count those in which the last term is 0. Then the

first $n-1$ terms can be any binary sequence of length $n-1$, with no two 1s adjacent. Their number is t_{n-1}. Now we count those in which the last term is 1. Then the second last term must be 0, and the first $n-2$ terms can be any binary sequence of length $n-2$, with no two 1s adjacent. Their number is t_{n-2}. It follows that $t_n = t_{n-1} + t_{n-2}$.

Problem 2a.
There is a Rook on the square at the bottom-left corner of a $3 \times n$ chessboard. It may pass through each square at most once, and may not pass through all four squares which form a 2×2 subboard. What is the number of different Rook paths which end on any square of the rightmost column?

By symmetry, we may assume that the destination is the square at the top-right corner, while the starting square is

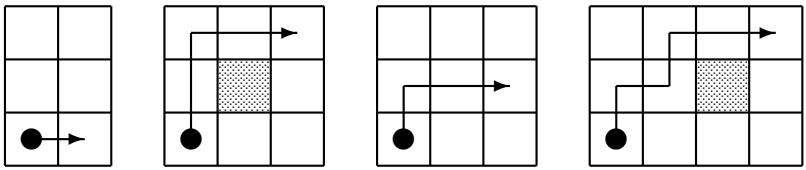
(c) at the bottom-left corner;
(d) at the top-left corner;
(e) in the middle of the leftmost column.

Call the numbers of such paths c_n, d_n and e_n respectively. Below are some initial data.

n	1	2	3	4	5	6	7	8	9	10
c_n	1	3	6	10	18	38	86	190	403	837
d_n	1	1	3	9	22	46	91	183	383	819
e_n	1	2	4	8	17	36	76	160	337	710

There are no clear patterns in these sequences. All we can do is to establish some recurrence relations for them.

Suppose the Rook is on the square at the bottom-left corner. Its first move may be to the right, and the number of ways of continuing from there is c_{n-1}. Suppose the first move is upward. If the second move is also upward, then the next two moves must be to the right. The number of ways of continuing from there is d_{n-2}. Suppose the second move is to the right. If the third move is also to the right, the number of ways of continuing from there is e_{n-2}. If it is upward, then the next two moves must be to the right. The number of ways of continuing from there is d_{n-3}.



We have

$$c_n = c_{n-1} + d_{n-2} + e_{n-2} + d_{n-3}, \tag{3}$$
$$d_n = d_{n-1} + c_{n-2} + e_{n-2} + c_{n-3}, \tag{4}$$
$$e_n = e_{n-1} + c_{n-2} + d_{n-2}. \tag{5}$$

The other two recurrence relations can be proved in an analogous manner.

Combining (3), (4) and (5), we have

$$\begin{aligned} e_{n+2} - e_{n+1} &= c_n + d_n \\ &= (c_{n-1} + d_{n-1}) + (c_{n-2} + d_{n-2}) + 2e_{n-2} + (c_{n-3} + d_{n-3}) \\ &= (e_{n+1} - e_n) + (e_n - e_{n-1}) + 2e_{n-2} + (e_{n-1} - e_{n-2}) \\ &= e_{n+1} + e_{n-2}. \end{aligned}$$

It follows that

$$e_n = 2e_{n-1} + e_{n-4}.$$

The recurrence relations for $\{c_n\}$ and $\{d_n\}$ are much more complicated. Subtracting (5) from (4), we have $c_n - d_n = c_{n-1} - d_{n-1} - c_{n-2} + d_{n-2} - c_{n-3} + d_{n-3}$. This is equivalent to

$$c_n - c_{n-1} + c_{n-2} + c_{n-3} = d_n - d_{n-1} + d_{n-2} + d_{n-3}. \tag{6}$$

Subtracting $c_{n-1} = c_{n-2} + d_{n-3} + e_{n-3} + d_{n-4}$ from (4), we have

$$\begin{aligned} d_n - c_{n-1} &= (d_{n-1} - c_{n-2}) + (c_{n-2} - d_{n-3}) + (c_{n-4} + d_{n-4}) + (c_{n-3} - d_{n-4}) \\ &= d_{n-1} - d_{n-3} + c_{n-4} + c_{n-3}. \end{aligned}$$

