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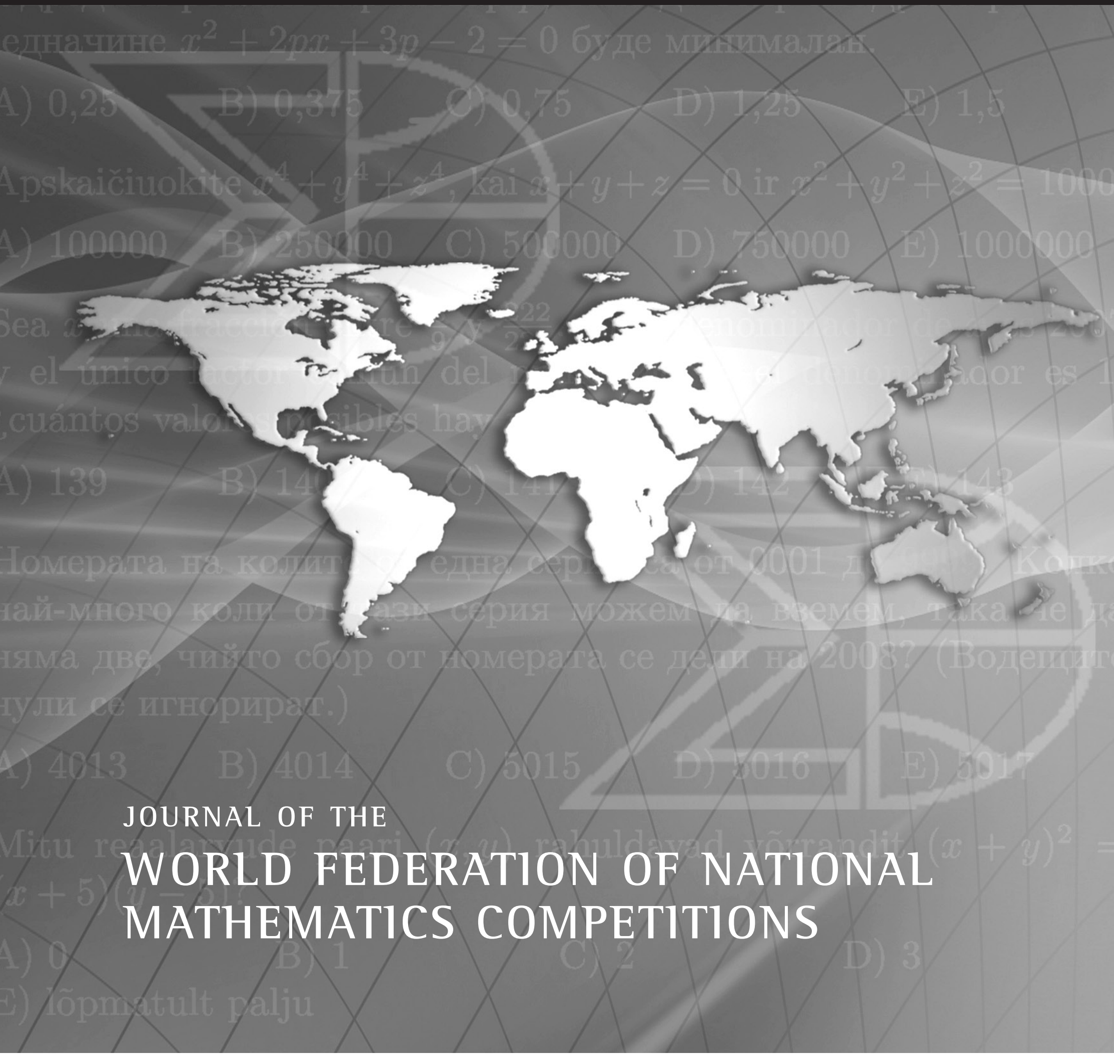
MATHEMATICS COMPETITIONS



JOURNAL OF THE

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MATHEMATICS COMPETITIONS VOLUME 23 NUMBER 2 2010

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The aims of the Federation are:–

1. ***to promote excellence in, and research associated with, mathematics education through the use of school mathematics competitions;***
2. ***to promote meetings and conferences where persons interested in mathematics contests can exchange and develop ideas for use in their countries;***
3. ***to provide opportunities for the exchanging of information for mathematics education through published material, notably through the Journal of the Federation;***
4. ***to recognize through the WFNMC Awards system persons who have made notable contributions to mathematics education through mathematical challenge around the world;***
5. ***to organize assistance provided by countries with developed systems for competitions for countries attempting to develop competitions;***
6. ***to promote mathematics and to encourage young mathematicians.***

From the President

It is a very exciting time to be participating in mathematics competitions or contributing to their organization. As plans mature for future conferences and congresses of WFNMC, it is important to get our collective bearings and move forward as we continue to provide meaningful mathematical experiences and opportunities for young pupils, students and teachers everywhere.

At our recent Congress in Latvia we enjoyed many excellent talks reminding all those present of the wealth of information, projects, problems, formats, emphases that very creative minds from a great variety of cultures and backgrounds are producing and sharing to inform and renew mathematics competitions for students and teachers all over the world. Ideas, initiatives and insights from that Congress are to be found in this latest issue of the World Federation journal.

Looking at the next four years, we have planned a mini-conference to be held on 7 July 2012 in conjunction with ICME-12 in Korea 8–15 July 2012 as well as the seventh WFNMC Congress which will take place in Beijing in 2014. Details of these events will be published in this journal as planning continues. We cordially invite all of our readers to prepare and participate with their innovations and experiences in mathematics competitions, problem creation, and other vehicles for promoting students' creativity and mathematical thinking.

It has frequently been said that Carl Friedrich Gauss (1777–1855) was the last mathematician to dominate all of the mathematics known in his time. Since then increasing specialization and a burgeoning corpus of knowledge has made it difficult, perhaps impossible, for young students and their teachers to participate actively in research on the frontiers of mathematical science.

These new circumstances led to a renewal of interest in mathematical puzzles, games and problem-solving challenges, activities that had existed for many centuries.

Several important movements in the second half of the 19th century quickly moved to address the impediment caused by the highly specialized nature of mathematical research, and to fill the gap between active young minds and aspiring mathematicians and the construction of new mathematical knowledge. A long line of creative and imaginative thinkers, from Lewis Carroll to Martin Gardner, resorted to proposing many new challenges on the level of recreational mathematics, a movement that continues in this 21^{st} century with international meetings of games and puzzles, the organization of math fairs and a host of activities that have proved an outlet for many creative puzzle proposers and captured the imagination of many young people throughout the world.

Mathematical Olympiads, inspired almost certainly by the same circumstances and founded roughly at the same time, have fulfilled a complementary necessity, that of spurring problem creation, essentially equivalent to research in elementary mathematics, and problem solving that permit students and teachers to hone other important aspects of mathematical thinking, not only imagination and the construction of a solution, but furthermore, the written communication and proving of results, that closely parallels research on the frontiers of mathematics. Problem solving in challenging mathematical competitions is not a simulation of doing mathematics, for each young competitor it is doing mathematics and it becomes a life-changing experience.

Furthermore, in the last half century the panorama has indeed altered as other groups, parents, governmental offices and international agencies, have become vitally interested in competitions and challenging mathematical education as a means of preparing future generations of scientists and citizens for a global society whose dynamics stem from knowledge-based innovation.

Our conferences and congresses permit each one of us to work toward perfecting our art of challenging young minds. These pages provide information, insight and ideas of how to do this. We invite all to read, to learn, to enjoy and to contribute to future issues of the WFNMC journal.

María Falk de Losada
President of WFNMC
Bogotá, December 2010

From the Editor

Welcome to *Mathematics Competitions* Vol. 23, No. 2.

First of all I would like to thank again the Australian Mathematics Trust for continued support, without which each issue (note the new cover) of the journal could not be published, and in particular Heather Sommariva, Bernadette Webster and Pavel Calábek for their assistance in the preparation of this issue.

Submission of articles:

The journal *Mathematics Competitions* is interested in receiving articles dealing with mathematics competitions, not only at national and international level, but also at regional and primary school level. There are many readers in different countries interested in these different levels of competitions.

- The journal traditionally contains many different kinds of articles, including reports, analyses of competition problems and the presentation of interesting mathematics arising from competition problems. Potential authors are encouraged to submit articles of all kinds.
- To maintain and improve the quality of the journal and its usefulness to those involved in mathematics competitions, all articles are subject to review and comment by one or more competent referees. The precise criteria used will depend on the type of article, but can be summarised by saying that an article accepted must be correct and appropriate, the content accurate and interesting, and, where the focus is mathematical, the mathematics fresh and well presented. This editorial and refereeing process is designed to help improve those articles which deserve to be published.

At the outset, the most important thing is that if you have anything to contribute on any aspect of mathematics competitions at any level, local, regional or national, we would welcome your contribution.

Articles should be submitted in English, with a black and white photograph and a short profile of the author. Alternatively, the article can be submitted on an IBM PC compatible disk or a Macintosh disk. We prefere LaTeX or TeX format of contributions, but any text file will be helpful.

Articles, and correspondence, can also be forwarded to the editor by mail to

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Jaroslav Švrček,
January 2011

First Announcement

WFNMC Mini-conference

In conjunction with ICME12 which will be held in Seoul, Korea from 8–15 July 2012, WFNMC will hold a mini-conference on 7 July at the same venue. María de Losada, President of WFNMC, and Sung Je Cho, Chair of the International Program Committee for ICME12, will serve as co-organizers of the WFNMC mini-conference.

At present all those interested in serving on the organizing committee (limited to three persons), contributing papers, organizing workshops or exhibitions at the mini-conference should contact María de Losada at mariadelosada@gmail.com. Details and a call for papers will follow in the next issue of the WFNMC Journal.

First Announcement

VII WFNMC Congress

The VII Congress of WFNMC will take place in Beijing in 2014. Please make sure to put the Congrress on your schedule and make Beijing your destination for the summer of 2014.

Further information and a first call for papers will appear in the December, 2011 issue of the WFNMC Journal.

First Announcement

Topic Study Group 34, ICME12, Seoul, Korea

WFNMC Vice President, Ali Rejali, and President, María de Losada, have been named co-chairs of Topic Study Group 34: *The role of mathematical competitions and other challenging contexts in the teaching and learning of mathematics* at ICME12.

Look for the first call for papers which will be issued in the first semestre of 2011.

Notes on an Agenda for Research and Action for WFNMC

María Falk de Losada

María Falk de Losada is a world famous authority in the work with mathematically gifted pupils. She was, first of all, one of the founders of the Mathematical Olympiad in Colombia. She has worked as a professor at the Universidad Antonio Narino in Bogotá for many years. In 2008 she was elected at the WFNMC mini-conference in Monterrey as the president of WFNMC.

Abstract. Two fundamental lines of research and action for the World Federation of National Mathematics Competitions are proposed complementing the traditional areas of problem creation and competition design. Links to the work of other international groups and to areas of research and development such as curriculum design and teacher education are discussed. Certain precepts to be adopted by WFNMC are proposed.

In the thoughts that we will try to convey this morning, there are at least two lines of research and action that we see as fundamental to the work of our Federation.

The first has to do with the academic planning, organization and carrying out of mathematical olympiads and similar events designed to allow each student who so wishes to pursue his or her optimal level of development in doing mathematics, which we define as solving problems that for him or her are new and original.

Every year new and exciting competitions are organized to cover new areas seen as crucial, be these by age group, by geographical region or by the type of problems—content, level of difficulty, format—included. In 2008 in Latin America we were told about the Brazilian Math Olympiad for Public Schools that had more than 18 million participants. In 2009 the Colombian Math Olympiads founded an interuniversity competition for Latin America similar to the IMC, with the participation of universities from 6 countries in its first version, and a second version to be held in Brazil in 2010. WFNMC has been instrumental in giving support and recognition to those doing this work, as well as serving as a multiplier through its congress and journal to inform others whose own work can benefit from analyzing the design, the activities and the problems posed.

Peru has been able to obtain the backing of the Ministry of Education for its math Olympiad, which has meant an increase to about 3 million students in the event. This is another important area—that of significantly increasing the number of students who benefit from the Olympiad experience—where the work of WFNMC continues to be influential.

Similar activity abounds in virtually all geographical regions around the globe and in the days to come we will hear from several speakers how new events have given new impulse to students and sparked favorable changes in math education in their countries or regions.

Our Federation should continue to be instrumental in bringing the Olympiad experience to students in all corners of the globe: accompanying new national, transnational and regional events, approaching new groups such as those organizing the primary or elementary school Olympiad on the international level, as well as the organizers of the IMC and similar university competitions, to extend the reach of mathematical competitions to ensure that they also enrich the mathematical experience of the youngest students and undergraduates alike.

WFNMC also serves as an avenue for competitions covering new academic territory. Two years ago we learned about an optimization competition in Russia named *Construct, Investigate, Explore* held in the same mode as distance learning and serving students from the sixth form to postgraduate level. In this congress we will hear about an International Internet Competition for University Students, the Australian Intermediate Mathematics Olympiad, a Statistics Olympiad in Iran and several others.

One area that must be strengthened is that of formal research to supplement our professional appreciation of the impact of competition activity.

The Federation should look to encourage its members to engage in research that can provide solid evidence of the impact of competitions on the student and on the educational system, as well as on the field of mathematics; research that provides a foundation for practice in a variety of ways.

The American Mathematical Society, for example, published in 2008 a study done by Jim Gleason, using models developed by psychologists, and thus pertinent and acceptable to educators in other areas of research in mathematics education such as PME (International Group for the Psychology of Mathematics Education), that shows that in fact designing an Olympiad with a first round that includes multiple-choice questions—as many of the popular Olympiads do—is a process that responds to the objectives and corresponds to the aims that such events profess, and that the great majority of the problems posed in a popular (unnamed) nationwide competition are in fact well designed to fulfill its aims and objectives.

