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MATHEMATICS COMPETITIONS

JOURNAL OF THE
WORLD FEDERATION OF NATIONAL MATHEMATICS COMPETITIONS

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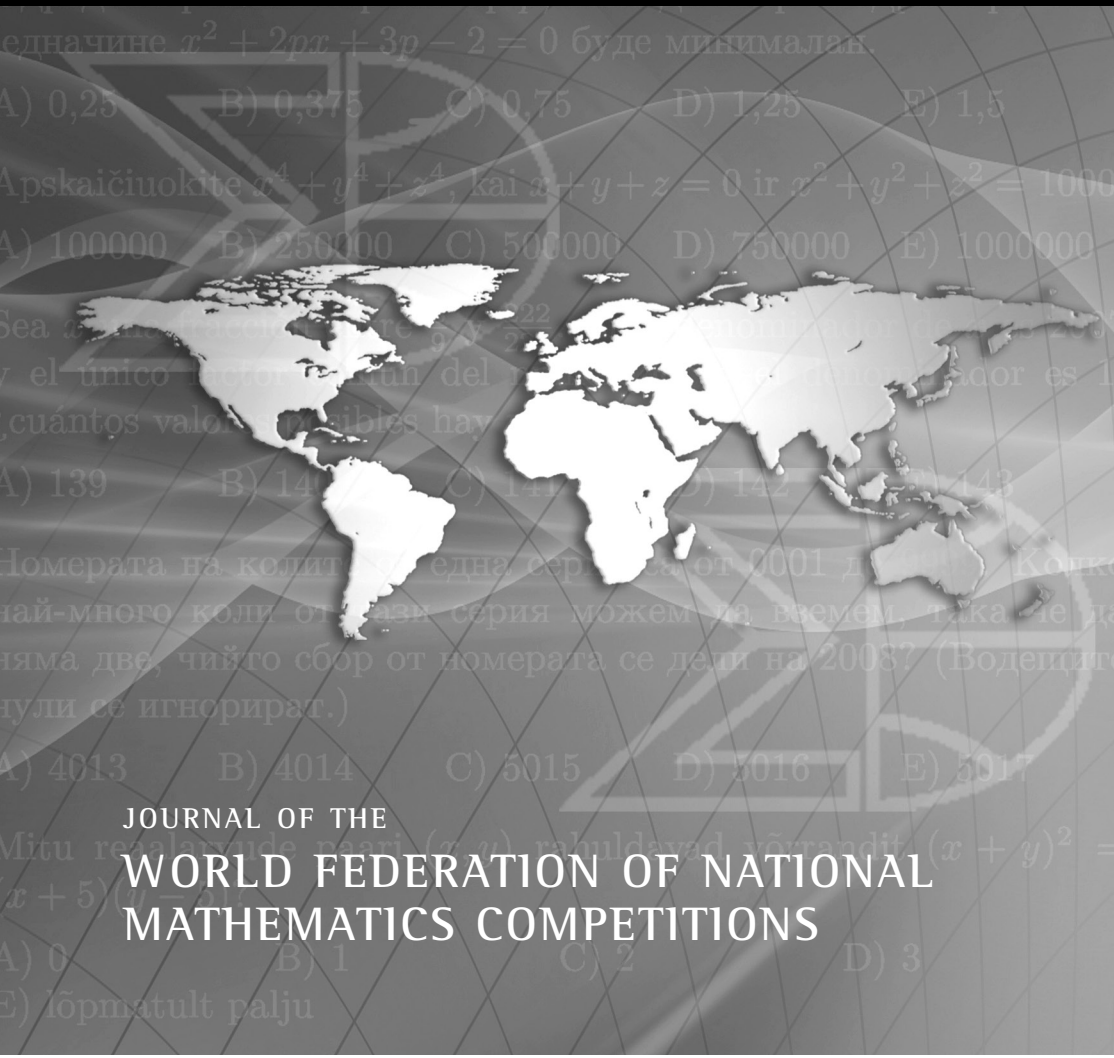

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The aims of the Federation are:

1. ***to promote excellence in, and research associated with, mathematics education through the use of school mathematics competitions;***
2. ***to promote meetings and conferences where persons interested in mathematics contests can exchange and develop ideas for use in their countries;***
3. ***to provide opportunities for the exchanging of information for mathematics education through published material, notably through the Journal of the Federation;***
4. ***to recognize through the WFNMC Awards system persons who have made notable contributions to mathematics education through mathematical challenge around the world;***
5. ***to organize assistance provided by countries with developed systems for competitions for countries attempting to develop competitions;***
6. ***to promote mathematics and to encourage young mathematicians.***

From the President

Dear Fellow Federalists!

I hope to see many of you at the VII Congress of our Federation in Barranquilla, Columbia, and enjoy our interaction.

It is urgent for interested individuals, organizations, and countries to submit to me proposals for hosting the VIII Congress of WFNMC, during the summer 2018.

We prefer the proposals to be backed by professional organizations and/or the host country. We expect the Congress to last for 5–7 days. We prefer the participants to stay together or in a very close proximity from each other. The proposers will closely estimate the cost of lodging, meals and registration fee per delegate; include cultural attractions of the Congress's locale; an excursion for all delegates; and tentative dates.

I realize that the expense of traveling to Colombia has prevented some of our active members from coming to the 2014 Congress. Therefore, they ought to give a thought of organizing the future congresses on their continent, perhaps, in their country!

We would like to receive all proposals *as soon as possible.* The future of the Federation is in our hands, and the role of our congresses is hard to overestimate. The Executive would be happy to work with the proposers and address any questions they may have.

With warm regards to everyone,

Alexander Soifer
President of WFNMC
June 2014

From the Editor

Welcome to *Mathematics Competitions* Vol. 27, No. 1.

First of all I would like to thank again the Australian Mathematics Trust for continued support, without which each issue of the journal could not be published, and in particular Heather Sommariva, Bernadette Webster and Pavel Calábek for their assistance in the preparation of this issue.

Submission of articles:

The journal *Mathematics Competitions* is interested in receiving articles dealing with mathematics competitions, not only at national and international level, but also at regional and primary school level. There are many readers in different countries interested in these different levels of competitions.

- The journal traditionally contains many different kinds of articles, including reports, analyses of competition problems and the presentation of interesting mathematics arising from competition problems. Potential authors are encouraged to submit articles of all kinds.
- To maintain and improve the quality of the journal and its usefulness to those involved in mathematics competitions, all articles are subject to review and comment by one or more competent referees. The precise criteria used will depend on the type of article, but can be summarised by saying that an article accepted must be correct and appropriate, the content accurate and interesting, and, where the focus is mathematical, the mathematics fresh and well presented. This editorial and refereeing process is designed to help improve those articles which deserve to be published.

At the outset, the most important thing is that if you have anything to contribute on any aspect of mathematics competitions at any level, local, regional or national, we would welcome your contribution.

Articles should be submitted in English, with a black and white photograph and a short profile of the author. Alternatively, the article can be submitted on an IBM PC compatible disk or a Macintosh disk. We prefer LaTeX or TeX format of contributions, but any text file will be helpful.

Articles, and correspondence, can also be forwarded to the editor by mail to

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June 2014

A Problem from an Unorthodox Mathematics Competition

Andy Liu

Andy Liu is a professor of mathematics at the University of Alberta in Canada. His research interests span discrete mathematics, geometry, mathematics education and mathematics recreations. He edits the Problem Corner of the MAA's magazine Math Horizons. *He was the Chair of the Problem Committee in the 1995 IMO in Canada. His contribution to the 1994 IMO in Hong Kong was a major reason for him being awarded a David Hilbert International Award by the World Federation of National Mathematics Competitions.*

In 1971, the Soviet Union Mathematical Olympiad was held in Riga, the home court of our eminent colleague Agnis Andjans. On the second day, the contestants at the highest school level were treated to a pleasant surprise. They were confronted with the usual three problems, but were asked to attempt only one of them. All three were somewhat open-ended, and were more like investigation topics than competition problems. They may have been the prototype of what were later used in the International Mathematics Tournament of Towns Summer Seminars. In this paper, we examine one of those three problems in some detail.

A switch receives two input signals and produces two output signals. When it is off, the signals pass straight through the switch. When the switch is on, the signals cross over. Figure 1 shows a network with three switches which can produce all 3! permutations of 3 input signals. We call this a 3-mixer.

What is the minimum number m of switches in a network which can produce all $n!$ permutations of n input signals? Since each of the m

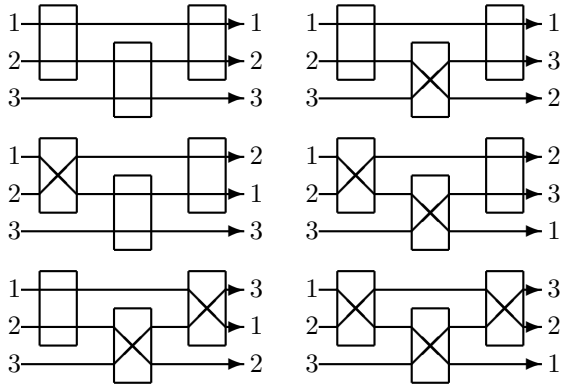

Figure 1

switches may be on or off, the network can handle at most 2^m input signals. Hence we must have $2^m \geq n!$. For $n = 3$, $2^2 < 3! < 2^3$. The minimum number of switches required is 3, so that the network in Figure 1 is optimal.

For $n = 4$, $2^4 < 4! < 2^5$. We seek a network with 5 switches which can produce all 4! permutations of 4 input signals. A 4-mixer is shown in Figure 2.

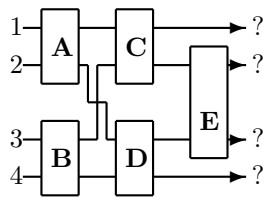

Figure 2

The chart below shows how each of the 4!=24 permutations of the 4 input signals may be obtained. A blank means that the particular switch is

off. A cross means that the particular switch is on, so that a cross-over of signals results.

A		×					×	×	×			
B			×				×			×	×	
C				×				×				×
D					×					×		×
E						×			×		×	
1	1	2	1	3	1	1	2	3	2	1	1	3
2	3	3	4	1	3	2	4	2	1	4	2	1
3	2	1	2	2	4	3	1	1	3	3	4	4
4	4	4	3	4	2	4	3	4	4	2	3	2

A	×	×	×	×			×	×	×	×		×
B	×	×	×		×		×	×	×		×	×
C	×			×	×	×	×	×		×	×	×
D		×		×	×	×	×		×	×	×	×
E			×			×		×	×	×	×	×
1	4	2	2	3	4	3	4	4	2	3	4	4
2	2	4	1	2	1	4	2	1	3	4	3	3
3	1	3	4	4	3	1	3	2	4	2	1	2
4	3	1	3	1	2	2	1	3	1	1	2	1

The reader is invited to investigate networks which can handle 5, 6 or 7 input signals. For $n = 8$, we have $2^{15} < 8! < 2^{16}$. As n increases, networks with the optimal number of switches are harder to find. An 8-mixer using 26 switches was given in the statement of the problem (see [4]). It is reproduced here as Figure 3. This network includes two 4-mixers constructed in Figure 2.

This diagram is unfortunately very confusing. There are apparently 16 input signals and 16 output signals until one realizes that the short segments represent dummy signals. However, this makes the switches in the right column meaningless. A switch in the left column merely directs an input signal to either the upper or the lower 4-mixer. They

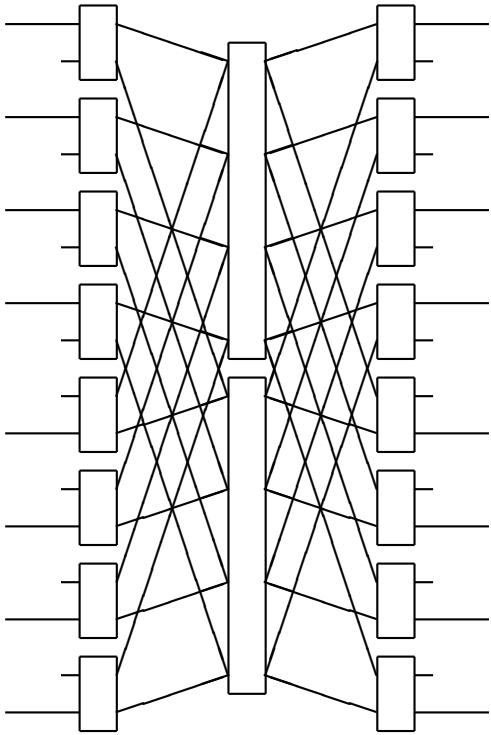

Figure 3

must be manually controlled to ensure that the right number of input signals enter each 4-mixer.

A greater problem occurs with the output signals from the two 4-mixers. The 4-mixers themselves have no control over which branch any of its output signals would follow immediately upon exit. This decision simply cannot be done without involving a switch.

The network in Figure 4 is modified from the preceding one. Apart from eliminating all the problems mentioned, it actually saves 8 switches, using a total of only 18, including 5 in each of the 4-mixers.

