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MATHEMATICS COMPETITIONS

JOURNAL OF THE

WORLD FEDERATION OF NATIONAL MATHEMATICS COMPETITIONS

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Please send articles to:

The Editor
Mathematics Competitions
World Federation of National Mathematics Competitions
University of Canberra Locked Bag 1
Canberra GPO ACT 2601
Australia

Fax:+61-2-6201-5052

or

Dr Jaroslav Švrček
Dept. of Algebra and Geometry
Faculty of Science
Palacký University
Tr. 17. Listopadu 12
Olomouc
772 02
Czech Republic

Email: svrcek@inf.upol.cz

MATHEMATICS COMPETITIONS VOLUME 24 NUMBER 2 2011

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World Federation of National Mathematics Competitions

Executive

President: Professor María Falk de Losada
Universidad Antonio Narino
Carrera 55 # 45-45
Bogotá
COLOMBIA

Senior Vice President: Professor Alexander Soifer
University of Colorado
College of Visual Arts and Sciences
P.O. Box 7150 Colorado Springs
CO 80933-7150
USA

Vice Presidents: Dr. Robert Geretschläger
BRG Kepler
Keplerstrasse 1
8020 Graz
AUSTRIA

Professor Ali Rejali
Isfahan University of Technology
8415683111 Isfahan
IRAN

Publications Officer: Dr Jaroslav Švrček
Dept. of Algebra and Geometry
Palacký University, Olomouc
CZECH REPUBLIC

North America: Professor Harold Reiter
Department of Mathematics
University of North Carolina at Charlotte
9201 University City Blvd.
Charlotte, NC 28223-0001
USA

Oceania: Professor Derek Holton
Department of Mathematics and Statistics
University of Otago
PO Box 56
Dunedin
NEW ZEALAND

South America: Professor Patricia Fauring
Department of Mathematics
Buenos Aires University
Buenos Aires
ARGENTINA

The aims of the Federation are:–

1. ***to promote excellence in, and research associated with, mathematics education through the use of school mathematics competitions;***
2. ***to promote meetings and conferences where persons interested in mathematics contests can exchange and develop ideas for use in their countries;***
3. ***to provide opportunities for the exchanging of information for mathematics education through published material, notably through the Journal of the Federation;***
4. ***to recognize through the WFNMC Awards system persons who have made notable contributions to mathematics education through mathematical challenge around the world;***
5. ***to organize assistance provided by countries with developed systems for competitions for countries attempting to develop competitions;***
6. ***to promote mathematics and to encourage young mathematicians.***

From the President

Once again I have been given the delightful opportunity of putting before our readers an enticing menu of writings, activities and events to refresh the spirit and set the mind to action.

The Journal

The present issue of the WFNMC journal once again offers a prime choice of articles that reflect a great variety of problems, competitions, puzzles, and other activities designed to challenge and entertain us. We invite our readers to peruse the issue thoroughly and take full advantage of all it has to offer.

Miniconference

WFNMC has organized a miniconference for July 7, 2012, at COEX, Seoul, Korea, the same venue that will host ICME-12 from July 8–15. We are looking forward to contributions from our members and readers. In particular, we hope to provide a star-studded group of speakers who will enlighten and engage all attendees with talks on new competitions, problem creation, development of students' and teachers' mathematical thinking through challenging mathematics, and new resources in challenging mathematics for students and teachers.

Please consult the third announcement and call for papers in this issue of the journal, and share your experience and analyses with other members of the Federation.

ICME-12

ICME-12 has recognized challenging mathematics as a field of research and action in mathematics education; the activity of ICME-12 pertaining to challenge will be centered on the sessions of Topic Study Group 34: *The role of mathematical competitions and other challenging contexts in the teaching and learning of mathematics.*

WFNMC will have two ninety-minute sessions at ICME-12 during which there will be time for lively academic sessions as well as a business meeting. Papers submitted for the miniconference will also be considered for presentation at the academic session of the Federation during ICME-12.

Our students

For all of us who work with competitions, museums, exhibitions, workshops, summer schools, fairs and the like, challenging elementary mathematics is elegant and engaging, and stepping up to meet these challenges is a totally satisfying experience. Yet for most of us, the focus is on our students, on letting them feel the power and see the beauty, on giving them a very special experience. Invited by Romas Kasuba to address the participants in the Lithuanian mathematics competitions last year, I tried to put into written words what I think we would all like to say to these fantastic young students. This is what resulted.

"It would be of great interest to me to know what thoughts are running through your minds at this time.

There are so many reasons for taking part in problem-solving competitions in mathematics, and so many different benefits to be expected. I will name only a few; and I invite you to share your own thoughts with me, as well as with your families, teachers, and friends.

There is the element of fun. Challenges, among them those found in the problems of math competitions, almost always are taken up because they have an element of fun in them. Fun can lead to engagement, and develop into a passionate calling.

There is the element of surprise. Surprises make life interesting, jogging our routines; persons, places, facts, problems that we already know can be precious to us, but surprises make us adjust our attitudes and actions in new ways that enrich our lives.

There is the element of beauty, a beautiful problem created by a beautiful mind, is more than satisfying, it is joyful and uplifting. And so is a beautiful solution such as those many of you will create and that will fill your teachers with pride.

There is the element of learning, learning about mathematics, about what our mathematical mentors expect of us, about what we are capable of doing, about adjusting our own expectations.

There is the element of belonging, belonging to a tradition of many millennia, a human endeavor, a community of thinkers, creators of knowledge and understanding.

There is the element of truth. When you are facing a problem there is no way to bluff, to fool yourself or others, you are alone, constantly assessing yourself and your own progress, you have a real opportunity to take Shakespeare's wonderful advice: 'This above all: to thine ownself be true, and it must follow as the night the day, thou canst not then be false to any man.'

I am sure you all have your eyes on the future, your future, the future of your country, friends and family. You have the power to open up new possibilities by enjoying all of these fine elements that taking part in mathematics competitions and solving challenging problems have in store for you. My best wishes for all, wishes that all of you will find in this singular experience everything that you are looking for and more!"

María Falk de Losada
President of WFNMC
Bogotá, December 2011

From the Editor

Welcome to *Mathematics Competitions* Vol. 24, No. 2.

First of all I would like to thank again the Australian Mathematics Trust for continued support, without which each issue (note the new cover) of the journal could not be published, and in particular Heather Sommariva, Bernadette Webster and Pavel Calábek for their assistance in the preparation of this issue.

Submission of articles:
The journal *Mathematics Competitions* is interested in receiving articles dealing with mathematics competitions, not only at national and international level, but also at regional and primary school level. There are many readers in different countries interested in these different levels of competitions.

- The journal traditionally contains many different kinds of articles, including reports, analyses of competition problems and the presentation of interesting mathematics arising from competition problems. Potential authors are encouraged to submit articles of all kinds.
- To maintain and improve the quality of the journal and its usefulness to those involved in mathematics competitions, all articles are subject to review and comment by one or more competent referees. The precise criteria used will depend on the type of article, but can be summarised by saying that an article accepted must be correct and appropriate, the content accurate and interesting, and, where the focus is mathematical, the mathematics fresh and well presented. This editorial and refereeing process is designed to help improve those articles which deserve to be published.

At the outset, the most important thing is that if you have anything to contribute on any aspect of mathematics competitions at any level, local, regional or national, we would welcome your contribution.

Articles should be submitted in English, with a black and white photograph and a short profile of the author. Alternatively, the article can be submitted on an IBM PC compatible disk or a Macintosh disk. We prefere LaTeX or TeX format of contributions, but any text file will be helpful.

Articles, and correspondence, can also be forwarded to the editor by mail to

The Editor, *Mathematics Competitions*
Australian Mathematics Trust
University of Canberra Locked Bag 1
Canberra GPO ACT 2601
AUSTRALIA

or to

Dr Jaroslav Švrček
Dept. of Algebra and Geometry
Palacky University of Olomouc
17. listopadu 1192/12
771 46 OLOMOUC
CZECH REPUBLIC

jaroslav.svrcek@upol.cz

Jaroslav Švrček
December 2011

The Passing of a Maestro

Another colleague has left us ...

Luis Davidson San Juan, 1921–2011

Having had the privilege of counting him among our friends and collaborators, it is our sad duty to inform our readers of the passing in the month of November in Havanna (La Habana) of Luis Davidson San Juan.

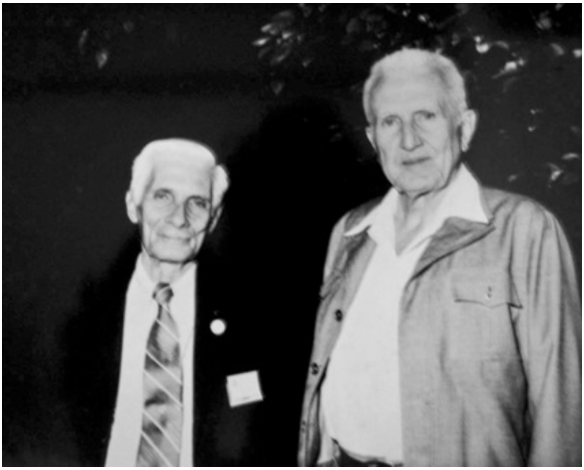
Luis was born in Havanna, Cuba, on September 10, 1921. At the Universidad de La Habana he studied Physical and Mathematical Sciences, and in 1944 obtained his PhD. In 1950 he was part of the Cuban delegation at the International Congress of Mathematicians organized at Harvard University, beginning his work on the international scene.

Beginning in 1963 Luis Davidson was among the organizers of mathematics competitions in Cuba; he first participated in the IMO as leader

With then Australian Prime Minister Robert Hawke,
Peter O'Halloran and others at the Closing Ceremony
of the International Mathematical Olympiad, Australia, July, 1988.

of the Cuban delegation in 1971. Luis was a member of the IMO Site Committee and designated its Vice President in 1988.

The OEI (Organización de Estados Iberoamericanos para la Ciencia, la Culture y la Educación) awarded Luis the distinction of Maestro Founder of Mathematics Competitions in Ibero-America.



With José Fernández, the Cuban Minister of Educación,
at the IV Iberoamerican Mathematics Olympiad, held in Cuba in 1989.

Luis was present at the First Congress of WFNMC in 1990, in Waterloo, and in 1992 at ICME-7 in Quebec, he received the Paul Erdös Prize from WFNMC.

Luis Davidson San Juan was the author of many books. His last, published in 2010, *Equations and mathematicians*, is part of a series he had planned to write and tells the story of mathematicians through the problems they posed and solved.

He will be missed.

María Falk de Losada
Bogotá
COLOMBIA

Announcements

Third Announcement and Call for Papers

WFNMC miniconference

In conjunction with ICME12 which will be held in Seoul, Korea, from 8–15 July 2012, WFNMC will hold a miniconference on July 7 at the same venue, COEX Seoul.

Call for Papers

The deadline for the reception of papers is February 8, 2012. These should be submitted in full, be of no more than 12 pages in length (single space, Times New Roman, 12pt) and be accompanied by an abstract of no more than 40 words.

Of special interest are papers dealing with the following topics:

- problem creation;
- the development of students' ability to think mathematically;
- the development of teachers' ability to think mathematically and creatively and of their capacity to create and/or use challenging problems in their teaching; and
- new resources for the student and the teacher (journals, books, webpages, etc.).

Results of the review process will be announced by 1 March 2012.

Second Announcement

VII WFNMC Congress

The VII Congress of WFNMC will take place in Beijing in 2014. Please make sure to put the Congress on your schedule and make Beijing your destination for the summer of 2014.

Further information and a call for papers will appear in subsequent issues of the WFNMC Journal.

Third Announcement

Topic Study Group 34, ICME 12, Seoul, Korea

WFNMC Vice President, Ali Rejali, and President, María de Losada, have been named co-chairs of Topic Study Group 34: *The role of mathematical competitions and other challenging contexts in the teaching and learning of mathematics at ICME12.*

The group will address the following areas of research and action:

- the development of strategies and tools for incorporating challenging mathematics in schools and regular classes and the way in which these are given direction by teachers, text books, educational systems and curricula, on extracurricular activities in schools (or in other contexts and aspects not listed here) as well as on research focusing on their impact;
- challenge developed for implementation beyond the classroom such as journals, books, competitions at different levels—including the different regional and international Olympiads—exhibitions, mathematics clubs, mathematics houses, lectures, camps, corresponding programs, mathematics days, fairs, family programs, as well as research regarding the above items and similar tools;
- preparation programs for competitions and different types of recognition given to outstanding students and teachers;
- innovative competitions or other innovative mathematical challenges throughout the world;
- new activities and research on programs that provide mathematical challenge, with students as their main focus;
- analysis of the ways in which competitions and other challenging activities contribute to the motivation of students towards the study of mathematics;
- analysis of relationships between technological environments and challenging tasks, and of the impact of challenges focusing on students, classroom practice, assessment, as well as research addressing such analysis;

- analysis of ways of providing an enveloping challenging atmosphere for students to learn mathematical subjects and to be engaged in mathematical studies;
- follow up research on the discussions which took place at previous ICME's, especially DG 19 at ICME 11.

(http://dg.icme11.org/tsg/show/20).

Make ICME 12 and TSG 34 part of your summer 2012 agenda!

The Italian Team Competition

Giuseppe Rosolini

Giuseppe Rosolini obtained his doctorate in Mathematics at the University of Oxford in 1986 with a thesis on categorical models of computation, and holds the chair of mathematical logic at the University of Genova. His main research interests are in logic, proof theory, and category theory.

He is past president of AILA, the Italian Association of Logic. As a member of the Italian Olympic Committee of UMI, the Italian Mathematics Union, he has been in charge of the organization of the Italian Team Competition since its inception.

1 Introduction

It is the week after Labour Day. Cesenatico, one among the many beautiful spots along the Italian Adriatic Coast, is bustling with high school students talking about mathematics. They have just come from all over the country to compete in the final round of the Italian Mathematical Olympiad and in that of the Italian Team Competition.

The Italian Mathematical Olympiad is the penultimate stage in the series of individual competitions organized by the Italian Olympic Committee to determine the six Italian representatives to the International Mathematical Olympiad (IMO). Some 300 competitors, after passing the local selections, will take part in it. It will run in the morning on Friday, the results will be announced on Sunday.

The Italian Team Competition is the final stage of another mathematical contest which involves high school teams. It consists of two rounds:

semifinals are on Friday afternoon, and on Saturday there is the final round in the morning, and the award ceremony in the afternoon. Some 700 students are involved in this: those on the teams, many of whom also take part in the individual contest, and their supporters.

It was some twelve years ago when Massimo Gobbino, deputy leader of the Italian team at the IMO, had the idea of organizing a leisure competition for teams of seven individual contestants trying to keep them busy in the two days' long wait for the results of the individual competition. And two years later, together with Alberto Arosio, another member of the Italian Olympic Committee, who thought of adding a twist to it: pride and enthusiasm. Instead of grouping individual contestants at Cesenatico, they invited high schools to perform in the team competition with a team of their students.

The idea caught on: after three years, there were some 100 schools who took part in the local contests to gain access to the final round of the Team Competition in Cesenatico.

In 2011, more than 700 Italian high schools took part in one of the 30 local contests, held in mid-March throughout the country. At each school the students interested to make the team train together in winter with the help of teachers and trainers; in many schools it is a group of more than 20 students.