The Mathematical Sciences Research Institute of UC Berkeley commissioned a study to analyze the educational history of students with outstanding results in the Putnam Competition, a case study whose 2005 report by Steve Olson revealed several critical moments in the attraction of young students to the field of mathematics.

There is no doubt that this line of research and action is seen, by IMU for example, and by all those present, as vital to the continuing task of attracting talented young people to the field of mathematics, not only enriching the lives of these future mathematicians, but also thus furthering mathematics itself. The afternoon of talks given by former IMO medalists at the 50$^{\text{th}}$ IMO last year in Bremen showed brilliantly the key contribution being made.

During the time Petar Kenderov and I had the opportunity to serve on the ICMI Executive Committee, this role of mathematics competitions was highlighted by a concern of IMU referred to as “the pipeline issue”. The math education community represented by ICMI has slowly focused on the issue, coming up with an intermediate case study (eight countries) that has yielded some very useful information. There will be a report at the IMU meeting in India in August. Nevertheless, the question of whether or not there is a real decline in the number of people choosing to follow a career in mathematics has not been answered by the study; and under the current design and objectives no attempt to gather complete information will be made.

As we all know this is the same concern that prompted Hilbert in the early twentieth century to draw up his famous list of problems yet to be solved. Young people must know that there are still open questions in mathematics so that they will find the field not only attractive but irresistible. Certainly events such as the IMO have led a large number of talented young students to devote themselves to doing mathematics. Its power and beauty are exposed; the student feels himself part of the sublime human endeavor that is mathematics.

The Federation must be active in proposing and supporting research regarding the solidity of practice and the depth and extent of impact of competitions, and in publishing its findings. Our journal must reflect these aspects of the work of the Federation as well as the fine job it has done offering articles on the design, academic planning, problem creation, organization and realization of competitions.

As we have seen in the past and will see again in the days to come, the problems posed in math competitions often link the work of the Federation and its members to research in mathematics. This is one of the most important aspects that link the Federation to research; the original problems created for math competitions often correspond to new results, that is to say, results of effectively doing research in elementary mathematics. Another important link stems from new results in research on the frontier of mathematics that can and have lead to the formulation of original problems for the highest level competitions, such as the IMO. More than a dozen of the talks given at this congress speak to results of this nature.

WFNMC can and should stress this aspect of its work and the work of its members before the international community, making clear its links to important areas of research in mathematics and mathematics education on many different levels.

There is a second line of research and development that we wish to suggest for future action of the Federation, that is, the designing, planning, organizing and carrying out of research relating to the nature of mathematical thinking and of how the experience of the average math class can be brought closer to developing the mathematical thinking of students in ways that will be personally satisfying and fun, and enable the student to leave all options open when making life choices, from career to personal finance to exercising the rights of a citizen.

This line of development will link the work of the Federation with other areas of research in mathematics education, and will involve looking at both teacher education and the curriculum.

The idea that *all* students can enjoy challenging and enriching experiences in mathematics is not new.

When beginning to prepare this talk, I remembered Plato's dialogue *Meno* in which Plato wishes to gain adherents to his explanation of how learning is possible, and in which Socrates' actions have been famously taken to illustrate what we call the Socratic method, however I wish to look at them as an example of mathematics teaching involving challenges.

In the *Meno*, the pupil's condition as a slave is meant to assure us that he has no prior mathematical knowledge, and that all for him is new. The problem set by Socrates is to draw (construct) a square whose area is twice the area of a given square.

The slave begins by suggesting that we double the length of the side and Socrates, drawing in the sand or dust shows that the resulting figure has four times the area of the original.

(We reproduce on the next pages from a copy of Plato's *Collected Dialogues and Letters* the only illustrations in more than 1700 pages of text.)

what I can since you ask me. I see you have a large number of retainers here. Call one of them, anyone you like, and I will use him to demonstrate it to you. b

MENO: Certainly. [*To a slave boy.*] Come here.

SOCRATES: He is a Greek and speaks our language?

MENO: Indeed yes—born and bred in the house.

SOCRATES: Listen carefully then, and see whether it seems to you that he is learning from me or simply being reminded.

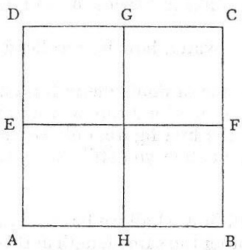
MENO: I will.

SOCRATES: Now boy, you know that a square is a figure like this?

(*Socrates begins to draw figures in the sand at his feet. He points to the square* ABCD.)

BOY: Yes.

SOCRATES: It has all these four sides equal? c



BOY: Yes.

SOCRATES: And these lines which go through the middle of it are also equal? [EF, GH.]

BOY: Yes.

SOCRATES: Such a figure could be either larger or smaller, could it not?

BOY: Yes.

SOCRATES: Now if this side is two feet long, and this side the same, how many feet will the whole be? Put it this way. If it were two feet in this direction and only one in that, must not the area be two feet taken once?

BOY: Yes.

SOCRATES: But since it is two feet this way also, does it not become twice two feet? d

BOY: Yes.

SOCRATES: And how many feet is twice two? Work it out and tell me.

BOY: Four.

The slave's second suggestion is to increase the sides a distance of half the length of the given side.

In each instance Socrates disproves the conjecture using diagrams or proofs without words, and then finally he draws the diagonal of the given square and leads the slave to see that (again proof without words) the square constructed on the diagonal as side fulfills the requirements of the problem posed.

The problem is fresh, requiring thought; the attempts by the slave to solve it are respected but firmly shown to be in error; the solution arrived at through questioning is diagramatic, conclusive and elegant.

We are not trying to promote the Socratic method, but rather illustrate how 2400 years ago it was thought that anybody—even a slave with no prior knowledge—can solve interesting and challenging problems in mathematics if and when that person is given a chance.

What have we here? An excellent problem, a superstar teacher, a superlative way of exhibiting the solution that instantly convinces the learner of its correctness.

Every child shows his or her originality and creativity outside the math classroom; how must we set our goals and build the possibility of their realization into our schools so that every child is given the chance to show his or her creativity within the confines of the mathematics classroom as well?

When we speak of giving a child or youngster the opportunity to meet more challenging mathematics as an essential component of school mathematics, we are in fact proposing three things: give the child/student exposure to a mathematical situation or problem that is beyond what he or she has already met and practiced, provide tools to grasp the problem and think about it, assist the child to find ways of expressing his or her thoughts, progress and solutions.

We cite four fundamental and interrelated reasons for establishing a role for the Federation in giving renewed impulse to the evolution of mathematics as presented to the student in school and indeed even at university—although we will develop only one of them—under the

SOCRATES: Now could one draw another figure double the size of this, but similar, that is, with all its sides equal like this one?

BOY: Yes.

SOCRATES: How many feet will its area be?

BOY: Eight.

SOCRATES: Now then, try to tell me how long each of its sides will be. The present figure has a side of two feet. What will be the side of the double-sized one?

BOY: It will be double, Socrates, obviously.

SOCRATES: You see, Meno, that I am not teaching him anything, only asking. Now he thinks he knows the length of the side of the eight-foot square.

MENO: Yes.

SOCRATES: But does he?

MENO: Certainly not.

SOCRATES: He thinks it is twice the length of the other.

MENO: Yes.

SOCRATES: Now watch how he recollects things in order—the proper way to recollect.

You say that the side of double length produces the double-sized figure? Like this I mean, not long this way and short that. It must be equal on all sides like the first figure, only twice its size, that is, eight feet. Think a moment whether you still expect to get it from doubling the side.

BOY: Yes, I do.

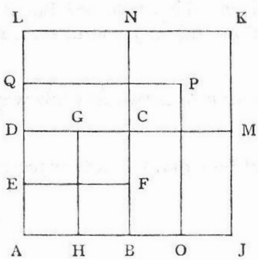
SOCRATES: Well now, shall we have a line double the length of this [AB] if we add another the same length at this end [BJ]?

BOY: Yes.

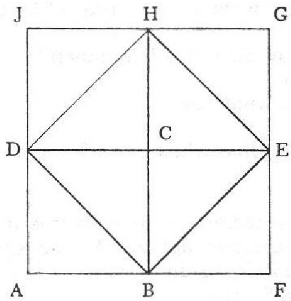
SOCRATES: It is on this line then, according to you, that we shall make the eight-foot square, by taking four of the same length?

BOY: Yes.

SOCRATES: Let us draw in four equal lines [*i.e., counting* AJ *and adding* JK, KL, *and* LA *made complete by drawing in its second half* LD], using the first as a base. Does this not give us what you call the eight-foot figure?



MENO 369



BOY: Yes.

SOCRATES: Now we can add another equal to it like this? [BCEF.]

BOY: Yes.

SOCRATES: And a third here, equal to each of the others? [CEGH.]

BOY: Yes.

SOCRATES: And then we can fill in this one in the corner? [DCHJ.]

BOY: Yes.

SOCRATES: Then here we have four equal squares?

BOY: Yes. e

SOCRATES: And how many times the size of the first square is the whole?

BOY: Four times.

SOCRATES: And we want one double the size. You remember?

BOY: Yes.

SOCRATES: Now does this line going from corner to corner cut each of these squares in half? 85

BOY: Yes.

SOCRATES: And these are four equal lines enclosing this area?

premise that math competitions and competition mathematics have shown how careful planning, analysis, original problems and non standard representations can counteract the separation that some sectors have attempted to foster between elementary mathematics and school mathematics, between mathematics and math education.

These four reasons concern the rights of the student, the nature of mathematics itself, the way technology is changing the way we do mathematics—and indeed the way we think, and the needs of the knowledge-based society.

Several countries and school systems have long since taken note of the fundamental change that must take place. Many have built a rich mathematical culture. For example, professor Konstantinov will be talking during this Congress of the Russian system under the title "Math Classes in Russia".

Other systems are beginning to take note of the desirability of structuring a more challenging curriculum for all.

In the USA the state of Connecticut has put forth the following precept:

> Every student needs and deserves a rich and rigorous mathematics curriculum that is focused on the development of concepts, the acquisition of basic and advanced skills and the integration of problem solving experiences. The Department of Education encourages educators to provide such challenging mathematics opportunities to foster the growth of intelligent, thoughtful and mathematically literate members of society.

What is interesting about the Connecticut statement is that it stresses the importance of a challenging mathematical education for the student and future citizen. This is an essential emphasis, it is what *every student needs and deserves*. Furthermore, in a recent (Summer, 2010) article *Ed.* The Magazine of the Harvard Graduate School of Education thought it appropriate to emphasize that access to a more exacting mathematics curriculum—albeit in the form of better ways of teaching algebra—should be thought of as *a new civil right*. Let's glance back at the *Meno*. Can the fact that the pupil is a slave allude to the belief that challenging mathematics education is the right of all?

The latter is essentially the gist of ICMI Study 16 in which many members of WFNMC took part, and Peter Taylor will be speaking to the topic during this Congress.

Just a few words that come to mind. In Study 16 several case studies of educational systems developing their commitment to a much more challenging mathematics curriculum were highlighted, among them that of Singapore. This is not the forum for discussing "Singapore Math" as it has been baptized, but to emphasize that its strength in treating more challenging problems for all students lies in the appropriate and, at the elementary level, graphical or visual representation of mathematical concepts and facts, a representation that permits the student to think about the concepts and facts without the intermediation of mathematical symbols that for the student unfortunately can come to resemble Hilbert's formalist philosophy—the manipulation of symbols devoid of meaning according to explicit rules of transformation.

The parallels with the method of Socrates in the *Meno* are nothing less than striking, but the message is clear. Present a challenging problem, provide the tools to grasp it and think about it, and accompany the student to find ways of expressing the solution in a convincing manner.

Another important area of research in this line of thought involves developing the capacity of all students to think mathematically. When debating new topics for its studies a few years ago, ICMI rejected a proposal to focus on the topic of mathematical thinking as too vague and research in the area too underdeveloped, choosing the topic of proof and proving instead. Thus we have an area of research—the development of mathematical thinking—prime for the intervention of WFNMC and its members. We have seen how problem solving, algorithms and formal mathematical proof all stem from the same thinking strategy, that of finding a way—however ingenious—to base each new step or result on one previously solved or proved. Several of the presentations we will hear in the days to come speak to analysis of mathematical thinking and strategies of problem solution.

Clearly, we hesitate to make a sweeping proposal having lived through the furor of the new math just some 50 years ago; we prefer to use the word evolution and not revolution; we insist nevertheless that curriculum change is long overdue.

1 The royal road and the curriculum

However, this evolution towards a more challenging math curriculum has not had the impact it merits on the situation and the policies related to mathematics education in general.

Again I wish to return to a famous incident in Greek mathematics, one that is frequently quoted but not necessarily interpreted fully with respect to its implications. We have all heard the tale of Menaecmus' saying to Alexander the Great words to the effect that "there is no royal road to geometry".

Now a recent headline in the *Los Angeles Times* read:

> America keeps looking for one simple solution for its education shortcomings. There isn't one.

Unfortunately, I believe that the great majority of teachers as well as those responsible for educational policy are still looking for the royal road. A fix-it, something that future teachers can digest in an instant and that can be transmitted to pupils without difficulty, but above all effortlessly.

I have not encountered an appropriate parallel, but I will venture this one. Many curricular proposals and textbooks present an excessively algorithmic approach to school mathematics, a sort of predigested or previously blended input, much like baby food, leaving totally unrecognizable the source elements in all their richness, color, shape, texture.

WFNMC and its members can participate in convincing society as a whole and political leaders in particular that there truly is no royal road to geometry; and it would be a tragedy if there were, supporting its efforts with research related to the success that more challenging mathematics has had in Singapore and in many other countries and regions of the world. Mathematics is captivating because it requires an effort, a sustained effort, an inspired effort, to do mathematics on every level of expertise. Proposals, projects, plans, design of more challenging mathematical curricula must be a focus of WFNMC and its members, jointly with research permitting a clear evaluation of their impact.