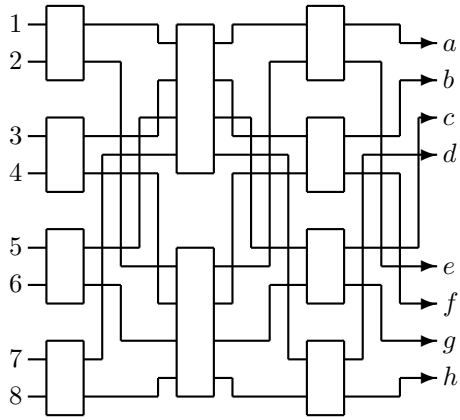

Figure 4

Let the input signals be 1, 2, 3, 4, 5, 6, 7 and 8. Let the desired permutation be a, b, c, d, e, f, g and h. If all the switches are off, as shown in Figure 4, the permutation we obtain would be 13572468. Suppose $\{a, b, c, d\}$ consists of one of each of the pairs (1,2), (3,4), (5,6) and (7,8), we only have to set the left column of switches to send the input signals to the correct 4-mixer, have them sorted out in the right order, and turn off all switches in the right column. Suppose $\{a, b, c, d\}$ contains a pair. For definiteness, say $a = 1$ and $b = 2$. Then $\{e, f, g, h\}$ must also contain at least one pair, say (7,8). We consider three non-equivalent cases.

Case 1. $e = 7$ and $f = 8$.
We set the relevant switches as shown in Figure 5.

Case 2. $e = 7$ and $g = 8$.
We set the relevant switches as shown in Figure 6.

Case 3. $g = 7$ and $h = 8$.
We set the relevant switches as shown in Figure 7.

The input signals 3, 4, 5 and 6 can be handled in a similar manner.

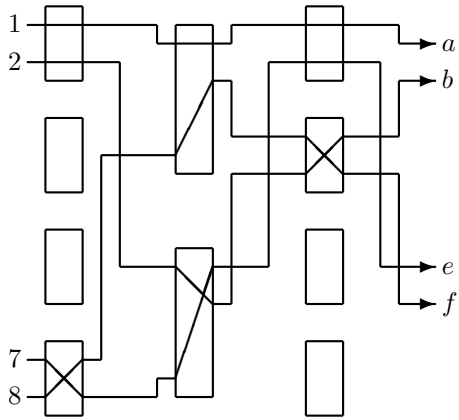

Figure 5

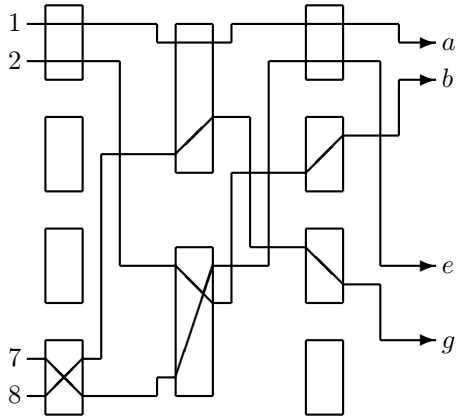

Figure 6

This completes the justification that the network in Figure 4 is indeed an 8-mixer.

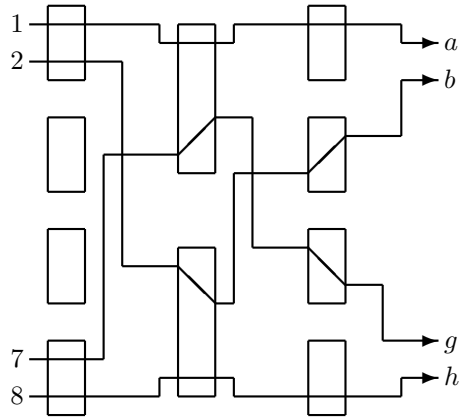

Figure 7

This solution is called an *adaptive* solution in that we have to be on hand to decide whether each switch should be on or off. A *non-adaptive* solution means that switches are on or off according to some predetermined criterion, and the network can work without supervision.

In [3], this is achieved at the expense of just one more switch. The automated 8-mixer is shown in Figure 8, which also includes two 4-mixers. Instead of starting with 1, 2, 3, 4, 5, 6, 7 and 8 and producing all 8! permutations, we start off with an arbitrary one of the 8! permutations and sort them in order. The two problems are equivalent. When two signals enters a switch in the 8-mixer in Figure 8, the signal with the smaller numerical value always exits via the upper branch, so that the relative values of the signals determine whether the switch should be on or off.

This network is based on a merge-sort parallel algorithm due to Batcher [1]. Let the sorted quartet which emerges from the upper 4-mixer be $a_1,\ a_2,\ a_3$ and a_4, and the sorted quartet which emerges from the lower 4-mixer be $b_1,\ b_2,\ b_3$ and b_4. The top two switches in the second column, along with the upper switch in the third column, sort $a_1,\ a_3,\ b_1$ and b_3 into (c_1, c_2, c_3, c_4). The bottom two switches in the second column, along

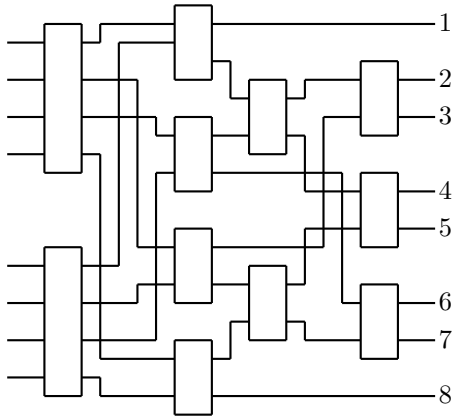

Figure 8

with the lower switch in the third column, sort a_2, a_4, b_2 and b_4 into (d_1, d_2, d_3, d_4). Clearly, we have $c_1 = 1$ and $d_4 = 8$. We claim that $\{c_2, d_1\} = \{2, 3\}$, $\{c_3, d_2\} = \{4, 5\}$ and $\{c_4, d_3\} = \{6, 7\}$, completing the sorting.

By symmetry, we may assume that $c_1 = a_1$. Then $c_2 = b_1$ or a_3 while $d_1 = b_2$ or a_2. Now only a_1 and possibly a_2 can be ahead of c_2. Similarly, only a_1 and possibly b_1 can be ahead of d_1. Hence $\{c_2, d_1\} = \{2, 3\}$. By symmetry again, $\{c_4, d_3\} = \{6, 7\}$, so that we must have $\{c_3, d_2\} = \{4, 5\}$.

It is a valuable exercise to write out a general argument for the Batcher merge-sort with 2^n input signals for an arbitrary positive integer n. For an alternative parallel algorithm, see [2]. Finally, it was asked in [3] whether an 8-mixer can consist only of 5 columns of switches. This may be done adaptively using the network in Figure 4.

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Predicting the Future: Four Classic Conjectures of Mathematics

Alexander Soifer

Born and educated in Moscow, Alexander Soifer has for 33 years been a Professor at the University of Colorado, teaching math, and art and film history. He has published over 200 articles, and a good number of books. In the past 3 years, 6 of his books have appeared in Springer: The Mathematical Coloring Book: Mathematics of Coloring and the Colorful Life of Its Creators*;* Mathematics as Problem Solving*;* How Does One Cut a Triangle?*;* Geometric Etudes in Combinatorial Mathematics*;* Ramsey Theory Yesterday, Today, and Tomorrow*; and* Colorado Mathematical Olympiad and Further Explorations. *He has founded and for 29 years ran the Colorado Mathematical Olympiad. Soifer has also served on the Soviet Union Math Olympiad (1970–1973) and USA Math Olympiad (1996–2005). He has been Secretary of WFNMC (1996–2008), and Senior Vice President of the World Federation of National Mathematics Competitions (2008–2012); from 2012 he has been the president of the WFNMC. He is a recipient of the Federation's Paul Erdős Award (2006). Soifer's Erdős number is 1.*

1 PART Naught: OVERTURE

What is creating conjectures about?

> Surely, it is the art of predicting the future.

Niels Bohr took such predicting jokingly:

> *Predicting is very difficult, especially the future.*

Albert Einstein—nonchalantly:

> I never think of the future – it comes soon enough.

In conjecturing, we use the power of intuition to envision the result without being able to prove it.

In 1989 after Paul Erdős' lecture here at the University of Colorado, Professor of Mathematics Gene Abrams was unimpressed, "What is a big deal about posing problems? Proving results is much more important." "Without someone posing problems, and moreover predicting results by conjectures, as Erdős has done, there will be nothing to prove," I replied. When we commence research, we do not know what is true, and let our intuition lead the way. We have to rely on insight or good luck in choosing a conjecture to prove. And if we choose a conjecture that is not true, it would take a very long time to prove it, a very, very long time!

I oppose discrimination of young high school and college mathematicians based on their youth and inexperience, as Ronald Reagan once remarked. Thus, I think we ought to share with young high school and college mathematicians unsolved problems and conjectures that are waiting for their conquerors. There is plenty of contemporary mathematics, hidden behind an elaborate maze of definitions, and sometimes consisting of merely juggling with them. This kind of juggling does not interest me much. I prefer classic problems. By "classic" I do not necessarily mean problems that are centuries old, but rather problems that are easy to understand by anyone, including a middle school student or a layman, but tantalizingly difficult if at all possible to solve. There are additional conditions on admission of a problem into the "classic" category: an aesthetic appeal of a problem is essential, as are expectations that the result will defy our intuition.

In this essay I will demonstrate interaction between Olympiad and research problems, and present "live" fragments of mathematics, centered on predicting the future by formulating classic conjectures. I am offering you a journey on a mathematical train of thought through problems, conjectures, and results. I hope you will enjoy the ride!

2 CONJECTURE I. Squares in a Square (Erdős 1932)[1]

In 1994 Paul Erdős shared with me one of his oldest conjectures.

We are in Budapest, Hungary, rolled back 82 years. In 1932, the 19-year old Paul Erdős poses the following problem. Inscribe in a unit square r squares, which have no interior points in common. Denote by $f(r)$ the maximum of the sum of side lengths of the r squares. (We allow side lengths to be zero.) The problem is to evaluate the function $f(r)$.

Open Problem 1 For every positive integer r find the value of $f(r)$.

In fact, this formulation came about later, when Paul and I renewed efforts to settle the problem. Originally Paul formulated the following narrow but surprisingly difficult conjecture. When Paul shared the conjecture with me, he offered a \$50 price for its first proof or disproof.

Fifty Dollar Squares in a Square Conjecture 2 (Paul Erdős, 1932) For any positive integer k,

$$f(k^2 + 1) = k.$$

The conjecture is still open today, in the year 2014, waiting, as Paul Erdős used to say, "for stronger arms, or, perhaps, brains" to be settled. However, Paul and I made some progress in a broader problem of describing the function $f(r)$. First of all we observed the following lower and upper bounds for $f(r)$. Symbol $\lfloor x \rfloor$ as usual denotes the largest integer that is not greater than x.

Result 3 (P. Erdős and A. Soifer, 1995, [3]). The following inequality is true for any positive integer r:

$$\lfloor \sqrt{r} \rfloor \leq f(r) \leq \sqrt{r}.$$

[1] This chapter is based on sections E14 and E15 of the author's book [16].

Proof.

1. *The Upper Bound.* The celebrated Cauchy Inequality states that

$$\left(\sum_{i=1}^{r} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{r} a_i^2\right)\left(\sum_{i=1}^{r} b_i^2\right).$$

Setting $b_i = 1$ for every $i = 1, 2, \ldots, r$ we get

$$\left(\sum_{i=1}^{r} a_i\right)^2 \leq \left(\sum_{i=1}^{r} a_i^2\right) r. \qquad (*)$$

Let r squares of side lengths a_i, $i = 1, 2, \ldots, r$ with no interior points in common be placed in a unit square. Then the combined area of the r squares does not exceed the area of the unit square $\sum_{i=1}^{r} a_i^2 \leq 1$, and we get from the inequality (*) above the required upper bound

$$f(r) = \sum_{i=1}^{r} a_i \leq \sqrt{r}.$$

2. *The Lower Bound.* Surely, the function $f(r)$ is non-decreasing, therefore, $r \geq \lfloor\sqrt{r}\rfloor^2$ implies $f(r) \geq f\left(\lfloor\sqrt{r}\rfloor^2\right)$.

Now let us partition the unit square into $\lfloor\sqrt{r}\rfloor^2$ congruent squares, each of the side length $\frac{1}{\lfloor\sqrt{r}\rfloor}$, and calculate the sum of side lengths of these $\lfloor\sqrt{r}\rfloor^2$ squares, we get

$$\frac{1}{\lfloor\sqrt{r}\rfloor} \times \lfloor\sqrt{r}\rfloor^2 = \lfloor\sqrt{r}\rfloor.$$

Observe that this partition and the calculation demonstrate the inequality $f\left(\lfloor\sqrt{r}\rfloor^2\right) \geq \lfloor\sqrt{r}\rfloor$. By combining the two inequalities of this and the preceding paragraphs, we get the required lower bound

$$f(r) \geq f\left(\lfloor\sqrt{r}\rfloor^2\right) \geqslant \lfloor\sqrt{r}\rfloor.$$

Result 3 has the following consequence.

Corollary 4 If $r = k^2$ for a positive integer k, then we get the equality $f(r) = k$.

Corollary 4 allows us to see the Erdős Fifty Dollar Squares in a Square Conjecture in a slightly different light.

Fifty Dollar Squares in a Square Conjecture 2, Second Version (P. Erdős) At perfect square numbers $r = k^2$ (k is an integer) the function $f(r)$ does not increase

$$f\left(k^2+1\right) = f\left(k^2\right).$$

Paul Erdős and I were able to prove that, surprisingly, the function $f(r)$ is strictly increasing everywhere else. But to prove that we needed to find a much sharper lower bound for $f(r)$.