This is equivalent to

$$d_n - d_{n-1} + d_{n-3} = c_{n-1} + c_{n-3} + c_{n-4}. \tag{7}$$

From (6) and (7), we have $d_{n-2} = (c_n - c_{n-1} + c_{n-2} + c_{n-3}) - (c_{n-1} + c_{n-3} + c_{n-4})$. This is equivalent to

$$d_n = c_{n+2} - 2c_{n+1} + c_n - c_{n-2}.$$

Substituting this into (7), we have

$$\begin{aligned} c_{n-1} + c_{n-3} + c_{n-4} &= d_n - d_{n-1} + d_{n-3} \\ &= (c_{n+2} - 2c_{n+1} + c_n - c_{n-2}) - (c_{n+1} - 2c_n + c_{n-1} - c_{n-3}) \\ &\quad +(c_{n-1} - 2c_{n-2} + c_{n-3} - c_{n-5}). \end{aligned}$$

This is equivalent to

$$c_n = 3c_{n-1} - 3c_{n-2} + c_{n-3} + 3c_{n-4} - c_{n-5} + c_{n-6} + c_{n-7}.$$

By symmetry, we also have

$$d_n = 3d_{n-1} - 3d_{n-2} + d_{n-3} + 3d_{n-4} - d_{n-5} + d_{n-6} + d_{n-7}.$$

Problem 2b.
There is a Rook on the square in the middle of the leftmost column of a $3 \times n$ chessboard. It may pass through each square at most once, and may not pass through all four squares which form a 2×2 subboard. What is the number of different Rook paths which ends on any square of the rightmost column?

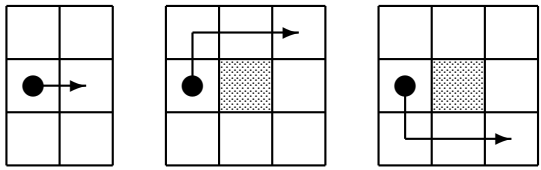
By symmetry, we may assume that the destination is the square in the middle of the rightmost column, while the starting square is

(e) at the bottom-left corner;
(e) at the top-left corner;
(f) in the middle of the leftmost column.

Note that the first two cases of Problem 2b are equivalent to the last case of Problem 2a. Hence both are also labelled (e). We only have to deal with the last case of Problem 2b. Call the numbers of such paths f_n. Below are some initial data.

n	1	2	3	4	5	6	7	8	9	10
e_n	1	2	4	8	17	36	76	160	337	710
f_n	1	1	3	7	15	31	65	137	289	609

The first move of the Rook may be to the right, and the number of ways of continuing from there is f_{n-1}. Suppose the first move is upward or downward. Then the next two moves must be to the right. The number of ways of continuing from either position is e_{n-2}.



It follows that

$$f_n = f_{n-1} + 2e_{n-2}.$$

This is equivalent to $e_{n-2} = \frac{1}{2}(f_n - f_{n-1})$. We already know that $e_n = 2e_{n-1} - e_{n-4}$. Substitution yields $\frac{1}{2}(f_{n+2} - f_{n+1}) = f_{n+1} - f_n - \frac{1}{2}(f_{n-2} - f_{n-3})$. This is equivalent to

$$f_n = 3f_{n-1} - 2f_{n-2} + f_{n-4} - f_{n-5}.$$

Actually, $\{f_n\}$ satisfies a simpler recurrence relation, in fact, the same one satisfied by $\{e_n\}$. We prove this by mathematical induction. The basis can easily be verified from the initial data. Suppose $f_n = 2f_{n-1} + f_{n-4}$ holds for some $n \geq 4$. Then

$$\begin{aligned} f_{n+1} &= 3f_n - 2f_{n-1} + f_{n-3} - f_{n-4} \\ &= 2f_n + (2f_{n-1} + f_{n-4}) - 2f_{n-1} + f_{n-3} - f_{n-4} \\ &= 2f_n + f_{n-3}. \end{aligned}$$

This completes the inductive argument.