2 Teachers

If we look at analyses in several different contexts we can see that outstanding student performance almost always leads us back to extraordinary teachers, and this is true both for students performing on the

Olympiad level and for students taking part in almost any mathematical activity measuring their performance.

This is stated unequivocally in the MSRI report alluded to earlier.

WFNMC and its members know how to work with teachers on an extracurricular basis, we can and should learn to work intensely and with inspiration on transforming pre-service and in-service teacher education, allowing teachers to experience the power and beauty of mathematics so aptly embodied in challenges in the elementary mathematics that forms the backdrop for school mathematics, and we must document the results obtained through research.

3 This may well be the moment to develop some precepts and work toward their implementation.

Precepts

- Every child has a right to confront mathematics that is challenging enough to develop his mathematical thinking and to elicit his creativity in response.
- Challenging mathematics is the only learning experience that is true to the nature of mathematics itself and to that kind of thinking that can be said to be mathematical.
- Students' mathematical growth is directly related to the education, talent and dedication of the teacher. Teachers who have not had themselves the experience of confronting challenging mathematics and of thinking mathematically will not be able to open the door to such experience for their students.

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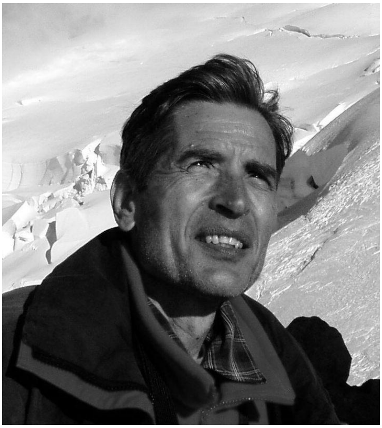
Partition of a Polygon into Triangles[1]

Kiril Bankov



Kiril Bankov is a professor of mathematics and mathematics education at the University of Sofia in Bulgaria. He teaches future mathematics teachers. Bankov has been involved in mathematics competitions in Bulgaria for more than 20 years as an author of contest problems and as a member of juries. Having a great experience in international mathematics education research, he works for the International Study Center of the Teacher Education and Development Study—Mathematics (TEDS-M) at Michigan State University (USA). Kiril Bankov is Secretary of WFNMC (since 2008).

A partition of a polygon into triangles means to represent the polygon as a union of triangles in such a way that any two triangles do not have a common interior point. If any two triangles have either a common side or a common vertex, or do not have a common point at all, the partition is called a triangulation. This paper deals with partitions that are not triangulations.

We may think of a triangulation as a regular way to partition a polygon into triangles. Main properties of triangulations are discussed in [1]. To calculate the number t of the triangles in any partition we may use the equation $180°t = 360°n + 180°(k-2)$ (Fig. 1), where k is the number of the sides (the vertexes) of the polygon, and n is the number of all vertexes of the triangles in the partition that are in the interior of the polygon.

There are contest problems dealing with triangulations. Here is an example from the Junior Balkan Mathematical Olympiad, 1999 [3].

[1]The work on this paper is partially supported by the Scientific Research Fund at the University of Sofia.

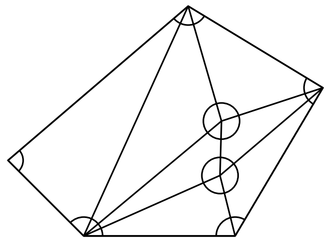


Fig. 1: Triangulation

Problem 1. Let S be a square with the side length 20 and let M be the set of points formed with the vertexes of S and another 1999 points lying inside S. Prove that there exists a triangle with vertexes in M and with area at most equal with $\frac{1}{10}$.

To solve the problem, one should only consider a triangulation of the square using the given points and to apply the above formulae for the number of the triangles.

Partitions that are not triangulations also appear in mathematical contests. Here is an example problem of a partition that is not a triangulation. It is from the Saint Petersburg mathematical Olympiad, 1968, [2], 1994, p. 50.

Problem 2. Prove that an equilateral triangle cannot be partitioned into equilateral triangles in such a way that any two triangles are not congruent.

The above problem, certainly, does not consider a triangulation, since for any triangulation of an equilateral triangle into equilateral triangles all triangles are congruent.

The common properties of such "not-regular" partitions are not very popular. This paper presents some of them.

First of all we need some definitions. Consider a partition of a polygon into triangles. A triangle of a partition is called a *regular triangle* if each side of the triangle does not contain a vertex of another triangle different from the end points of this side. Each triangle of any triangulation is a regular triangle. Partitions that are not triangulations have at least one triangle having a side that contains a vertex of another triangle different from the end points of this side. Let us call such a triangle a *non-regular triangle* (Fig. 2). Consider a partition of a polygon into triangles in such a way that every segment of the partition that lies in the interior of the polygon is not a common side of any two triangles. Such a partition is called an *anti-triangulation*.

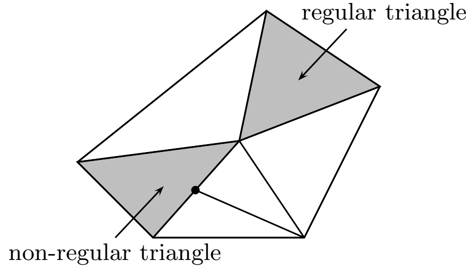


Fig. 2: Regular and nonregular triangle

All polygons in this paper are convex polygons, although some of the results are also true for non-convex polygons. For any partition of a polygon into triangles we will use the following notations: k—the number of the sides (the vertexes) of the polygon; t—the number of all triangles in the partition; n—the number of all vertexes of the triangles in the partition (including the vertexes of the polygon).

The points that are vertexes of triangles but not the vertexes of the polygon can be of three different types: Type A point is a vertex of several triangles and does not lie on a side of a triangle (Fig. 3). Each Type A point lies in the interior of the polygon. Type B point is a vertex of several triangles and lies on a side of another triangle (Fig. 4). Each Type B point lies in the interior of the polygon. Type C point lies on a side of the polygon, but is not a vertex of the polygon (Fig. 5).

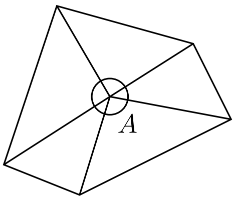


Fig. 3: Type A point

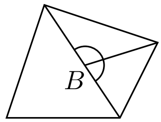


Fig. 4: Type B point

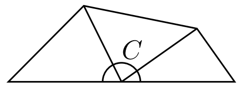


Fig. 5: Type C point

These three types of vertexes are important because if we calculate the sum of the angles of the triangles that have a vertex in one of these points, the result is the following: for any Type A point the sum is $360°$, for any Type B point it is $180°$, and for any Type C—also $180°$. For any partition of a polygon into triangles denote by n_A, n_B and n_C the number of points of Type A, B, and C, respectively. Obviously,

$$n = n_A + n_B + n_C + k. \tag{1}$$

Lemma. The number of the triangles in any partition is

$$t = 2n_A + n_B + n_C + k - 2. \tag{2}$$

Proof. Denote by S the sum of the interior angles of all triangles in the partition. From one hand, $S = t \cdot 180°$. From the other hand,

$$S = n_A \cdot 360° + n_B \cdot 180° + n_C \cdot 180° + (k-2) \cdot 180°.$$

Therefore, $t = 2n_A + n_B + n_C + k - 2$.

Corollary 1. For any partition of a convex polygon into triangles the inequality $t > n_B$ holds.

This follows from (2) because n_A and n_C are not negative and $k \geq 3$.

Theorem 1. Any partition of a convex polygon into triangles contains a regular triangle.

Proof. Assume there is a partition of a polygon into triangles that does not contain a regular triangle. This means that every triangle of the partition has a side having a point of Type B in it. Since for every two different triangles these Type B points are also different, it follows that $n_B \geq t$. This is a contradiction to corollary 1.

Corollary 2. For any partition of a convex polygon into triangles the inequality $t \geq n - 2$ holds.

Proof. From (1) and (2) it follows that $t = n + n_A - 2$. Therefore, $t \geq n - 2$, since $n_A \geq 0$.

Theorem 2. There is an anti-triangulation of any triangle, but there is not an anti-triangulation of a convex polygon having more than 3 sides.

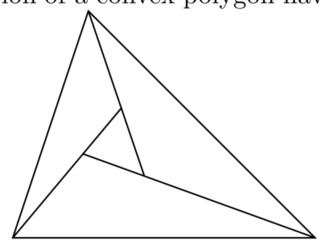


Fig. 6: Anti-triangulation

Proof. Fig. 6 shows an anti-triangulation of a triangle. Assume there is an anti-triangulation of a polygon having more than 3 sides, $k \geq 4$. Denote by c the sum of the sides of all triangles in the anti-triangulation, $c = 3t$. There are two types of such sides: Type I side lies in the interior of the polygon, and Type E side lies on a side of the polygon. Denote their numbers by c_I and c_E, respectively, $3t = c = c_I + c_E$. If we circle around the polygon, points of Type C, or the vertexes of the polygon, and sides of Type E appear consecutively. Therefore, $c_E = n_C + k$. Since each point of Type B is an end point of at least 3 sides of Type I, and for every two different points of Type B these triples of sides of Type I are different, it follows that $3n_B \geq c_I$. Therefore,

$$\begin{aligned}3t = c = c_I + c_E &\leq 3n_B + n_C + k = 3(n_B + n_C + k) - 2n_C - 2k \\ &\leq 3n - 2(n_C + k) \leq 3n - 2k.\end{aligned}$$

Since $k > 3$, it follows that $t \leq n - \frac{2}{3}k < n - \frac{2}{3} \cdot 3 = n - 2$. This contradicts to Corollary 1.

From Theorem 2 immediately follows:

Corollary 3. Let each interior angle of a hexagon be 120°. Any partition of this hexagon into equilateral triangles contains at least two congruent triangles.

This is because any such partition contains at least two equilateral triangles sharing a common side.

The solution to Problem 2 above can easily be obtained from the Corollary 3 the following way.

Solution to the Problem 2: Obviously, each vertex of the given equilateral triangle is a vertex of one of the equilateral triangles of the partition. This means that if we cut the three triangles at the vertices (shaded in Fig. 7), the remaining figure (not shaded in Fig. 7) is a hexagon with interior angles of 120° each. This hexagon is partitioned into equilateral triangles. According to the Corollary 3, at least two of these triangles are congruent.

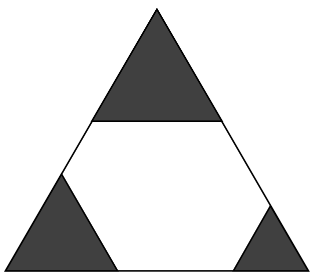


Fig. 7: Equilateral triangle

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[1] Bankov, K., *Triangulation*, Mathematics & Informatics, Vol. 1, (1991), 1.

[2] Fomin, D. V., *Saint Petersburg Mathematical Olympiads*, Polytehnika, St. Petersburg 1994 (in Russian).

[3] http://www.artofproblemsolving.com/Forum/resources.php?c=1&cid=21&year=1999&sid=847d8b6b76c563cf21770b5af111b9da (last visited August 2010).

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Between the Line and the Plane: Chromatic Étude in 6 Movements

Alexander Soifer

Born and educated in Moscow, Alexander Soifer has for 31 years been a Professor at the University of Colorado, teaching Mathematics, Art History and European Cinema courses. He has written 7 books and over 200 articles, including the following books published by Springer: The Mathematical Coloring Book *(2009)*, Mathematics as Problem Solving *(2009)*, How Does One Cut a Triangle? *(2009)*, Geometric Etudes in Combinatorial Mathematics *(2010)*, Ramsey Theory Yesterday, Today and Tomorrow *(2010, editor)*, The Colorado Mathematical Olympiad and Further Explorations *(2011). Soifer founded and for 27 years has run the Colorado Mathematical Olympiad. He served on the Soviet Union Math Olympiad (1970–1973) and USA Math Olympiad (1996–2005). Soifer has been Secretary (1996–2008) and Senior Vice President (2008–present) of the World Federation of National Mathematics Competitions; he is a recipient of the 2006 Federation's Paul Erdős Award. Soifer is the founding editor of the research quarterly* Geombinatorics *(1991–present), whose other editors include Ronald L. Graham, Branko Grünbaum, Heiko Harborth, Peter D. Johnson Jr., and Janos Pach. Paul Erdős was an editor and active author of this journal. Soifer's Erdős number is 1. Soifer's other coauthors include Saharon Shelah, John H. Conway, Vladimir Boltyanski, and Dmytro Karabash.*

> The teaching of mathematics has sometimes degenerated into empty drill in problem solving, which may develop formal ability but does not lead to real understanding or to greater intellectual independence…
>
> Understanding of mathematics cannot be transmitted by painless entertainment any more than education in music can be brought by the most brilliant journalism to those who have never listened

intensively. Actual contact with the content of living mathematics is necessary.

Richard Courant
What is Mathematics? Preface to the 1st ed., 1941

Mathematics, rightly viewed, possesses not only truth, but supreme beauty… capable of a stern perfection such as only the greatest art can show.

Bertrand Russell
The Study of Mathematics, 1902

Abstract. In this talk, I will start with the famous open problem of finding *the chromatic number* χ *of the plane* (the minimum number of colors for coloring the plane so that distance 1 does not appear between points of the same color). We will then reduce the problem to the line, in fact to just coloring of positive integers.

Surprisingly, there is much fun to be had on the line.

Not surprisingly, Paul Erdős initiated all the fun! We will then move back to new chromatic problems on the plane.