Theorem 6 (P. Erdős and A. Soifer, 1995, [3]). Any positive integer r can be presented in a form $r = k^2 + m$, where $0 \le m \le 2k$. Accordingly, the following inequalities hold

(A) If $m = 2t+1$, where $0 \le t < k$, then $f(r) \ge k + \frac{t}{k}$.

(B) If $m = 2t$, where $0 \le t \le k$, then $f(r) \ge k + \frac{t}{k+1}$.

Proof. Given a positive integer r, we can present it in a form $r = k^2+m$, where $0 \le m \le 2k$. Indeed, it suffices to choose $k = \lfloor\sqrt{r}\rfloor$. If r is a perfect square, $r = k^2$, then $m = t = 0$, and Corollary 4 provides the exact value $f(r) = k$, which is a part of the required inequality (B). We can assume now that r is *not* a perfect square, i.e., $m \ne 0$. The parity of m dictates two cases.

(A) $m = 2t+1$ and $0 \le t < k$. Let us first partition the unit square into k^2 congruent squares; we get a $k \times k$ square grid, call it G; and then replace a $t \times t$ subgrid of the grid G with a $(t+1)\times(t+1)$ square grid of the same total size as the removed subgrid (Figure 1). We end up with a partition of the unit square into $k^2 - t^2 + (t+1)^2 = k^2 + 2t + 1$ little squares, some of which (the original ones) have side length $\frac{1}{k}$, and others (the squares of the inserted $(t+1)\times(t+1)$ square grid)

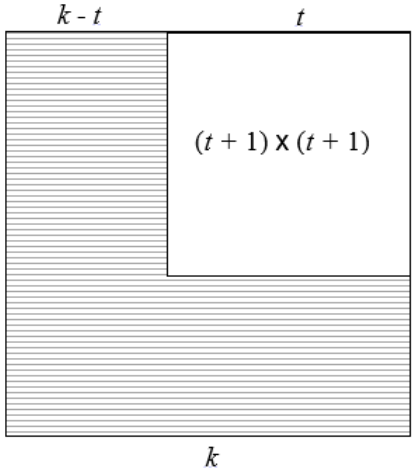


Figure 1

have side length $\frac{t}{k(t+1)}$. Let us calculate the sum of the side lengths of all these $k^2 + 2t + 1$ little squares, we get

$$\frac{1}{k}k^2 - \frac{1}{k}t^2 + \frac{t}{k\,(t+1)}(t+1)^2 = k + \frac{t}{k}.$$

This partition and the calculation deliver the following lower bound for $f(r)$

$$f\,(r) \geq k + \frac{t}{k}.$$

(B) $m = 2t$ and $0 < t \leq k$. We first partition the unit square into $(k+1)^2$ congruent squares; we get a $(k+1) \times (k+1)$ square grid, call it G; and then replace a $(k-t+1) \times (k-t+1)$ subgrid of the grid G with a $(k-t) \times (k-t)$ square grid of the same total size as the removed subgrid (Figure 2).

We end up with a partition of the unit square into $(k+1)^2 - (k-t+1)^2 + (k-t)^2 = k^2 + 2t$ little squares, some of which [the original ones] have side length $\frac{1}{k+1}$, and others [the squares of the inserted $(k-t) \times (k-t)$ square grid] of the side length $\frac{k-t+1}{(k+1)(k-t)}$.

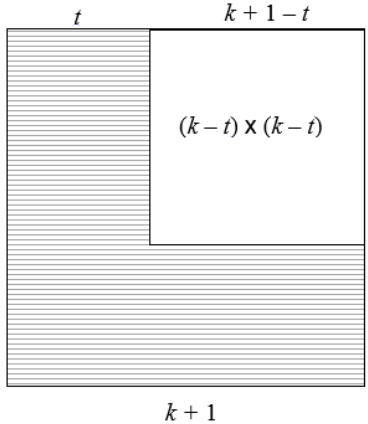


Figure 2

Let us calculate the sum of side lengths of all these $k^2 + 2t$ little squares, we get

$$\frac{1}{k+1}(k+1)^2 - \frac{1}{(k+1)(k-t+1)}(k-t+1)^2 + \frac{k-t+1}{(k+1)(k-t)}(k-t)^2 = k + \frac{t}{k+1}.$$

This partition and calculation deliver the following lower bound for $f(r)$

$$f(r) \geqslant k + \frac{t}{k+1}.$$

Done!

Result 7 (P. Erdős and A. Soifer, 1995, [3]) The function $f(r)$ is strictly increasing everywhere except possibly at perfect square points, i.e., if $r \neq k^2$ for an integer k, then $f(r+1) > f(r)$.

Proof. Once again parity of m and Theorem 6 dictate two cases.

(A) $m = 2t+1$ and $0 \le t < k$. In this case $t+1 \le k$, and by substituting $t+1$ for t in the lower bound found in part 2 of Result 3, we get

$$f\left(k^2+2t+2\right) \ge k + \frac{t+1}{k+1}.$$

This inequality and Result 1 deliver the necessary chain of inequalities

$$\begin{aligned} f(r) = f\left(k^2+2t+1\right) &\leqslant \sqrt{k^2+2t+1} \\ &< k + \frac{t+1}{k+1} \leqslant f\left(k^2+2t+2\right) = f(r+1). \end{aligned}$$

(B) $m = 2t$ and $0 < t \le k$. By using Result 1 and the lower bound found in part 1 of Result 3 above, we get the necessary chain of inequalities

$$\begin{aligned} f(r) = f\left(k^2+2t\right) &\leqslant \sqrt{k^2+2t} \\ &< k + \frac{t}{k} \leqslant f\left(k^2+2t+1\right) = f(r+1). \end{aligned}$$

Result 7 is proven.

Note. In the proof above I omitted a demonstration of two inequalities

$$\sqrt{k^2+2t+1} < k + \frac{t+1}{k+1}$$

and

$$\sqrt{k^2+2t} < k + \frac{t}{k}.$$

I hope their verification would be a welcome exercise in secondary algebra for the reader.

Paul Erdős and I believed that the lower bounds in Theorem 6 were quite good, and conjectured that they just may be the best possible.

Conjecture 8 (P. Erdős and A. Soifer, 1995, [3]) Any positive integer r can be presented in a form $r = k^2 + m$, where $0 \leq m \leq 2k$. Accordingly, we conjecture the following equalities

(A) If $m = 2t + 1$, where $0 \leq t < k$, then $f(r) = k + \frac{t}{k}$.

(B) If $m = 2t$, where $0 < t \leq k$, then $f(r) = k + \frac{t}{k+1}$.

We also observed that our examples in Theorem 6 completely tiled the unit square, and thus posed the following open problem.

Open Problem 9 (P. Erdős and A. Soifer, 1995, [3]) Is it true that for any positive integer r, the value of $f(r)$ can be attained by a set of r squares that form a complete tiling of the unit square by themselves or with an addition of at most *one* extra square?

As I thought about Paul Erdős' problem, it appeared natural for me to pose a "dual" problem, and thus give birth to the *New Squares in a Square Problem.*

Let $\square$ stand for a square shape and $r > 1$ a positive integer. Denote by $\mathbf{S}(\square, r)$ the smallest area of a square Q such that any r squares whose areas add up to at most 1, can be packed in Q (i.e., embedded in Q with no interior points in common).

In 1997 I offered this conjecture for small values of r at the 14th Colorado Mathematical Olympiad.

14.5 Squares in A Square (A. Soifer, [16])

(A) Prove that any two squares whose areas add up to 1 can be inscribed with no interior points in common in a square of area 2.

(B) Prove that any four squares whose areas add up to 1 can be inscribed with no interior points in common in a square of area 2.

(C) Prove that any five squares whose areas add up to 1 can be inscribed with no interior points in common in a square of area 2.

This Olympiad assertion shows that for any r in the range $2 \leq r \leq 5$, $\mathbf{S}(\square, r) = 2$. I then formulated the following conjecture.

New Squares in a Square Conjecture 10 (A. Soifer, 1996, [11]) For any positive integer $r > 1$, $\mathbf{S}(\square, r) = 2$, or to simplify notations, $\mathbf{S}(\square) = 2$.

Two years have passed since I created this conjecture. In May 1997, I was in Lincoln, Nebraska, grading papers of the USA Mathematical Olympiad, together with other members of USA Mathematics Olympiad subcommittee. During a break, I put the New Squares in a Square Conjecture on the board. Later the same day Richard Stong told me "I proved your conjecture." Indeed, he did! Richard devised a simple "greedy" algorithm and a nice, clever proof that his algorithm works[2], and thus New Squares in a Square Conjecture became a theorem, which I happily published in *Geombinatorics*.

Theorem 11 (R. Stong, 1997, [17]) Any finite set of squares of the combined area 1 can be packed in a square of area 2.

Later I discovered that Conjecture 10 and Theorem 11 were not new, and although Stong's proof was better, he was preceded by 30 years by J. W. Moon and Leo Moser of Edmonton, Alberta, Canada [8]. Ecclesiastes (1:9–14 NIV) comes to mind:

> What has been will be again, what has been done will be done again;
> there is nothing new under the sun.

On a positive side, I brought a new excitement and new players to the problem. Moreover, I was already riding toward the next station on my train of thoughts, the one, it seems, no one has traveled before. I conjectured [12] that the identical result was true for circular discs (I will use here the word "disc" to mean a circular disc). This 1998 conjecture is still open today, 16 years later.

Discs in a Disc Conjecture 12 (A. Soifer, [12]). Any finite set of circular discs of combined area 1 can be packed in a circular disc of area 2: $\mathbf{S}(\bigcirc) = 2$.

[2]Space considerations prevent me from including this proof here, but I recommend everyone to read it in *Geombinatorics*.

What about triangles? In working with similar to each other triangles we encounter issues that had not existed for circular discs—limitations on the way "clones" are embedded. We can limit embedding to translations, and thus define a function $\mathbf{S}_T(\triangle)$. Or we can place no limitations on embedding at all and end up with our original $\mathbf{S}(\triangle)$. Of course, $\mathbf{S}(\triangle) \leq \mathbf{S}_T(\triangle)$. It was not at all obvious whether these two values are equal. In 1995 T. J. Richardson calculated the easier of the two values.

Packing Triangles Theorem 13 (T. J. Richardson, 1995 [10]) Any finite set of similar to each other triangles of combined area 1 can be packed in a similar to them triangle of area 2, i.e., in my notations $\mathbf{S}(\Delta) \leq 2$.

In 1999 I pointed out the difference in triangular embedding case and posed these "triangular problems" in *Geombinatorics* [14]. In 2003 the Polish geometer Janusz Januszewski improved Richardson's result on the pages of *Geombinatorics*.

Theorem 14 (J. Januszewski [5]) $\mathbf{S}(\Delta) = 2$ if and only if the triangle Δ is equilateral.

On January 27, 2009, Januszewski informed me that he calculated the harder value $\mathbf{S}_T(\triangle)$ that I asked for in 1999 [6].

Packing Triangles by Translations Theorem 15 (J. Januszewski, 2009, [6]). For any triangle $\triangle$, $\mathbf{S}_T(\triangle) = 2$, i.e., any finite set of similar to each other triangles of combined area 1 can be packed in a similar to them triangle of area 2 *by translations alone*.

Let us roll the time back 15 years. By 1998 I felt it was time to generalize these observations to include *all* geometric figures in our "games." I got busy.

Definition [13] Given figures f and F; it is convenient to call a figure f a *F-clone* if f is homothetic to F.

Definition [13] Given a figure F. Let $\mathbf{S}(F)$ be the minimum real number such that any finite set of F-clones of the combined area 1 can be packed in an F-clone of area $\mathbf{S}(F)$.

Theorems 11 and 13–14 can be written in these notations as follows

a) For a square $\square$, $\mathbf{S}(\square) = 2$.

b) For any triangle $\triangle$, $\mathbf{S}(\triangle) = 2$.

However, it is easy to see that numbers $\mathbf{S}(F)$ are not even bounded if we impose no limitation on figures F in the study.

Result 16 (A. Soifer, 1998, [13]) For any number r, there is a figure F such that $\mathbf{S}(F) > r$.

Proof. Indeed, for any r, we can construct a cross C thin enough so that only one of the two C-clones of area $\frac{1}{2}$ can be inscribed in a C-clone of area r (Figure 3).

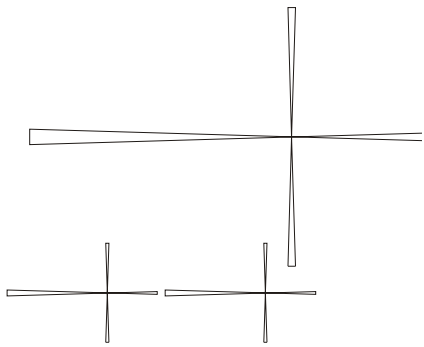


Figure 3

Thus, it makes sense to limit the scope of our games to *convex* figures. The main problem then can be formulated as follows:

Main Open Problem 17 (A. Soifer, 1998, [13]) For any convex figure F, find $\mathbf{S}(F)$.

This is a difficult problem that in full generality may withstand centuries. However, partial solutions are possible and welcome.

I hope you have enjoyed the ride. Out of the window of our train of mathematical thought you may have noticed the terrain that has been continuously changing, with one problem giving birth to another. The journey is not over: in fact, it has only begun. We got a sense of packing some particular shapes, and are now ready to commence a search for its essence, a result encompassing all convex figures. In 1998 I came up with a bold conjecture.