Yu-Cheng Daniel Chiu
National Taichung First Senior High School
Taichung
CHINA

Wei Ting Kelvin Shih
Lanya Junior High School
Taipei
CHINA

Tournament of Towns Selected Problems, Fall 2014

Andy Liu

All solutions here are provided by Po-Sheng Wu, a Taiwanese student. He was one of three students to have obtained a perfect score in the 2014 International Mathematical Olympiad in Cape Town.

1. Peter prepares a list of all possible words consisting of m letters each of which is T, O, W or N, such that the numbers of Ts and Os is the same in each word. Betty prepares a list of words consisting of $2m$ letters each of which is T or O, such that the numbers of Ts and Os is the same in each word. Whose list contains more words?

Solution.
For each word Peter writes down, convert every T to TT, every O to OO, every W to OT and every N to TO. Since Peter's word has an equal number of Ts and Os, the new word has an equal number of TTs and OOs. Regardless of the numbers of OTs and TOs, the new word has an equal number of Ts and Os overall, and is therefore on Betty's list. By reversing this conversion process, every word Betty writes down is on Peter's list. It follows that the two lists have equal length.

2. A *uniform* number is a positive integer in which all digits are the same. Prove that any n-digit positive integer can be expressed as the sum of at most $n+1$ uniform numbers.
Example: The numbers 4, 111 and 999999 are uniform.

Solution.
Denote by d_n the n-digit number in which every digit is d. We prove by induction on n that every positive integer less than or equal to 1_n is the sum of at most n uniform numbers. The result is trivial for $n = 1$. Suppose it holds for some $n \geq 1$. Consider a number $m \leq 1_{n+1}$. Subtract from m the largest uniform number $u \leq m$. If $m = 1_{n+1}$, then $u = m$ and $m-u = 0$. If $9_n \leq m < 1_{n+1}$, then $u = 9_n$ and $m - u \leq 1_n$. If $d_n \leq m < (d+1)_n$ for some d,

$0 \leq d \leq 8$, then $u = d_n$ and $m - u \leq 1_n$. In all cases, $m - u \leq 1_n$ and is a sum of at most n uniform numbers by the induction hypothesis. It follows that m is the sum of at most $m + 1$ uniform numbers. Since an n-digit number is less than 1_{n+1}, it is also the sum of at most $n + 1$ uniform numbers.

3. Gregory writes down 100 numbers on a blackboard and calculates their product. In each move, he increases each number by 1 and calculates their product. What is the maximum number of moves Gregory can make if the product after each move does not change?

Solution.
Suppose $a_1, a_2, \ldots, a_{100}$ are real numbers for which there exists a real number k such that $(k+a_1)(k+a_2)\cdots(k+a_{100}) = a_1 a_2 \cdots a_{100}$. Treating this as an equation for k, there are at most 100 real roots, one of which is 0. It follows that Gregory can make at most 99 moves. This maximum can be attained if Gregory starts with the numbers from -99 to 0. The initial product is 0, and this value is maintained for the next 99 moves, until he hits 100! on the 100th move.

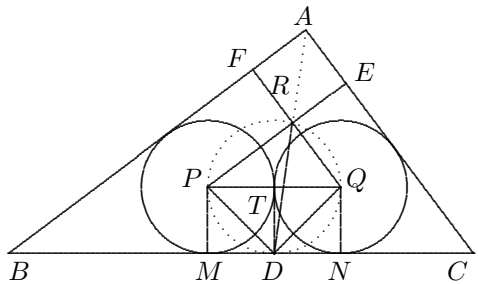
4. On a circular road there are 25 equally spaced booths, each with a patrolman numbered from 1 to 25 in some order. The patrolman switch booths by moving along the road, so that their numbers are from 1 to 25 in clockwise order. If the total distance travelled by the patrolmen is as low as possible, prove that one of them remains in the same booth.