1 Overture: How Does One Convey in a Classroom What Mathematics Is?

Once upon a time, the Mathematical & Mechanical College of the Leningrad State University called a "Conference on Work with [High School] Students." This January 31–February 5, 1974 event was truly a major fare, a Soviet Union national event, attended by 172 delegates from 55 universities of the country [5]. Quite tellingly, half of the delegates were undergraduate students, who were traditionally actively involved in the work with gifted high school students. One of the undergraduate students in attendance is well known to you all: his name is Agnis Andžāns!

The organizers surprised me by putting the following plenary lineup on one of the early days of the conference: D. K. Faddeev, V. G. Boltysanski, A. Soifer, and I. M. Yaglom. I was 25, with a brand new Ph.D. and was honored to speak between great leaders of combinatorial geometry Boltyanski and Yaglom. In my talk "Combinatorial Geometry for

Students," I presented a few ideas about mathematical education, and backed them by solutions of problems that much later comprised my "How Does One Cut a Triangle?" book. In spite of what math education researchers lead us to believe, good original ideas are hard to come by, and so let me repeat my 1974 ideas here:

We all agree that *problem solving* is the most important component of math learning. We disagree on what problem solving should amount to. It is mostly used as a synonym of *analysis*, search for a solution of a *single-idea* problem. A typical problem would be: "given legs 3 and 4 of a right triangle, compute the hypotenuse by using Pythagorean Theorem." (Now, in 2010, due to misunderstood progress, we see problems like "given legs 3.1 and 4.2 of a right triangle, compute the hypotenuse by using Pythagorean Theorem *and your calculator*.") This does not do for a student the most important thing of math instruction, and that is to show *what mathematics is and what mathematicians do.*

We ought to use *multi-idea* problems, requiring for their solution *synthesis* of ideas and methods, preferably from different areas of mathematics. We ought to introduce students not only to analytical reasoning but also *construction* of counterexamples. As I. M. Gelfand said: '

"Theories come and go; examples stay forever."

In order to show what mathematics is and what mathematicians do, we ought to present *real fragments of mathematics* with their analytical proofs and constructions of counter examples; with *open-ended and open problems*; with *mathematical intuition* leading research like a light at the end of a tunnel; with synthesis of ideas from algebra, geometry, trigonometry, linear algebra, mathematical analysis, combinatorial geometry.

If we are to succeed in passing the baton to future generations, we must show mathematics with its *beauty, elegance, and results challenging our intuition.* We ought to try and show that *mathematics is alive*, that every solved problem gives birth to a myriad of unsolved ones. We want to *stop discrimination based on age*, and offer young talented mathematicians who are still in high school or early college the opportunity to be the first to solve a problem. As the South Carolina Reflector put it:

"Think like a tea bag. You don't know your strength until you get in hot water."

I can add only one more thought to what the 25-year old me presented in Saint Petersburg: *mathematics cannot be taught; it can only be learned by students doing math with our gentle guidance.*

I hope you agree with me and your hottest question is: *where will we get such problems, such appropriate fragments of live mathematics?*

This is what I have been working on ever since that 1974 Saint Petersburg Conference—minus a few years taken by my crossing the Atlantic as a refugee and settling down in the Land of the Free. I have written 6 books and over 200 articles. I have founded and for 27 years have run the Colorado Mathematical Olympiad, and have founded and for 19 years served as the publisher of the quarterly *Geombinatorics* dedicated to problem posing essays that allow young mathematicians engage in real research.

The greatest source of fine Olympiad problems I found in *creating bridges between problems of mathematical Olympiads and research problems of mathematics*. Today I will show you a brand new example of such a bridge, used in the April 23, 2010 Colorado Mathematical Olympiad.

2 From Coloring the Plane to Coloring Integers

The **chromatic number** $\chi\left(E^2\right)$ **of the plane** E^2 is the minimum number of colors required for coloring the plane in such a way that no monochromatic pair of points is distance 1 apart. From my previous WFNMC's talks and articles you may recall [11] that this problem was created by the 18-year old American student Edward Nelson in November of 1950, who together with his fellow 20-year old college student John Isbell, right then established the bounds for this number:

$$4 \leq \chi\left(E^2\right) \leq 7.$$

During the following 60 years mathematicians tried but have not been able to narrow down this wide range in general case. The legendary mathematician and leading promoter of this problem Paul Erdős posed a good number of related problems (see [11]). Let us look at one of them, likely posed in the 1960s, although I only found a much later written record [2] and a mention "I asked [this question] long ago" in the video recording of Paul Erdős's 1994 talk [3] which I attended. I have included this open problem in my book [11, p. 38]:

Erdős's Open Problem 1. Given a set S of positive numbers, a "*set of forbidden distances*" if you will. Find the minimum number $\chi_S\left(E^2\right)$ of colors required for coloring the plane is such a way that no monochromatic pair of points is distance r apart for *any* r from S.

It is natural to call $\chi_S\left(E^2\right)$ the **S-chromatic number of the plane**.

How difficult this problem is judge for yourselves: for the 1-element set S this is the chromatic number of the plane problem!

All right, let us enter the shallower waters. The **chromatic number** $\chi(E^1)$ **of the line** E^1 (i.e., of the set of real numbers with the usual definition of distance) is defined as the minimum number of colors required for coloring the line in such a way that no monochromatic pair of points is distance 1 apart.

Problem 2. Find $\chi(E^1)$.

Solution. Color semi-open segments $[n, n+1)$ in color 0 or 1 depending upon the parity of n. Therefore, $\chi(E^1) = 2$. $\square$

We can reduce our games further: just replace the set of real numbers E^1 by the set N of positive integers and ask the same question:

Problem 3. Find $\chi(N)$.

Solution. $\chi(N) = 2$ immediately follows from problem 2. $\square$

We want hard but solvable problems, do we not? OK, let us stay with coloring the set N, but increase the set S of forbidden distances.

Let us introduce notations for a few popular sets that will soon appear:

$$\begin{aligned} ODD &= \{1, 3, \ldots, 2n-1, \ldots\}; \\ EVEN &= \{2, 4, \ldots, 2n, \ldots\}; \\ BIN &= \{2^0, 2^1, \ldots, 2^n, \ldots\}; \\ FACT &= \{1!, 2!, \ldots, n!, \ldots\}. \end{aligned}$$

Even Problem 4. Find $\chi_{EVEN}(N)$.

Solution. Look at the subset E of N consisting of all even numbers. Since the distance between any pair of even numbers is even, i.e., forbidden, all even numbers must be colored in different colors. Thus, we can say that $\chi_{EVEN}(N)$ does not exist, or *stretch the definition to allow the infinity*, which is what I prefer:

$$\chi_{EVEN}(N) = \infty. \qquad \square$$

Odd Problem 5. Find $\chi_{ODD}(N)$.

Solution. Partition N into two subsets: the odds and the evens, and color all odds red and all evens blue. Since the distance between any monochromatic pair is even, we are done:

$$\chi_{ODD}(N) = 2. \qquad \square$$

Is it too easy for you? All right, let us go back in time, to the year 1987, when Paul Erdős boarded this train of thought again.

3 The Erdős–Katznelson Theorem

In 1987, Paul Erdős posed the following problem to the well known Israeli born mathematician Yitzhak Kaznelson, a Stanford professor, who recollects 14 years later [5]:

> In 1987 Paul Erdős asked me if the Cayley graph defined on Z by a lacunary sequence [defined in the theorem below] has necessarily

> a finite chromatic number. Below is my answer [in the positive], delivered to him on the spot but never published [until 2001]. . . ”

Erdős’s intuition did not fail to envision the positive answer. As usual for me, I am naming this result after both, the author of the conjecture and its prover.

The 1987 Erdős-Kaznelson Theorem 6. [5] Let $\varepsilon > 0$ be fixed and suppose that $S = \{n_1, n_2, \ldots, n_j, \ldots\}$ is a *lacunary sequence* of positive integers, i.e., such that $n_{j+1} > (1 + \varepsilon)\, n_j$ for all $j \geq 1$ and a fixed ε. Define a graph $G = G(S)$ with vertex set Z (the integers) by letting the pair (n, m) be an edge if and only if $|n - m| \in S$. Then the chromatic number $\chi(G)$[1] finite.

Is the condition of exponential growth in the Erdős-Kaznelson Theorem necessary? No, Odd Problem 5 above delivers the desired counterexample. Dreaming about this problem during the night of April 20–21, 2010, I understood how to construct infinitely many counterexamples:

Infinity of Counterexamples 7. Let $0 < k < n$ be a pair of positive integers. If $S = \{x : x \in N; x \equiv k \pmod{n}\}$, then $\chi_S(N) = n$.

Proof. As René Descartes put it, I “leave to others the pleasure of discovery.”

You may have noticed, one case remained unaddressed in problem 7; I will c it as all it problem 8 and leave for you to prove:

Infinity for Infinity of Examples 8.
If $S = \{x : x \in N; x \equiv 0 \pmod{n}\}$, then $\chi_S(N) = \infty$.

But we diverted to mathematics—let us return to history! :-)

Yitzhak Katznelson presented the Erdős conjecture and his proof in the positive at a 1991 seminar attended by the young Israeli mathematician Yuval Peres.

[1] The minimal number of colors required for coloring the vertices of G so that the endpoints of every edge are differently colored.

Yuval Peres (currently a professor at the University of California Berkeley and a researcher at Microsoft) improved Katznelson's upper bound for the chromatic number in Erdős's problem, and has just now, in 2010, published it in a joint paper [9] with Wilhelm Schlag. But long before this publication, Yuval spoke about the problem and his proof at a seminar attended by the 1990 and 1991 winner of Colorado Mathematical Olympiad Matthew Kahle, who at the time of Peres's talk was a Ph.D. student at the University of Washington Seattle. Matthew has now completed his post-doctoral fellowship at Stanford University and is starting a new fellowship at the Institute for Advance Study in Princeton. I let Matt Kahle contribute his part of the story. Matt writes to me in March 16, 2010 e-mail [7]:

> Regarding authorship: I believe Erdős proposed the following problem: If S is a sequence growing at least exponentially fast (i.e. there is a constant $C > 1$ such that the n^{th} term of S is eventually greater than C^n), then the graph on the integers with edges for every difference in S has finite chromatic number. I saw a proof of this conjecture by Yuval Peres in a seminar talk when I was a graduate student at UW [University of Washington]. Then I wondered about the exact chromatic numbers for a few special cases ($S = 2^n, 3^n, n!$) and quickly worked them out by hand. But for $n!$ I could only establish that the chromatic number was either 4 or 5. I proposed it to a small group of high school kids at Canada/USA Mathcamp (without telling them how I already knew that the chromatic number was either 4 or 5), and offered \$20 for a solution. Adam Hesterberg [who is presently a student at Princeton University] claimed my prize the next day, having improved on my upper bound of 5, rediscovering the basic technique himself and then improving on my argument.

Matt submitted to me what became "problems 5A and 5B" of the 27^{th} Colorado Mathematical Olympiad on April 23, 2010. He wrote up Adam Hesterberg's solution of problem 5B. While Adam's main idea of the existence of an "r" was nothing short of brilliant, the nesting segments argument required corrections and a non-trivial induction, which was done by Dr. Bob Ewell. Finally, I polished (Polished? :-)) their contributions.

4 Matthew Kahle's Problem

Problem 9. Coloring Integers A (Matthew Kahle)**.** Find $\chi_{BIN}(N)$, i.e., the minimum number of colors necessary for coloring the set of positive integers so that any two integers which differ by a power of 2 are colored in different colors.

Solution. Clearly 3 colors are necessary, since 1, 2, 3 require different colors. But coloring cyclically modulo 3 does the trick because a multiple of 3 is never equal to a power of 2. So, 3 colors are also sufficient. $\chi_{BIN}(N) = 3$. □

Problem 10. Coloring Integers B (Matthew Kahle)**.** Find $\chi_{FACT}(N)$, i.e., the minimum number of colors necessary for coloring the set of positive integers so that any two integers which differ by a factorial are colored in different colors.

Solution. Since $1! = 1$ and $2! = 2$, any three consecutive integers must be colored in 3 different colors. Assume 3 colors suffice. Then from 1 through 6, we have colors a, b, c, a, b, c. Accordingly, number 7 must be colored with color a, but it is not allowed because $7 - 1 = 3!$ This contradiction proves that at least 4 colors are needed.

Suppose for a moment that there exists of a number r, (necessarily irrational), such that $n!r$ is in the interval $[1, 3] \pmod 4$, for *every* positive integer n. Then we can determine which of the 4 colors to use on an integer k by looking at $kr \pmod 4$: the intervals $[0, 1)$ $[1, 2)$ $[2, 3)$ $[3, 4)$ mod 4 determine our 4 colors.

Thus defined 4-coloring satisfies the conditions of the problem. Indeed, suppose $|i - j| = n!$ for some n. By multiplying through by r, we get $|ri - rj| = rn!$, which is between 1 and 3 $\pmod 4$. In particular ri and rj belong to different color-defining-intervals mod4, and thus i and j received different colors.

All that is left to prove is *the existence* of the desired r. We will do it by induction in n.

For $n = 1, 2$ we find intervals $[r_1, r_2]$ such that for any r, $r_1 \le r \le r_2$, we get $1 \le rn! \le 3 \pmod 4$. The desired r_1, r_2 are listed in the table:

n	$n!$	r_1	r_2
1	1	1	3
2	2	$2\frac{1}{2}$	$3\frac{1}{2}$

Observe that for $n = 2$ the width of the interval $[r_1, r_2]$ is $r_2 - r_1 = \frac{1}{n!}$. Assume that for some $n \geq 2$ there is an interval $[r_1, r_2]$ such that

1. $r_2 - r_1 = \frac{1}{n!}$;
2. for any r, $r_1 \leq r \leq r_2$, we have $1 \leq rm! \leq 3 \pmod 4$ for any $m \leq n$; and
3. this interval is nested inside the $[r_1, r_2]$ interval obtained for $n - 1$.