And the teams from 96 high schools are present in Cesenatico to compete in one of the three semifinals on Friday afternoon. In each semifinal, 32 teams will compete for 10 places in the final.

2 The team competition

The team competition (*Gara di Matematica a Squadre*, in Italian) takes place in a indoor stadium, in front of spectators and fans who can—and usually do—cheer for their team.

The seven students on a team sit together at an assigned table. They can collaborate, using only pen and paper, to solve 24 problems. Each problem requires an integral answer between 0 and 9999—more precisely, a sequence of four digits. The jury will not want to know by which means

Figure 1: The stadium, the day before

the answer was obtained: they shall simply verify if it is correct. If that is so, the team gains points; otherwise, they will lose points, but they can continue to work on the problem to find the correct answer. After 120 minutes from the beginning of the contest, the team with the top score wins.

Physical contact between teams is kept to the minimum: only a specified member of a team, the messenger, can deliver the team's answers to the jury, all others cannot move away from the team's table. But among the messengers, anything goes.

The scores are computed and updated in real time from the beginning to the end, and are communicated to the messengers by computer displays—as well as to those on the stands by computer projections—so that everybody at every moment knows which team is ahead and how far behind the others are trailing.

A team need not solve the 24 problems in sequence; an answer to any of the 24 problems can be given at any moment during the competition.

Figure 2: A messenger delivers an answer to the jury, others check results on the computer displays

The problems are of various levels of difficulty and, of course, of different kinds, but the score for a correct answer is the same for each problem at the beginning of the contest: 20 points. At each tick of a minute, the score of an unanswered problem increases by 1 point. And an incorrect answer will push up the score of an unanswered problem by 2 points. Other computer displays inform the teams (but not the public) about the updated scores of the 24 problems and about each team's answering record.

Each team has a captain, usually the best in mathematics on the team. A captain should decide which teammates tackle which problems in the list. But, after a few minutes into the game, it is the messenger who really develops the team strategy because he/she is the only member of the team who can figure out, from the displays, which problems the other teams tackle and how well they perform.

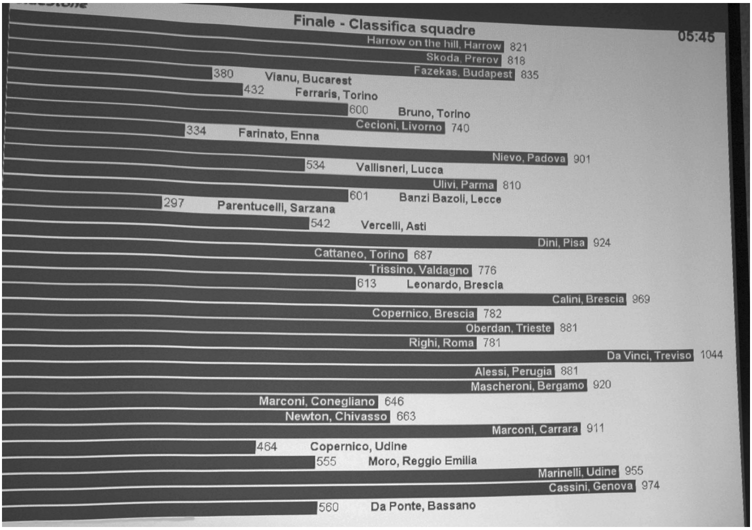


Figure 3: A computer projection

3 The final: some problems, some results

On Saturday, at 9 o'clock the final for the Team Competition begins. There are four guest teams from the Czech Republic, Hungary, Romania, and the United Kingdom.[1]

At the start of the competition, each messenger runs to the team's table with seven copies of 24 problems. To ensure that all answers are considered the following conditions are stated at the top of the text:

- if the answer is not an integer, mark the greatest integer less than it

[1] The Czech team lead by Prof. Jaroslav Švrček is from Jakub Škoda Gymnázium, Přerov; the Hungarian team lead by Prof. Janos Pataki is from Fazekas Mihály Gimnázium; the Romanian one, lead by Professor Monica Dumitrache is from Colegiul National de Informatica Tudor Vianu; and the team from the UK is from Harrow School, lead by Prof. James Hanson.

- if the answer is a negative integer, or if the problem has no solution, mark `0000`
- if the answer is an integer greater than `9999`, mark the last four digits.

The 24 problems were all set with reference to the comics saga of Asterix and Obelix. They were divided in two halves: one entitled **An easy trip to Britain**, the other **A less easy mission to Egypt**, to suggest to the competitors which problems would present more difficulties.

A solution to a typical problem would not only require abstract reasoning, but also some lengthy calculations.

Most of the problems for the Team Competition are prepared by the young mathematicians who collaborate with the Italian Olympic Committee—many appear in fig. 4 in their costumes—under the supervision of Francesco Morandin, Federico Poloni and Marco Romito of the Italian Olympic Committee. In the following we list two problems from each part.

Figure 4: All invigilators wore costumes in tone with the problems: there were Asterix, Obelix, Gauls, Romans, Egyptians, pirates,. . .

The coins of the Britons: On the boat headed for Britain, Abelix asks about the kind of money used there, and Cantorax answers, "It's very simple! We have iron bars worth 3 and a half sestertii plus 4 zinc pieces. A zinc piece is worth 1 and a half copper pieces. To make a sestertium you need 12 bronze pieces or, alternatively, 6 and a half copper pieces." If Abelix already has 18 bronze pieces, how many copper pieces will he need to reach the value of one iron bar?

Answer: `0019`

Math-use-lemmix's birthday: Little Etothex's seventh birthday is on the same day as the birthday of old Math-use-lemmix, the most envied man in the village (because of his wife). When the child asks how old Math-use-lemmix is, the old man replies, "My age is a positive integer n less than 300 whose digits either do not divide n or are zero, such that, if you subtract your age, it becomes a number whose only prime divisors are 7 and the first digit of n". What is old Math-use-lemmix's age?

Answer: `0203`

Figure 5: Of course, there were Caesar and Cleopatra

Chaos at the building site: The construction of Cleopatra's monument has been speeding up, and now there are 11808 workers, among whom, unfortunately, there may be many Roman spies. When Abelix asks each worker to tell him what he knows, he receives the following answers, in order:

"There is at least 1 Egyptian among us."
"There are at least 2 Egyptians among us."
"There are less than 3 Egyptians among us."
"The workers at the construction site are at least 4."
"The workers at the construction site are at least 5."
"The workers at the construction site are less than 6."

continuing with the seventh worker answering as the first, the eighth as the second and so on, the n-th worker talks about n Egyptians or n workers. Clearly, the Egyptian workers will always tell the truth, while the Roman spies will always lie. How many spies are there among the workers? (None of the true Egyptians is a spy.)

Answer: `4728`

The pyramid of Cleopatra: Cleopatra has commissioned another pyramid, this time with a square base. Its lateral sides are equilateral triangles whose sides measure 100 cleopaces (a unit of measurement determined by the length of Cleopatra's legs). The queen demanded that the architects surround the pyramid, situated in a perfectly flat desert, with a wall so that, from every point on the internal wall, Cleopatra can reach the summit of the pyramid in exactly 200 (cleo)paces, walking on the desert and the surface of the pyramid. What will the perimeter of the wall be, in cleopaces?

Answer: `1047`

Without doubt, collaboration on the team has proved to be the most important strategy to attack the problems: the winner of the 2011 Italian *Gara di Matematica a Squadre* is the team of Liceo Scientifico Leonardo Da Vinci from Treviso. The teams from Liceo Scientifico Annibale Calini, Brescia, and Liceo Scientifico Gian Domenico Cassini, Genova, complete the podium.

Certainly, the two gold medalists and the bronze medalist (in the individual competition) on the winning team helped to beat the others, but it is important to note that the other two teams on the podium collectively enlisted just three bronze medalists. And there were many gold and silver medalists on teams which trailed behind those two.

In the last few years, together with the growing success of the Team Competition, which presses students to join forces and to collaborate to solve problems on a wide spectrum of mathematical subjects, the average number of high school students that train constantly on olympic mathematics has increased. The comments from the teachers suggest that the two facts are related.

As for the individual competition, the winner was Federico Borghese from Liceo Scientifico Farnesina in Rome with 31 points. Roberto Dvornicich, head of the Italian Olympic Committee, had the following comments on the competition and, in particular, on the results:

> "The final round of the Italian Mathematical Olympiad has become harder and harder in recent years, as the students are far better prepared than years ago. This year, some very difficult problems in the exam paper produced a rather low average score so that already the score of 40 was to be considered as absolutely outstanding. The guest competitors from the Czech Republic, Hungary, and Romania, produced some excellent solutions collecting gold and silver medals. In my experience, the best in such a competition have very good chances to become excellent professional mathematicians."

Giuseppe Rosolini
Dipartimento di Matematica
Università degli Studi di Genova
ITALY

The Iberoamerican Mathematics Competition for University Students

Maria Losada

Maria E. Losada participated as a young teenager in the Colombian National Mathematics Olympiad and has been involved in the project on and off ever since she obtained an undergraduate and a doctorate degree in Mathematics, the latter in the area of Set Theory. For the last six years she has been the director of the Colombian Mathematics Olympiad Project. She has been a team leader, deputy leader and coordinator at various International Mathematical Olympiads and created the Regional Math Olympiads in Colombia in 2006 and the Iberoamerican Interuniversity Mathematics Competition (CIIM) in 2009.

This year during the first week of October, 48 university students from 11 Latin American university teams and four national olympiad teams from 7 countries met in Quito, Ecuador, to compete in the Third Iberoamerican Mathematics Competition for University Students (CIIM). The competition was founded in Colombia in 2009 as an effort to improve the competitive level of university students of the region and in particular to increase the participation in the International Mathematics Competition for university students (IMC). Although the CIIM was first organized by the Colombian Math Olympiad subsequent hosts have been the IMPA and Military Engineering Institute in Rio de Janeiro, Brazil, under the leadership of Carlos Tamm de Araujo and Paulo Maranhão, and the San Francisco University of Quito in Ecuador directed by Eduardo Alba. Considering the fact that the IMPA has offered summer scholarships for the medalists of the competition, and that the CIMAT in Guanajuato, Mexico, has put in a bid for organizing it next year, there are a good indications that the CIIM is on its way to becoming a well-established university competition in Latin America. Complementing the Iberoamerican Mathematics Olympiad for University students (OIMU) also begun in Colombia, the CIIM offers the opportunity for universities to get more involved in the preparation of their students hopefully, increasing the overall mathematics level in the region.

1 Short History

The first CIIM was organized in Colombia (http://oc.uan.edu.co/ciim/) in the year 2009 with a participation of 55 students, of 15 teams from 12 universities and two national olympiad teams from 6 Latin American countries: Brazil, Colombia, Costa Rica, Ecuador, Mexico and Venezuela. There were two four and a half hour exams held on two consecutive days, each consisting of three problems worth 10 points each. Medals were given to two thirds of the students without honorable mentions in an effort to boost the participation in the contest. For this same reason the special gold medal was awarded to the two most outstanding students.

In 2010 the CIIM was organized in Brazil with 56 students from 15 universities and 2 national olympiad teams from 5 countries: Brazil, Colombia, Ecuador, Mexico and Venezuela. The exam format was continued and medals were given more selectively to 28 students while 5 honorable mentions were given to students that did not obtain a medal but solved a problem completely. Two special gold medals were once again awarded. The IMPA also gave summer school scholarships to all medalists, a gesture they continued in the third version of the CIIM.

2 Structure of the CIIM

In contrast to the OIMU, which is by correspondence, the CIIM emulates the IMC by offering venues for the university teams to meet and exchange experiences. Although the proposed participants are mainly university teams, individual participants and national teams are also welcome. Teams generally consist of four students and a professor who is their team leader. Professors form a jury to choose the problems for the exam from a short list provided in principle from original problems contributed by the participating teams.

To ensure international standards the CIIM began the competition with special invited guests Géza Kós and Jószef Pelikán from Hungary, both well-known experts in competitions. The former is also a main figure in problem selection at the IMC, and his contribution allows the CIIM to strive to work with original problems of a good international level; for this reason he has been one of the special invited guests of the CIIM every year. Another special guest is Carlos Tamm de Araujo, known to his acquaintances as Gugu who, together with Alexander Fomin in Colombia has become a strong fundamental force in the university competition scene in Ibero-America. When creating the CIIM the author proposed a union with these problem-creating specialists with the goal of creating a team of support for the university competitions that would allow both the CIIM and the OIMU to be organized in different places throughout Ibero-America. This is a relatively novel idea on the competition scene, although it is a bit similar to the reality of other competitions throughout the world. After all, experts like Géza Kós, Gugu and Alexander Fomin are not easy to come across.

There is no fixed format for the exams at the CIIM, but to date (see below) they have consisted of two exams of three problems each in a four and half hour period. Each problem is worth ten points.

As is the case in the IMC, the solutions of the exams are kept anonymous and are read and graded by all the members of the jury first. Once marks are proposed for all the papers, they are handed over to the team leaders for coordination. If leaders agree with the proposed marks they sign an acceptance form and coordination is not needed. Any discrepancy between the proposed marks from the jury and those of the leader is handled first by those who read the papers anonymously, by discussing the mathematical elements of the student's solution in what constitutes coordination. Any further disagreement is then handled by the jury or its representative.

Similar to other competitions, half of the participants receive a gold, silver or bronze medal with a gold:silver:bronze ratio of 1:2:3. Students that do not receive a medal and solve a problem with complete score receive an honorable mention.

The CIIM also follows the main logistical component of the IMC in that the hosts choose and contract accommodation, meal venues and local transportation and charge the students a fixed amount to cover most of these costs. Additional costs such as the costs of materials, exam sites or travel for special guests are assumed by the host institution. Special guests include the problem specialists and also representatives of the next host institution.

3 III CIIM

The teams that participated in the III CIIM organized by the San Francisco University of Quito, Ecuador, were from the following seven countries: Brazil, Colombia, Costa Rica, Ecuador, Guatemala, Mexico and Peru. Brazil, Ecuador, Guatemala and Mexico brought national olympiad teams. The other teams were from the Instituto Militar de Ingeniería in Rio de Janeiro (Brazil), the Universidad Industrial de Santander in Bucaramanga (Colombia), the Universidad Antonio Nariño and the Universidad de los Andes in Bogotá (Colombia), the Universidad del Valle del Cauca in Cali (Colombia), the Universidad de Costa Rica in San José, the Escuela Politécnica Nacional in Guayaquil (Ecuador), the host institution, the Universidad San Francisco de Quito (Ecuador), the Universidad Nacional Autónoma de México, the Universidad Nacional de Ingeniería in Lima (Perú) and the Pontificia Universidad Católica de Perú.

This year's problem set or short list had a fairly good contribution of proposed problems from both the expert group and the participating universities. The resulting exam was very accessible to the students and was also a good instrument for differentiating them. One student from Peru, Daniel Chen Soncco from the Universidad Nacional de Ingeniería, obtained a perfect score. The four other gold medalists, three from the Brazilian Olympiad team and the last from the Universidad de Costa Rica, solved five of the six problems and managed good headway on the sixth problem.

4 Problems

The problem set follows below.

First Day. Tuesday October 4th, 2011

1. (Universidad de los Andes) Find all real numbers a for which there exist distinct real numbers b, c, d different from a such that the four tangents to the curve $y = \sin(x)$ at the points $(a, \sin(a))$, $(b, \sin(b))$, $(c, \sin(c))$ and $(d, \sin(d))$ form a rectangle.