Clones in Convex Figures Conjecture 18 (A. Soifer, 1998 [13]) For *any* convex figure F, any set of $F-$clones $F_1, F_2, \ldots, F_n$ whose areas add up to at most 1, can be packed in a clone F_0 of area 2, i.e.,

$$\mathbf{S}(F) \leq 2.$$

However, when I wrote this conjecture up for *Geombinatorics* [13], I inadvertently put it as $\mathbf{S}(F) = 2$. As the Russian proverb has it, *There is no bad without some good in it!* Three years later, in 2001, the Slovak geometer Pavel Novotný constructed a counterexample to the published equality $\mathbf{S}(F) = 2$.

Novotný's Example 19 ([9], 2001) For rectangle R_0 of size $\sqrt[8]{\frac{3}{2}} \times \sqrt[8]{\frac{2}{3}}$ we get $\mathbf{S}(R_0) = \sqrt{\frac{8}{3}} < 2$.

Novotný understood my typo, as he wrote "Soifer's conjecture could be changed to $\mathbf{S}(F) \leq 2$."

A year later, in 2002, Janusz Januszewski [5], beautifully completed the above result of Novotný.

Januszewski's Theorem 20 ([5], 2002) For any rectangle R, $\mathbf{S}(R) \leq 2$. Moreover, $\mathbf{S}(R) = 2$ if and only if the rectangle R is a square.

Januszewski gave the main conjecture its final attribution:

The Soifer–Novotný Conjecture 21 ([5], 2002) For any convex figure F, $\mathbf{S}(F) \leq 2$.

Finally, Januszewski posed a natural problem that is a particular case of problem 17.

Januszewski's Problem 22 [5] Classify convex figures F for which $\mathbf{S}(F) = 2$.

He among others noticed [5] that, perhaps, the Soifer–Novotný Conjecture 21 can be generalized to n-dimensional Euclidean spaces.

n-Dimensional Conjecture 23 Let F be a convex body in an n-dimensional Euclidean space. Then any set of F-clones can be packed in an F-clone of volume 2^{n-1}.

I would also like to know the minimum value of $\mathbf{S}(F)$.

Open Problem 24 Find $\min \mathbf{S}(F)$ over all convex figures F and classify figures F for which this minimum is attained.

Most of these series of results appeared on pages of *Geombinatorics*, a quarterly dedicated to problem posing essays in combinatorial and discrete geometry (hence its title). Peter Winkler of Dartmouth University dedicated a section of his book [18, pp. 146 and 157] to the Discs in a Disc Conjecture.

> This lovely conjecture is due to Alexander Soifer of the University of Colorado, Colorado Springs. It and its relatives have been the subject of a dozen of articles in the journal *Geombinatorics*; it is known, for example, that squares of total area 1 can be packed into a square of total area 2. The generalization to higher dimension was suggested by your author, among others; the case of two balls, each of volume 1/2, shows that 2^{d-1} is best possible.

I hope you have enjoyed your ride on this train of mathematical thought. However, the original 1932 conjecture is still open 82 years later, in the year 2014. It is time to double the prize:

CONJECTURE I. Hundred Dollar Squares in a Square Conjecture (P. Erdős). At perfect square numbers the function $f(r)$ does not increase: $f(k^2+1) = f(k^2)$.

3 CONJECTURE II. The Happy End Problem (Erdős–Szekeres 1933)

During the winter of 1932–33, two young friends, mathematics student Paul Erdős, aged 19, and chemistry student George (György) Szekeres, 21, solved the problem posed by their young lady friend Esther Klein, 22, but did not send it to a journal for a year and a half [4].

Erdős–Szekeres's Theorem 25 [4] For any positive integer $n \geq 3$ there is an integer $ES(n)$ such that any set of at least $ES(n)$ points in the plane in general position[3] contains n points that form a convex polygon.

The authors knew only two values, obtained by the members of their group of Jewish-Hungarian friends.

a) Esther Klein: $ES(4) = 5$.

b) E. Makai and Paul Turán independently: $ES(5) = 9$.

It is fascinating how sure Erdős and Szekeres were of their conjecture. In one of his last, posthumously published problem papers [2], Erdős attached the prize and modestly attributed the conjecture to Szekeres: "I would certainly pay \$500 for a proof of Szekeres's conjecture."

CONJECTURE II: The Erdős–Szekeres Happy End \$500 Conjecture. $ES(n) = 2^{n-2} + 1$.

In 1933 Erdős and Szekeres proved lower and upper bounds for $ES(n)$; the conjectured value is their lower bound. The upper bound has only recently been improved first by Ronald L. Graham and Fan Chung and then by others, but is still pretty far from the conjectured value. Paul

[3] I.e., no three points lie on a line.

Erdős named it *The Happy End Problem.* He explained the name often in his talks. On June 4, 1992 in Kalamazoo I took notes of his talk:

> I call it The Happy End Problem. Esther captured George, and they lived happily ever after in Australia. The poor things are even older than me.

This paper also convinced George Szekeres to become a mathematician. For Paul Erdős the paper had a happy end too: it became one of his early mathematical gems, the first of Paul's numerous contributions to and leadership of the Ramsey Theory and, as Szekeres put it, of "a new world of combinatorial set theory and combinatorial geometry."

The personages of The Happy End Problem appear to me like heroes of Shakespeare's plays. Paul, very much like *Tempest's* Prospero, gave up all his property, including books, to be free. George and Esther were so close, that they ended their lives together, like Romeo and Juliet. In the late summer 2005 e-mail, Tony Guttmann conveyed to the world the sad news from Adelaide:

> George and Esther Szekeres both died on Sunday morning [August 28, 2005]. George, 94, had been quite ill for the last 2–3 days, barely conscious, and died first. Esther, 95, died an hour later. George was one of the heroes of Australian mathematics, and, in her own way, Esther was one of the heroines.

On May 28, 2000, during a dinner in the restaurant of the Rydges North Sydney Hotel in Australia, George Szekeres told me "my student and I proved Esther's Conjecture for 17 with the use of computer," i.e., $ES(6) = 17$. "Which computer did you use?" asked I. "I don't care how pencil is made," answered George.

4 CONJECTURE III. Chromatic Number Conjecture (Soifer 2008)

In 1986, the Third Colorado Mathematical Olympiad included the following problem from our typical category: its solution is easy to see, especially when somebody has shown it to you.

3.2 Santa Claus and his elves paint the plane in two colors, red and green. Prove that the plane contains two points of the same color exactly one mile apart.

Solution. Toss on the plane an equilateral triangle with side lengths equal to one mile. Since its three vertices (pigeons) are painted in two colors (pigeonholes), there are two vertices painted in the same color (at least two pigeons in a hole). These two vertices *are* one mile apart.

You can prove the same result about a 3-colored plane:

No matter how the plane is 3-colored, it contains two points of the same color exactly one mile apart.

One may think that we know everything about the Euclidean plane. What else can there be after Pythagoras and Euclid, Steiner and Hilbert? First of all the Ancient Greeks did not think about these kinds of problems, where nothing is known about the coloring. Secondly, these simple problems are the starting points of a deep and still unresolved train of thought. For instance, try to push the problem to the next natural step, a 4-colored plane.

Is it true that no matter how the plane is 4-colored, it contains two points of the same color exactly one mile apart?

Imagine, nobody knows!

Chromatic Number of the Plane Problem 26 What is the smallest number of colors with which we can color the plane in such a way that no color contains two points distance 1 apart?

This number is called *the chromatic number of the plane* and is often denoted by χ. We can easily show (do) that 7 colors suffice, and thus $\chi \leq 7$. And so we know that $\chi = 4$ or 5 or 6 or 7. Which is it?

In August 1987 I attended an inspiring talk by the member of the U.S. National Academy of Sciences Paul Halmos at Chapman College in California. It was entitled "Some problems you can solve, some you cannot." This problem was an example of a problem "you cannot solve." So far Halmos is correct.

While writing *The Mathematical Coloring Book* [15], in ca. 2007, I formulated the following conjecture.

Chromatic Number of the Plane Conjecture 27 (Soifer 2007) $\chi = 7$.

If you are familiar with 3- and generally n-dimensional Euclidean space E^n, you will readily see that this problem straightforwardly generalizes to n dimensions, and we can ask a more general question of the chromatic number of E^n. In [15] I have conjectured a simple formula for the chromatic number of the n-dimensional Euclidean space E^n.

CONJECTURE III: Chromatic Number of n-Space $\chi(E^n) = 2^{n+1} - 1$.

Ss Paul Erdős used to say, "If true, this conjecture may take centuries to prove, but we shall see!"

5 CONJECTURE IV. Triangular Covering (Conway–Soifer 2004)

In 2004 we held the 21st Colorado Mathematical Olympiad, for which I created the following problem [16].

21.4. To Have a Cake

(A) We need to protect from the rain a cake that is in the shape of an equilateral triangle of side 2.1. All we have are identical tiles in the shape of an equilateral triangle of side 1. Find the smallest number of tiles needed.

(B) Suppose the cake is in the shape of an equilateral triangle of side 3.1. Will 11 tiles be enough to protect it from the rain?

Solution.

(A) Mark 6 points in the equilateral triangle of side 2.1: its vertices and midpoints of the sides (Fig. 4). A tile can cover at most one such point, therefore we need at least 6 tiles.

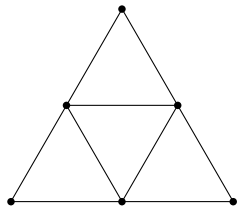


Figure 4

On the other hand, 6 tiles can do the job. There are different ways to achieve it. Here is one. We can first cover the three corners (Fig. 5 left), and then use 3 more tiles to cover the remaining hexagon (Fig. 5 right).

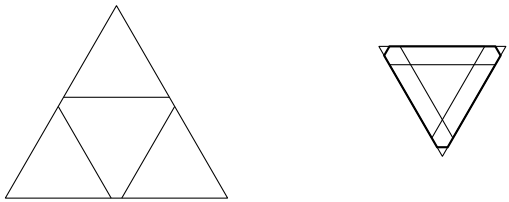


Figure 5

(B) We can use 4 tiles to cover the top triangle of side 2, and then use the remaining 7 tiles for a bottom trapezoid (Fig. 6).

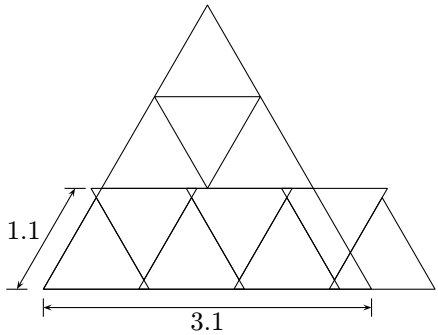


Figure 6

> Have you noticed that I did not ask the Olympians to prove that 11 covering tiles are necessary? At the Olympiad, I could only ask what I can prove myself!

Upon my return to Princeton, where I worked at the time, I shared a more general form of this problem with John H. Conway. Imagine, we both found proofs of the sufficient condition, which were markedly different. And so John and I decided to set a world record: to publish an article containing just one word in its text. Let me reproduce here our submission to *The American Mathematical Monthly.*

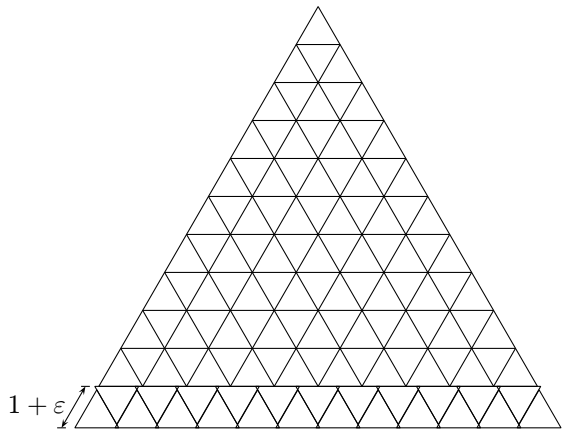
Can $n^2 + 1$ unit equilateral triangles cover an equilateral triangle of side $> n$, say $n + \varepsilon$?

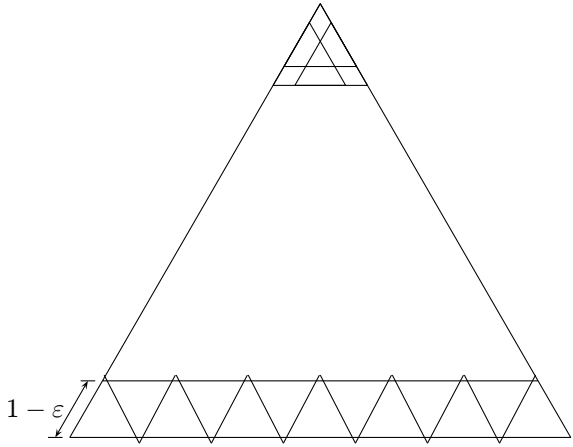
John H. Conway & Alexander Soifer

Princeton University, Mathematics, Fine Hall, Princeton, NJ 08544, USA

conway@math.princeton.edu asoifer@princeton.edu

$n^2 + 2$ can:





The American Mathematical Monthly was puzzled. On April 30, 2004, Editorial Assistant of the Monthly Margaret A. Combs sent me an e-mail.

> The Monthly publishes exposition of mathematics at many levels, and it contains articles both long and short. Your article, however, is a bit too short to be a good Monthly article ... A line or two of explanation would really help.

Having learned from me about *The Monthly* reply, John Conway exclaimed, "Do not give up too easily!" And so, I replied the same day as follows.