Now consider $n + 1$. Clearly, for one value of $k = 0, 1, 2,$ or 3, we get

$$r_1' = \left(r_1 + \frac{k}{(n+1)!}\right)(n+1)! \equiv 1 \pmod 4. \tag{$*$}$$

We now define r_2' and observe that

$$r_2' = \left(r_1 + \frac{k+2}{(n+1)!}\right)(n+1)! \equiv 3 \pmod 4. \tag{$**$}$$

We get $r_1 \leq \left(r_1 + \frac{k}{(n+1)!}\right) = r_1'$. We also get $r_2' = r_1' + \frac{2}{(n+1)!} = r_1 + \frac{k+2}{(n+1)!} < r_1 + \frac{2}{n!} = r_2$. (The last inequality is true because $n \geq 2$, $k + 2 \leq 5$, and thus $\frac{k+2}{n+1} < 2$.) By the definition of ${r_2}'$, we have $r_2' - r_1' = \frac{2}{(n+1)!}$.

Finally, since the new interval, $[r_1', r_2']$, is nested inside the interval $[r_1, r_2]$ obtained for n for which we had the following condition satisfied:

for any r, $r_1 \leq r \leq r_2$, we have $1 \leq rm! \leq 3 \pmod 4$, for any $m \leq n$,

this condition is certainly satisfied for the interval $[r_1', r_2']$ and any $m \leq n$. Relations (*) and (**) show that this condition is also satisfied for $m = n + 1$. $\square$

This problem raised a great deal of interest after the Olympiad. I received distinct solutions from Russell Shaffer, the 1st prize winner of the first 1984 Olympiad; Brad Arnold, the father of the Olympiad's current and past winners; and from Ed Gardner, the father of this year's 1^{st} prize winner. While we needed only one value of r in our solution, it was noticed that a continuum of workable r's exists.

Before the Olympiad I posed the following conjecture:

Soifer Conjecture 11. There is an analytic function that expresses r in terms of e.

It was quickly resolved in the positive by Brad Arnold, who reported that our solution's value of r can be calculated as follows: $r = 2\cosh 1 + \sinh 1 - \cos 1 - \sin 1$, where hyperbolic trigonometric functions are defined as follows: $\sinh x = \mathrm{e}^x - \mathrm{e}^{-x})/2$; $\cosh x = (\mathrm{e}^x + \mathrm{e}^{-x})/2$.

Therefore, we get: $r = 3\mathrm{e}/2 + \mathrm{e}/2 - \cos 1 - \sin 1$. Euler's Formula $\mathrm{e}^{\mathrm{i}x} = \cos x + \mathrm{i}\sin x$ allows us to figure out $\cos 1 + \mathrm{i}\sin 1 = \mathrm{e}^{\mathrm{i}}$; $\cos 1 - \mathrm{i}\sin 1 = \mathrm{e}^{-\mathrm{i}}$; and get:

$$\begin{aligned}\cos 1 &= \left(\mathrm{e}^{\mathrm{i}} + \mathrm{e}^{-\mathrm{i}}\right)/2;\\ \sin 1 &= \left(\mathrm{e}^{\mathrm{i}} - \mathrm{e}^{-\mathrm{i}}\right)/2\mathrm{i} = \left(\mathrm{i}\mathrm{e}^{-\mathrm{i}} - \mathrm{i}\mathrm{e}^{\mathrm{i}}\right)/2\end{aligned}$$

Finally:

$$\begin{aligned}r &= 3\mathrm{e}/2 + \mathrm{e}/2 + \mathrm{e}^{\mathrm{i}}(\mathrm{i}-1)/2 - \mathrm{e}^{-\mathrm{i}}(\mathrm{i}+1)/2\\ &= 2.87958917259825278978475416303 63\ldots\end{aligned}$$

Brad's alternaitive value for r was

$$\begin{aligned}r_2 &= 2\cosh 1 - 2 = (\mathrm{e}^x + \mathrm{e}^{-x})/2 - 2\\ &= 1.086161269630487556955811241514 1\ldots\end{aligned}$$

It is natural to ask now what would happen with the chromatic number if we go back and expand our set from positive integers N to the line of all real numbers E^1:

Problem 11. Find $\chi_{FACT}(E^1)$.

I am leaving the pleasure of discovering this solution to you.

5 From Coloring Integers back to Coloring the Plane

Now that we gained insight in games of integers, let us go back to coloring the points of the plane.

Even Problem 12. Find $\chi_{EVEN}(E^2)$.

Solution. Already in coloring of merely positive integers N (Even Problem 4), we got the infinity: $\chi_S(N) = \infty$. The more $\chi_{EVEN}(E^2) = \infty$. $\square$

Odd Problem 13. Find $\chi_{ODD}(E^2)$.

Surprisingly, this problem proved to be not so easy. In fact, it is still open!

This problem, quite independently from our logical train of thought, was posed by the Israeli born American mathematician Moshe Rosenfeld on Wednesday, March 9, 1994. How do I know this? On the Day of Creation (of the problem :-), Moshe told the problem to Paul Erdős, who minutes later, introduced me to Moshe and shared with me the brand new problem. All three of us were attending the South-Eastern International Conference on Combinatorics, Graph Theory and Computing at Florida Atlantic University in Boca Raton, Florida. The following morning Paul mentioned the new problem in his talk (9:30–10:30 A.M.).

In creating this problem, Moshe was inspired by the following William Lowell Putnam Competition problem; he even published in our journal *Geombinatorics* his own new proof [10] of it:

Putnam Competition Problem 14 (Problem B-5, December 1993)**.** Show there do not exist four points in Euclidean plane such that the pairwise distances between the points are all odd integers.

In my opinion, this problem should have never been offered at this prestigious competition because it was published by Ronald L. Graham in Vol. 2, No. 3, p. 156, 1979–1980 of a very popular *Mathematical Intelligencer* magazine.

Worse yet, this was a special case of a result published [4] in a very popular magazine by R. L. Graham, B. L. Rothschild, and E. G. Straus ("Are there $n+2$ points in E^n with odd integral distances? ", *Amer. Math. Monthly* 81 (1974), 21–25) which asserts that the answer to the title question is "yes" if and only if $n \equiv 14 \pmod 1$6.

What is known about Odd Problem 9? All we know is the lower bound of 5, which was published in September 2009 by the powerhouse of American, Canadian and Israeli mathematicians Hayri Ardal, Ján Maňuch, Moshe Rosenfeld, Saharon Shelah and Ladislav Stacho. In fact, Moshe Rosenfeld first submitted this article to me for *Geombinatorics*'s special issue dedicated to the passing of the fine geometer and man Victor Klee, but then the group changed its collective mind and settled on *Discrete and Computational Geometry*, a Springer journal.

Lower Bound Theorem 15 [1]. $5 \leq \chi_{ODD}(E^2)$.

However, the group wanted—and did not succeed—to find out whether $\chi_{ODD}(E^2)$ is finite.

Odd Open Problem 13'. Is $\chi_{ODD}(E^2)$ finite?

What we do know is that under an additional condition requiring all monochromatic sets to be (Lebesgue) measurable, we get the infinity, as shown independently by various authors, including the M.I.T. undergraduate student Jacob Steinhardt in his 2009 paper [12]. Let us add a superscript "m" for this measurable case.

Odd Measurable Theorem 16 [12]. $\chi^m_{ODD}\left(E^2\right) = \infty$.

The plane versions of our Colorado Mathematical Olympiad problems remain open as well:

Open Problem 17. Find $\chi_{BIN}(E^2)$.

Open Problem 18. Find $\chi_{FACT}(E^2)$.

It would be worthwhile to at least determine whether these chromatic numbers are finite:

Open Problem 17'. Is $\chi_{BIN}(E^2)$ finite?

Open Problem 18'. Is $\chi_{FACT}(E^2)$ finite?

Since here we are actually coloring graphs, whose vertices are points in the plane and edges have prescribed by the set S integral lengths, I have got to bring your attention a beautiful result published in 1997 by the three Japanese mathematicians Hiroshi Maehara, Katsuhiro Ota and Nonrihide Tokushige:

Integral Distance Theorem 19 [8]**.** Every finite graph can be drawn in the plane so that any two vertices have an integral distance if and only if they are adjacent.

6 Coda

I have presented here one "Colorful Adventure: Between the Line and the Plane." Those whose whetted appetite now demands more colorful adventures, will find, I hope, much pleasure in perusing *The Mathematical Coloring Book* [11], my 18-year project that Springer published in 2009. Here I will conclude with what the great British writer Somerset Maugham used to call "Summing Up".

Research problems are a treasure trove of beautiful ideas that could inspire new generations of Olympiad problems. And conversely, Olympiad problems and their solutions give birth to new open problems and thus inspire new research, as you have witnessed here. To paraphrase the American Oath of Allegiance, problems of mathematical Olympiads and research problems of mathematics are indivisible with inspiration and opportunity for all!

We, research mathematicians, can learn much from young high school and college mathematicians. After all, the most beautiful and daring idea presented here came from the high school student Adam Hesterberg!

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Grammars and Finite-State Automata

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The mathematics of computer science (informatics) is a rich source of problems for competitions at many levels. The aim of this paper is to provide a brief introduction to a few of these ideas.

Let's begin with a problem.

Problem 1. Consider the following method for creating strings of 1s. Begin with the symbol x, and apply the following replacement rules as many times as desired and in any order:

$$x \mapsto 111x \tag{1}$$

$$x \mapsto 11111x \tag{2}$$

$$x \mapsto 111 \tag{3}$$

The process ends when only 1s remain. For example, we may obtain a string of 16 1s using rules 2, 1, 2, and 3, in that order:

$$x \overset{2}{\mapsto} 11111x \overset{1}{\mapsto} 11111111x \overset{2}{\mapsto} 1111111111111x \overset{3}{\mapsto} 1111111111111111.$$

How many strings of one or more 1s CANNOT be obtained by applying these rules?

(A) 4 (B) 6 (C) 8 (D) 10 (E) 15

Clearly, only strings of 1s may be formed. The set of rules which generates these strings is called a **grammar,** while the set of all strings which may be created using these rules is called the **language** produced by that grammar.

The symbol "x" is called a **non-terminal symbol,** since as long as there is an x present anywhere in the string, we must continue applying rules until only 1s remain. Similarly, the symbol "1" is called a **terminal symbol.**

This informal definition of a grammar will suffice for this paper. Keep in mind that in writing contest problems, it is desirable to keep the problem statement as brief as possible. In addition, as the above problem shows, it is possible to give the contestant an informal feel for what a grammar is without including details of a formal definition.

How does one approach Problem 1? Of course one may begin by simply writing out several strings in the language, looking for patterns. However, rules (1) and (2) are suggestive of the postage stamp problem: given an infinite supply of three- and five-cent stamps, what denominations of postage are possible?

It is well known that the maximum number of 1s which cannot be produced by (1) and (2) only is $(3-1)(5-1)-1=7$, so that a quick inspection shows that only strings of 1, 2, 4, or 7 ones cannot be obtained by using (1) and (2). Since the process ends with an application of (3), strings of 4, 5, 7, or 10 ones cannot be obtained. Further, since we *must* end by using (3), the strings 1 and 11 are also impossible to generate.

Thus, only strings of 1, 2, 4, 5, 7, or 10 ones cannot be produced by using the above rules. Hence there are six such strings.

Of course it is not necessary to be familiar with the postage stamp problem in order to be able to solve this problem. But hopefully the solution to this problem serves as an illustration of the application of mathematical ideas to the analysis of languages produced by grammars.

One beautiful aspect of working with grammars is that they are very easy to write – simply jot down any set of rules which comes to mind, and explore the grammar produced. This is the easy part! Sometimes, it is difficult to determine whether the rules can produce any strings at all. By way of example, consider the following two rules:

$$a \mapsto 1a1a1 \tag{1}$$
$$11a \mapsto 1 \tag{2}$$

As before, we begin with the non-terminal a and apply rules until only 1s remain. A first thought is that it is not possible to ever eliminate all the non-terminal a's – since any time you used rule (1), the second a produced could never be removed because in order to do so, two 1s would need to precede it in order to use rule (2).

This turns out to be incorrect. A simple attempt at creating a string in the language yields

$$a \overset{1}{\mapsto} 1a1a1 \overset{1}{\mapsto} 11a1a11a1 \overset{2}{\mapsto} 11a11a1 \overset{2}{\mapsto} 111a1 \overset{2}{\mapsto} 111.$$

Thus, in applying rule (1) more than once, the additional 1s needed to use rule (2) are thereby produced.

So now the problem becomes more interesting! In general, rules which allow strings to shrink, such as rule (2), can allow for some very unusual behavior. What language is produced by this grammar? The curious reader may think for a few moments before continuing to the solution.

It should be clear that applying rule (1) increases the length of the string by 4, while applying rule (2) decreases the length of the string by 2. Since we begin with a – a string of length 1 – any string produced by this grammar must have odd length.

But we can never produce a single 1, since in order to do so, we would have to obtain the string $11a$ immediately before. It is clear that after the initial a, applying either of the rules results in a string ending in 1. Thus $11a$ can never be produced, and thus neither can a string consisting of a single 1.

Can strings of all odd lengths greater than one be obtained? It turns out that this is indeed possible. We use a standard proof technique here:

derive additional rules which may be used to give the desired result. In particular, we will show that the following rules may be derived from the original two:

$$a \mapsto 111a1 \tag{3}$$

$$a \mapsto 11111a1 \tag{4}$$

Certainly rule (3) is easy to see, since we have already derived in on our way to producing 111 earlier. Rule (4) may be similarly derived by applying rule (1) three times, and then using rule (2) three times. Note that a fourth application of rule (2) produces the string 11111, and so suggests a pattern: to produce a string of $2n-1$ ones $(n > 1)$, apply rule (1) n times, followed by rule (2) $n+1$ times.