2. (Brazilian Mathematics Olympiad) Let k be a positive integer and let a be an integer such that $a-2$ is a multiple of 7 and a^6-1 is a multiple of 7^k. Show that $(a+1)^6-1$ is also a multiple of 7^k.

3. (Géza Kós) Let $f(x)$ be a rational function with complex coefficients whose denominator does not have multiple roots. Let $u_0, u_1, \ldots, u_n$ be the complex roots of f and $w_1, w_2, \ldots, w_m$ be the roots of f'. (Each root is considered as many times as its multiplicity). Suppose that u_0 is a single root of f. Show that

$$\sum_{k=1}^{m} \frac{1}{w_k - u_0} = 2\sum_{k=1}^{n} \frac{1}{u_k - u_0} .$$

Note: A rational function is the quotient of two polynomials.

Second Day. Wednesday October 5th, 2011

4. (Universidad de San Francisco de Quito)

Define $(b_0, b_1, \ldots, b_{n-1}) = (1, 1, 1, 0, \ldots, 0)$ for $n \geq 3$. Let $C_n = (c_{i,j})_{n\times n}$ be the matrix defined by $c_{i,j} = b_{(j-i) \bmod n}$. Show that $\det(C_n) = 3$ if n is not a multiple of 3 and $\det(C_n) = 0$ if n is a multiple of 3.

Note: $m \bmod n$ is the remainder of the division of m by n.

5. (Universidad Nacional Autónoma de México) Let n be a positive integer with d nonzero digits. For $k = 0, \ldots, d-1$, define n_k as the number that is obtained by moving the last k digits of n to the front. For example, if $n = 2184$ then $n_0 = 2184$, $n_1 = 4218$, $n_2 = 8421$ and $n_3 = 1842$. For m a positive integer, define $s_m(n)$ as the number of values k such that n_k is a multiple of m. Finally,

define a_d as the number of integers n with d nonzero digits for which $s_2(n) + s_3(n) + s_5(n) = 2d$. Find

$$\lim_{d\to\infty} \frac{a_d}{5^d} .$$

6. (Brazilian Mathematics Olympiad) Let Γ be the branch of the hyperbola $x^2 - y^2 = 1$ where $x > 0$. Let $P_0, P_1, \ldots, P_n, \ldots$ be distinct points of Γ with $P_0 = (1; 0)$ and $P_1 = \left(\frac{13}{12}, \frac{5}{12}\right)$. Let t_i be the tangent line to Γ at P_i. Suppose that for all $i \geq 0$ the area of the region delimited by t_i, t_{i+1} and Γ is a constant that does not depend on i. Find the coordinates of the points P_i in terms of i.

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Maria Elizabeth Losada
Olimpiadas Colombianas de Matemáticas
Universidad Antonio Nariño
Cra 38 No. 58A-77
Bogotá
COLOMBIA
director.olimpiadas@uan.edu.co

A New Theorem on any Right-angled Cevian Triangle

G. W. Indika Shameera Amarasinghe

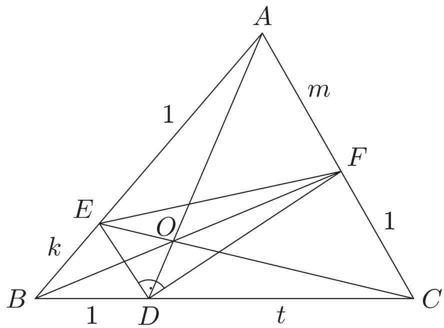
G. W. Indika Shameera Amarasinghe is still an undergraduate student(24 years) studying Mathematics mainly at the Faculty of Science, University of Kelaniya, Sri Lanka. He has already published more than 15 mathematics research papers with short articles in peer-reviewed international & local mathematics & science journals such as Math Horizons *and* Mathematical Spectrum.

1 Introduction

In this paper an exact formula is revealed by the author considering the certain correlations existing between the bases of the main triangle and the inscribed right-angled Cevian triangle, to some significant extent indicating a new generalized theorem with respect to two particular angle bisectors generated by the two side lengths of any right-angled Cevian triangle.

2 Proposed Theorem

Let ABC be any triangle in which a right-angled Cevian triangle such as DEF is inscribed, such that the angle EDF is a right angle and D, E and F are any points on BC, AB and AC respectively. Consequently, under these general conditions DE and DF must only be the bisectors of angle ADB and angle ADC respectively whenever any right-angled Cevian triangle is generated.



$\frac{BD}{DC} = \frac{1}{t}$, $\frac{AE}{BE} = \frac{1}{k}$, $\frac{CF}{AF} = \frac{1}{m}$. The constants $t, k, m > 0$, $BC = a$, $AC = b$, $AB = c$, O is the Cevian center.

The exact generalized formula which is relevant to the DEF Cevian triangle presented by the author can be denoted as follows according to the above figure.

$$k = \frac{a}{\sqrt{(t+1)(b^2+tc^2)-a^2t}}$$

Subsequently, a prominent generalized corollary is divulged as an interlocutory consequence with respect to the hypotenuse of any right-angled Cevian triangle with the liason of the Cevian segments and the main Cevian which connects the right-angled vertex of the Cevian triangle and the opposite vertex of the main triangle as follows.

$$EF^2 = AD \cdot BC - AE \cdot BE - AF \cdot FC$$

A Significant Remark Although the converse of this theorem can very easily be proved using Ceva's Theorem and the angle bisector theorem in any point on BC such as D by initially considering DE and DF as the bisectors of angle ADB and angle ADC respectively, from the converse it *cannot exactly be adduced* and *deduced* that the right-angled Cevian triangle DEF with DE and DF angle bisectors, is the only right-angled Cevian triangle existing as the cevians AD, BF and CE are concurrent at O (particulary since D point is moving). Therefore via the converse it cannot be proved that DE and DF *must only be* the bisectors of angle ADB and angle ADC respectively. Hence the proposed theorem of the author cannot be exactly and utterly proved through the converse of it.

Prior to the commencement of the proof of the proposed theorem, the following lemma which exists on any triangle is given as the principle foundation of the proof of the theorem.

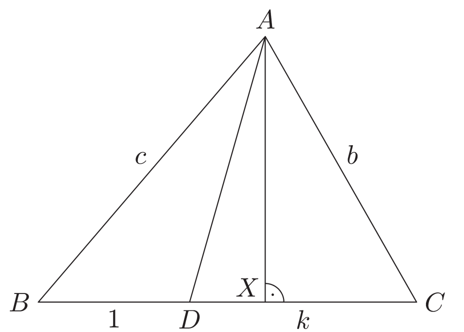
3 Lemma

Let ABC be a triangle in which any D point is located on the BC length such that $\frac{BD}{DC} = \frac{1}{k}$, $k > 0$, then

$$AD^2 = \frac{(k+1)(b^2 + kc^2) - a^2k}{(k+1)^2}.$$

k is a constant and $BC = a$, $AC = b$, $AB = c$.

4 Proof of the Lemma



$BC = a$, $AC = b$, $AB = c$. $\frac{BD}{DC} = \frac{1}{k}$ (k is a constant). Therefore $BD = \frac{a}{(k+1)}$, $DC = \frac{ka}{k+1}$. The line AX is perpendicular to BC. Angle $ADB > 90°$. Moreover $c^2 = AX^2 + BX^2 = AD^2 - DX^2 + (BD + DX)^2 = AD^2 + BD^2 + 2 \cdot BD \cdot DX$. So

$$2 \cdot BD \cdot DX = c^2 - AD^2 - BD^2 \tag{1}$$

Hence it can be easily proved that

$$2 \cdot DC \cdot DX = AD^2 + DC^2 - b^2 \tag{2}$$

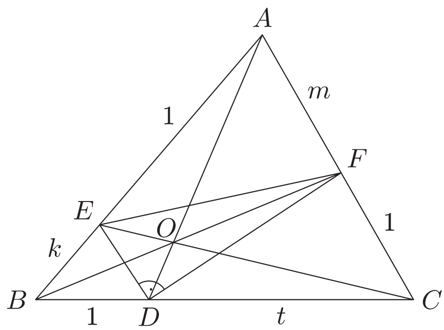
Dividing (1) by (2)

$$\frac{BD}{DC} = \frac{1}{k} = \frac{c^2 - AD^2 - BD^2}{AD^2 + DC^2 - b^2}.$$

Substituting for $BD = \frac{a}{k+1}$ and $DC = \frac{ka}{k+1}$, after easy manipulation

$$AD^2 = \frac{(k+1)(b^2 + kc^2) - a^2 k}{(k+1)^2} \tag{3}$$

5 Proof of the Theorem



$$\frac{BD}{DC} = \frac{1}{t}, \quad \frac{AE}{BE} = \frac{1}{k}, \quad \frac{CF}{AF} = \frac{1}{m}.$$

Angle $EDF = 90°$, $k, t, m > 0$, $BC = a$, $AC = b$, $AB = c$, AD, BF and CE Cevians are concurrent at O.

Ceva's theorem yields

$$\frac{AE}{BE} \cdot \frac{BD}{DC} \cdot \frac{CF}{AF} = 1,$$

hence

$$\frac{1}{k} \cdot \frac{1}{t} \cdot \frac{1}{m} = 1,$$

so

$$m = \frac{1}{kt}. \tag{4}$$

Using (3) for ABC, the length of AD is found as

$$AD^2 = \frac{(t+1)(b^2+tc^2) - a^2t}{(t+1)^2},$$

thus

$$AD = \frac{\sqrt{(t+1)(b^2+tc^2) - a^2t}}{(t+1)} \tag{4.1}$$

$$DE^2 = \frac{(k+1)(BD^2+kAD^2) - c^2k}{(k+1)^2}$$

(using (3) for ADB). Replacing for BD^2 and AD^2

$$DE^2(k+1)^2 = (k+1)\left(\frac{a^2}{(t+1)^2} + \frac{k(t+1)(b^2+tc^2) + kta^2}{(t+1)^2}\right)$$

$$DE^2 = \frac{a^2(1-kt) + k(t+1)(b^2+tc^2)}{(k+1)(t+1)^2} - \frac{c^2k}{(k+1)^2} \tag{5}$$

$$DF^2(m+1)^2 = (m+1)(AD^2+mDC^2) - b^2m$$

(using (3) for ADC). Replacing for m by (4) and replacing for DC^2 and AD^2 we obtain

$$DF^2\frac{(kt+1)^2}{k^2t^2} = \frac{kt+1}{kt}\left(\frac{(t+1)(b^2+tc^2) - a^2t}{(t+1)^2 + \frac{a^2t}{k(t+1)^2}}\right) - \frac{b^2}{kt}$$

$$DF^2 = \frac{t\left(a^2t(1-k) + k(t+1)(b^2+tc^2)\right)}{(t+1)^2(kt+1)} - \frac{b^2kt}{(kt+1)^2} \tag{6}$$

$$EC^2 = \frac{(k+1)(a^2+kb^2) - c^2k}{(k+1)^2}$$

(using (3) for ABC).

$$EF^2(m+1)^2 = (m+1)(AE^2 + mEC^2) - b^2m$$

(using (3) for AEC). Replacing m by (4) and by replacing EC^2 and AE^2

$$EF^2\frac{(kt+1)^2}{k^2t^2} = \frac{(kt+1)}{kt}\left(\frac{c^2}{(k+1)^2} + \frac{(k+1)(a^2+kb^2)-c^2k}{kt(k+1)^2}\right) - \frac{b^2}{kt}$$

$$EF^2 = \frac{c^2k(t-1)+(k+1)(a^2+kb^2)}{(kt+1)(k+1)^2} - \frac{b^2kt}{(kt+1)^2} \tag{7}$$

$$EF^2 = DE^2 + DF^2$$

(using the Pythagoras' Theorem for DEF). We can use (5), (6), (7), simplify and get

$$k = \frac{a}{\sqrt{(t+1)(b^2+tc^2)-a^2t}}.$$

Since $t > 0$ and by (4.1) $(t+1)(b^2+tc^2)-a^2t > 0$ the ratio k ($k > 0$ exists such that $k \in \mathbb{R}$).

$$\frac{BE}{AE} = k = \frac{a}{t+1}\cdot\frac{t+1}{\sqrt{(t+1)(b^2+tc^2)-a^2t}} = \frac{BD}{AD} \tag{9}$$

Therefore DE is the angle bisector of ADB.

$$\frac{AF}{FC} = m = \frac{1}{kt} = \frac{\sqrt{(t+1)(b^2+tc^2)-a^2t}}{at} \tag{9.1}$$

Since $t > 0$ and by (4.1), $(t+1)(b^2+tc^2)-a^2t > 0$ the ratio m ($m > 0$ exists such that $m \in \mathbb{R}$)

$$\frac{AF}{FC} = m = \frac{(t+1)}{at}\cdot\frac{\sqrt{(t+1)(b^2+tc^2)-a^2t}}{t+1} = \frac{AD}{DC}.$$

Therefore DF is the bisector of angle ADC. (10)

Likewise from (9) and (10) it can be successively adduced that DE and DF are the bisectors of the angle ADB and the angle ADC respectively. Likewise the bases includindg the right angle of any right-angled Cevian triangle behave as angle bisectors of the adjacent angles generated by the main Cevian(AD) which joins the right-angled vertex of the Cevian triangle and the opposite vertex of the main triangle.

6 Corollary

The length of the hypotenuse EF of any right-angled Cevian triangle DEF can be adduced as $EF^2 = AD \cdot BC - AE \cdot BE - AF \cdot FC$. The value of EF^2 has been obtained in (7) with the liaison of a, b, c, t and k.

7 Proof of the Corollary

As DE is the angle bisector of ADB, it is well known that,

$$DE^2 = AD \cdot BD - AE \cdot BE \qquad (10.1)$$

As DF is the angle bisector of ADC, it is well known that,

$$DF^2 = AD \cdot DC - AF \cdot FC \qquad (10.2)$$

$$\begin{aligned} DE^2 + DF^2 &= AD(BD + DC) - AE \cdot BE - AF \cdot FC \\ EF^2 &= AD \cdot BC - AE \cdot BE - AF \cdot FC \qquad (11) \end{aligned}$$

8 Significant Conclusions of the Theorem

(1) Readers are encouraged to consider very carefully the fact that the author has *never used trigonometry or vector algebra methods* on the proof in the above theorem and corollaries, accenting that the indepth Euclidean length calculations can be thoroughly accomplished using only advanced Euclidean plane geometry, without the assistance of trigonometry.

(2) If D is a fixed point which adduces further and if k could be obtained from a, b, and c, then the proof of this theorem would be much easier since the proof can be given using some elementary Euclidean techniques. Nevertheless within this theorem all the 3 points or edges of the right-angled Cevian triangle are *movable as freely as you wish* and that's why a more generalized and intricate proof has been given for the theorem.

(3) Note that since k, m can be obtained with the liason of a, b, c and t using (8) and (9.1) for all $t \in \mathbb{R}$, $t > 0$, any right-angled Cevian triangle

can be constructed or inscribed on any arbitary point of BC; similarly on AB and AC as well.

(4) If either E or F is a fixed point intending that if k or m is a constant ratio, then by using (8) and (9.1) it can be easily adduced that t must also be a constant ratio revealing that the D point or the right-angled edge of the Cevian triangle must also be a fixed point; whence consequently there cannot be two right-angled Cevian triangles existing if either E or F is a fixed point.