> I respectfully disagree that a short paper in general—and this paper in particular—merely due to its size must be "a bit too short to be a good Monthly article." Is there a connection between quantity and quality? ... We have posed a fine (in our opinion) open problem and reported two distinct "behold-style" proofs of our advance on this problem. What else is there to explain?

The Monthly published our article [1], but spoiled our single-word world record by unilaterally including our title in the body of the article!

John Conway believed that since his and my coverings were so vastly distinct, the problem was too hard to continue fighting with it. However,

shortly after, the Columbia University undergraduate student Dmytro Karabash joined me in working on this problem. We generalized the problem to covering an arbitrary triangle T. Tiling triangles will be similar to T and their corresponding sides will be $n+\varepsilon$ times smaller—let us call them $1/(n+\varepsilon)$-clones of T.

Result 28 (Karabash–Soifer, 2005) Any non-equilateral triangle T can be covered by n^2+1 defined above $1/(n+\varepsilon)$-clones of T.

Proof. An appropriate affine transformation maps equilateral triangle on Figure 9 onto T. This transformation gives a covering of T with n^2+2 tiling clones, but now we can cover the transformed top triangle (see Figure 9) with 2 clones instead of 3 as shown in Figure 10, thus reducing the total number of covering clones to n^2+1.

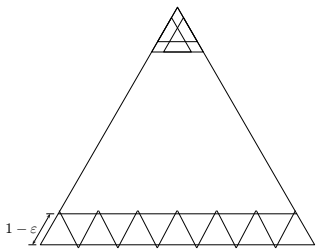


Figure 9

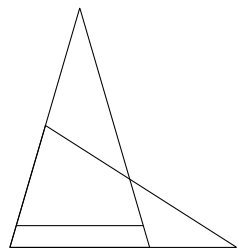


Figure 10

Dmytro and I also generalized the problem by introducing *trigons* [7]. But we were unable to prove the Conway–Soifer Conjecture that an equilateral triangle requires n^2+2 covering triangles. Imagine, the equilateral triangle proved to be the hardest of all! You have a chance to prove it yourselves—sharpen your pencils!

Triangular Covering Conjecture IV (Conway–Soifer 2004) An equilateral triangle of side n cannot be covered by n^2+1 equilateral triangles of side $1-\varepsilon$.

The "smallest" open case is the following conjecture.

The Hexagon Conjecture (Karabash–Soifer 2005) Seven equilateral triangles of side $1 - \varepsilon$ cannot cover an equilateral hexagon of side 1.

Acknowledgment. I thank Robert Ewell for cleaning up my illustrations.

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An Innovative Contest

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In 2008, the International Mathematical Competition (IMC) was formed from the merger of two contests. One of them was the World Youth Mathematics Intercity Competition (WYMIC), described in [1]. It has now become the Junior High School Division of the IMC.

The IMC is well supported by governments of participating countries as well as academic and other non-profit institutions. There are no registration fees, and all local expenses for all leaders and students, such as room and board, transportation and official tours, are covered by the host city. Payment is only required for accompanying persons, usually parents or observers.

Apart from the contest sessions and the official tours, which are typical of most of the competitions, the IMC proudly features a cultural evening when participating teams give short performances highlighting their native heritage. Much mutual understanding and appreciation is fostered. The IMC also organizes a public lecture for all leaders and students, a seminar on mathematics education for the leaders, and a mechanical puzzle carnival for the students.

There are two contests within the IMC, an Individual Contest and a Team Contest. The Individual Contest is traditional, and its format does not change from what was described for the WYMIC in [1]. The main purpose of this paper is to present and discuss the Team Contest.

Its format has been evolving, and has only become standardized in recent years.

The primary intent of the Team Contest is to encourage the students not to treat mathematics as well-versed algorithms and not to regard mathematics as a solely solitary endeavour. We want them to enhance their ability to have original thoughts and creative ideas. We also want them to learn to work with one another.

There is no point in putting in the Team Contest only problems which are just harder or longer than those in the Individual Contest. Naturally, we cannot avoid including some such problems, but we want them to have rather unorthodox settings, preferably with elegant ideas behind them. We give four examples.

Problem 1. (from the 2010 Team Contest)
Determine all ordered triples (x, y, z) of positive real numbers such that each of

$$x + \frac{1}{y}, \quad y + \frac{1}{z} \quad \text{and} \quad z + \frac{1}{x}$$

is an integer.

Problem 2. (from the 2012 Team Contest)
A positive real number is given. In each move, one can do one of the following: add 3 to it, subtract 3 from it, multiply it by 3 and divide it by 3. Determine all the numbers such that after exactly three moves, the original number comes back.

Problem 3. (from the 2011 Team Contest)
Three avenues, of respective widths 15 m, 14 m and 13 m, converge on Red Triangle in the outskirts of Moscow. Traffic is regulated by three swinging gates hinged at the junction points of the three avenues. As shown in the diagram below, the gates at A and B close off one avenue while the gate at C is pushed aside to allow traffic between the other two avenues through the Red Triangle. Calculate the lengths, in m, of the three gates if each pair closes off one avenue exactly. (See Fig. 1.)

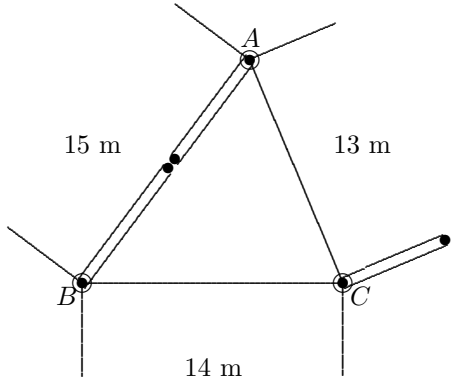


Figure 1

Problem 4. (from the 2013 Team Contest)
Four different stamps are in a 2×2 block. The diagram below shows the 13 different connected subblocks which can be obtained from this block by removing 0 or more of the stamps. The shaded squares represent stamps that have been removed. How many different connected subblocks of stamps can be obtained from a 2×4 block of eight different stamps? (See Fig. 2, page 44.)

We also encourage students not to take competitions too seriously. So we introduce problems with an element of play, which are unlikely to be found in standard mathematics contests. We give six examples.

Problem 5. (from the 2007 Team Contest)
Four teams take part in a week-long tournament in which every team plays every other team twice, and each team plays one game per day. The diagram below on the left shows the final scoreboard, part of which has broken off into four pieces, as shown on the diagram below on the right. These pieces are printed only on one side. A black circle indicates a victory and a white circle indicates a defeat. Which team wins the tournament? (Problem 56 in [2].) (See Fig. 3, page 44.)

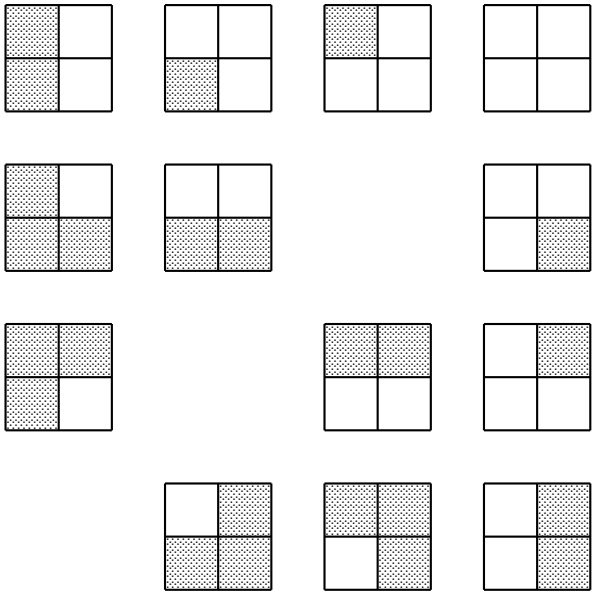


Figure 2

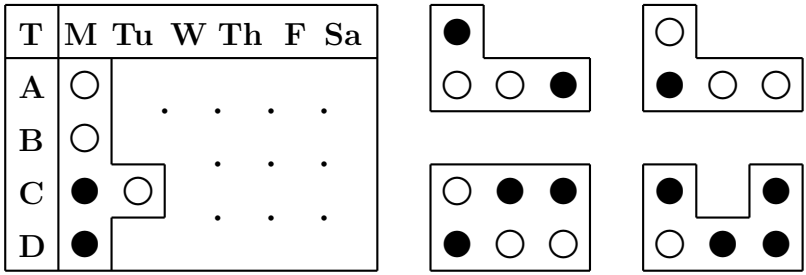


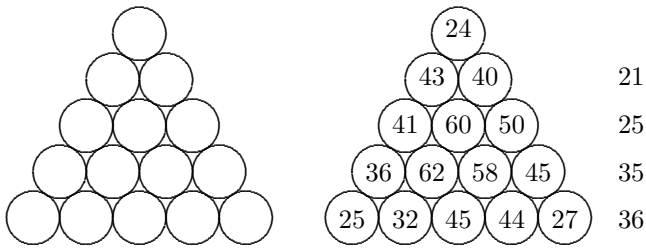
Figure 3

Problem 6. (from the 2012 Team Contest)
Use straight and circular cuts to dissect a circle into congruent pieces. There must be at least one piece which does not contain the centre of the circle in its interior or on its perimeter.

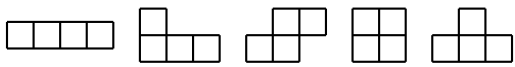
Problem 7. (from the 2010 Team Contest)
Put each of the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14 and 15 into a different one of the fifteen circles in the diagram below on the left, so that

(1) for each circle, the sum of the numbers in it and in all circles touching it is as given by the diagram below on the right;

(2) for each row except the first, the sum of the numbers in the circles on it is as given by the diagram below on the right.



Problem 8. (modified from the 2006 and 2012 Team Contests)
There are five tetrominoes, each consisting of four unit squares joined edge to edge, as shown in the diagram below. They are called the I-, L-, N-, O- and T-Tetrominoes respectively. When two pieces are put next to each other, their unit squares must share an entire edge.



(a) Use two different pieces to construct a connected figure with reflectional symmetry. Find seven solutions using different combinations of pieces.

(b) Use three different pieces to construct a connected figure with reflectional symmetry. Find five solutions using different combinations of pieces.

(c) Use three different pieces to construct a figure with rotational symmetry. Find one solution.

Problem 9. (from the 2002 Team Contest)
Robbie the robot is locked in a solar panel and must get out through the hatch located at the central square of the panel, which is shaded. Locked in with him are other dummy robots under his control. Each robot is mobile, but it can only move along a row or a column directly toward another robot, and can only stop when it bumps into the target robot, stopping in the empty space in front. In each scenario, four moves are allowed, where a continuous sequence of motions by the same robot counts as one move. Robbie is denoted by R.

			A			
B						
R	C			D		

As an example, C-D-A-B and R-D-A is a two-move solution to the above scenario.

A						B
						R
D						E

			A			B
C					R	
			D			E

A		B				C
R	D					E

A			B			C
	R	D				E

	A		B			C
D						
		R			E	

A				B		C
		R	D			E

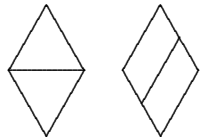
	A				B	
						C
	D		R		E	

A						B
C						
R	D					E

Problem 10. (from the 2001 Team Contest)
There are seven shapes formed of three or four equilateral triangles connected edge to edge, as shown in the 2×5 chart below.

	1	**6**
	2	**7**
	3	**8**
	4	**9**
	5	**10**

For each of the numbered spaces in the chart, find a figure which can be formed from copies of the shape at the head of the row, as well as from copies of the shape at the head of the column. The pieces may be rotated or reflected. The problem in Space 1 has been solved in the diagram below as an example.



Apart from contents, we also use the format of the Team Contest to overcome another perceived problem. We have encountered teams with superstars who dominate other teammates, often tackling the problems all by themselves. In recent years, our Team Contest has had the following structure.

There are ten problems to be solved within an hour. For odd-numbered problems, only answers are required, but for even-numbered problems, full solutions must be provided. In the first ten minutes, the four team members are presented with the first eight problems. They will divide them among themselves. Each must be responsible for at least one problem. They may discuss the problems and share solving strategies, but no writing is allowed at this stage.

For the next thirty-five minutes, the four team members go their separate way and write down answers or solutions to the problems they have been allotted. No communication among them is allowed at this stage. In the last fifteen minutes, the four team members get back together and tackle the last two problems jointly. This has worked well, but we are open to further experimentation.

To give every team a chance of winning something, we divide the participating teams into six groups at random. The top team in each group wins a gold trophy. Each of the next two teams wins a silver trophy. Each of the next three teams wins a bronze trophy. However, as the number of teams in each group is usually greater than twelve, it is still necessary to put out a creditable performance even if a team is drawn into a relatively weak group. A second set of trophies is presented based

on performance in the Individual Contest. The team score is the sum of the best three of the individual scores.

For the Individual Contest itself, the number of recipients of gold, silver and bronze medals are in the ratio 1:2:3, just as in the International Mathematical Olympiad, but the total number of medalists is set at two-fifths of the total number of contestants, rather than one-half. A student among the top half of the non-medalists may receive a certificate of merit.

A book by the three authors of this article, containing all problems of the Individual and Team Contests from 1999 to 2013, will have been published by the time this article appears. It is in English, and runs to over 350 pages. Inquiries may be directed to the Chiu Chang Mathematics Publishers of Taiwan at ccmp@seed.net.tw.