The reader is welcome to attempt an induction proof using this observation – but it is not so easy to guarantee that when applying rule (2), there are *always* two 1s just preceding the occurrence of an a.

However, using rules (3) and (4) allow for a direct proof. For it is clear that applying rule (3) p times (where $p > 1$) followed by rule (2) results in a string of length $4p-1$. Moreover, applying rule (4) once, rule (3) q times, and finally rule (2) results in a string of length

$$6 + 4q - 1 = 4q + 5.$$

Note that q may be 0 here, as we may use rule (2) immediately after rule (4). It is clear that allowing p and q to vary over their ranges will produce strings of ones of all odd lengths greater than 1.

Now there is no reason that we are limited to either one terminal symbol or one non-terminal symbol. The complexity of a problem can increase significantly with multiple terminal and non-terminal symbols.

Problem 2. Consider the following method for creating strings of 0s and 1s. Begin with the symbol x, and apply the following replacement rules as many times as desired and in any order.

$x \mapsto 1$	(1)	$y \mapsto 1x$	(5)
$x \mapsto 0x$	(2)	$z \mapsto 0$	(6)

$$x \mapsto 1y \qquad (3) \qquad\qquad z \mapsto 0y \qquad (7)$$
$$y \mapsto 0z \qquad (4) \qquad\qquad z \mapsto 1z \qquad (8)$$

The process ends when only 0s and 1s remain. For example,

$$x \overset{3}{\mapsto} 1y \overset{4}{\mapsto} 10z \overset{8}{\mapsto} 101z \overset{6}{\mapsto} 1010.$$

Which of the following numbers, interpreted as a binary string, can be obtained using the above rules?

(A) 5^{2011} (B) 6^{2011} (C) 7^{2011} (D) 8^{2011} (E) 9^{2011}

How does one approach a problem like this? The key lies in examining what it means for a string to end in x, y, or z. The rules make sense if we interpret a string ending in x to mean "the string created so far, interpreted as a binary number, is congruent to 0 mod 3." Similarly, that a string ends in y means "the string created so far, interpreted as a binary number, is congruent to 1 mod 3," and that a string ends in z means "the string created so far, interpreted as a binary number, is congruent to 2 mod 3."

Each rule is consistent with this interpretation. For example, (5) indicates that if the string so far ends in y (the string is congruent to 1 mod 3), adding a 1 on the end multiplies the number by 2 and adds 1, leaving a string which is congruent to $2 \cdot 1 + 1 \equiv 0$ mod 3, so that it must now end in x.

Now notice that (3) and (7) leave the string ending in y, so that after applying one of these rules, the string so far may be interpreted as a binary number which is equivalent to 1 mod 3. Since (1) and (6) are similar to (3) and (7) except for adding the trailing "y," we see that when the last x, y, or z is replaced by a 0 or a 1, a binary number congruent to 1 mod 3 remains.

With the same interpretation of the above rules, it is not difficult to see that given a binary number congruent to 1 mod 3, a sequence of rules may be found which produce this number. For example, we obtain 10011 as follows:

$$x \overset{3}{\mapsto} 1y \overset{4}{\mapsto} 10z \overset{7}{\mapsto} 100y \overset{5}{\mapsto} 1001x \overset{1}{\mapsto} 10011.$$

It is not difficult to make the above arguments rigorous by using induction on the length of the binary strings produced. Thus, the rules produce precisely those binary strings which, when interpreted as binary numbers, are congruent to 1 mod 3. This immediately gives the correct answer to the multiple-choice question.

This solution technique, while differing substantially from the first, is typical in analyzing grammars. We may consider x, y and z as intermediate "states" in the construction of our string, and interpret rule (2) as meaning "after reading in a 0, we are in state x." This leads to the construction of what is called a **finite-state machine** for this grammar; however, given limitations of space, this topic cannot be addressed in any depth. Also closely related is a discussion of **regular expressions,** but again, the interested reader will necessarily need to explore that topic on his or her own.

It should be clear that number theory is playing a rather prominent role in our discussion so far. However, problems involving grammars allow for non-traditional and novel applications of number theory – so that as far as contest problems are concerned, something new may be added to the repertoire. The following is an example of an open-ended problem requiring a written solution. While not involving number theory in a deep way, the problem is certainly more involved than the postage stamp problem we began with.

Problem 3. Consider the following method for creating strings of 0s and 1s. Begin with the symbol x, and apply the following replacement rules as many times as desired and in any order.

$$x \mapsto 11x \tag{1}$$

$$x \mapsto y \tag{2}$$

$$y \mapsto y000 \tag{3}$$

$$111y00 \mapsto y \tag{4}$$

$$y \mapsto 10 \tag{5}$$

For example,

$$x \overset{1}{\mapsto} 11x \overset{2}{\mapsto} 11y \overset{3}{\mapsto} 11y000 \overset{5}{\mapsto} 1110000.$$

Describe all strings which can be produced using these rules.

Solution. Note that the strings accepted by this language consist of some number of 1s followed by some number of 0s. Given that a string must end with rule (5), there must be at least one of each symbol. We will show that given $m, n \geq 1$, the string of m 1s followed by n 0s belongs to this language.

Let R_i represent the number of times rule (i) is applied. Then it is evident that there are $2R_1 - 3R_4 + 1$ 1s in the string followed by $3R_3 - 2R_4 + 1$ 0s, with the "+1" terms coming from rule (5). Thus, given m and n, we must simultaneoulsly solve

$$2R_1 - 3R_4 + 1 = m, \quad 3R_3 - 2R_4 + 1 = n.$$

This system may be solved in terms of R_4, giving

$$R_1 = \frac{3R_4 + m - 1}{2}, \quad R_3 = \frac{2R_4 + n - 1}{3}.$$

By choosing R_4 to be a positive integer simultaneously satisfying

$$R_4 \equiv (m-1) \bmod 2, \quad R_4 \equiv (n-1) \bmod 3,$$

then R_1 and R_3 are seen to be positive integers. But of course this is always possible since 2 and 3 are relatively prime.

Note that since $m, n \geq 1$, we always have

$$R_1 \geq \frac{3}{2}R_4, \quad R_3 \geq \frac{2}{3}R_4,$$

so that there will always be enough 0s or 1s to apply Rule (4) as many times as is necessary—presuming that we begin by applying rule (1) R_1 times, then apply rule (2), and then apply rule (3) R_3 times.

Thus, the strings produced by these rules consist of those strings consisting of one or more 1s followed by one or more 0s.

Now for our last problem.

Problem 4. Consider the following method for creating strings of 0s and 1s. Begin with the symbol a, and apply the following replacement rules as many times as desired and in any order.

$$a \mapsto 1a1 \tag{1}$$
$$a \mapsto 11a \tag{2}$$
$$a \mapsto 1 \tag{3}$$
$$11a1 \mapsto b0 \tag{4}$$
$$b \mapsto b0 \tag{5}$$
$$b000 \mapsto a \tag{6}$$

The process ends when only 0s and 1s remain. For example, the string 11 may be obtained as follows:

$$a \overset{1}{\mapsto} 1a1 \overset{2}{\mapsto} 111a1 \overset{4}{\mapsto} 1b0 \overset{5}{\mapsto} 1b00 \overset{5}{\mapsto} 1b000 \overset{6}{\mapsto} 1a \overset{3}{\mapsto} 11.$$

Describe all binary strings obtained by these rules.

Solution: First, notice that it is not possible using these rules to produce a string beginning with a 0. To be completely rigorous, this may be established by a brief induction argument on the length of strings containing the symbols a, b, 0, and 1.

It is, however, possible to produce any string beginning with a 1. First note that it is possible to replace the symbol a with $a1$ as follows:

$$a \overset{1}{\mapsto} 1a1 \overset{1}{\mapsto} 11a11 \overset{4}{\mapsto} b01 \overset{5}{\mapsto} b001 \overset{5}{\mapsto} b0001 \overset{6}{\mapsto} a1.$$

It is also possible to replace the symbol a with $a0$, as follows:

$$a \overset{2}{\mapsto} 11a \overset{1}{\mapsto} 111a1 \overset{4}{\mapsto} 1b0 \overset{5}{\mapsto} 1b00 \overset{5}{\mapsto} 1b000 \overset{5}{\mapsto} 1b0000$$
$$\overset{6}{\mapsto} 1a0 \overset{1}{\mapsto} 11a10 \overset{4}{\mapsto} b00 \overset{5}{\mapsto} b000 \overset{5}{\mapsto} b0000 \overset{6}{\mapsto} a0.$$

Thus, we may repeatedly add either a 0 or 1 at the end of any string. A final application of (3) finishes the process and adds a 1 to the beginning of the string. Thus, any string beginning with a 1 can be produced. (Note that a single 1 may be produced just by an application of (3).)

Certainly no number theory is involved here, but it is not at all obvious what language is produced by this grammar just by glancing at the rules.

It is my hope that this brief introduction into grammars (and the related subjects of finite-state machines and regular expressions) gives the reader a feel for a class of interesting contest problems requiring a minimum of background knowledge. Moreover, while there are a few standard techniques for solving such problems, it should be evident that a slight change in the rules of a grammar might require entirely different solution methods. Finally, it bears repeating that it is very easy to generate open-ended questions regarding the language produced by a particular grammar. It is not so easy to create *interesting* problems.

As a final remark, there is a very famous puzzle closely related to grammars – the MU puzzle in Douglas Hofstadter's *Gödel, Escher, Bach.* Hofstadter weaves a discussion of this problem throughout his book, which is an excellent introduction to mathematical thinking, number theory, and logic. It bears witness to the simple elegance which may be associated with problems involving the mathematics of computer science.

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Applications of Semi-Invariants in Solving Math Competition Problems

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Abstract. This paper is a reflection of my fourteen years' experience working with students that have particular interest in mathematics and preparing them for competitions. For them it is necessary to cover math topics not taught in the school curriculum. An example of such a topic is the so called semi-invariants. Problems related to this notion are given in math competitions. The problems vary in difficulty but have common methods and principles for their solution.

An invariant in mathematics is a property that does not change after the application of a certain transformation. In this paper we are dealing with properties that change after the application of transformations but the changes are always by one and the same way.

1 Introduction: Definitions and methods.

Consider some main definitions.

A *position* is the way in which the elements (points, lines, creatures, etc.) of a specific problem are connected (or ordered, situated, etc.) at a given moment.

The set M of all possible positions $\alpha_1, \alpha_2, \ldots$ for a specific problem is called a *conditional set*.

A rule in the context of a specific problem is a condition (or a set of conditions) under which a given position $\alpha_1 \in M$ can be moved to another position $\alpha_2 \in M$. The set of all rules of a problem is denoted by G.

Sets M and G are finite.

The movement from α_1 to α_2 is called *a transformation*.

Usually the set G of the transformations, as well as the rules, are not defined in the statement of the problem. They need to be found as a part of the solution. Suppose a specific problem is given the sets M and G of all positions and all transformations respectively are known.

A semi-invariant S for a problem is a numeric function, defined on the set M of all positions, such that the value of S always decreases (or always increases) after the application of each transformation of G. Since M is a finite set S has a finite number of values. It follows from the definition that every semi-invariant has the least and the greatest value.

The method for solving problems using semi-invariants follows a procedure like this. Usually, the set M of all positions is (more or less) easy to define from the statement of the problem. Finding the useful set of transformations G that transform every position from M to another position from M, as well as the semi-invariant S need a lot of inventiveness and experience. As long as they are properly determined, we may consider the position α in which the semi-invariant S has an extreme value (either the least or the greatest). If a transformation from G can be applied to α, we will find a value of S that is beyond the extreme value, which is a contradiction. In this case α is the required position that leads to the solution to the problem.

The rest of the paper discusses variety of problems that use the most common types of semi-invariants.

2 Semi-invariant—the sum of the lengths of segments

Problem 1 is a typical example for such semi-invariant.

Problem 1 [2]. $2n$ $(n > 1)$ different points on the plane are given. Prove that n not intersecting segments joining these points can be drawn in such a way that each point is the end point of only one segment.

Solution. It is important in this problem to consider configurations that occurs by an arbitrary partition of these $2n$ points into n pairs and connecting the points in each pair with a segment. Any such configuration is a position, that defines the set M of all positions. We

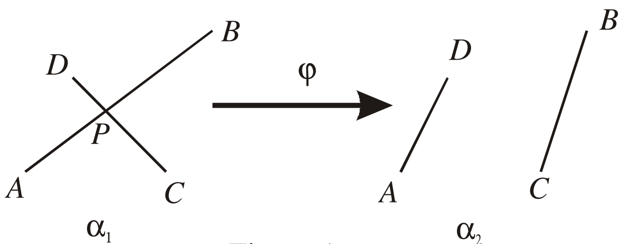

Figure 1

move from one position to another the following way: if there is a position α_1 in which four points A, B, C, and D are connected by intersecting segments AB and CD, $AB \cap CD = P$ (Fig. 1), we can move to another position α_2 by substituting the segments AB and CD with AD and BC that do not intersect (we do not change any other segments connecting the other points). A useful semi-invariant for the problem is the sum of all n segments in each position. We know the rules, the set G of all transformations and the semi-invariant S. We need to prove that after the application of any transformation the value of S decreases. This is because (Fig. 1) $AP + PD > AD$ (from $\triangle APD$), $BP + PC > BC$

(from $\triangle CPB$), therefore $AB + CD > AD + BC$. Since there are a finite numbers of positions, the values of S are also a finite set. Let S_1 be the least value of S and it is reached for position α_1. If α_1 contains intersecting segments, we may apply a transformation from G to move α_1 to another position. The value of S in this new position would be less than S_1, which is a contradiction. Therefore, α_1 consists of n not intersecting segments joining the points.