(5) Eventually this felicitous theorem further adduces the complexity and beauty of the advanced geometry as well as the challenge of being resolved as an advanced Euclidean geometry problem, although this method is being rapidly collapsed and understated in most university mathematics curriculums at present.

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G. W. Indika Shameera Amarasinghe
Goddallawatta, Piyadigama, Ahangama,
SRI LANKA
GL 80650
indshamat@gmail.com
http://www.researchgate.net/profile/Indika_Shameera_Amarasinghe

The Problem from a Mathematical Camp

Pavel Calábek & Jaroslav Švrček

Pavel Calábek is a Senior Lecturer, Department of Algebra and Geometry, Palacký University of Olomouc. He deals with a problematic of educating mathematically gifted students. He is a member of Czech Mathematical Olympiad committee and its problem committee. He has been deputy leader in international competitions (IMO, MEMO, etc.)

Jaroslav Švrček is a Senior Lecturer Department of Algebra and Geometry, Palacký University of Olomouc. He is Vice-Chair of the Czech Mathematical Olympiad and member of its problems committee. He works as the editor of our journal Mathematics Competitions*. He has been many times leader or deputy leader of the Czech IMO and MEMO teams. He is one of the founders and organizers of the local international mathematical competition DUEL.*

The aim of this article is to present some interesting results gained on the basis of long-range working of both authors with mathematically gifted pupils of secondary schools in the Czech Republic. In the school-year 2010/2011 there was assigned for pupils on special seminarium (maths camp) the following problem:

Problem 1. Let a, b, c be pairwisely different positive real numbers. Prove that the quadratic equation

$$(a+b+c)x^2 + 2\left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a}\right)x + \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) = 0 \tag{1}$$

with unknown x has two different real roots.

This problem was such an incentive for the pupils, that they worked out several interesting solutions which will be presented in this contribution. Most of solvers have tried to prove that the discriminant of given quadratic equation (1) must be (under given conditions) positive. This is a necessary and sufficient condition for satisfying the given task. Their consideration led to the solving of a problem given to contestants of the second round of the British Mathematical Olympiad (BMO) in the year 2005.

Problem 2. Prove that for arbitrary positive real numbers a, b, c the following inequality

$$\left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a}\right)^2 \geq (a+b+c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right). \tag{2}$$

holds.

For successful solving of Problem 1 we need to prove the following statement: For arbitrary positive real numbers a, b, c which are mutually different cannot the equality (in the Problem 2) hold. Then a discriminant of the given quadratic equation is a positive number.

All achieved solutions of the given problem we can divide in some types.

1 Proofs involving a multiplying form of the inequality (2)

After multiplying both sides of (2) we obtain the following (equivalent) form of this inequality

$$\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}+2\left(\frac{a}{c}+\frac{b}{a}+\frac{c}{b}\right)\geq 3+\left(\frac{a}{c}+\frac{b}{a}+\frac{c}{b}\right)+\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right).$$

After a short manipulation this inequality implies

$$\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}+\frac{a}{c}+\frac{b}{a}+\frac{c}{b}\geq 3+\frac{a}{b}+\frac{b}{c}+\frac{c}{a}. \qquad (3)$$

Several ways of proving the last inequality came from solvers.

Sum of squares

Let us denote $x = a/b$, $y = b/c$, $z = c/a$, then x, y, z are positive real numbers. Then the inequality (3) is in the form

$$x^2+y^2+z^2+\frac{1}{z}+\frac{1}{x}+\frac{1}{y}\geq 3+x+y+z,$$

which implies (after some manipulation)

$$\frac{(x+1)(x-1)^2}{x}+\frac{(y+1)(y-1)^2}{y}+\frac{(z+1)(z-1)^2}{z}\geq 0.$$

The inequality is fulfilled for any triple of positive real numbers x, y, z, because all summands on its left side are non-negative numbers. The equality holds in this case if and only if $x = y = z = 1$, i.e. $a = b = c$.

Using a sum of squares and the AM–GM means inequality

It is easy to see that

$$\left(\frac{a}{b}-1\right)^2+\left(\frac{b}{c}-1\right)^2+\left(\frac{c}{a}-1\right)^2\geq 0$$

for any positive real numbers. By the AM–GM means inequality it holds also

$$\frac{a}{b}+\frac{b}{a}\geq 2\sqrt{\frac{ab}{ba}}=2$$

and similarly

$$\frac{b}{c}+\frac{c}{b}\geq 2,\qquad \frac{c}{a}+\frac{a}{c}\geq 2.$$

Adding up all four of these inequalities we have

$$\left(\frac{a}{b}-1\right)^2+\left(\frac{b}{c}-1\right)^2+\left(\frac{c}{a}-1\right)^2+$$
$$+\left(\frac{a}{b}+\frac{b}{a}\right)+\left(\frac{b}{c}+\frac{c}{b}\right)+\left(\frac{c}{a}+\frac{a}{b}\right)\geq 6.$$

Using some algebraical manipulation we have the inequality (3). It is easy to see again, that the equality holds if and only if $a=b=c$.

Combining the Cauchy-Schwarz inequality and the AM–GM inequality

By the AM–GM means inequality we get

$$\frac{a}{c}+\frac{b}{a}+\frac{c}{b}\geq 3\sqrt[3]{\frac{abc}{cab}}=3.$$

To complete the proof of the inequality (3) it is necessary to prove the following inequality

$$\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}\geq\frac{a}{b}+\frac{b}{c}+\frac{c}{a}.$$

But the last inequality is a consequence of the Cauchy-Schwarz inequality and AM–GM inequality of the form

$$(1^2+1^2+1^2)\left(\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}\right)\geq\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)^2\geq 3\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right).$$

The equality holds if and only if $a=b=c$.

Using the AM–GM inequality only

The solution based on another way of rearranging inequality (2) was presented by a member of the mentioned maths camp—*Jakub Solovský*.

We can multiply both sides of the inequality (2) by the positive real number $(abc)^2$. Further we can factorize the terms on both sides of the inequality. Then this inequality is equivalent to the following inequality

$$\begin{gathered} a^4c^2 + b^4a^2 + c^4b^2 + 2a^3b^2c + 2b^3c^2a + 2c^3a^2b \geq \\ \geq a^3b^2c + b^3c^2a + c^3a^2b + a^3bc^2 + b^3ca^2 + c^3ab^2 + 3a^2b^2c^2. \end{gathered}$$

By some algebraical manipulation we get the equivalent inequality

$$a^4c^2+b^4a^2+c^4b^2+a^3b^2c+b^3c^2a+c^3a^2b \geq a^3bc^2+b^3ca^2+c^3ab^2+3a^2b^2c^2.$$

By the AM–GM inequality we have

$$a^3b^2c + b^3c^2a + c^3a^2b \geq 3a^2b^2c^2. \tag{4}$$

Using the same inequality we can further obtain

$$\frac{4}{6}a^4c^2 + \frac{1}{6}b^4a^2 + \frac{1}{6}c^4b^2 \geq a^3bc^2, \tag{5}$$

and similarly

$$\frac{4}{6}b^4a^2 + \frac{1}{6}c^4b^2 + \frac{1}{6}a^4c^2 \geq b^3ca^2, \tag{6}$$

$$\frac{4}{6}c^4b^2 + \frac{1}{6}a^4c^2 + \frac{1}{6}b^4a^2 \geq c^3ab^2. \tag{7}$$

The inequality (1) is then a sum of inequalities (4)–(7).

Simultaneously, the equality in (4) will be true, if and only if $a^3b^2c = b^3c^2a = c^3a^2b$, thus in the case $a = b = c$. Similarly, the equality in (5)–(7) holds, if and only if $a^4c^2 = b^4a^2 = c^4b^2$, thus $a = b = c$. Therefore we can finally see that the equality in (1) holds, if and only if $a = b = c$.

2 Proofs without using a multiplying form of the inequality (2)

Other solutions immediately used inequality (2). Some of them involved also two identities

$$(a+b+c) = \left(\frac{1}{bc} + \frac{1}{ca} + \frac{1}{ab}\right) abc$$

and

$$\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) = (bc+ca+ab)\frac{1}{abc}.$$

After multiplying these inequalities we can find, that the identity

$$(a+b+c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) = \left(\frac{1}{bc} + \frac{1}{ca} + \frac{1}{ab}\right)(bc+ca+ab) \tag{8}$$

holds.

Cauchy–Schwarz inequality

By the Cauchy–Schwarz inequality it holds that

$$\left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a}\right)(ab+bc+ca) \geq (a+b+c)^2. \tag{9}$$

Similarly, by the same inequality we have

$$\left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a}\right)\left(\frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca}\right) \geq \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)^2. \tag{10}$$

Multiplying of (9) and (10) we get after easy manipulation (2). Similarly as in previous solutions we can see, that the equality in (9) and (10) holds, if and only if $a = b = c$.

Jensen's inequality

On the base of the identity (8) we can employ also well-known Jensen's inequality.

Since the function $y = \frac{1}{x}$ is convex on the set of positive real numbers, for positive real numbers

$$\alpha_1 = \frac{a}{a+b+c}, \quad \alpha_2 = \frac{b}{a+b+c}, \quad \alpha_3 = \frac{c}{a+b+c}$$

it holds $\alpha_1 + \alpha_2 + \alpha_3 = 1$. By Jensen's inequality we have

$$\frac{a}{a+b+c} \cdot \frac{1}{b} + \frac{b}{a+b+c} \cdot \frac{1}{c} + \frac{c}{a+b+c} \cdot \frac{1}{a} \geq$$
$$\geq \frac{1}{\frac{a}{a+b+c} \cdot b + \frac{b}{a+b+c} \cdot c + \frac{c}{a+b+c} \cdot a}.$$

The last inequality can be easily rearranged to the inequality (9).

Similarly, for the same convex function $y = \frac{1}{x}$ and positive real numbers

$$\alpha_1 = \frac{\frac{1}{a}}{\frac{1}{a} + \frac{1}{b} + \frac{1}{c}}, \quad \alpha_2 = \frac{\frac{1}{b}}{\frac{1}{a} + \frac{1}{b} + \frac{1}{c}}, \quad \alpha_3 = \frac{\frac{1}{c}}{\frac{1}{a} + \frac{1}{b} + \frac{1}{c}}$$

with a sum 1 it holds (by the Jensen's inequality)

$$\frac{\frac{1}{a}}{\frac{1}{a} + \frac{1}{b} + \frac{1}{c}} \cdot \frac{1}{\frac{1}{c}} + \frac{\frac{1}{b}}{\frac{1}{a} + \frac{1}{b} + \frac{1}{c}} \cdot \frac{1}{\frac{1}{a}} + \frac{\frac{1}{c}}{\frac{1}{a} + \frac{1}{b} + \frac{1}{c}} \cdot \frac{1}{\frac{1}{b}} \geq$$
$$\geq \frac{1}{\frac{\frac{1}{a}}{\frac{1}{a} + \frac{1}{b} + \frac{1}{c}} \cdot \frac{1}{c} + \frac{\frac{1}{b}}{\frac{1}{a} + \frac{1}{b} + \frac{1}{c}} \cdot \frac{1}{a} + \frac{\frac{1}{c}}{\frac{1}{a} + \frac{1}{b} + \frac{1}{c}} \cdot \frac{1}{b}}.$$

This inequality can be easily rearranged to the inequality (10) and further we can complete the proof similarly as in the last solution.

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Pavel Calábek
Faculty of Science
Palacký University
Olomouc
CZECH REPUBLIC
pavel.calabek@upol.cz

Jaroslav Švrček
Faculty of Science
Palacký University
Olomouc
CZECH REPUBLIC
jaroslav.svrcek@upol.cz

The difficulties in the search for solutions of functional inequalities

Peter Samovol, Valery Zhuravlev & Tal Kagalovsky

Peter Samovol (Ph.D) is a Lecturer at the Department of Mathematics, Kaye Academic College of Education and at the Mathematical Club, Ben Gurion University, Be'er-Sheva, Israel. He has a high level of experience preparing students for regional and international Mathematics Olympiads. His area of interest includes Development of the Mathematical Thinking and Numbers Theory.

Valery Zhuravlev is a Russian mathematician. He is living and working in Moscow. Much of his time is spent working with mathematically talented students of secondary schools in Russia.

Tal Kagalovsky has finished high school at Eshel Hanasi last year (2010). Tal won several mathematical Olympiads in Israel, and has participated in different research projects, such as Young Scientists project.

1 Introduction

Among various problems of modern school mathematics there are always problems incomprehensible for the majority of good pupils and teachers. Sometimes such problems can cause serious difficulties even for professional mathematicians. A typical example of such difficulties is in a topic "Functional equations and inequalities".

The first reason for those difficulties is the absence of elementary solution algorithms. To overcome difficulties in the search for the solution one needs frequently to use well developed creative thinking. In addition, there is a need to use abstract definitions when solving functional equations and inequalities, or the ability to generalize. It is not a secret that abstract in mathematics can scare a lot of pupils and push them away. A lot of times it happens because the science of generalization was not presented in a proper way. In reality, "... to abstract in order to ascend, to fly up, and to look on a crowd of separate facts from a bird's-eye view and to see the correlation between them, it is impossible to achieve without abstraction. This view from above can be very beautiful and useful, and it is reward for a work of understanding of abstract ideas. It is important to feel the advantage of abstractiveness as early as possible. Then this feeling will stimulate the search of the solution, and will keep us out of despair."[1]

2 Functional inequalities

In contemporary school literature it is difficult to find general approaches to the solution of functional inequalities. We will try to describe possible logical and technical difficulties below.

Let us consider a set of examples.

Example 1. Let us define the value of $f(f(x))$ as $f \circ f$. Let $f(x) = x^2 + x - \frac{1}{4}$. Solve inequality $\underbrace{f \circ f \circ f \circ \cdots \circ f}_{2011} \geq 0$.

Solution. It is easy to notice, that

$$f(x) = x^2 + x - \frac{1}{4} = \left(x + \frac{1}{2}\right)^2 - \frac{1}{2} \;\Rightarrow$$

$$\begin{aligned} f(f(x)) &= \left(f(x) + \frac{1}{2}\right)^2 - \frac{1}{2} \\ &= \left(\left(\left(x + \frac{1}{2}\right)^2 - \frac{1}{2}\right) + \frac{1}{2}\right)^2 - \frac{1}{2} = \left(x + \frac{1}{2}\right)^4 - \frac{1}{2} \end{aligned}$$

In a similar way,

$$\begin{aligned} f \circ f \circ f &= f(f(f(x))) = \left(f(f(x)) + \frac{1}{2}\right)^2 - \frac{1}{2} \\ &= \left(\left(x + \frac{1}{2}\right)^4 - \frac{1}{2} + \frac{1}{2}\right)^2 - \frac{1}{2} = \left(x + \frac{1}{2}\right)^8 - \frac{1}{2} \end{aligned}$$

Therefore we conclude that

$$\underbrace{f \circ f \circ f \circ \cdots \circ f}_{n} = f \circ \left(\underbrace{f \circ f \circ f \circ \cdots \circ f}_{n-1}\right) = \left(x + \frac{1}{2}\right)^{2^n} - \frac{1}{2}$$

[1] Alberto P. Calderon (1920–1998). Published in: Bull. ICMI, 47 (1999), 56–62

(The proof by the mathematical induction by n is trivial).