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1 Appendix: Solutions to Problems

Problem 1.
Let $x = \frac{a}{b}$, $y = \frac{c}{d}$ and $z = \frac{e}{f}$, where a and b are relatively prime positive integers, as are c and d, and also e and f. Since $x + \frac{1}{y} = \frac{a}{b} + \frac{d}{c}$ is an integer, we must have $b = c$. Similarly, $d + e$ and $f = a$. Moreover, c divides $a + e$ so that it divides $a + c + e$. Similarly, e and a also divide $a + c + e$. Since a, c and e are pairwise relatively prime, ace divides $a + c + e$, so that $acek = a + c + e$ for some positive integer k. We may assume that e is the largest among a, c and e. Then $acek = a+c+e \leq 3e$ so that $ack \leq 3$. Suppose $k = 3$. Then $a = c = 1$ and $3e = e + 2$ so that $e = 1$. This leads to $(x, y, z) = (1, 1, 1)$. Suppose $k = 2$. Since $2ac \leq 3$, we must still have $a = c = 1$. Then $2e = e + 2$ so that $e = 2$.

This leads to $(x, y, z) = (1, \frac{1}{2}, 2)$ and its cyclic permutations $(\frac{1}{2}, 2, 1)$ and $(2, 1, \frac{1}{2})$. Finally, suppose $k = 1$. Since $ac \leq 3$, one of a and c must be 1. Suppose $a = 1$ so that $ce = c + e + 1$. If $c = 1$, then $e = e + 2$, which is impossible. If $c = 2$, then $2e = e + 3$ so that $e = 3$. This leads to $(x, y, z) = (\frac{1}{2}, \frac{2}{3}, 3)$ and its cyclic permutations $(\frac{2}{3}, 3, \frac{1}{2})$ and $(3, \frac{1}{2}, \frac{2}{3})$. If $c = 3$, then $3e = e + 4$ so that $e = 2$, but this contradicts $c \leq e$. If $c = 1$ instead of $a = 1$, then $a = 2$ and $e = 3$. This leads to $(x, y, z) = (2, \frac{1}{3}, \frac{3}{2})$ and its cyclic permutations $(\frac{1}{3}, \frac{3}{2}, 2)$ and $(\frac{3}{2}, 2, \frac{1}{3})$.

Problem 2.
The operations of adding 3 and subtracting 3 are inverses of each other, as are the operations of multiplying by 3 and dividing by 3. If the same number is obtained after three operations, we can also achieve the same result by performing the inverses of these operations in reverse order. Since the number of operations is odd, we cannot perform only additions and subtractions, nor can we perform only multiplications and divisions. Let the given number be x. We consider two cases.
Case I: Only one operation is multiplication or division.
By symmetry, we may assume that this operation is multiplication. There are three subcases.
Subcase I(a). The multiplication is the first operation.
The last two must both be subtractions. From $3x - 3 - 3 = x$, we have $x = 3$.
Subcase I(b). The multiplication is the second operation.
If the first operation is addition, then the third operation cannot bring the number back to x. Hence after two operations, we have $3(x - 3)$. If the third operation is addition, we have $3(x-3)+3 = x$ and we get $x = 3$ again. If the third operation is subtraction, we have $3(x - 3) - 3 = x$ so that $x = 6$.
Subcase I(c). The multiplication is the third operation.
The first two operations must both be subtractions. From $3(x-3-3) = x$, we have $x = 9$.
Case II. Only one operation is addition or subtraction.
By symmetry, we may assume that this operation is subtraction. There are three subcases.
Subcase II(a). The subtraction is the first operation.
The last two operations must both be multiplications. From $3(3(x-3)) = x$, we have $x = \frac{27}{8}$.

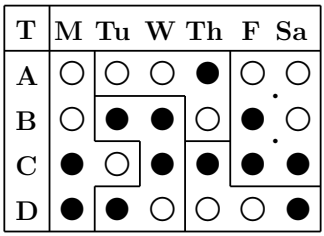
Subcase II(b). The subtraction is the second operation.
If the first operation is division, then the third operation cannot bring the number back to x. Hence after two operations, we have $3x - 3$. The third operation must also be multiplication. From $3(3x - 3) = x$, we have $x = \frac{9}{8}$.
Subcase II(c). The subtraction is the third operation.
The first two operations must both be multiplications. From $3(3x) - 3 = x$, we have $x = \frac{3}{8}$.
In summary, the possible values are $\frac{3}{8}$, $\frac{9}{8}$, $\frac{27}{8}$, 3, 6 and 9.

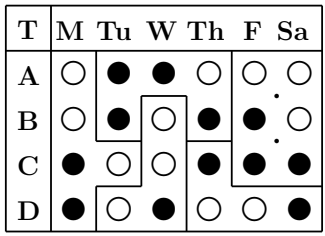
Problem 3.
Let the swinging gates at A, B and C be of lengths a, b and c m respectively. Then $a + b = 15$ m, $b + c = 14$ m and $c + a = 13$ m. Hence $2a = (a + b) + (a + c) - (b + c) = 15 + 13 - 14 = 14$ m so that $a = 7$ m. Similarly, $b = 8$ m and $c = 6$ m.

Problem 4.
Consider the last column in a $2 \times n$ block. Let the number of different connected subblocks which contain both stamps be a_n, the number of those which contains only the top stamp be b_n, the number of those which contain only the bottom stamp be c_n, and the number of those which contain neither stamps be d_n. It is easy to see that $a_1 = b_1 = c_1 = 1$ while $d_1 = 0$. The diagram in the statement of the problem shows that $b_2 = c_2 = d_2 = 3$ while $a_2 = 4$. For $n \geq 3$, we can add a column of two stamps to any connected subblock of the $2 \times (n - 1)$ block, along with the empty block. Hence we have $a_n = a_{n-1} + b_{n-1} + c_{n-1} + 1$. Similarly, $b_n = a_{n-1} + b_{n-1} + 1$, $c_n = a_{n-1} + c_{n-1} + 1$ as well as $d_n = a_{n-1} + b_{n-1} + c_{n-1} + d_{n-1}$. Using these recurrence relations, we obtain $a_3 = 11$, $b_3 = c_3 = 8$, $d_3 = 13$, $a_4 = 28$, $b_4 = c_4 = 20$ and $d_4 = 40$. It follows that the desired number is given by $a_4 + b_4 + c_4 + d_4 = 108$.

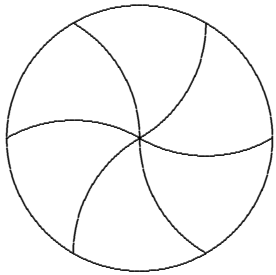
Problem 5.
When reconstructing the broken scoreboard, there are two positions where the U-shaped piece can be placed, so as to leave room for the 3×2 rectangle. Once it is in place, the positions for the remaining pieces are determined. They are placed so that there are two black circles and two white circles in each column. There are two possibilities, as shown in the diagrams below. In the former, A and B play C and D on Monday and Saturday, A and D play B and C on Wednesday and Friday, while

A and C play on Tuesday and Thursday. In the latter, A and B play C and D on Monday and Saturday, A and D play B and C on Thursday and Friday, while A and C play on Tuesday and Wednesday. In either case, the winner of the tournament is Team C.

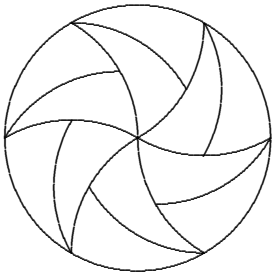


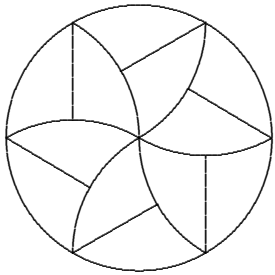


Problem 6.
The diagram below shows a six-piece dissection in which every piece contains the centre of the circle on its perimeters. Thus it is not a solution. However, it is a good first step towards a solution.



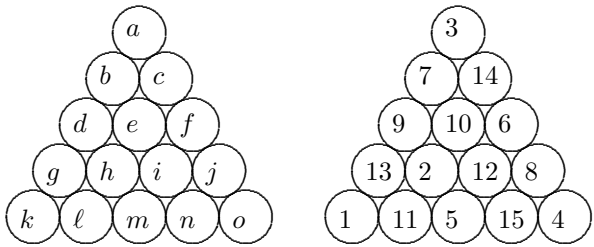
The second step of two different twelve-piece dissections are shown in the diagram below.





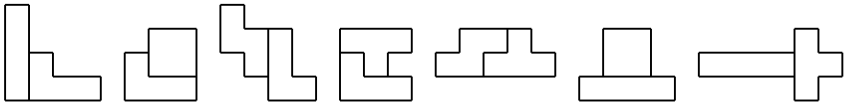
Problem 7.

Label the numbers as shown in the diagram below on the left. From $a+b+c=24$ and $b+c=21$, we deduce that $a=3$. From $a+b+c+d+e=43$, $a+b+c+e+f=40$ and $d+e+f=25$, we have $e=10$ so that $d=9$ and $f=6$. From $d+g+h+k+\ell=36$, $f+i+j+n+o=45$, $g+h+i+j=35$ and $k+\ell+m+n+o=36$, we have $m=5$. From $b+c+d+e+f+h+i=60$, we have $h+i=14$. Now $d+e+g+h+i+\ell+m=62$. Hence $g+\ell=24$ so that $k=1$. Similarly, $e+f+h+i+j+m+n=58$. Hence $j+n=23$ so that $o=4$. From $d+g+h+k+\ell=36$, we have $h=2$ so that $i=12$. From $h+i+\ell+m+n=45$, we have $\ell+n=26$. Since we have $i=12$ already, one of ℓ and n is 11 and the other is 15. Suppose $\ell=15$. Then $g=9$, but we already have $d=9$. Hence $\ell=11$, $g=13$, $n=15$ and $j=8$. Finally, from $c+e+f+i+j=50$, we have $c=14$ so that $b=7$. The completed diagram is shown below on the right.

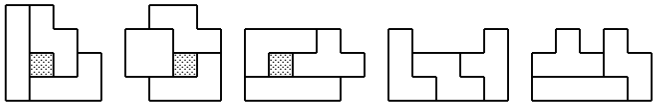


Problem 8.

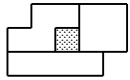
(a) Seven constructions are shown in the diagram below, using the combinations IL, LO, LN, LT, NT, IO and IT. For some combinations, there are other constructions.



(b) Five constructions are shown in the diagram below, using the combinations LIN, LON, LIT, LNT and TIN. For some combinations, there are other constructions.



(c) A construction is shown in the diagram below, using the combination ION.



Problem 9.
The moves are given below for each scenario.

A						B
			C			R
D						E

E-D,B-A-E,C-B,R-C

			A			B
C					R	
			D			E

E-D,D-A-C,B-A-E-D
R-B

A		B				C
R	D					E

E-C,B-A-D-E,C-A-D-B,
R-A-C

A			B			C
	R	D				E

A-B-D,C-B,D-E,
R-D-C-A

	A		B			C
D						
		R			E	

C-B-E,D-C,B-A,R-B-D

A				B		C
		R	D			E

E-D-B,B-A,C-B,R-C-E

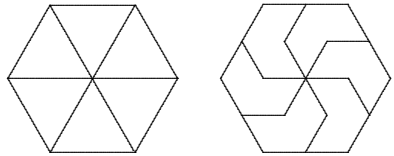
	A				B	
						C
	D		R		E	

D-A,A-B,C-D,R-E-A-C

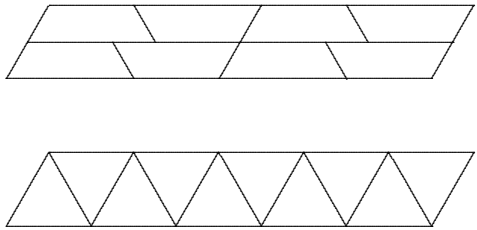
A						B
C						
R	D					E

E-B,B-A-D-E,C-B,R-A-C

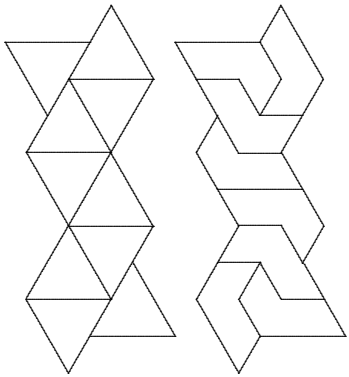
Problem 10.
The diagram below shows possible constructions for the problem in Space 2.



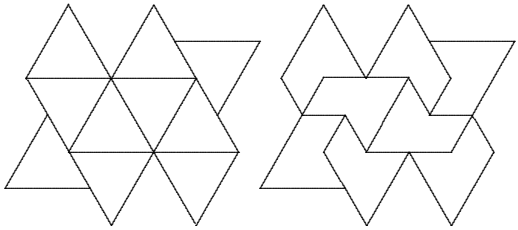
The diagram below shows possible constructions for the problem in Space 3.



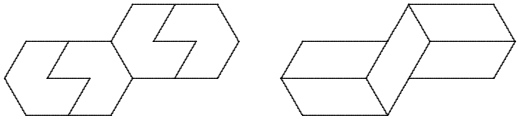
The diagram below shows possible constructions for the problem in Space 4.



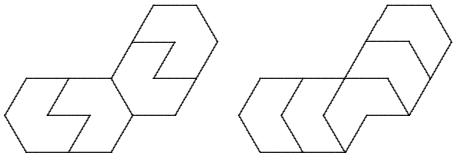
The diagram below shows a possible construction for the problem in Space 5.