Solution.
Consider the motion plan which accomplishes the desired result in which the total distance covered by the patrolmen is minimum. Suppose all of them move. Let m be the number of those who move clockwise and n be the number of those who move counterclockwise. Then $m \neq n$ since $m + n = 25$. By symmetry, we may assume that $m < n$. Ask each patrolman who moves clockwise to go one booth farther and each patrolmen who moves counterclockwise to stop one booth earlier. Then the patrolmen's numbers will still be in order. However, the total distance they have covered will be reduced by $n - m$ times the distance between two booths.

This contradicts the minimality assumption on the original motion plan.

5. In triangle ABC, $\angle A = 90°$. Two equal circles tangent to each other are such that one is tangent to BC at M and to AB, and the other is tangent to BC at N and to CA. Prove that the midpoint of MN lies on the bisector of $\angle A$.

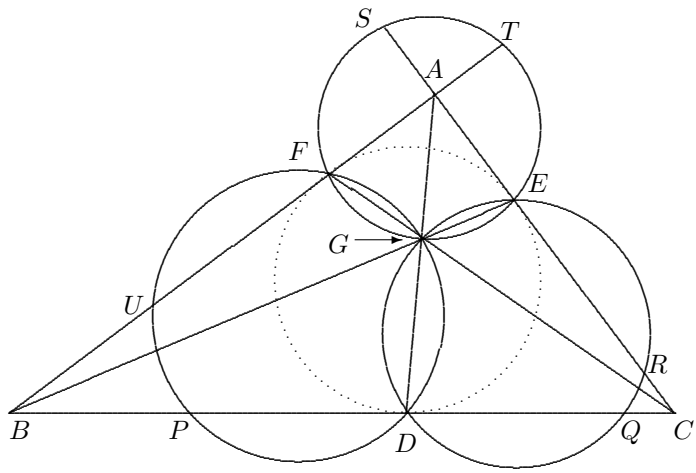
Solution.
Let P be the centre circle tangent to BC at M and Q be the centre of the circle tangent to BC at N. Let T be the point of tangency of these two circles and let D be the midpoint of MN. Drop perpendiculars from P to CA at E and from Q to AB at F, intersecting each other at R. Then $AERF$ is a square whose side length is equal to the common radii of the two circles. Hence $\angle RAF = \angle ARF = 45°$. Now $DMPT$ and $DNQT$ are also squares. Hence $\angle PDQ = 90° = \angle QRP$. Hence $DPQR$ is a cyclic quadrilateral and $\angle DRQ = \frac{1}{2}\angle DTQ = 45°$. It follows that A, R and D are collinear, so that D lies on the bisector of $\angle A$.



6. The incircle of triangle ABC is tangent to BC, CA and AB at D, E and F respectively. It is given that AD, BE and CF are concurrent at a point G, and that the circumcircles of triangles GDE, GEF and GFD intersect the sides of ABC at six distinct points other than D, E and F. Prove that these six points are concyclic.

Solution.
Let the circumcircle of triangle FGD intersect the line AB at U and the line BC at P. Let the circumcircle of triangle DGE intersect the line BC at Q and the line CA at R. Let the circumcircle

of triangle EGF intersect the line CA at S and the line AB at T. Since $PDFU$ is cyclic, $BP \cdot BD = BU \cdot BF$. Since $BD = BF$, we have $BP = BU$. Since $QEDR$ is cyclic, $BQ \cdot BD = BG \cdot BE$. Since $TFGE$ is cyclic, $BG \cdot BE = BF \cdot BT$. From $BD = BF$, we have $BQ = BT$. It follows that $PQTU$ has a circumcircle. Similarly, we can show that $PRQU$ and $PSTU$ have circumcircles too. The three circles cannot be distinct as otherwise PQ, RS and TU are their pairwise radical axes and will be concurrent at the radical centre of the three circles. However, these three lines are the sides of ABC, and cannot be concurrent. It follows that two of these three circles must coincide. Then all three will coincide, and P, Q, R, S, T and U all lie on this common circle.