In our case we get

$$\left(x+\frac{1}{2}\right)^{2^{2011}}-\frac{1}{2}\geq 0\Leftrightarrow\left(x+\frac{1}{2}\right)^{2^{2011}}\geq\frac{1}{2}\Leftrightarrow\left|x+\frac{1}{2}\right|\geq\sqrt[2^{2011}]{0.5}$$
$$\Leftrightarrow x\in(-\infty-\sqrt[2^{2011}]{0.5}-0.5]\cup[\sqrt[2^{2011}]{0.5}-0.5,+\infty).$$

In general case we find the following problem:

Problem. Let us define the value of $f\left(f\left(x\right)\right)$ as $f\circ f$.

a) Let $f\left(x\right)=x^2+ax+b$, $b=\frac{a^2}{4}-\frac{a}{2}$. Solve the inequality

$$\underbrace{f\circ f\circ f\circ\cdots\circ f}_{n}\geq c.$$

b) Find the coefficients for cubic polynomial $f\left(x\right)=x^3+ax^2+bx+c$ and solve the inequality

$$\underbrace{f\circ f\circ f\circ\cdots\circ f}_{n}\geq d$$

in the same way.

Example 2. Does there exist a real function of a real variable $f\colon\mathbb{R}\to\mathbb{R}$ such that for any real x,y it is true

$$f\left(x-f\left(y\right)\right)\leq y\cdot f\left(x\right)+x. \tag{1}$$

Solution. The first difficulty is that when the equation was substituted by the inequality, it immediately limited our technical abilities, e.g. we can use the transitivity law only in one direction. However, we can bypass this difficulty, e.g. by the use of enumerative technique, but one needs to be very careful. The statement (1) uses the sign ($\leq$), i.e. one can present the problem as a combination of two independent ones E2.1 and E2.2.

E2.1. Does there exist a real function of a real variable $f\colon \mathbb{R} \to \mathbb{R}$ such that for any real x, y it is true that

$$f(x - f(y)) = y \cdot f(x) + x \tag{2}$$

Solution. We define $f(0) = c$. For $y = 0$, $x \in \mathbb{R}$ we get:

$$\begin{aligned} f(x - f(0)) &= 0 \cdot f(x) + x \\ f(x - c) &= x \\ f(x) &= x + c \end{aligned}$$

We now use the last equation in the statement (2),

$$(x - f(y)) + c = f(x - f(y)) = y \cdot f(x) + x = y \cdot (x + c) + x$$

hence

$$\begin{aligned} x - f(y) + c &= y \cdot (x + c) + x \\ x - y - c + c &= y(x + c) + x \\ 0 &= y(x + c + 1). \end{aligned}$$

We obtain a contradiction for $x \neq -c - 1, y \neq 0$.

E2.2. Does there exist a real function of a real variable $f\colon \mathbb{R} \to \mathbb{R}$ such that for any real x, y it is true that

$$f(x - f(y)) < y \cdot f(x) + x \tag{3}$$

Solution. For $y = 0$, $x \in \mathbb{R}$ we obtain: $f(x - f(y)) < y \cdot f(x) + x$

$$\begin{aligned} f(x - f(0)) &< 0 \cdot f(x) + x \Leftrightarrow \\ f(x - c) &< x \Rightarrow f(x) < x + c. \end{aligned}$$

For $x = 0$, we get $c < c$ which is a contradiction.

A reader can look for a more brief solution of problems E2.1 and E2.2. However, the following question immediately arises: are the separate proofs of impossibilities of E2.1 and E2.2 enough for the proof of impossibility of (1)? We would like to answer "Why not?", but this is an incorrect answer. The point is that the solution of a functional inequality is usually a set of pairs of numbers. And then if we claim that a pair (x_0, y_0) contradicts E2.1, and does not contradict to E2.2, then it does not contradict to the total statement (1). In a similar way, if a pair (x_1, y_1) contradicts E2.2, and does not contradict to E2.1, then it does not contradict to the total statement (1), Therefore, contradictions to E2.1 and E2.2 in two different points does not lead to contradiction to statement (1). To understand this fact is a real logical challenge for pupils. We present below a correct solution.

Suppose that such function exists.

I. We define $f(0) = c$. Consider for $y = 0, x \in \mathbb{R}$ the following inequality: $f(x - f(y)) \leq y \cdot f(x) + x$. We find: $f(x - f(0)) \leq 0 \cdot f(x) + x$

$$\begin{aligned} f(x-c) &\leq x \\ f(x) &\leq x + c. \end{aligned} \tag{4}$$

The following true statement follows from inequality (4). Y_1: There exists the sequence of real numbers $x_1, x_2, \ldots, x_i, \ldots$, such that $x_i \to (-\infty)$ starting from some i, and also $f(x_i) \leq (x_i + c) \to (-\infty)$.

II. It follows from the problem that for all $y \in \mathbb{R}$, $f(y) = x$

$$\begin{gathered} \mathrm{c} = f(0) \leq y \cdot f(f(y)) + f(y) \\ \mathrm{c} \leq y \cdot f(f(y)) + f(y) \leq y \cdot f(f(y)) + y + c \\ 0 \leq y \cdot [f(f(y)) + 1]. \end{gathered} \tag{5}$$

III. We define $f(1) = a$.

III_1. If $f(1) = a > 0$, then we consider all ordered pairs $(x, y) = (1, y)$

$$\begin{aligned} f(x - f(y)) &\leq y \cdot f(x) + x \\ f(1 - f(y)) &\leq y \cdot a + 1. \end{aligned} \tag{6}$$

According to statement Y_1 we find that for $y_i \to (-\infty)$ it follows $f(y_i) \to (-\infty)$. Then $m_i = [1 - f(y_i)] \to (+\infty)$. From inequalities (5) we obtain:

$$0 \leq m_i \cdot [f(f(m_i)) + 1] \Rightarrow \begin{cases} -1 \leq f(f(m_1)) \\ m_i > 0 \end{cases}$$

Using statement Y_1 and inequalities (5) and (6) we get:

$$-1 \leq f\left(f\left(m_i\right)\right) < f\left(m_i\right) + c = f\left(1 - f\left(y_i\right)\right) + c \leq y_i \cdot a + 1 + c$$
$$m_i = 1 - f\left(y_i\right) > 0, y_i \to (-\infty)$$
$$-1 \leq y_i \cdot a + 1 + f\left(0\right), a > 0, y_i \to (-\infty).$$

Contradiction.

III$_2$. Consider $f\left(1\right) = a = 0$: Then for $y = 1$ and $x < 0$ we find:

$$f\left(x - f\left(y\right)\right) \leq y \cdot f\left(x\right) + x,$$
$$f\left(x\right) \leq 1 \cdot f\left(x\right) + x$$
$$0 \leq x.$$

Contradiction.

III$_3$. Consider $f\left(1\right) = a < 0$. From the original inequality of the problem it immediately follows:

$$f\left(x - f\left(y\right)\right) \leq y \cdot f\left(x\right) + x, \text{ for } x = a,\ y = 1$$
$$c = f\left(0\right) = f\left(a - a\right) \leq 1 \cdot f\left(a\right) + a$$
$$c \leq f\left(a\right) + a \leq (a + c) + a$$
$$0 \leq a.$$

Contradiction.

We therefore conclude that $f\left(1\right) = a$ is not defined, i.e. such a function does not exist.

Conclusion. For any real function $f\colon \mathbb{R} \to \mathbb{R}$, one can find such real x, y that satisfy the inequality

$$f\left(x - f\left(y\right)\right) > y \cdot f\left(x\right) + x.$$

As we have seen transition to infinity was very useful. Such transition we shall apply in the next examples.

Example 3. (52nd IMO, 2011) Let $f\colon \mathbb{R} \to \mathbb{R}$ be a real-valued function defined on the set of real numbers that satisfies

$$f\,(x+y) \le y \cdot f\,(x) + f(f(x)) \tag{7}$$

for all real numbers x and y. Prove that $f(x) = 0$ for all $x \le 0$.

Solution.

I. We define $f\,(0) = c$. To start we get some inequalities which we use later. For $x = 0$ we have for all real y from (7),

$$\begin{aligned} f\,(y) \le y \cdot f\,(0) + f(f(0)) = c \cdot y + f(c) \Leftrightarrow \\ f\,(y) \le c \cdot y + f(c). \end{aligned} \tag{8}$$

If we make substitutions $y = x$ and $y = f\,(x)$ then from (8) we get respectively:

$$f(x) \le c \cdot x + f(c) \tag{9}$$

and

$$f(f(x)) \le c \cdot f(x) + f(c). \tag{10}$$

For substitutions $y = -x$ from (7) and using (10) we get

$$c = f\,(0) \le -x \cdot f\,(x) + f(f(x)) \le -x \cdot f\,(x) + c \cdot f(x) + f(c)$$

hence

$$c - f(c) \le (c - x) f\,(x). \tag{11}$$

II. Now we want to prove that $f\,(0) = c = 0$.

II_1. Suppose that $f\,(0) = c > 0$. Then for $x < 0 < c$ we have

$$c - x > 0.$$

Multiply both sides of (9) by $c - x$ and we get from (11)

$$c - f(c) \le (c - x) f\,(x) \le (c - x)(c \cdot x + f(c))$$

hence

$$c - f(c) \le (c - x)(c \cdot x + f(c)).$$

The left side of this inequality is a constant. But for the right side we have $(c-x)(c\cdot x+f(c))\to -\infty$ if $x\to -\infty$. Contradiction.

II_2. Suppose that $f(0)=c<0$. For $y=0$ we have for all real x from (7),

$$f(x)\le f(f(x)). \tag{12}$$

In particular, for $x=0$ and for $x=c$ we have

$$c=f(0)\le f(f(0))=f(c)\Leftrightarrow c\le f(c) \tag{13}$$
$$f(c)\le f(f(c)). \tag{14}$$

Further, by (10) we get

$$f(c)\le f(f(c))\le c\cdot f(c)+f(c)\Rightarrow 0\le c\cdot f(c).$$

Since $c<0$ then $0\ge f(c)$. If $f(c)=0$, then from (14)

$$0=f(c)\le f(f(c))=f(0)=c<0.$$

Contradiction.

Hence $f(c)<0$. Then from (9) for all $x\ge 0$

$$f(x)\le c\cdot x+f(c)<0\Rightarrow f(x)<0.$$

If exist $x_0<0$ that $f(x_0)\ge 0$ then we get $0>f(f(x_0))\ge f(x_0)\ge 0$. Contradiction. Therefore $f(x)<0$ for all $x\in\mathbb{R}$.

If we make substitution $y=f(x)-x$ into (7) then we get:

$$f(f(x))\le (f(x)-x)\cdot f(x)+f(f(x))$$

Hence for all x, $0\le (f(x)-x)\cdot f(x)$.

Since $f(x)<0$ then $x\ge f(x)$ for all $x\in\mathbb{R}$. Remark that if $x\to-\infty$ then $f(x)\to-\infty$. Substituting $f(x)$ in place of x in the last inequality we get the chain inequalities by (12), $f(f(x))\ge f(x)\ge f(f(x))$. Hence $f(x)=f(f(x))$ for all $x\in\mathbb{R}$.

Substitute $y=-x$ into (7) then we get:

$$\mathrm{c}=f(0)\le -x\cdot f(x)+f(f(x))=(1-x)\cdot f(x)\Leftrightarrow\ \mathrm{c}\le(1-x)\cdot f(x).$$

The left side of this inequality is a constant. But for the right side we have $(1-x)f(x) \to -\infty$ if $x \to -\infty$. Contradiction.

Thus we have only one possibility: $f(0) = c = 0$.

III. If $f(0) = c = 0$ then from (9) we get for all real x, $f(x) \le c{\cdot}x + f(c) = f(0) = 0 \Leftrightarrow f(x) \le 0$. From (11) for all real x

$$0 = c - f(c) \le (c-x)f(x) = -x \cdot f(x) \Leftrightarrow x \cdot f(x) \le 0.$$

Hence for all $x < 0$ we have $0 \le f(x) \le 0 \Leftrightarrow f(x) = 0$. Q.E.D.

Example 4. Does there exist a real function of a real variable f, such that for any real (x, y) inequality (15) is true?

$$\frac{f(x) + f(y)}{2} \ge f\left(\frac{x+y}{2}\right) + |x-y|. \tag{15}$$

Solution. We use the proof by contradiction.

I. If $f(x) = const = c_0$, then for $(x-y) \ne 0$ $\frac{c_0+c_0}{2} \ge c_0 + |x-y|$, $0 \ge |x-y|$. We conclude that $f(x) \ne const$.

II. One of the difficulties in the solution of this problem is the fact that all the data is abstract. It is useful in such a situation to consider a graph of the function. Let us draw points $(x, y) \subset [a, b]$. We divide interval $[a, b]$ on n equal parts. The length of each part is $m = \frac{b-a}{n}$. Thus, we obtain $1 \le k \le n$ $a = x_0$, $x_k = a + k \cdot m$.

III. Without loss of generality, suppose that $f(x_0) \le f(x_1)$. Let us prove that $f(x_2) > f(x_1)$. Indeed, if $f(x_2) \le f(x_1)$, then from (15) it follows:

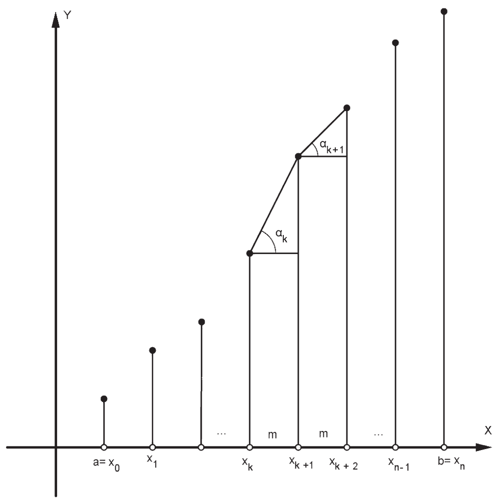
$$\frac{f(x_0) + f(x_2)}{2} \ge f\left(\frac{x_0+x_2}{2}\right) + 2m$$

$$\frac{f(x_0) + f(x_2)}{2} \ge f(x_1) + 2m$$

$$0 \ge \frac{f(x_0) - f(x_1)}{2} + \frac{f(x_2) - f(x_1)}{2} \ge 2m > 0.$$

Contradiction.

Therefore, $f(x_2) > f(x_1)$. In a similar way, we find that for values $\{x_1; x_2; x_3\}$, if $f(x_2) > f(x_1)$, then from (15) it follows: $f(x_3) > f(x_2)$. Continuing these considerations, we find $f(x_0) < f(x_1) < f(x_2) < \cdots < f(x_n)$.