Problem 2 [2]. n points and n lines on the plane are given, $n > 1$. None three of the points lie on a straight line and non two of the lines are parallel. Prove that from these points n not intersecting perpendiculars to the lines can be drawn in such a way that there is exactly one perpendicular to each of the lines.

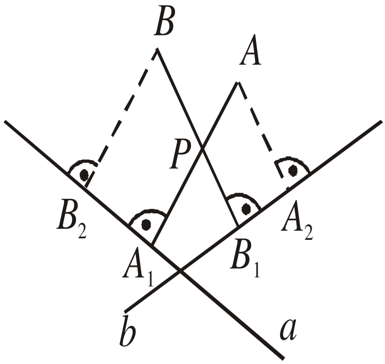

Figure 2

Solution. Let's draw from each point a perpendicular to one of the lines in such a way that there is exactly one perpendicular to each of the lines. This is the starting position. If any two perpendiculars do not intersect, the problem is solved. Otherwise, consider the two intersecting perpendiculars AA_1 and BB_1 from the points A and B to the lines a and b respectively (Fig. 2). Let P be their point of intersection. The transformation φ is defined that replaces the perpendiculars AA_1 and BB_1 with the perpendiculars AA_2 and BB_2 respectively, as shown on Figure 2. This operation decreases the sum of the lengths of the perpendiculars. Indeed, $AA_2 < AP + PB_1$ and $BB_2 < BP + PA_1$. Adding these inequalities we get $AA_2 + BB_2 < AA_1 + BB_1$.

Let the set M of all positions be the set of the configurations that occur by drawing from each point a perpendicular to one of the lines in such a way that there is exactly one perpendicular to each of the lines. The rule and the transformations are described above. The semi-invariant S is the sum of all drawn perpendiculars in a given position. Since M is a final set, the values of S are also a final set. We have already proven that the value of S decreases after the application of a transformation. The solution to the problem can be completed the same way as the solution to Problem 1.

3 Semi-invariant—the number of the pairs of elements

Here we consider problems dealing with processes of recoloring or reordering of elements using given rules. Sometimes it is useful to arrange the elements on pairs (of adjacent or of differently colored elements). In this situation the number of such pairs may play the role of a semi-invariant.

Problem 3 [1]**.** Several red and several blue points are given. Some of them are connected by segments. One of these points is called "a special point" if more than a half of the points with which it is connected are the opposite color. If there is "a special point", it is recolored in the opposite color. If there is another "special point" we can apply the same operation (recoloring) again, and so on. Prove that after several application of this operation (recoloring of "a special point"), no one "special point" remains.

Solution. A position is the configuration of the given points as they are colored in a particular moment and the connecting segments. A transformation is a recoloring of "a special point" in the opposite color. From the definition of "a special point" it follows that after each application of a transformation the number of the segments with differently colored end points decreases. It is easy to check that the number S of the segments with differently colored end points at a particular moment is a semi-invariant. Let S_1 be the least value of S and it is reached for position α_1. If α_1 contains "a special point", we may apply a transformation to move α_1 to another position. The value of S in this new position would be less than S_1, which is a contradiction. Therefore, α_1 does not contain "a special point".

Problem 4 [3]**.** In the City of Flowers a square garden 5×5 is partitioned into 25 square gardens 1×1. Each garden 1×1 is cultivated by an elf. Each elf is at enemy of not more than three other elves. Prove that there is a way to allocate the gardens 1×1 among the elves in such a way that no two enemy elves are neighbors. (Neighbors are elves whose gardens have a common side.)

Solution. If elf X is an enemy of his neighbor Y we will look for elf Z among the friends (not an enemy) of Y, who could exchange his garden with X, without creating a new conflict.

Note that X has four neighbors, each of whom has at most three enemies. To avoid creating new conflict Z must not be among the 12 potential enemies of the neighbors (including X). Also, if Z is a neighbor of an enemy of X, exchanging Z and X, will create new conflict between X and his enemy. This is why Z must not be among the 12 neighbors of enemies of X (including X). So Z must not be among the $2 \times 12 - 1 = 23$ elves (subtracting one because X is counted twice). Even if we add Y to them, there is still an elf Z, who could be exchanged with X. We call such an exchange a transformation. Let's in the beginning arbitrarily assign the 1×1 gardens between the elves. Denote by S the number of the couples of neighbors who are enemies. After each application of a transformation the value of S decreases by 1. Also, S has a finite number of values. Therefore, S is a semi-invariant. Then for the minimum value of S there is not a pair of neighbors who are enemies, because if there are any, it is possible to apply a transformation again and get a smaller value of S, which is a contradiction. Thus, after a finite number of operations, no neighbors who are enemies remain.

4 Semi-invariant—the value of an expression

In the following problems the semi-invariant is the value of an expression. The difficulty sometimes is to determine the expression that could play the role of the semi-invariant. Here are the most common expressions involving the numbers $a_1, a_2, \ldots, a_n$:

(i) $a_1 + a_2 + \cdots + a_n$

(ii) $a_1 \cdot a_2 \cdot \cdots \cdot a_n$

(iii) $a_1 - a_2 + a_3 - a_4 + \cdots + (-1)^{n-1}a_n$

(iv) $a_1 + 2a_2 + 3a_3 + \cdots + na_n$

(v) ${a_1}^2 + {a_2}^2 + \cdots + {a_n}^2$

(vi) $a_1a_2 + a_2a_3 + a_3a_4 + \cdots + a_{n-1}a_n + a_na_1$

Problem 5. (Russian Mathematical Olympiad, 1961). In each cell of a $m \times n$ matrix an arbitrary number is written. It is allowed to simultaneously change the signs of all numbers in a chosen row or a column. Prove that after the application of this operation several times it is possible to obtain a position in which the sum of the numbers in every row and every column is non-negative.

Solution. Let S be the sum of all numbers in the matrix. Choose a row (or a column) B that has a negative sum of the numbers, $b < 0$. The change of the signs of the numbers of B changes the sum from S to $S + 2\,|b|$. Therefore, the sum of all numbers in the matrix increases. For each cell there are two possibilities for the sign of the number written in it—a positive or a negative. All possible matrixes that can be obtained are 2^{mn}, i.e. a finite number. Then S is a semi-invariant. Let S_1 is the largest possible value of S. If the matrix for which S_1 is obtained contains a row or a column with a negative sum of the numbers, then we can apply the operation again and this will increase the value of S, which is a contradiction.

Problem 6 (selected by the Problem Selected Committee for the 32nd IMO, proposed from Bulgaria). Two students A and B are playing the following game: Each of them writes down on a sheet of paper a positive integer and gives the sheet to the referee. The referee writes down on a blackboard two integers, one of which is the sum of both integers, written by the players. After that the referee asks student A: "Can you tell the integer, written by the other student?" If A answers "no", the referee puts the same question to the student B. If B answers "no", the referee puts the question back to A etc. Assume that both students are intelligent and truthful. Prove that after a finite number of questions, one of the students will answer "yes".

Solution. Let a and b be the numbers written by A and B, respectively, and let $x < y$ be the numbers written by the referee. First, we will prove the following statements:

1. If both A and B know, that $\lambda < b < \mu$ and if the answer of A is "no", then both A and B know that $y - \mu < a < x - \lambda$.
2. If both A and B know that $\lambda < a < \mu$ and the answer of B is "no", then both A and B know that $y - \mu < b < x - \lambda$.

Indeed, if $a < y - \mu$, then $a + b < a + \mu \leq y$. Therefore, $a + b = x$ and A can answer "yes". Moreover, if $a \geq x - \lambda$, then $a + b > a + \lambda \geq x$. It follows that that $a + b = y$ and A can answer "yes". This proves 1. The same way we can prove 2.

Assume that neither A nor B ever answer "yes". In the beginning both A and B know that $0 < b < y$. Let $\alpha_0 = 0$ and $\beta_0 = y$. In a particular moment A and B would know that $\alpha_k < b < \beta_k$ and it is a turn of A to be asked. According to the assumption, he should answer "no". According to 1 both A and B know that $y - \beta_k < a < x - \alpha_k$. Again, according to the assumption B should answer "no". According to 2 both A and B know that $y-(x - \alpha_k) < b < (y - \beta_k)$, i.e. $\alpha_k+(y-x) < b < \beta_k-(y-x)$. Let $\alpha_{k+1} = \alpha_k + (y - x)$ and $\beta_{k+1} = \beta_k - (y - x)$. We will continue this way.

Since $y - x > 0$, the sequence $\{\alpha_k\}$ increases and the sequence $\{\beta_k\}$ decreases. According to the assumption, the process continues to the infinity, therefore we have $\alpha_{k_0} \geq \beta_{k_0}$ for any k_0. This is impossible since $\alpha_k < b < \beta_k$ for each k. We reach a contradiction, which proves that after a finite number of questions, one of the students will answer "yes".

Acknowledgment. The author thanks *Kalina Ilieva* and *Zlatina Tsvetkova* for their contribution for the preparation of the article.

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The 51st International Mathematical Olympiad Astana, Kazakhstan, 2010

The 51st International Mathematical Olympiad (IMO) was held from 2–14 July 2010 in Astana, Kazakhstan. Kazakhstan is a relatively large country, the ninth largest in area. It borders Russia to the north and China to the east. The other six "stans" lie to the south. A total of 523 high school students arrived from 97 countries in order to participate.

The first task facing the Team Leaders at the IMO is to set the two competition papers. During this period the Leaders and their observers are trusted to keep all information about the contest problems confidential. This took place in a health retreat in Almaty, a city some 1000 km south of Astana. The local Problems Selection Committee had already shortlisted 28 problem proposals submitted from around the world. After some discussion of the problem proposals the *Jury* of Team Leaders finalized the exam questions. The six questions are described as follows:

1. An easy functional equation which explores the interplay between multiplicativity and the floor function. It was proposed by France.
2. A medium classical geometry problem. This lovely problem was proposed by Hong Kong and turned out to be a little meatier than the Jury anticipated.
3. A difficult number theoretic functional equation proposed by the United States. This gem of a problem has a simple, but hard to think of idea, required to crack it.
4. An easy classical geometry problem. The original problem was an "if and only if" type question proposed by Poland. But the Jury only asked for the easier direction, which after an initial observation can be done by a pure angle chase.
5. An algorithmic combinatorics problem reminiscent of the Ackerman function proposed by The Netherlands. It was left to the student to discover that using just two very innocuous looking operations, it is possible to construct mind boggling huge numbers. This was meant to be a medium problem, but the majority of students found it to be rather difficult.

6. A difficult problem about sequences of real numbers proposed by Iran. While the ideas for solving this question are not difficult to think of, the actual solution of the problem will keep escaping the would be solver unless the ideas are drawn together in a subtle way.

These six questions were posed in two exam papers held on consecutive days. Each paper contained three of the problems and contestants were allowed $4\frac{1}{2}$ hours to attempt each paper. This year the competition days were Wednesday July 7 and Thursday July 8.

The day before the first day of competition, there was the Opening Ceremony held at the Palace of Independence, Astana. There was a welcoming speech by the Minister of Education and Science, who also read a letter from Kazakhstan's President. József Pelikán, Chairman of the IMO Advisory Board, then spoke to the audience reminding the contestants that the nature of a competition means that some will do better than others, but that this is not the deciding factor as to whether or not one turns out to be an excellent mathematician. This was followed by the parade of the Teams. There were various other music and dance shows, including the performance of the music group Ulytau.

After the exams the Leaders and their Deputies spent about two days assessing the work of the students from their own countries, guided by marking schemes discussed earlier. A local team of markers called *Coordinators* also assessed the papers. They too were guided by the marking schemes but are allowed some flexibility if, for example, a Leader brings something to their attention in a contestant's exam script which is not covered by the marking scheme. The Team Leader and Coordinators have to agree on scores for each student of the Leader's country in order to finalize scores. Some irregularities were identified in the exam scripts of North Korea, and this eventually led to their disqualification.

Questions 1 and 4 turned out to be rather easy as expected, both averaging over 5 marks. Questions 3, 5 and 6 were difficult, averaging 0.5, 0.9 and 0.4 marks respectively. There were 266 (=51.5 %) medals awarded. The distributions being 115 (=22.2 %) Bronze, 104 (=20 %) Silver and 47 (=9.1 %) Gold. One student, Zipei Nie from China, was declared the absolute overall winner of the IMO, being the only contestant to achieve a perfect score of 42. The medal cuts were set at 27 for Gold (the lowest in IMO history), 21 for Silver and 15 for Bronze. Most Gold medalists solved about four questions, most Silver medalists solved three questions and most Bronze medalists solved two and a bit questions. Of those who did not get a medal, a further 160 contestants received an Honourable Mention for solving at least one question perfectly. In fact 37 of these could be called "Double Honourable Mentions", because that is how many students solved exactly two problems perfectly, scoring 7, 0, 0, 7, 0, 0, just one point short of the Bronze cut.

The awards were presented at the Closing Ceremony. There was grand fanfare at the beginning with all Teams walking up the red carpet to the sound of horns and drums to enter the Palace of Independence. The Prime Minister attended in person and presented a Gold medal to the absolute winner of the IMO. Scholarships were awarded to members of the Kazakh Team, giving them the right to enter any university in the country. Furthermore, each Gold medalist was also given a laptop as an extra prize.

The 2010 IMO was supported and organised by DARYN, the Republican scientific-practical centre of Kazakhstan, with further sponsorship from Exxon Mobil and the Kazakh National Welfare Fund.

The 2011 IMO is scheduled to be held in Amsterdam, The Netherlands.

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1 IMO Papers

Wednesday, July 7, 2010
Language: English

First Day

Problem 1. Determine all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that the equality

$$f\big(\lfloor x \rfloor y\big) = f(x)\big\lfloor f(y) \big\rfloor$$

holds for all $x, y \in \mathbb{R}$.