7. Let PQR be a given triangle. $AFBDCE$ is a non-convex hexagon such that the interior angles at D, E and F all have measure $181°$. Moreover, $BD + DC = QR$, $CE + EA = RP$, $AF + FB = PQ$, $\angle EAF = \angle RPQ - 1°$, $\angle FBD = \angle PQR - 1°$ and $\angle DCE = \angle QRP - 1°$. Prove that

$$\frac{BD}{DC} = \frac{CE}{EA} = \frac{AF}{FB}.$$

Solution.
Let $BD + DC = QR$, $CE + EA = RP$ and $AF + FB = PQ$. We

claim that

$$\frac{BD}{DC} = \frac{CE}{EA} = \frac{AF}{FB}$$

if and only if $\angle EAF = \angle RPQ - 1^\circ$, $\angle FBD = \angle PQR - 1^\circ$ and $\angle DCE = \angle QRP - 1^\circ$. We first assume that

$$\frac{BD}{DC} = \frac{CE}{EA} = \frac{AF}{FB}.$$

Then triangles BDC, CEA and AFB are similar to one another. Hence

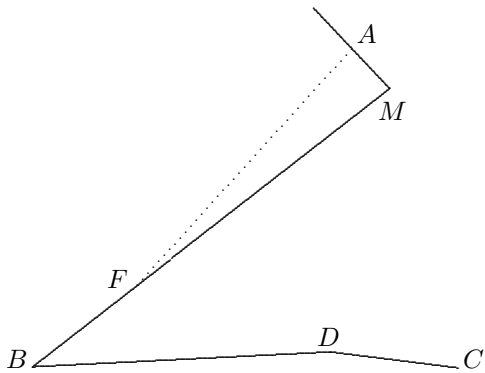
$$\frac{BD + DC}{BC} = \frac{CE + EA}{CA} = \frac{AF + FB}{AB}.$$

It follows that

$$\frac{QR}{BC} = \frac{RP}{CA} = \frac{PQ}{AB},$$

so that triangles ABC and PQR are similar to each other. Then

$$\begin{aligned}\angle EAF &= \angle CAB - \angle CAE - \angle ECA \\ &= \angle RPQ - \angle CAE - \angle ECA = \angle RPQ - 1^\circ.\end{aligned}$$



Similarly, $\angle FBD = \angle PQR - 1^\circ$ and $\angle DCE = \angle QRP - 1^\circ$. We now prove the converse, assuming that $\angle EAF = \angle RPQ - 1^\circ$, $\angle FBD = \angle PQR - 1^\circ$ and $\angle DCE = \angle QRP - 1^\circ$. Fix the points B, C and D. Let M be the fixed point on the same side of BC as D such that $MB = PQ$ and $\angle MBD = \angle PQR - 1^\circ$. Then

F lies on the segment MB and we have $FM = BM - BF = PQ - BF = AF$. Moreover, $\angle AFM = 180° - \angle AFB = 1°$. Hence $\angle FMA = \frac{1}{2}(180-1)° = 89\frac{1}{2}°$. It follows that A lies on a fixed line through M. Let the fixed point N be be defined in an analogous way, using E instead of F and interchanging B and C. Then A also lies on a fixed line through N. Hence there is at most one possible position for A, and A can only exist if we indeed have

$$\frac{BD}{DC} = \frac{CE}{EA} = \frac{AF}{FB}.$$

This result holds if $181°$ and $1°$ are replaced respectively by $180°+\theta$ and θ, as long as $AFBDCE$ is still a non-convex hexagon. The diagram above illustrates the case $\theta = 10°$, which is much easier to see than if $\theta = 1°$.

8. A spiderweb is a square grid with 100×100 nodes, at 100 of which flies are stuck. Starting from a corner node of the web, a spider crawls from a node to an adjacent node in each move. A fly stuck at the node where the spider is will be eaten. Can the spider always eat all the flies in no more than

(a) 2100 moves;

(b) 2000 moves?