IV. We conclude, that for all $1 \le k \le n$ in $\{x_k; x_{k+1}; x_{k+2}\}$:

$$\frac{f(x_{k+1}) - f(x_k)}{2} - \frac{f(x_k) - f(x_{k-1})}{2} \ge 2m, k \ge 1$$

$$\frac{f(x_{k+1}) - f(x_k)}{m} - \frac{f(x_k) - f(x_{k-1})}{m} \ge 4 \Leftrightarrow$$

$$\tan \alpha_k - \tan \alpha_{k-1} \ge 4.$$

Then

$$
\begin{aligned}
\tan\alpha_1 - \tan\alpha_0 &\geq 4\\
\tan\alpha_2 - \tan\alpha_1 &\geq 4\\
\tan\alpha_3 - \tan\alpha_2 &\geq 4\\
&\cdots\\
\underline{\tan\alpha_{\text{n-1}} - \tan\alpha_{n-2}} &\underline{\geq 4}\\
\tan\alpha_{n-1} - \tan\alpha_0 &\geq 4\cdot(n-1) \Rightarrow\\
\tan\alpha_{n-1} &\geq \tan\alpha_0 + 4\cdot(n-1) > 4\cdot(n-1)\,.
\end{aligned}
$$

V. From the construction of the function graph, we find:

$$
\begin{aligned}
f(x_1) &= f(x_0) + m\cdot\tan\alpha_0\\
f(x_2) &= f(x_1) + m\cdot\tan\alpha_1\\
f(x_3) &= f(x_2) + m\cdot\tan\alpha_2\\
&\cdots\\
f(x_n) &= f(x_{n-1}) + m\cdot\tan\alpha_{n-1}
\end{aligned}
$$

So

$$
\begin{aligned}
f(b) = f(x_n) &= f(x_0) + m(\tan\alpha_0 + \tan\alpha_1 + \tan\alpha_2 + \cdots + \tan\alpha_{n-1})\\
&\geq f(a) + m(4 + 4\cdot 2 + 4\cdot 3 + \cdots + 4\cdot(n-1))\\
&\geq f(a) + 4\cdot\frac{b-a}{n}\cdot\frac{n\cdot(n-1)}{2}\\
&= f(a) + 2\cdot(b-a)\cdot(n-1) \Rightarrow\\
f(b) &\geq f(a) + 2\cdot(b-a)\cdot(n-1) \to \infty.
\end{aligned}
$$

We conclude that $f(x_n) = f(b)$ is undefined. Contradiction.

Therefore, such function f of a real variable does not exist.

Remark. If increasing n we obtain $f(x_0) \geq f(x_1)$, then for this case we consider a symmetrical interval $[2a-b, a]$. As in a previous case, we divide it into n equal parts, and prove that

$$f(t_1) > f(x_0)$$
$$t_k - t_{k-1} = \frac{b-a}{n} = m$$
$$f(x_0) < f(t_1) < f(t_2) < \cdots < f(t_n)$$

$0 < 90^\circ - \beta_{n-1} < \varepsilon \to 0$. Therefore, the function is undefined $f(t_n) = f(2a-b)$, which contradicts the condition of the problem.

3 When the technique helps

Sometimes difficult abstract functional inequalities or equations can be solved by special technical tricks. The difficulty is that it is impossible to define the technical trick from the very beginning. Successful ideas appear when the data is under consideration.

Example 5. (Grossman Olympiad, Israel, 2000) Find all functions of integer variables that fulfill

$$f\colon \mathbb{Z} \to \mathbb{Z}$$
$$3f(z) - 2f(f(z)) = z$$

Solution.

I. For some $z_0 \in \mathbb{Z}$ we define,

$$\begin{aligned} &x_0 = z_0, x_1 = f(z_0), x_2 = f(f(z_0)) = f \circ f, \\ &x_3 = f(f(f(z_0))) = f \circ f \circ f, \ldots, \\ &x_n = \underbrace{f(f(f\ldots(f(z_0))\ldots))}_{n} = \underbrace{f \circ f \circ \cdots \circ f}_{n}. \end{aligned}$$

We first show, that if $f(z) = z$, then this function is a solution. Indeed, we get an identity

$$\begin{matrix} f\colon \mathbb{Z} \to \mathbb{Z} \\ 3 \cdot f(z) - 2 \cdot f(f(z)) = z \end{matrix} \quad \Leftrightarrow \quad \begin{matrix} f\colon \mathbb{Z} \to \mathbb{Z} \\ 3z - 2z = z \end{matrix}$$

II. Then from the conditions of the problem we have

$$f\colon \mathbb{Z} \to \mathbb{Z}$$
$$f(z) - z = 2f(f(z)) - 2f(z).$$

We find

$$f \circ f - f(z_0) = \frac{1}{2}\left(f(z_0) - z_0\right),$$
$$f \circ f \circ f - f \circ f = \frac{1}{2}\left(f \circ f - f(z_0)\right),$$
$$\dots$$
$$\underbrace{f \circ f \circ \dots \circ f}_{n} - \underbrace{f \circ f \circ \dots \circ f}_{n-1} = \frac{1}{2}\left(\underbrace{f \circ f \circ \dots \circ f}_{n-1} - \underbrace{f \circ f \circ \dots \circ f}_{n-2}\right) \tag{16}$$

We get the chain of the equalities

$$\underbrace{f \circ f \circ \dots \circ f}_{n} - \underbrace{f \circ f \circ \dots \circ f}_{n-1} = \frac{1}{2}\left(\underbrace{f \circ f \circ \dots \circ f}_{n-1} - \underbrace{f \circ f \circ \dots \circ f}_{n-2}\right)$$
$$= \frac{1}{2}\left(\frac{1}{2}\left(\underbrace{f \circ f \circ \dots \circ f}_{n-2} - \underbrace{f \circ f \circ \dots \circ f}_{n-3}\right)\right) = \dots$$
$$= \frac{1}{2}\left(\frac{1}{2}\left(\dots\left(\frac{1}{2}\left(f \circ f \circ f - f \circ f\right)\right)\right)\right) = \frac{1}{2^{n-2}}\left(f \circ f - f(z_0)\right)$$
$$= \frac{1}{2^{n-1}}\left(f(z_0) - z_0\right).$$

Hence

$$\underbrace{f \circ f \circ \dots \circ f}_{n} - \underbrace{f \circ f \circ \dots \circ f}_{n-1} = \frac{f(z_0) - z_0}{2^{n-1}}\ . \tag{17}$$

The left side of the last equality is a integer number for any natural $n \geq 2$. Therefore, equality (17) is possible only if $f(z_0) - z_0 = 0$ for any z_0. It means we have the only possible solution: $f(x) = x$, $x \in \mathbb{Z}$.

Answer: $f(x) = x$, $x \in \mathbb{Z}$.

Remark. The equality (17) we can get after multiplying all equalities in (16) too. In this case we have:

$$\Bigg(\underbrace{f \circ f \circ \cdots \circ f}_{n} - \underbrace{f \circ f \circ \cdots \circ f}_{n-1}\Bigg) \cdot A = \frac{f(z_0) - z_0}{2^{n-1}} A\ ,$$

where

$$A = (f \circ f - f(z_0))\,(f \circ f \circ f - f \circ f) \cdot \ldots \cdot \Bigg(\underbrace{f \circ f \circ \cdots \circ f}_{n-1} - \underbrace{f \circ f \circ \cdots \circ f}_{n-2}\Bigg).$$

We should consider two possibilities.

If $A \neq 0$, then it is possible to cancel $A \neq 0$ on both sides and we get (17).

If $A = 0$, then one of the factors is equal to zero. We have for all natural $n \geq 2$:

$$\begin{array}{c} f\colon \mathbb{Z} \to \mathbb{Z} \\ 3 \cdot x_{n-1} - 2 \cdot x_n = x_{n-2} \end{array} \quad \Leftrightarrow \quad \begin{array}{c} f\colon \mathbb{Z} \to \mathbb{Z} \\ 2 \cdot (x_{n-1} - x_n) = x_{n-2} - x_{n-1} \end{array}$$

It is clear that if for some natural $k \geq 2$ and some z_0 it is true $x_{k-1} - x_k = 0$, then we immediately obtain $x_{k-2} - x_{k-1} = 0$. Finally, "going back", we find: $x_0 = x_1$, i.e. $f(z) = z$.

4 Instead of conclusion

As we have mentioned above, solution of any functional inequality or equation is always a small mathematical research, which may become a small discovery for a pupil. The level of difficulties corresponds to the level of the achievement. The limits of this paper do not allow us to analyze other interesting examples. But we hope that the small number examples described above will let the reader to feel more comfortable when solving functional equations and inequalities. The problems of this topic can serve as a measure of creative thinking.

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Peter Samovol, Ph.D.
School "Eshel Hanasi" (teacher)
Kaye Academic College of Education, Beer-Sheva
Ben-Gurion University of the Negev, Beer-Sheva
Max Boren Street, 21
Beer-Sheva 84834
ISRAEL
email: Pet12@012.net.il

Tal Kagalovsky
Kahal 7
Lehavim
ISRAEL
email: spintal@gmail.com

Valery M. Zhuravlev
Kuskovskaya street, 17/32
Moscow, 111141
RUSSIA
e-mail: Zhuravlevvm@mail.ru

The 52nd International Mathematical Olympiad, Amsterdam, The Netherlands, 2011

The 52nd International Mathematical Olympiad (IMO) was held on 12–24 July in Amsterdam, The Netherlands.

The Netherlands is known for its unusual geographical make up, with about 20 % of its land lying below the sea level, protected by an elaborate system of dikes. The word "Netherlands" literally means "low-lying lands". The capital Amsterdam is a vibrant, modern city permeated with museums, bicycles and canals. It is the ideal location to accommodate 564 of the world's best high school mathematics students, representing 101 countries. This is the second largest IMO in terms of contestant numbers, with only one less contestant than the record achieved two years ago in Bremen.

Just like the Olympic Games, the IMO began with an Opening Ceremony. Nazar Agakhanov, Chairman of the IMO Advisory Board, highlighted the importance of mathematics and welcomed all the participants using different languages. The teams then marched on stage according to continental groupings in a parade of nations. The parade was interspersed by energetic local dancers, acrobats, skaters and a freestyle BMX rider, creating a youthful and lively atmosphere. The IMO was officially opened in a comical fashion, with a banging of the gong by Robbert Dijkgraaf, Chairman of the Board of IMO 2011, and Eberhard van der Laan, Mayor of Amsterdam.

A few days before the IMO, the Team Leaders gathered in a conference centre named NH Koningshof, situated in a town called Veldhoven, in the Eindhoven Region. Incidentally, Eindhoven was recently crowned as the smartest region in the world, a title certainly reassured by the presence of the *Jury* of IMO Team Leaders. The task was to select the contest problems from a shortlist of 30 problem proposals. Understandably, this process was kept secret from the students and far away from the contest location. The shortlist was of the highest quality due to excellent preparation by the locally organised Problems Selection Committee. Despite being spoilt for choice, the Jury eventually decided on six problems, as follows.

1. An easy algebraic exercise with shades of number theory. The underlying ideas are quite simple, but the problem demands a fair level of organisation. It was proposed by Mexico.
2. A beautiful combinatorial geometry problem involving windmills, a Dutch cultural icon. It puts a fresh spin on the concept of *dividing lines*, or lines with an almost equal number of points on either side. Observations of these lines lead to a solution incorporating a delightful invariant. The problem was proposed by Geoff Smith of the United Kingdom. Geoff was a former UK Team Leader and a current member of the IMO Advisory Board.
3. A tricky functional inequality originated from Belarus. Just like any other functional inequality, the key is to apply analytic techniques repeatedly with the hope of obtaining sensible bounds. In this particular case, much algebraic manipulation is required, so perseverance is a must.
4. An easy combinatorics problem proposed by Iran, concerning a balance and a selection of weights. Many approaches are possible, ranging from nice yet simple bijection arguments, to possibly tedious inductive calculations, to applications of sophisticated identities such as the generating function of Stirling numbers.
5. A medium level number theory problem exploring the divisibility of a positive integer-valued function. Insights on the underlying structure of the function are useful and reasonably succinct, but delicate bounding arguments are possible. This problem was also proposed by Iran.
6. An incredibly intimidating classical geometry problem from Japan, and an absolute gem at the same time. The sheer elegance of the result is only outmatched by its overwhelming difficulty. A great deal of technical proficiency and pure ingenuity is required to crack this one.

This set of problems have broken a couple of noteworthy trends. There hasn't been an IMO paper with two combinatorics problems since 2001, and there hasn't been one with only one geometry problem since 1997. This is a refreshing change, in the interest of balance amongst the four areas of Olympiad mathematics. Remarkably, the windmill problem barely outlasted a geometric counterpart by a single vote, so both aforementioned trends could have easily continued.

The contest itself consisted of two exams, held on Monday July 18 and Tuesday July 19. Each exam had three problems and lasted for four and a half hours. During the first half hour of each exam, the contestants were allowed to ask the Jury, through written means, for clarifications on the problems. On the second day, the Jury received nearly 190 questions from students, as the wording of problem 4 turned out to be particularly problematic.

After the exams the Leaders and their Deputies spent about two days assessing the work of the students from their own countries, guided by marking schemes discussed earlier. A local team of markers called *Coordinators* also assessed the papers. They too were guided by the marking schemes but are allowed some flexibility if, for example, a Leader brought something to their attention in a contestant's exam script which was not covered by the marking scheme. The Team Leader and Coordinators have to agree on scores for each student of the Leader's country in order to finalise scores. This year, the marking schemes were comprehensive and the Coordinators were well prepared, so the entire process went smoothly without any major disputes.

The outcome was not entirely as expected. The supposedly medium level question 2 averaged 0.65 marks, compared to the supposedly difficult question 3 which averaged 1.05 marks. The easiest was question 1 with an average of 5.35 marks, followed by question 4 then question 5, which averaged 4.06 and 3.26 marks respectively. Question 6 was the most difficult, averaging only 0.32 marks. There were 281 (=49.8 %) medals awarded, the distributions being 137 (=24.3 %) Bronze, 90 (=16 %) Silver and 54 (=9.6 %) Gold. The medal cuts were set at 28 for Gold, 22 for Silver and 16 for Bronze. Most Gold medallists solved about four questions, most Silver medallists solved three questions and most Bronze medallists solved two and a bit questions. Of those who did not get a medal, a further 121 contestants received an Honourable Mention for solving at least one question perfectly.

There were a couple of outstanding performers worth mentioning. The first was Lisa Sauermann from Germany, who was the only contestant to achieve a perfect score of 42. This caps off an illustrious IMO career for Lisa. With 4 Gold and 1 Silver, she is the most decorated contestant in the history of the competition, displacing her fellow countryman Christian Reiher (4 Gold and 1 Bronze) at the top of the IMO Hall of Fame. The other outstanding performer is the 13-year-old Peruvian Raúl Arturo Chávez Sarmiento, who solved 5 questions for a score of 35, placing sixth overall. With 1 Gold, 1 Silver and 1 Bronze so far, the young Raúl has a bright future in front of him, including the possibility of overtaking Lisa if he chooses to keep coming back.

The awards were presented at the Closing Ceremony. Various media personnel, sponsor representatives and the Mayor of Amsterdam were there to congratulate the medal winners for their accomplishments. Special recognition was given to Lisa Sauermann for her extraordinary IMO record. She was presented with a laurel wreath and a personal congratulation from Christian Reiher. Lisa has become a role model for other girls aspiring to do well at the IMO.

Many thanks to members of IMO 2011 Organising Committee, guides, coordinators and many behind the scenes staff, crew members and volunteers for their outstanding efforts in putting together a truly wonderful IMO.

The 2011 IMO was supported and organised by the Dutch Ministry of Education, the National Platform Science & Technology and Google.

The 2012 IMO is scheduled to be held in Mar del Plata, Argentina.

Ivan Guo
Australian IMO Team Leader
AUSTRALIA

1 IMO Papers

Monday, July 18, 2011
Language: English

First Day

Problem 1. Given any set $A = \{a_1, a_2, a_3, a_4\}$ of four distinct positive integers, we denote the sum $a_1 + a_2 + a_3 + a_4$ by s_A. Let n_A denote the number of pairs (i, j) with $1 \leq i < j \leq 4$ for which $a_i + a_j$ divides s_A. Find all sets A of four distinct positive integers which achieve the largest possible value of n_A.