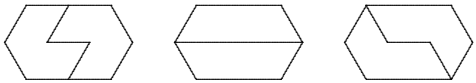
The diagram below shows possible constructions for the problem in Space 6.



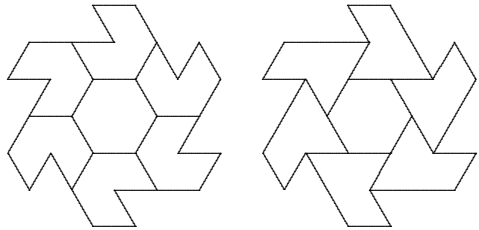
The diagram below shows possible constructions for the problem in Space 7.



The diagram below shows possible constructions for the problems in Spaces 8 and 9.



The diagram below on the next page shows a possible construction for the problem in Space 10.



Remark.
A polyiamond is a figure formed of many unit equilateral triangles joined edge to edge. There is one moniamond, one diamond and one triamond. The seven figures in our problem are the three tetriamonds and the four pentiamonds. Two polyiamonds are said to be compatible if there exists a figure which both can tile. There are twenty-one pairs of these seven tetriamonds and pentiamonds, and all pairs are compatible except for one, consisting of the two figures at the heads of the two rows in the chart in the statement of the problem. We leave as exercises finding possible constructions for the other ten compatible pairs.

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Polyominoes on a Multicoloured Infinite Grid

Hans Hung-Hsun Yu

A polyomino is a figure consisting of unit squares joined edge to edge. The polyominoes with 1, 2, 3, 4 and 5 squares are called monominoes, dominoes, trominoes, tetrominoes and pentominoes, and their numbers are 1, 1, 2, 5 and 12, respectively. They are shown in Figure 1.

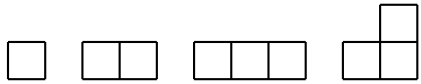
Polyominoes have been made popular by Henry Dudeney [1], Solomon Golomb [6] and Martin Gardner [2,3,4,5]. Fabulous acrylic sets may be ordered from Kadon Enterprises at their website http://gamepuzzles.com. Many fascinating problems are based on polyominoes, and they are also the focus of our investigation.

We divide the infinite plane into unit squares by two sets of grid lines which are mutually perpendicular and evenly spaced. Each square is to be painted in some colour. For a given polyomino, we seek a scheme using the minimum number of colours so that wherever the polyomino is placed, as long as each of its squares covers one square of the grid, the covered squares all have different colours. We now state this formally.

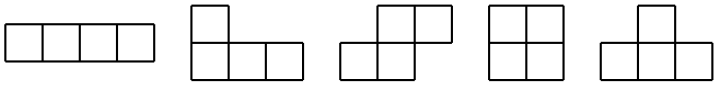
Problem.
For each polyomino, determine the minimum number n of colours for which there exists an n-colour infinite grid so that wherever the polyomino is placed, the covered squares all have different colours.

For a polyomino consisting of n unit squares, clearly n colours are necessary. Clearly, 1 colour is sufficient for the Monomino. The checkerboard pattern extended to the infinite grid shows that 2 colours are sufficient for the Domino. The 3-colour infinite grid in Figure 2 shows that 3 colours are sufficient for the I-Tromino.

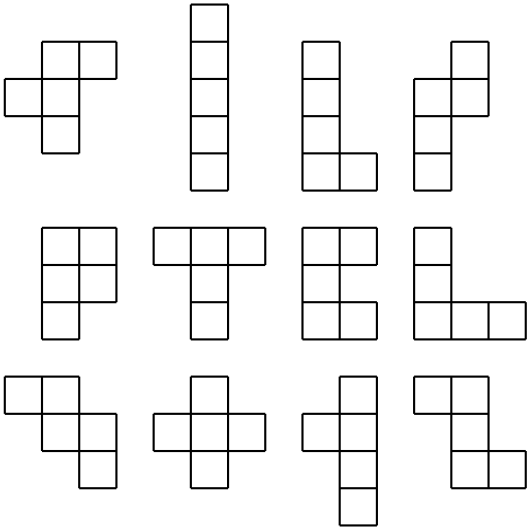
For the V-Tromino, 3 colours are not sufficient. Consider a 2×2 grid. If any 2 of its 4 squares have the same colour, the V-tromino can cover both of them and violate the condition of the problem. This shows that



Monomino, Domino, I- and V-Trominoes



I-, L-, N-, O- and T-Tetrominoes



F-, I-, L-, N-, P-, T-, U-, V-, W-, X-, Y- and Z-Pentominoes

Figure 1

$n \geq 4$. The 4-colour infinite grid in Figure 3 shows that 4 colours are sufficient for the V-Tromino.

We now turn our attention to the tetrominoes. Since each of them has 4 squares, $n \geq 4$. The 4-colour infinite grid in Figure 3 establishes that $n = 4$ for the N-Tetromino and the O-Tetromino, while the 4-colour infinite grid in Figure 4 establishes that $n = 4$ for the I-Tetromino.

1	2	3	1	2	3	1	2	3	1
2	3	1	2	3	1	2	3	1	2
3	1	2	3	1	2	3	1	2	3
1	2	3	1	2	3	1	2	3	1
2	3	1	2	3	1	2	3	1	2
3	1	2	3	1	2	3	1	2	3
1	2	3	1	2	3	1	2	3	1
2	3	1	2	3	1	2	3	1	2
3	1	2	3	1	2	3	1	2	3
1	2	3	1	2	3	1	2	3	1

Figure 2

1	2	1	2	1	2	1	2	1	2
3	4	3	4	3	4	3	4	3	4
1	2	1	2	1	2	1	2	1	2
3	4	3	4	3	4	3	4	3	4
1	2	1	2	1	2	1	2	1	2
3	4	3	4	3	4	3	4	3	4
1	2	1	2	1	2	1	2	1	2
3	4	3	4	3	4	3	4	3	4
1	2	1	2	1	2	1	2	1	2
3	4	3	4	3	4	3	4	3	4

Figure 3

Consider the region in Figure 5 with 5 unit squares. Every two of them may be covered by a suitable placement of the T-Tetromino. This shows that $n \geq 5$.

4	1	2	3	4	1	2	3	4	1
1	2	3	4	1	2	3	4	1	2
2	3	4	1	2	3	4	1	2	3
3	4	1	2	3	4	1	2	3	4
4	1	2	3	4	1	2	3	4	1
1	2	3	4	1	2	3	4	1	2
2	3	4	1	2	3	4	1	2	3
3	4	1	2	3	4	1	2	3	4
4	1	2	3	4	1	2	3	4	1
1	2	3	4	1	2	3	4	1	2

Figure 4

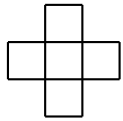


Figure 5

Figure 6 shows a 5-colour infinite grid which establishes that $n = 5$ for the T-Tetromino.

Consider the region in Figure 7, with 12 unit squares. Every two of the 4 central squares may be covered by a suitable placement of the L-Tetromino. Thus 4 colours, 1, 2, 3 and 4, are needed there. Moreover, any of these squares and any of the peripheral squares may be covered by a suitable placement of the L-Tetromio. Thus these 4 colours may not be used again for the 8 peripheral squares. If only 7 colours are available, then one of the 3 additional colours, say 5, must be used on at least 3 peripheral squares. Paint any peripheral square in colour 5. We may assume by symmetry that it is the one shown below. Then the 4 squares marked with crosses cannot be painted in colour 5. Now 2 of the

4	1	5	2	3	4	1	5	2	3
2	3	4	1	5	2	3	4	1	5
1	5	2	3	4	1	5	2	3	4
3	4	1	5	2	3	4	1	5	2
5	2	3	4	1	5	2	3	4	1
4	1	5	2	3	4	1	5	2	3
2	3	4	1	5	2	3	4	1	5
1	5	2	3	4	1	5	2	3	4
3	4	1	5	2	3	4	1	5	2
5	2	3	4	1	5	2	3	4	1

Figure 6

3 blank squares must be in colour 5, but any 2 of them may be covered by a suitable placement of the L-Tetromino. This shows that $n \geq 8$.

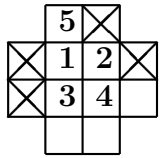


Figure 7

Figure 8 shows an 8-colour infinite grid which establishes that $n = 8$ for the L-Tetromino.

We conclude our investigation by studying the pentominoes. Since each of them has 5 squares, $n \geq 5$. The 5-colour infinite grid in Figure 6 establishes that $n = 5$ for the X-Pentomino, while the 5-colour infinite grid in Figure 9 establishes that $n = 5$ for the I-Pentomino.

For the V- and Z-Pentominoes, consider a 3×3 grid. If any 2 of its 9

1	2	5	6	1	2	5	6	1	2
3	4	7	8	3	4	7	8	3	4
5	6	1	2	5	6	1	2	5	6
7	8	3	4	7	8	3	4	7	8
1	2	5	6	1	2	5	6	1	2
3	4	7	8	3	4	7	8	3	4
5	6	1	2	5	6	1	2	5	6
7	8	3	4	7	8	3	4	7	8
1	2	5	6	1	2	5	6	1	2
3	4	7	8	3	4	7	8	3	4

Figure 8

1	2	3	4	5	1	2	3	4	5
2	3	4	5	1	2	3	4	5	1
3	4	5	1	2	3	4	5	1	2
4	5	1	2	3	4	5	1	2	3
5	1	2	3	4	5	1	2	3	4
1	2	3	4	5	1	2	3	4	5
2	3	4	5	1	2	3	4	5	1
3	4	5	1	2	3	4	5	1	2
4	5	1	2	3	4	5	1	2	3
5	1	2	3	4	5	1	2	3	4

Figure 9

squares have the same colour, the V- or Z-Pentomino can cover both of them and violate the condition of the problem. This shows that $n \geq 9$.

The 9-colour infinite grid in Figure 10 shows that 9 colours are sufficient for the V- or Z-Pentomino.

1	2	3	1	2	3	1	2	3	1
4	5	6	4	5	6	4	5	6	4
7	8	9	7	8	9	7	8	9	7
1	2	3	1	2	3	1	2	3	1
4	5	6	4	5	6	4	5	6	4
7	8	9	7	8	9	7	8	9	7
1	2	3	1	2	3	1	2	3	1
4	5	6	4	5	6	4	5	6	4
7	8	9	7	8	9	7	8	9	7
1	2	3	1	2	3	1	2	3	1

Figure 10

We now consider the F-, L-, N-, P-, T-, U- and Y-Pentominoes as a group. Each of them contains the L-Tetromino, so that $n \geq 8$. The 8-colour infinite grid in Figure 8 establishes that $n = 8$ for all of them.

Consider the region in Figure 7. Every two of the 4 central squares may be covered by a suitable placement of the W-Pentomino. Thus 4 colours, 1, 2, 3 and 4, are needed there. Moreover, if any of the 8 peripheral squares are painted in any of these 4 colours, there is only one choice for each of them, as shown in Figure 11. None of these 4 colours can be used twice on the 8 peripheral squares, as those two squares may be covered by a suitable placement of the W-Pentomino. By symmetry, we may assume that the square marked 3 on the top row is painted in colour 5. Then neither of the peripheral squares marked 4 may be painted in colour 5, and another colour is needed. This shows that $n \geq 6$.

Figure 12 shows an 6-colour infinite grid which establishes that $n = 6$ for the W-Pentomino.

	3	4	
2	**1**	**2**	1
4	**3**	**4**	3
	1	2	

Figure 11

1	2	1	2	1	2	1	2	1	2
3	4	3	4	3	4	3	4	3	4
5	6	5	6	5	6	5	6	5	6
1	2	1	2	1	2	1	2	1	2
3	4	3	4	3	4	3	4	3	4
5	6	5	6	5	6	5	6	5	6
1	2	1	2	1	2	1	2	1	2
3	4	3	4	3	4	3	4	3	4
5	6	5	6	5	6	5	6	5	6
1	2	1	2	1	2	1	2	1	2

Figure 12

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Tournament of Towns Selected Problems, Fall 2013

Andy Liu

1. Can the ten digits 0, 1, 2, 3, 4, 5, 6, 7, 8 and 9 be arranged in a row so that no matter which six digits are removed, the remaining four digits, without changing their order, form a composite number?

Solution by Wen-Hsien Sun:
Put 5 and the even digits in any order in the last six positions. If any of them remains, then the four-digit number is either divisible by 5 or by 2, and is hence composite. If all of them are removed, then the digits 1, 3, 7 and 9, which are in the first four positions in some order, must form a composite number. A simple way is to arrange them in the order 1, 3, 9 and 7, because the number 1397 is divisible by 11, and is hence composite.

2. (a) Denote by (a, b) the greatest common divisor of a and b. Let n be a positive integer such that $(n, n+1) < (n, n+2) < \cdots < (n, n+35)$. Prove that $(n, n+35) < (n, n+36)$.

(b) Denote by $[a, b]$ the least common multiple of a and b. Let n be a positive integer such that $[n, n+1] > [n, n+2] > \cdots > [n, n+35]$. Prove that $[n, n+35] > [n, n+36]$.

Solution:

(a) Clearly, $(n, n+1) = 1$. Hence $(n, n+2) \geq 2$. On the other hand, since $(n, n+2)$ must divide $(n+2) - n = 2$, we must have $(n, n+2) = 2$. Similarly, $(n, n+k) = k$ for $3 \leq k \leq 35$. It follows that n is divisible by the least common multiple of 2, 3, ..., 35. In particular, n is divisible by $4 \times 9 = 36$, which implies that $n + 36$ is also divisible by 36. It follows that $(n, n+36) = 36 > 35 = (n, n+35)$.