Solution.
The answers to both parts are affirmative. We may assume that the spider starts from the top-left corner. Partition the spiderweb into ten 100×10 vertical strips, which it will comb through one by one from left to right. All vertical moves are along either the first or the last column of a strip, downwards on odd-numbered strips and upwards on even-numbered strips. The total number of vertical moves is $10 \times 99 = 990$. All horizontal moves are within the same strip back and forth between the first and the last columns, at the horizontal levels which contain at least one fly. When the spider reaches the bottom row in an odd-numbered strip or the top row in an even-numbered strip, it moves over to the next strip. The number of horizontal moves used for gobbling up flies is at most $9 \times 100 = 900$ since there are 100 flies. The number of horizontal moves used for changing strips is 9. The number of horizontal

moves used for getting to the correct column for changing strips is at most $9 \times 10 = 90$. It follows that the total number of moves required is at most 990+900+9+90=1989.

9. Each day, positive integers m and n are chosen by the government. On such day, x grams of gold may be exchanged for y grams of platinum such that $mx = ny$. Initially, $m = n = 1001$. On each subsequent day, the government reduces exactly one of m and n by 1, and after 2000 days, both numbers are equal to 1. Someone has 1 kilogram each of gold and platinum initially. Without knowing in advance which of m and n will be reduced the next day, can this person have a sure way of performing some clever exchanges, and end up with at least 2 kilograms each of gold and platinum after these 2000 days?

Solution.
On a day with given m and n, the virtual value of x grams of gold and y grams of platinum is taken to be $mx+ny$. This is well defined since it remains constant regardless of any exchange between the precious metals within the day. Consider an amount of precious metal with virtual value 1 on this day. After $m+n-2$ days, both m and n are 1. Without knowing in advance the order in which m and n are reduced during this period, let $f(m,n)$ be the maximum virtual value which can be guaranteed by performing some clever exchanges. We prove by induction on $m+n$ that $f(m,n) = \frac{m+n-1}{mn}$. Note that $f(1,1) = 1$ since no exchange matters, and indeed $\frac{1+1-1}{1^2} = 1$. For any $m > 1$, we have $f(m,1) = 1$. This is because only m can be reduced, and the virtual value is maximized by converting all platinum into gold. Indeed, $\frac{m+1-1}{m} = 1$. Similarly, $f(1,n) = 1$ for any $n > 1$. Suppose $m > 1$ and $n > 1$. Exchange the precious metals to end up with $\frac{\lambda}{m}$ grams of gold and $\frac{1-\lambda}{n}$ grams of platinum, where λ is some parameter to be determined. Note that we indeed have $m\frac{\lambda}{m} + n\frac{1-\lambda}{n} = 1$. On the following day, there are two possible scenarios. If m is reduced by 1, the new virtual value is $(m-1)\frac{\lambda}{m} + n\frac{1-\lambda}{n} = 1 - \frac{\lambda}{m}$. Then $f(m,n) = f(m-1,n)(1-\frac{\lambda}{m})$. However, if n is reduced by 1, the new virtual value is

$$m\,\frac{\lambda}{m} + (n-1)\,\frac{1-\lambda}{n} = 1 - \frac{1-\lambda}{n}.$$

Then

$$f(m,n) = f(m,n-1)(1-\frac{1-\lambda}{n}).$$

Since we do not know which scenario will take place, we choose the parameter λ so that $f(m-1,n)(1-\frac{\lambda}{m}) = f(m,n-1)(1-\frac{1-\lambda}{n})$. By the induction hypothesis,

$$\frac{m+n-2}{(m-1)n}\left(\frac{m-\lambda}{m}\right) = \frac{m+n-2}{m(n-1)}\left(\frac{n-1+\lambda}{n}\right).$$