Problem 2. Let $\mathcal{S}$ be a finite set of at least two points in the plane. Assume that no three points of $\mathcal{S}$ are collinear. A *windmill* is a process that starts with a line ℓ going through a single point $P \in \mathcal{S}$. The line rotates clockwise about the *pivot* P until the first time that the line meets some other point belonging to $\mathcal{S}$. This point, Q, takes over as the new pivot, and the line now rotates clockwise about Q, until it next meets a point of $\mathcal{S}$. This process continues indefinitely.

Show that we can choose a point P in $\mathcal{S}$ and a line ℓ going through P such that the resulting windmill uses each point of $\mathcal{S}$ as a pivot infinitely many times.

Problem 3. Let $f : \mathbb{R} \to \mathbb{R}$ be a real-valued function defined on the set of real numbers that satisfies

$$f(x + y) \leq y f(x) + f(f(x))$$

for all real numbers x and y. Prove that $f(x) = 0$ for all $x \leq 0$.

Time allowed: 4 hours 30 minutes
Each problem is worth 7 points

Tuesday, July 19, 2011
Language: English

Second Day

Problem 4. Let $n > 0$ be an integer. We are given a balance and n weights of weight $2^0, 2^1, \ldots, 2^{n-1}$. We are to place each of the n weights on the balance, one after another, in such a way that the right pan is never heavier than the left pan. At each step we choose one of the weights that has not yet been placed on the balance, and place it on either the left pan or the right pan, until all of the weights have been placed.

Determine the number of ways in which this can be done.

Problem 5. Let f be a function from the set of integers to the set of positive integers. Suppose that, for any two integers m and n, the difference $f(m) - f(n)$ is divisible by $f(m - n)$. Prove that, for all integers m and n with $f(m) \leq f(n)$, the number $f(n)$ is divisible by $f(m)$.

Problem 6. Let ABC be an acute triangle with circumcircle Γ. Let ℓ be a tangent line to Γ, and let ℓ_a, ℓ_b and ℓ_c be the lines obtained by reflecting ℓ in the lines BC, CA and AB, respectively. Show that the circumcircle of the triangle determined by the lines ℓ_a, ℓ_b and ℓ_c is tangent to the circle Γ.

Time allowed: 4 hours 30 minutes
Each problem is worth 7 points

2 Results

Some Country Scores			Some Country Scores		
Rank	Country	Score	Rank	Country	Score
1	China	189	15	Ukraine	136
2	U.S.A.	184	17	Canada	132
3	Singapore	179	17	U.K.	132
4	Russia	161	19	Italy	129
5	Thailand	160	20	Bulgaria	121
6	Turkey	159	20	Brazil	121
7	North Korea	157	22	Mexico	120
8	Romania	154	23	India	119
8	Taiwan	154	23	Israel	119
10	Iran	151	25	Australia	116
11	Germany	150	25	Hungary	116
12	Japan	147	25	Serbia	116
13	South Korea	144	28	Netherlands	115
14	Hong Kong	138	29	Indonesia	114
15	Poland	136	29	New Zealand	114

Mark Distribution by Question						
Mark	Q1	Q2	Q3	Q4	Q5	Q6
0	29	391	394	94	106	443
1	17	124	57	120	92	103
2	63	14	34	31	127	7
3	52	2	13	16	20	2
4	18	4	7	8	20	0
5	17	2	3	8	9	3
6	14	5	5	20	20	0
7	354	22	51	267	170	6
Total	564	564	564	564	564	564
Mean	5.35	0.65	1.05	4.06	3.26	0.32

Distribution of Awards at the 2011 IMO

Country	Total	Gold	Silver	Bronze	H.M.
Albania	24	0	0	0	1
Argentina	77	1	0	0	4
Armenia (5 members)	61	0	1	0	3
Australia	116	0	3	3	0
Austria	110	0	2	2	2
Azerbaijan	61	0	1	1	1
Bangladesh	50	0	0	1	1
Belarus	113	0	2	3	1
Belgium	88	0	0	4	1
Bolivia (4 members)	17	0	0	0	1
Bosnia and Herzegovina	64	0	0	1	4
Brazil	121	0	3	3	0
Bulgaria	121	0	2	3	1
Canada	132	1	2	3	0
Chile	48	0	0	1	1
China	189	6	0	0	0
Colombia	73	0	0	1	4
Costa Rica (4 members)	57	0	1	0	3
Croatia	110	0	1	5	0
Cyprus	51	0	0	1	1
Czech Republic	101	0	1	3	2
Denmark	76	0	1	1	2
Ecuador	32	0	0	1	0
El Salvador (2 members)	11	0	0	0	0
Estonia	76	0	0	2	3
Finland	68	0	1	0	3
France	111	0	1	4	1
Georgia	68	0	0	2	2
Germany	150	1	3	2	0
Greece	99	1	0	3	1
Guatemala (4 members)	8	0	0	0	0
Honduras (3 members)	21	0	0	0	1
Hong Kong	138	2	1	3	0
Hungary	116	0	2	3	1

Distribution of Awards at the 2011 IMO					
Country	Total	Gold	Silver	Bronze	H.M.
Iceland	48	0	0	0	3
India	119	1	1	2	2
Indonesia	114	0	2	4	0
Iran	151	2	4	0	0
Ireland	26	0	0	0	0
Israel	119	1	0	4	1
Italy	129	1	3	1	1
Ivory Coast	34	0	0	0	2
Japan	147	2	2	2	0
Kazakhstan	105	0	1	3	2
Kosovo	22	0	0	0	1
Kuwait (5 members)	1	0	0	0	0
Kyrgyzstan (5 members)	14	0	0	0	1
Latvia	68	0	1	1	1
Liechtenstein (1 member)	4	0	0	0	0
Lithuania	87	0	0	4	2
Luxembourg	48	0	0	1	2
Macau	71	0	0	2	3
Macedonia (FYR)	38	0	0	1	0
Malaysia	93	1	1	1	2
Mexico	120	0	2	4	0
Moldova	86	0	1	0	4
Mongolia	69	0	0	2	3
Montenegro (4 members)	13	0	0	0	1
Morocco	64	0	1	1	2
Netherlands	115	0	2	3	1
New Zealand	114	0	2	2	2
Nigeria	40	0	0	1	0
North Korea	157	3	3	0	0
Norway	67	0	1	0	3
Pakistan (4 members)	35	0	0	1	1
Panama (1 member)	6	0	0	0	0
Paraguay (5 members)	38	0	0	0	1
Peru	113	1	0	2	3

Distribution of Awards at the 2011 IMO					
Country	Total	Gold	Silver	Bronze	H.M.
Philippines (5 members)	69	0	0	3	0
Poland	136	2	2	1	1
Portugal	86	1	0	2	1
Puerto Rico (4 members)	32	0	0	0	2
Romania	154	1	5	0	0
Russian Federation	161	2	4	0	0
Saudi Arabia	53	0	0	2	0
Serbia	116	1	2	1	1
Singapore	179	4	1	1	0
Slovakia	111	0	2	3	1
Slovenia	64	0	0	1	3
South Africa	93	0	1	2	2
South Korea	144	2	3	0	1
Spain	83	0	0	3	1
Sri Lanka	49	0	0	1	2
Sweden	69	0	1	0	3
Switzerland	88	0	2	1	1
Syria	14	0	0	0	1
Taiwan	154	2	4	0	0
Tajikistan	68	0	1	0	2
Thailand	160	3	2	1	0
Trinidad and Tobago	29	0	0	0	1
Tunisia	46	0	0	1	1
Turkey	159	3	2	1	0
Turkmenistan	64	0	0	3	1
Ukraine	136	1	2	3	0
United Arab Emirates (5 members)	1	0	0	0	0
United Kingdom	132	2	1	2	1
United States of America	184	6	0	0	0
Uruguay (4 members)	29	0	0	0	2
Uzbekistan (5 members)	62	0	0	1	2
Venezuela (2 members)	21	0	0	0	2
Vietnam	113	0	0	6	0
Total (564 contestants)		**54**	**90**	**137**	**121**

Tournament of Towns (Selected Problems from the Spring 2011 papers)

Andy Liu

Andy Liu is a professor of mathematics at the University of Alberta in Canada. His research interests span discrete mathematics, geometry, mathematics education and mathematics recreations. He edits the Problem Corner of the MAA's magazine Math Horizons. *He was the Chair of the Problem Committee in the 1995 IMO in Canada. His contribution to the 1994 IMO in Hong Kong was a major reason for him being awarded a David Hilbert International Award by the World Federation of National Mathematics Competitions.*

1. Each diagonal of a convex quadrilateral divides it into two isosceles triangles. The two diagonals of the same quadrilateral divide it into four isosceles triangles. Must this quadrilateral be a square?

2. Worms grow at the rate of 1 metre per hour. When they reach their maximum length of 1 metre, they stop growing. A full-grown worm may be dissected into two new worms of arbitrary lengths totalling 1 metre. Starting with 1 full-grown worm, can one obtain 10 full-grown worms in less than 1 hour?

3. Four perpendiculars are drawn from four vertices of a convex pentagon to the opposite sides. If these four lines pass through the same point, prove that the perpendicular from the fifth vertex to the opposite side also passes through this point.

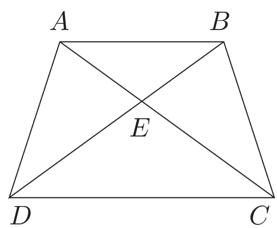
4. Two ants crawl along the sides of the 49 squares of a 7×7 board. Each ant passes through all 64 vertices exactly once and returns to its starting point. What is the smallest possible number of sides covered by both ants?

5. In a country, there are 100 towns. Some pairs of towns are joined by roads. The roads do not intersect one another except meeting at towns. It is possible to go from any town to any other town by road. Prove that it is possible to pave some of the roads so that the number of paved roads at each town is odd.

6. Among a group of programmers, every two either know each other or do not know each other. Eleven of them are geniuses. Two companies hire them one at a time, alternately, and may not hire someone already hired by the other company. There are no conditions on which programmer a company may hire in the first round. Thereafter, a company may only hire a programmer who knows another programmer already hired by that company. Is it possible for the company which hires second to hire ten of the geniuses, no matter what the hiring strategy of the other company may be?

Solutions

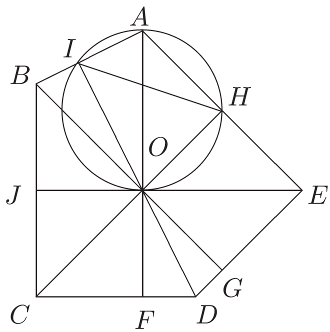
1. The answer is no. In the convex quadrilateral $ABCD$ in the diagram below, where the diagonals intersect at E, we have $\angle ADB = \angle BDC = \angle DCA = \angle ACB = \angle BAC = \angle ABD = 36°$. Then $\angle ADC = \angle DAE = \angle DEA = \angle CEB = \angle CBE = \angle BCD = 72°$. It follows that all of the triangles ABC, BAD, ACD, BDC, ABE, BCE, CDE and DAE are isosceles, and yet $ABCD$ is not a square.



2. If we do nothing, we will have 1 full-grown worm with 1 hour to spare. If we cut the given worm into two of lengths $\frac{1}{2}$ and $\frac{1}{2}$, we will have two full-grown worms with half an hour to spare. If we cut the given worm into two of lengths $\frac{1}{4}$ and $\frac{3}{4}$, we will have one full-grown worm and one half-grown worm in a quarter of an hour. As in the preceding case, we will have three full-grown worms in another half an hour, so that we have a quarter of an hour to spare. In the same manner, if we cut the given worm into two of lengths $\frac{1}{512}$ and $\frac{511}{512}$, we will have ten full-grown worms with $\frac{1}{512}$ of an hour to spare.

3. Solution by Scott Wang

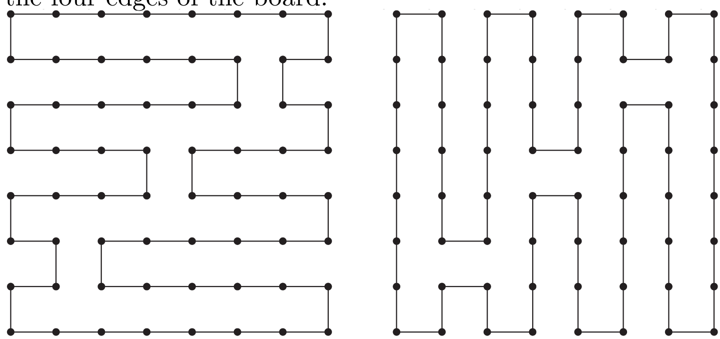
Let $ABCDE$ be the pentagon. Let BG, CH, DI and EJ be the altitudes concurrent at O. Let F be the point of intersection of AO with CD.



Since the right triangles COJ, JOB and BOI are similars to the right triangles EOH, HOD and DOG, we have $OC \cdot OH = OE \cdot OJ = OB{\cdot}OG = OD{\cdot}OI$. Hence triangles HOI and DOC are also similar, so that $\angle OHI = \angle ODC$. Since $\angle OHA = 90° = \angle OIA$, $AHOI$ is a cyclic quadrilateral. Hence $\angle AHI = \angle AOI = \angle DOF$. Now $\angle OFD = \angle 180° - \angle DOF - \angle ODF = 180° - \angle OHA - \angle OHI = 90°$.

4. Solution by Desmond Sisson

The eight sides at the four corner vertices must be traversed by both ants. Along each of the four edges of the board, the middle four vertices all have degree 3, and must lie on a side traversed by both ants. To minimize the number of such sides, they must cover these sixteen vertices in pairs. Hence their number cannot be less than 8+8=16. The following diagram shows the paths of the two ants with exactly 16 sides covered by both, every other side along the four edges of the board.



5. Solution by Adrian Tang

Let $\mathcal{F}$ be the set of towns with an odd number of paved roads and $\mathcal{G}$ be the set of towns with an even number of paved roads. Note that $|\mathcal{F}|$ is even at any time. Initially, $|\mathcal{F}| = 0$. If we have $|\mathcal{F}| = 100$ at some point, the task is accomplished. Suppose $|\mathcal{F}| < 100$. Then there are at least 2 towns A and B in $\mathcal{G}$. Since the graph is connected, there exists a tour from A to B, going along the roads without visiting any town more than once. Interchange the status of each road on this tour, from paved to unpaved and vice versa. (This is of course done on the planning map, before any actual paving is carried out.) Then A and B move from $\mathcal{G}$ to $\mathcal{F}$ while all other towns stay in $\mathcal{F}$ or $\mathcal{G}$ as before. Hence we can make $|\mathcal{F}|$ increase by 2 at a time, until it reaches 100.

6. Solution by Central Jury

Let there be eleven attributes on which the companies rank the candidates. The ranking of each attribute for each candidate is a non-negative integer. It turns out that for each candidate, the sum of the eleven rankings is exactly 100. Moreover, no two candidates have exactly the same set of rankings, and for each possible set of rankings, there is such a candidate. The eleven geniuses are those with a ranking of 100 in one attribute and a ranking of 0 in every other attribute. Two candidates know each other if their sets of rankings differ only in two attributes, and those two rankings differ by 1. Consider candidate A who is the first hired by the first company. By the pigeonhole principle, the ranking of at least one attribute for A is at least 10, and we may assume that this is the first attribute. The second company hires the candidate whose ranking in the first attribute is exactly 10 lower than that of A, but exactly 1 higher in each of the other ten attributes. At this point, the first company has a big edge in hiring the genius of the first attribute, but the second company has a small edge in hiring the genius of each of the other ten attributes. The second company concedes the genius of the first attribute to the first company, but aims to hire the other ten geniuses by maintaining these small advantages. Note that among the candidates hired by each company, the highest ranking in any attribute can only increase by 1 with each new hiring. Whenever the first company makes a hiring, the second company will respond by hiring a candidate whose rankings change in the same attributes and in the same directions.