(b) Note that $(a, b)[a, b] = ab$. Since $(n, n+1) = 1$, we have $[n, n+1] = n(n+1)$. If $(n, n+2) = 1$ also, then $[n, n+2] = n(n+2) > n(n+1) = [n, n+1]$ is a contradiction. Hence $(n, n+2) \geq 2$. On the other hand, since $(n, n+2)$ must

divide $(n+2)-n=2$, we must have $(n,n+2)=2$ so that $[n,n+2]=\frac{n(n+2)}{2}$. If $(n,n+3)\leq 2$, then $[n,n+3]\geq \frac{n(n+3)}{2}>\frac{n(n+2)}{2}=[n,n+2]$. Hence $(n,n+3)=3$. Similarly, $(n,n+k)=k$ for $4\leq k\leq 35$. It follows that n is divisible by the least common multiple of 2, 3, ..., 35. In particular, n is divisible by $4\times 9=36$, which implies that $n+36$ is also divisible by 36. Hence $(n,n+36)=36$ and $[n,n+36]=\frac{n(n+36)}{36}<\frac{n(n+35)}{35}=[n,n+35]$.

3. There are 100 red, 100 yellow and 100 green sticks. One can construct a triangle using any three sticks of different colours. Prove that we can choose one of the three colours and construct a triangle using any three sticks of that colour.

Solution:
Let the shortest stick overall have length s, and we may assume that it is red. Let the longest stick among the yellow and green ones have length ℓ, and we may assume that it is green. Then a triangle can be formed using the red stick of length s, the green stick of length ℓ and any yellow stick. Let the lengths of any three yellow sticks be $x\leq y\leq z$. Then $x+y\geq s+y>\ell\geq z$. Hence they can form a triangle.

4. Let n be a positive integer such that $3n+1$ is prime. When

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots+\frac{1}{2n-1}-\frac{1}{2n}$$

is expressed as an irreducible fraction, prove that the numerator is a multiple of $3n+1$.

Solution:
Note that n must be even, so that the number of integers from

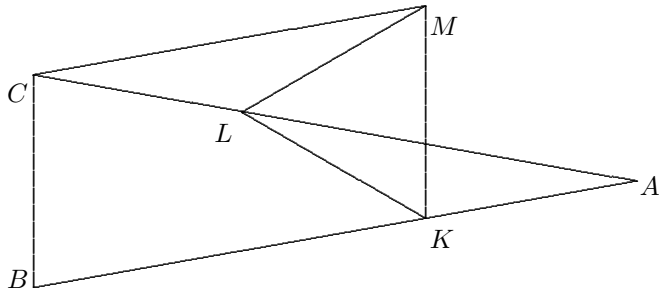
$n+1$ to $2n$ is even. We have

$$\begin{aligned}
&1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots+\frac{1}{2n-1}-\frac{1}{2n}\\
&=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{2n-1}+\frac{1}{2n}-2\left(\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2n}\right)\\
&=\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n-1}+\frac{1}{2n}\\
&=\left(\frac{1}{n+1}+\frac{1}{2n}\right)+\left(\frac{1}{n+2}+\frac{1}{2n-1}\right)+\cdots+\left(\frac{1}{\frac{3n}{2}}+\frac{1}{\frac{3n}{2}+1}\right)\\
&=(3n+1)\left(\frac{1}{2n(n+1)}+\frac{1}{(2n-1)(n+2)}+\cdots+\frac{1}{(\frac{3n}{2}+1)(\frac{3n}{2})}\right).
\end{aligned}$$

When combined as a single fraction, the denominator is the least common multiple of $2n(n+1)$, $(2n-1)(n+2)$, $\ldots$, $\frac{3n}{2}(\frac{3n}{2}+1)$. All of its factors are less than $3n+1$. During the reduction process, the factor $3n+1$ in the numerator, being prime, cannot be cancelled. When expressed as an irreducible fraction, the numerator is a multiple of $3n+1$.

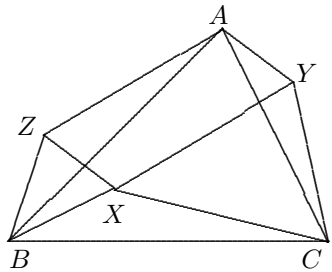
5. In ABC, $AB=AC$. K and L are points on AB and AC respectively such that $AK=CL$ and $\angle ALK+\angle LKB=60^\circ$. Prove that $KL=BC$.

Solution:
Complete the equilateral triangle KLM, with M on the side of the line AC opposite to B. Note that $\angle ALM=60^\circ-\angle ALK=\angle LKB$. Now $AK=CL$, $KL=LM$ and we have $\angle AKL=180^\circ-\angle LKB=180^\circ-\angle ALM=\angle CLM$. Hence triangles AKL and CLM are congruent, so that $AL=CM$ and $\angle LAK=\angle MCL$. It follows that KB is parallel to MC, and $KB=AB-AK=AC-CL=AL=MC$. Hence $BCMK$ is a parallelogram, so that $KL=KM=BC$.



6. ABC is an arbitrary triangle. X is a point on the same side of the line BC as A. Y is a point on the side of the line CA opposite to B. Z is a point on the side of the line AB opposite to C. The triangles XBC, YAC and ZBA are similar, with X, B and C corresponding to Y, A and C, and to Z, B and A respectively. Prove that $AYXZ$ is a parallelogram.

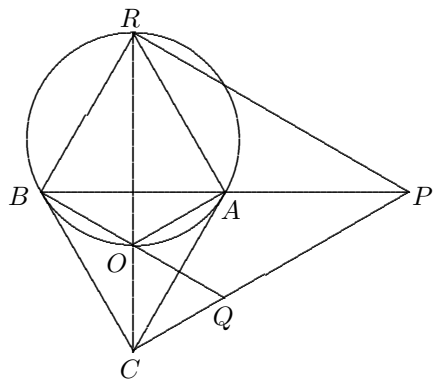
Solution:
Since triangle ZBA is similar to triangle XBC, $\frac{ZB}{XB} = \frac{AB}{CB}$ and $\angle ZBA = \angle XBC$. Hence $\angle ZBX = \angle ZBA + \angle ABX = \angle XBC + \angle ABX = \angle ABC$, so that triangle ZBX is similar to triangle ABC. Similarly, triangle YXC is also similar to triangle ABC. It follows that $\frac{XZ}{ZB} = \frac{AC}{AB} = \frac{CY}{XY}$. On the other hand, triangle ZBA is similar to triangle YAC. Hence $\frac{YA}{ZB} = \frac{AC}{AB} = \frac{CY}{ZA}$. It follows that $XZ = YA$ and $XY = ZA$, so that $AYXZ$ is a parallelogram.



7. ABC is an equilateral triangle with centre O. A line through C intersects the circumcircle of triangle BOA at points D and E. Prove that A, O and the midpoints of BD and BE are concyclic.

Solution:
Extend BA to P so that $BA = BP$. Extend BO to cut CP

at Q. Extend CO to cut the circumcircle of BOA again at R. Since $\angle AOB = 120°$, $\angle ARB = 60°$. Since OR is a diameter perpendicular to AB, we have $RA = RB$, so that BAR is an equilateral triangle. From $AR = AB = AC = AP$, $\angle BCP = 90° = \angle CRP$. Hence $\angle RCP = 60° = \angle CRP$, so that CPR is also an equilateral triangle. Hence $\angle OCQ = 60°$, and since $\angle COQ = \angle BOR = 60°$, COQ is another equilateral triangle. It follows that $OQ = OC = OB$ and $CQ \cdot CP = CO \cdot CR$. Moreover, $CO \cdot CR = CD \cdot CE$ since OR and DE are both chords of the circumcircle of BOA. Hence $CQ \cdot CP = CD \cdot CE$. This means that P, Q, D and E all lie on the same circle. When this circle is contracted towards B by a factor of $\frac{1}{2}$, the smaller circle passes through A, O, the midpoint of BD and the midpoint of BE.



8. Peter and Betty are playing a game with 10 stones in each of 11 piles. Peter moves first, and turns alternate thereafter. In his turn, Peter must take 1, 2 or 3 stones from any one pile. In her turn, Betty must take one stone from 1, 2 or 3 piles. Whoever takes the last stone overall is the winner. Which player has a winning strategy?

Solution by Yunhao Fu:
Draw an 11×11 grid, with each row representing a pile. Place stones on (i, j) if and only if $i \neq j$. Then there are 10 stones in each of the 11 rows, none lying on the main diagonal where $i = j$. The entire configuration is symmetric about this main diagonal.

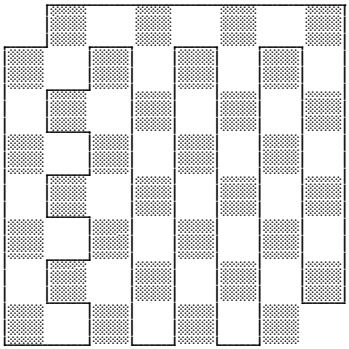
Betty's strategy is to take (j, i) whenever Peter takes (i, j). By symmetry, Betty gets the last stone and wins.

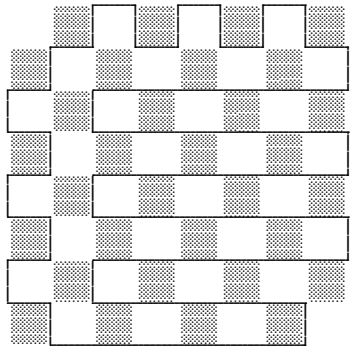
9. A spaceship lands on an asteroid, which is known to be either a sphere or a cube. The spaceship sends out an explorer which crawls on the surface of the asteroid. The explorer continuously transmits its current position back to the the spaceship, until it reaches the point which is symmetric to the landing site relative to the centre of the asteroid. Thus the spaceship can trace the path along which the explorer is moving. Can it happen that these data are not sufficient for the spaceship to determine whether the asteroid is a sphere or a cube?

Solution:
Suppose a sphere intersects each of the six faces of a cube in its incircle. Then these six circles are hinged to one another at the midpoints of the edges of the cube. If the spaceship lands at some point on one of these circles, the destination is on the circle on the opposite face. The explorer can get there by going from circle to circle. Since its path lies on the intersection of the surfaces of the cube and the sphere, the spaceship does not have enough information to determine the shape of the asteroid.

10. Penny chooses an interior point of one of the 64 cells of a standard chessboard. Basil draws a subboard consisting of one or more cells such that its boundary is a single closed polygonal line which does not intersect itself. Penny will then tell Basil whether the chosen point is inside or outside this subboard. What is the minimum number of times Basil has to do this in order to determine whether the chosen point is black or white?

Solution by Yunhao Fu:
The diagram below shows two subboards which work. If the chosen point is inside a white cell, it is either inside both subboards or outside both subboards. If the chosen point is inside a black cell, it is inside one subboard and outside the other subboard.





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Classical Geometry: Euclidean, Transformational, Inversive, and Projective

IE Leonard, JE Lewis, ACF Liu, GW Tokarsky

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Geometry has declined significantly as a component of high school mathematics syllabi over the last three decades. The reasons for this may be complex, but my feedback is that the powers that be often failed to see its relevance. It seems syllabus makers were looking at direct application, and failed to see the real use, other than its contribution to understanding space around us; that is its role in developing disciplined thinking, which is of course a generic asset for any student.

Possibly we as mathematicians were our worst enemies in publicising the subject. Those of us who loved and appreciated the subject were all too happy to accept a delivery which to many nonbelievers was an unattractive mystery. Text books were dry and there were not many of them anyway capable of exposing the subject, certainly in the English language. Of course there were some. The works by Coxeter (*Introduction to Geometry* and *Projective Geometry*) and by Coxeter with Greitzer (*Geometry Revisited*) were technical masterpieces, which took the role of references for many mathematicians and training manuals for younger students building up an armoury of skills and knowledge as preparation for Olympiads. I also include the MAA books by Yaglom on Transformations in this category.

Albeit, some of the more remote tools, such as the barycentric coordinates devised by Möbius, while generally discussed in books such as those of Coxeter, despite being revived in some quarters as valuable modern-day problem solving tools, often became lost.

It is very refreshing to see this new book on the market, which not only gives a broad coverage on Euclidean Geometry, but also extends this into other related or derivative topics. This book is primarily directed at university students, assuming they have had limited exposure to Euclidean Geometry at school and filling the void. The authors have been teaching geometry at university, and this book is based on the materials they developed.

The book does cover in its initial chapters the Euclidean topics of congruence and similarity, a general coverage of the interesting aspects of concurrence, area, the theorems of Ceva and Menelaus and a good number of miscellaneous topics such as the nine-point circle, Euler Line, polygon construction and the Circle of Apollonius.

It then moves on to transformations, with the Euclidean transformations, rotations, reflections and translations, moving on to isometries, symmetries, groups, homotheties and tessellations.

Finally topics move on to inversive and projective geometries, with chapters on reciprocation and cross ratios.

The book is well laid-out, written in a useful explanatory style. Topics challenging in other books are accessible here. A student who can master this book will be very well equipped indeed, not just for the knowledge in the book, but to be able self-explore deeper into the subject.

This book is designed as a text for a university course, however this is also a must for all mathematicians who train students for Olympiads, not only for their own reference, but also to help students in their training. If one needs only a small number of geometry books for reference this should at least be one of them.

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