This simplifies to $\frac{m-\lambda}{m-1} = \frac{n-1+\lambda}{n-1}$, which yields $\lambda = \frac{n-1}{m+n-2}$. It follows that

$$\begin{aligned} f(m,n) &= \frac{m+n-2}{(m-1)n}\left(\frac{m-\frac{n-1}{m+n-2}}{m}\right) \\ &= \frac{m^2+mn-2m-n+1}{mn(m-1} \\ &= \frac{(m-1)^2-n(m-1)}{mn(m-1)} \\ &= \frac{m+n-1}{mn}. \end{aligned}$$

This completes the inductive argument. Initially, $m = n = 1001$ and $x = y = 1000$, so that the virtual value is 2002000. Hence the final virtual value which can be guaranteed is

$$2002000 f(1001,1001) = 2002000 \times \frac{2001}{1001^2} = \frac{4002000}{1001} < 4000.$$

Since we have $m = n = 1$ now, $x+y < 4000$ so that it is impossible to obtain 4 kilograms of precious metals in any combination.

Andy Liu
University of Alberta
CANADA
email: acfliu@gmail.com

Subscriptions

Journal of the World Federation of National Mathematics Competitions

2 Issues Annually

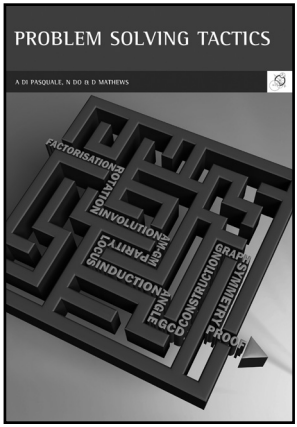
Current subscribers will receive a subscription notice after the publication of the second issue each year.

For new subscribers, information can be obtained from:

Australian Mathematics Trust
University of Canberra Locked Bag 1
Canberra GPO ACT 2601
AUSTRALIA
Tel: +61 2 6201 5137
Fax:+61 2 6201 5052
Email: publications@amt.edu.au

or from our website:
www.amt.edu.au

AMT Publications



Problem Solving Tactics

Lessons from the Australian Mathematical Olympiad Committee Training Program

Authors: A Di Pasquale, N Do & D Mathews

An exciting new addition to our catalogue, *Problem Solving Tactics* is a compilation of tricks and tactics useful in solving mathematical problems at the Olympiad level.

More than 150 ideas are illustrated in the fields of number theory, geometry, algebra and combinatorics.

With an informal style, clear diagrams and hundreds of practice problems, this book will be attractive to those aspiring to Olympiad training, mathematically able students and others interested in problem solving.

The authors, all research mathematicians and past Australian IMO medallists, are members of the training team for the Australian Mathematical Olympiad Committee's School of Excellence.

The Australian Mathematics Trust publishes an array of mathematical materials under its publishing arm, AMT Publishing.

The materials are valuable resources for the school library shelf, students wanting to improve their understanding and competence in mathematics, and teachers looking for relevant, interesting and challenging questions and enrichment material.

All the materials are available from AMT Publishing and can be purchased through the online bookshop at: **www.amt.edu.au** or contact the following:

Australian Mathematics Trust
University of Canberra Locked Bag 1
Canberra GPO ACT 2601
AUSTRALIA

Tel: +61 2 6201 5137

Fax: +61 2 6201 5052

Email: mail@amt.edu.au

The Australian Mathematics Trust

The Trust, of which the University of Canberra is Trustee, is a not-for-profit organisation whose mission is that all young Australians have the opportunity to realise their intellectual potential in mathematics and informatics. Its strengths are based upon:

- a network of dedicated mathematicians and teachers who work in a voluntary capacity supporting the activities of the Trust
- the quality, freshness and variety of its questions in the Australian Mathematics Competition, the Mathematics Challenge for Young Australians, and other Trust contests
- the production of valued, accessible mathematics materials
- dedication to the concept of solidarity in education
- credibility and acceptance by educationalists and the community in general whether locally, nationally or internationally
- a close association with the Australian Academy of Science and professional bodies.

AMT Publishing
Australian Mathematics Trust
University of Canberra Locked Bag 1
Canberra GPO ACT 2601
Australia
Tel: +61 2 6201 5137