Andy Liu
University of Alberta
CANADA
E-mail: aliumath@telus.net

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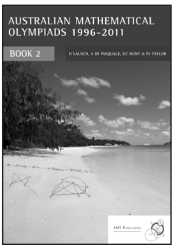

Australian Mathematical Olympiads Book 2 1996 – 2011
H Lausch, A Di Pasquale, DC Hunt & PJ Taylor

This book is the second in the series and contains a complete collection of all Australian Mathematical Olympiad papers from 1996 to 2011. Solutions to all problems are included and in a number of cases alternative solutions are offered.

Bundles of Past AMC Papers

Past Australian Mathematics Competition papers are packaged into bundles of ten identical papers in each of the Junior, Intermediate and Senior divisions of the Competition. Schools find these sets extremely valuable in setting their students miscellaneous exercises.

AMC Solutions and Statistics
Edited by PJ Taylor

This book provides, each year, a record of the AMC questions and solutions, and details of medallists and prize winners. It also provides a unique source of information for teachers and students alike, with items such as levels of Australian response rates and analyses including discriminatory powers and difficulty factors.

Australian Mathematics Competition Book 1 1978-1984
Edited by W Atkins, J Edwards, D King, PJ O'Halloran & PJ Taylor

This 258-page book consists of over 500 questions, solutions and statistics from the AMC papers of 1978-84. The questions are grouped by topic and ranked in order of difficulty. The book is a powerful tool for motivating and challenging students of all levels. A must for every mathematics teacher and every school library.

Australian Mathematics Competition Book 2 1985-1991
Edited by PJ O'Halloran, G Pollard & PJ Taylor

Over 250 pages of challenging questions and solutions from the Australian Mathematics Competition papers from 1985-1991.

Australian Mathematics Competition Book 3 1992-1998
W Atkins, JE Munro & PJ Taylor

More challenging questions and solutions from the Australian Mathematics Competition papers from 1992-1998.

Australian Mathematics Competition Book 4 1999-2005
W Atkins & PJ Taylor

More challenging questions and solutions from the Australian Mathematics Competition papers from 1999-2005.

Australian Mathematics Competition Primary Problems & Solutions Book 1 2004–2008
W Atkins & PJ Taylor
This book consists of questions and full solutions from past AMC papers and is designed for use with students in Middle and Upper Primary. The questions are arranged in papers of 10 and are presented ready to be photocopied for classroom use.

Problem Solving via the AMC
Edited by Warren Atkins

This 210-page book consists of a development of techniques for solving approximately 150 problems that have been set in the Australian Mathematics Competition. These problems have been selected from topics such as Geometry, Motion, Diophantine Equations and Counting Techniques.

Methods of Problem Solving, Book 1
Edited by JB Tabov & PJ Taylor

This book introduces the student aspiring to Olympiad competition to particular mathematical problem solving techniques. The book contains formal treatments of methods which may be familiar or introduce the student to new, sometimes powerful techniques.

Methods of Problem Solving, Book 2
JB Tabov & PJ Taylor

After the success of Book 1, the authors have written Book 2 with the same format but five new topics. These are the Pigeonhole Principle, Discrete Optimisation, Homothety, the AM-GM Inequality and the Extremal Element Principle.

Mathematical Toolchest
Edited by AW Plank & N Williams

This 120-page book is intended for talented or interested secondary school students, who are keen to develop their mathematical knowledge and to acquire new skills. Most of the topics are enrichment material outside the normal school syllabus, and are accessible to enthusiastic year 10 students.

International Mathematics — Tournament of Towns (1980-1984)
Edited by PJ Taylor

The International Mathematics Tournament of the Towns is a problem-solving competition in which teams from different cities are handicapped according to the population of the city. Ranking only behind the International Mathematical Olympiad, this competition had its origins in Eastern Europe (as did the Olympiad) but is now open to cities throughout the world. This 115-page book contains problems and solutions from past papers for 1980-1984.

International Mathematics — Tournament of Towns (1984-1989)
Edited by PJ Taylor

More challenging questions and solutions from the International Mathematics Tournament of the Towns competitions. This 180-page book contains problems and solutions from 1984-1989.

International Mathematics — Tournament of Towns (1989-1993)
Edited by PJ Taylor

This 200-page book contains problems and solutions from the 1989-1993 Tournaments.

International Mathematics — Tournament of Towns (1993-1997)
Edited by PJ Taylor

This 180-page book contains problems and solutions from the 1993-1997 Tournaments.

International Mathematics — Tournament of Towns (1997-2002)
Edited by AM Storozhev

This 214-page book contains problems and solutions from the 1997-2002 Tournaments.

International Mathematics — Tournament of Towns (2002-2007)
Edited by A Liu & PJ Taylor

This 222-page book contains problems and solutions from the 1997-2002 Tournaments.

Challenge! 1991 – 1998
Edited by JB Henry, J Dowsey, AR Edwards, L Mottershead, A Nakos, G Vardaro & PJ Taylor

This book is a major reprint of the original *Challenge! (1991-1995)* published in 1997. It contains the problems and full solutions to all Junior and Intermediate problems set in the Mathematics Challenge for Young Australians Challenge Stage, exactly as they were proposed at the time. It is expanded to cover the years up to 1998, has more advanced typography and makes use of colour. It is highly recommended as a resource book for classes from Years 7 to 10 and also for students who wish to develop their problem-solving skills. Most of the problems are graded within to allow students to access an easier idea before developing through a few levels.

Challenge! 1999—2006 Book 2
JB Henry & PJ Taylor
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Australians Challenge Stage, exactly as they were proposed at the time. They are highly recommended as a resource book for classes from Years 7 to 10 and also for students who wish to develop their problem-solving skills. Most of the problems are graded within to allow students to access an easier idea before developing through a few levels.

USSR Mathematical Olympiads 1989 – 1992
Edited by AM Slinko

Arkadii Slinko, now at the University of Auckland, was one of the leading figures of the USSR Mathematical Olympiad Committee during the last years before democratisation. This book brings together the problems and solutions of the last four years of the All-Union Mathematics Olympiads. Not only are the problems and solutions highly expository but the book is worth reading alone for the fascinating history of mathematics competitions to be found in the introduction.

Australian Mathematical Olympiads Book 1 1979 – 1995
H Lausch & PJ Taylor

This book is a complete collection of all Australian Mathematical Olympiad papers from the first competition in 1979-1995. Solutions to all problems are included and in a number of cases alternative solutions are offered.

Chinese Mathematics Competitions and Olympiads Book 1 1981-1993
A Liu

This book contains the papers and solutions of two contests, the Chinese National High School Competition and the Chinese Mathematical Olympiad. China has an outstanding record in the IMO and this book contains the problems that were used in identifying the team candidates and selecting the Chinese team. The problems are meticulously constructed, many with distinctive flavour. They come in all levels of difficulty, from the relatively basic to the most challenging.

Asian Pacific Mathematics Olympiads 1989-2000
H Lausch & C Bosch-Giral

With innovative regulations and procedures, the APMO has become a model for regional competitions around the world where costs and logistics are serious considerations. This 159 page book reports the first twelve years of this competition, including sections on its early history, problems, solutions and statistics.

Polish and Austrian Mathematical Olympiads 1981-1995
ME Kuczma & E Windischbacher

Poland and Austria hold some of the strongest traditions of mathematical Olympiads in Europe even holding a joint Olympiad of high quality. This book contains some of the best problems from the national Olympiads. All problems have two or more independent solutions, indicating their richness as mathematical problems.

Seeking Solutions
JC Burns

Professor John Burns, formerly Professor of Mathematics at the Royal Military College, Duntroon, and Foundation Member of the Australian Mathematical Olympiad

Committee, solves the problems of the 1988, 1989 and 1990 International Mathematical Olympiads. Unlike other books in which only complete solutions are given, John Burns describes the complete thought processes he went through when solving the problems from scratch. Written in an inimitable and sensitive style, this book is a must for a student planning on developing the ability to solve advanced mathematics problems.

101 Problems in Algebra from the Training of the USA IMO Team
Edited by T Andreescu & Z Feng

This book contains one hundred and one highly rated problems used in training and testing the USA International Mathematical Olympiad team. The problems are carefully graded, ranging from quite accessible towards quite challenging. The problems have been well developed and are highly recommended to any student aspiring to participate at National or International Mathematical Olympiads.

Hungary Israel Mathematics Competition
S Gueron

The Hungary Israel Mathematics Competition commenced in 1990 when diplomatic relations between the two countries were in their infancy. This 181-page book summarizes the first 12 years of the competition (1990 to 2001) and includes the problems and complete solutions. The book is directed at mathematics lovers, problem-solving enthusiasts and students who wish to improve their competition skills. No special or advanced knowledge is required beyond that of the typical IMO contestant and the book includes a glossary explaining the terms and theorems which are not standard that have been used in the book.

Chinese Mathematics Competitions and Olympiads Book 2 1993-2001
A Liu
This book is a continuation of the earlier volume and covers the years 1993 to 2001.

Bulgarian Mathematics Competition 1992-2001
BJ Lazarov, JB Tabov, PJ Taylor & A Storozhev

The Bulgarian Mathematics Competition has become one of the most difficult and interesting competitions in the world. It is unique in structure combining mathematics and informatics problems in a multi-choice format. This book covers the first ten years of the competition complete with answers and solutions. Students of average ability and with an interest in the subject should be able to access this book and find a challenge.

International Mathematical Talent Search Part 1 and Part2
G Berzsenyi

George Berzsenyi sought to emulate KöMaL (the long-established Hungarian journal) in fostering a problem-solving program in talent development, first with the USA Mathematical Talent Search and then the International Mathematical Talent Search (IMTS). Part 1 contains the problems and solutions of the first five years (1991-1996) of the IMTS, plus an appendix of earlier problems and solutions of the USAMTS. Part 2 contains the problems and solutions of rounds 21-44 of the IMTS. These books are aimed at advanced, senior students at Year 10 level and above.

Mathematical Contests – Australian Scene
Edited by PJ Brown, A Di Pasquale & K McAvaney

These books provide an annual record of the Australian Mathematical Olympiad Committee's identification, testing and selection procedures for the Australian team at each International Mathematical Olympiad. The books consist of the questions, solutions, results and statistics for: Australian Intermediate Mathematics Olympiad (formerly AMOC Intermediate Olympiad), AMOC Senior Contest, Australian Mathematics Olympiad, Asian-Pacific Mathematics Olympiad, International Mathematical Olympiad, and Mathematical Challenge for Young Australians Challenge Stage.

Mathematics Competitions
Edited by J Švrcek

This bi-annual journal is published by AMT Publishing on behalf of the World Federation of National Mathematics Competitions. It contains articles of interest to academics and teachers around the world who run mathematics competitions, including articles on actual competitions, results from competitions, and mathematical and historical articles which may be of interest to those associated with competitions.

Problems to Solve in Middle School Mathematics
B Henry, L Mottershead, A Edwards, J McIntosh, A Nakos, K Sims, A Thomas & G Vardaro

This collection of problems is designed for use with students in years 5 to 8. Each of the 65 problems is presented ready to be photocopied for classroom use. With each problem there are teacher's notes and fully worked solutions. Some problems have extension problems presented with the teacher's notes. The problems are arranged in topics (Number, Counting, Space and Number, Space, Measurement, Time, Logic) and are roughly in order of difficulty within each topic. There is a chart suggesting which problem-solving strategies could be used with each problem.

Teaching and Assessing Working Mathematically Book 1 & Book 2
Elena Stoyanova

These books present ready-to-use materials that challenge students understanding of mathematics. In exercises and short assessments, working mathematically processes are linked with curriculum content and problem-solving strategies. The books contain complete solutions and are suitable for mathematically able students in Years 3 to 4 (Book 1) and Years 5 to 8 (Book 2).

A Mathematical Olympiad Primer
G Smith

This accessible text will enable enthusiastic students to enter the world of secondary school mathematics competitions with confidence. This is an ideal book for senior high school students who aspire to advance from school mathematics to solving olympiad-style problems. The author is the leader of the British IMO team.

ENRICHMENT STUDENT NOTES

The Enrichment Stage of the Mathematics Challenge for Young Australians (sponsored by the Dept of Innovation, Industry, Science and Research) contains formal course work as part of a structured, in-school program. The Student Notes are supplied to students enrolled in the program along with other materials provided to their teacher. We are making these Notes available as a text book to interested parties for whom the program is not available.

NEWTON: Recommended for students of about Year 5 and 6, topics include polyominoes, arithmetricks, polyhedra, patterns and divisibility.

DIRICHLET: Recommended for students of Year 6 or 7, topics include problem-solving techniques, tessellations, base five arithmetic, pattern seeking, rates and number theory.

EULER: Recommended for students of about Year 7, topics include elementary number theory and geometry, counting and pigeonhole principle.

GAUSS: Recommended for students of about Year 8, topics include Pythagoras' Theorem, Diophantine equations, counting techniques and congruences.

NOETHER: Recommended for students of about Year 9, topics include number theory, sequences and series, inequalities and circle geometry.

PÓLYA: Recommended for students of about Year 10, topics include polynomials, algebra, inequalities and Euclidean geometry.

T-SHIRTS

T-shirts of the following six mathematicians are made of 100% cotton and are designed and printed in Australia. They come in white, Medium (Turing only) and XL.

Leonhard Euler T–shirt

The Leonhard Euler t-shirts depict a brightly coloured cartoon representation of Euler's famous Seven Bridges of Königsberg question.

Carl Friedrich Gauss T–shirt

The Carl Friedrich Gauss t-shirts celebrate Gauss' discovery of the construction of a 17-gon by straight edge and compass, depicted by a brightly coloured cartoon.

Emmy Noether T–shirt

The Emmy Noether t-shirts show a schematic representation of her work on algebraic structures in the form of a brightly coloured cartoon.

George Pólya T–shirt

George Pólya was one of the most significant mathematicians of the 20th century, both as a researcher, where he made many significant discoveries, and as a teacher and inspiration to others. This t-shirt features one of Pólya's most famous theorems, the Necklace Theorem, which he discovered while working on mathematical aspects of chemical structure.

Peter Gustav Lejeune Dirichlet T–shirt

Dirichlet formulated the Pigeonhole Principle, often known as Dirichlet's Principle, which states: "If there are *p* pigeons placed in *h* holes and $p>h$ then there must be at least one pigeonhole containing at least 2 pigeons." The t-shirt has a bright cartoon representation of this principle.

Alan Mathison Turing T–shirt

The Alan Mathison Turing t-shirt depicts a colourful design representing Turing's computing machines which were the first computers.

ORDERING

All the above publications are available from AMT Publishing and can be purchased online at:

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- the production of valued, accessible mathematics materials;
- dedication to the concept of solidarity in education;
- credibility and acceptance by educationalists and the community in general whether locally, nationally or internationally; and
- a close association with the Australian Academy of Science and professional bodies.