(Here $\lfloor z \rfloor$ denotes the greatest integer less than or equal to z.)

Problem 2. Let I be the incentre of triangle ABC and let Γ be its circumcircle. Let the line AI intersect Γ again at D. Let E be a point on the arc $\widehat{BDC}$ and F a point on the side BC such that

$$\angle BAF = \angle CAE < \frac{1}{2}\angle BAC.$$

Finally, let G be the midpoint of the segment IF.

Prove that the lines DG and EI intersect on Γ.

Problem 3. Let $\mathbb{N}$ be the set of positive integers. Determine all functions $g\colon \mathbb{N} \to \mathbb{N}$ such that

$$\big(g(m) + n\big)\big(m + g(n)\big)$$

is a perfect square for all $m, n \in \mathbb{N}$.

Time allowed: 4 hours 30 minutes
Each problem is worth 7 points

Thursday, July 8, 2010
Language: English

Second Day

Problem 4. Let P be a point inside the triangle ABC. The lines AP, BP and CP intersect the circumcircle Γ of triangle ABC again at the points K, L and M respectively. The tangent to Γ at C intersects the line AB at S. Suppose that $SC = SP$.

Prove that $MK = ML$.

Problem 5. In each of six boxes $B_1, B_2, B_3, B_4, B_5, B_6$ there is initially one coin. There are two types of operation allowed:

Type 1: Choose a nonempty box B_j with $1 \le j \le 5$. Remove one coin from B_j and add two coins to B_{j+1}.

Type 2: Choose a nonempty box B_k with $1 \le k \le 4$. Remove one coin from B_k and exchange the contents of boxes B_{k+1} and B_{k+2}.

Determine whether there is a sequence of such operations that results in boxes B_1, B_2, B_3, B_4, B_5 being empty and with box B_6 containing exactly $2010^{2010^{2010}}$ coins. (Note that $a^{b^c} = a^{(b^c)}$.)

Problem 6. Let $a_1, a_2, a_3, \ldots$ be a sequence of positive real numbers. Suppose that for some positive integer s, we have

$$a_n = \max\{a_k + a_{n-k} \mid 1 \le k \le n-1\}$$

for all $n > s$. Prove that there exist positive integers ℓ and N, with $\ell \le s$ and such that $a_n = a_\ell + a_{n-\ell}$ for all $n \ge N$.

Time allowed: 4 hours 30 minutes
Each problem is worth 7 points

2 Results

Rank	Country	Score	Rank	Country	Score
Some Country Scores			**Some Country Scores**		
1	China	197	16	Iran	127
2	Russia	169	18	Peru	124
3	U.S.A	168	19	Taiwan	123
4	South Korea	156	20	Hong Kong	121
5	Kazakhstan	148	21	Bulgaria	118
5	Thailand	148	22	Singapore	117
7	Japan	141	22	Ukraine	117
8	Turkey	139	24	Poland	116
9	Germany	138	25	U.K.	114
10	Serbia	135	26	Uzbekistan	112
11	Vietnam	133	27	Belarus	110
11	Italy	133	28	Azerbaijan	109
13	Canada	129	29	New Zealand	106
13	Hungary	129	30	Indonesia	105
15	Australia	128	30	France	105
16	Romania	127	32	Croatia	103

Mark Distribution by Question

Mark	Q1	Q2	Q3	Q4	Q5	Q6
0	39	223	429	84	353	471
1	17	93	48	2	85	14
2	27	23	10	10	26	4
3	16	8	4	47	3	3
4	34	2	4	2	0	0
5	35	4	4	4	2	6
6	54	2	2	2	11	4
7	295	162	16	366	37	15
Total	517	517	517	517	517	517
Mean	5.45	2.59	0.46	5.35	0.93	0.37

Distribution of Awards at the 2010 IMO					
Country	Total	Gold	Silver	Bronze	H.M.
Albania (4 members)	11	0	0	0	0
Argentina	92	0	1	2	2
Armenia	63	0	0	1	3
Australia	128	1	3	1	1
Austria	87	1	0	1	4
Azerbaijan	109	0	3	2	1
Bangladesh (5 members)	54	0	0	1	3
Belarus	110	0	2	3	1
Belgium	64	0	0	0	6
Bolivia (4 members)	8	0	0	0	0
Bosnia & Herzegovina	80	0	0	2	4
Brazil	99	0	2	1	3
Bulgaria	118	1	2	3	0
Cambodia	58	0	0	0	5
Canada	129	2	1	2	1
China	197	6	0	0	0
Colombia (4 members)	60	0	0	3	1
Costa Rica (3 members)	32	0	0	1	1
Croatia	103	0	2	3	1
Cuba (1 member)	26	0	1	0	0
Cyprus	62	0	0	1	3
Czech Republic	84	0	0	2	3
Denmark	90	1	0	2	2
Ecuador	39	0	0	1	1
El Salvador (3 members)	25	0	0	0	2
Estonia	61	0	0	0	4
Finland	52	0	0	1	4
France	105	0	3	1	1
Georgia	101	0	2	2	2
Germany	138	1	3	2	0
Greece	95	0	2	0	4
Guatemala (2 members)	12	0	0	0	1

Distribution of Awards at the 2010 IMO

Country	Total	Gold	Silver	Bronze	H.M.
Honduras (1 member)	21	0	1	0	0
Hong Kong	121	1	2	3	0
Hungary	129	2	2	1	1
Iceland	53	0	0	0	4
India	98	0	2	1	3
Indonesia	105	0	1	4	1
Iran	127	0	4	2	0
Ireland	18	0	0	0	0
Israel (5 members)	76	0	1	1	2
Italy	133	1	3	2	0
Ivory Coast (5 members)	54	0	1	0	2
Japan	141	2	3	0	1
Kazakhstan	148	3	2	0	1
Kuwait (5 members)	2	0	0	0	0
Kyrgyzstan	61	0	1	1	3
Latvia	72	0	0	2	2
Lithuania	92	0	1	3	2
Luxembourg (3 members)	31	0	0	0	2
Macedonia (FYR)	69	0	1	0	2
Malaysia	75	0	1	1	3
Mexico	102	0	1	4	1
Moldova	92	0	1	3	2
Mongolia	79	0	0	2	4
Montenegro (4 members)	7	0	0	0	1
Morocco	55	0	0	1	3
Netherlands	94	0	0	5	1
New Zealand	106	0	2	4	0
Nigeria (5 members)	27	0	0	1	0
Norway	41	0	0	0	1
Pakistan (5 members)	19	0	0	0	1
Panama (2 members)	32	0	0	2	0
Paraguay (4 members)	26	0	0	0	1
Peru	124	1	3	1	1

Distribution of Awards at the 2010 IMO					
Country	Total	Gold	Silver	Bronze	H.M.
Philippines (3 members)	45	0	1	0	2
Poland	116	2	1	1	2
Portugal	75	0	0	1	4
Puerto Rico (2 members)	34	1	0	0	0
Romania	127	2	1	2	1
Russia	169	4	2	0	0
Saudi Arabia	55	0	0	2	2
Serbia	135	1	3	2	0
Singapore	117	0	4	1	1
Slovakia	92	1	0	2	3
Slovenia	78	0	0	2	3
South Africa	69	0	0	2	4
South Korea	156	4	2	0	0
Spain	89	0	1	2	2
Sri Lanka	78	0	0	1	5
Sweden	52	0	0	0	3
Switzerland	92	0	0	3	3
Syria	29	0	0	0	2
Taiwan	123	1	3	1	1
Tajikistan	73	0	1	0	5
Thailand	148	1	5	0	0
Trinidad & Tobago (5 m.)	37	0	1	0	0
Tunisia (2 members)	31	0	1	0	1
Turkey	139	1	3	2	0
Turkmenistan	91	0	1	2	3
Ukraine	117	1	2	3	0
United Kingdom	114	1	1	2	2
United States of America	168	3	3	0	0
Uzbekistan	112	0	4	1	1
Venezuela (2 members)	16	0	0	0	1
Vietnam	133	1	4	1	0
North Korea	DQ				
Total (517 contestants)		**47**	**104**	**115**	**160**

Tournament of Towns (Spring, 2010)

Andy Liu

Andy Liu is a professor of mathematics at the University of Alberta in Canada. His research interests span discrete mathematics, geometry, mathematics education and mathematics recreations. He edits the Problem Corner of the MAA's magazine Math Horizons. *He was the Chair of the Problem Committee in the 1995 IMO in Canada. His contribution to the 1994 IMO in Hong Kong was a major reason for him being awarded a David Hilbert International Award by the World Federation of National Mathematics Competitions.*

Below are my favourite three problems from the Spring 2010 International Mathematics Tournament of the Towns.

1. (Junior O-Level Problem 4.)
At a party, each person knows at least three other people. Prove that an even number of them, at least four, can sit at a round table such that each knows both neighbours.

2. (Junior A-Level Problem 3.)
Is it possible that the sum of the digits of a positive integer n is 100 while the sum of the digits of the number n^3 is 100^3?

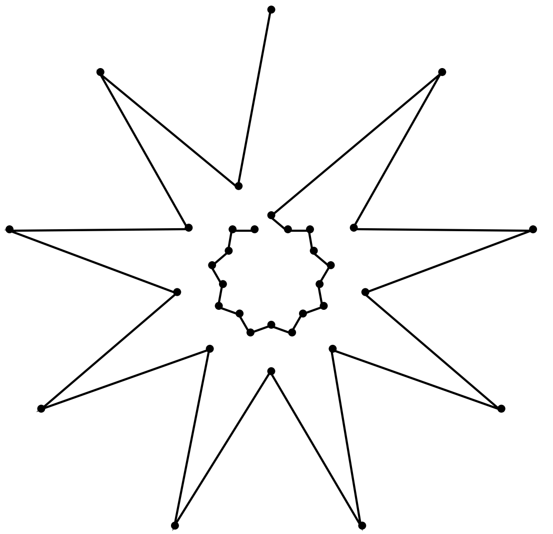
3. (Junior A-Level Problem 6.)
A broken line consists of 31 segments joined end to end. It does not intersect itself, and has distinct end points. Adjacent segments are not on the same straight line. What is the smallest number of straight lines which can contain all segments of such a broken line?

Here are the solutions.

1. **Solution by Central Jury.**
   Form the longest possible line of the people at the party so that each knows the next one. Let the first person be A. Then all the acquaintances of A must be in the line, as otherwise any missing one could be put in front of A to form a longer line. Let B, C and D be the first three acquaintances of A down the line. Suppose there are an odd number of people between B and C. Then we can put the segment of the line from B to C at a round table and insert A between B and C. This will meet the condition of the problem. Similarly, if there are an odd number of people between C and D, the condition can also be met. Finally, if there is an even number, possibly 0, of people both between B and C and between C and D, then there are an odd number of people betweeen B and D.

2. **Solution by Daniel Spivak.**
   Let $n = 10^{4^1} + 10^{4^2} + \cdots + 10^{4^{100}}$. Then the sum of the digits of n is 100. Consider n^3. It is the sum of 100^3 terms each a product of three powers of 10. We claim that if two such terms are equal, they must be products of the same three powers of 10. If $4^a + 4^b + 4^c = 4^x + 4^y + 4^z$, where $a \leq b \leq c$ and $x \leq y \leq z \leq c$, we must have $z = c$. Otherwise, even if $x = y = z = c - 1$, we still have $3(4^{c-1}) < 4^c$. Similarly, we must have $y = b$ and $x = a$, justifying the claim. Now a product of the same three powers of 10 can occur at most 3!=6 times. Hence there is no carrying in adding these 100^3 terms, so that the sum of the digits of n^3 is exactly 100^3.

3. **Solution by Daniel Spivak and Central Jury.**
   We first show that 9 lines are necessary. If we only have 8 lines, they generate at most 28 points of intersection. Since the broken line can only change direction at these points, it can have at most 29 segments. The diagram on the next page shows a broken line with 34 segments all lying on 9 lines. Hence 9 lines are also sufficient in this case.

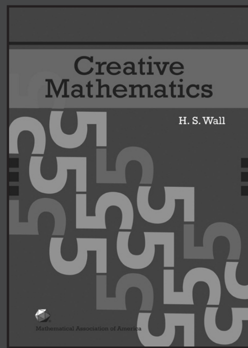


Andy Liu
University of Alberta
CANADA
E-mail: aliumath@telus.net

New from the Mathematical Association of America

Creative Mathematics

by H.S. Wall



Professor H.S. Wall wrote ***Creative Mathematics*** with the intention of leading students to develop their mathematical abilities, to help them learn the art of mathematics, and to teach them to create mathematical ideas. ***Creative Mathematics***, according to Wall, “is not a compendium of mathematical facts and inventions to be read over as a connoisseur of art looks over paintings.” It is, instead, a sketchbook in which readers try their hands at mathematical discovery.

The book is self contained, and assumes little formal mathematical background on the part of the reader. Wall is earnest about developing mathematical creativity and independence in students. In less than two hundred pages, he takes the reader on a stimulating tour starting with numbers, and then moving on to simple graphs, the integral, simple surfaces, successive approximations, linear spaces of simple graphs, and concluding with mechanical systems. The student who has worked through ***Creative Mathematics*** will come away with heightened mathematical maturity.

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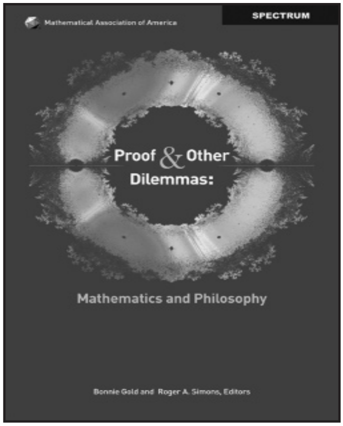

During the first 75 years of the twentieth century almost all work in the philosphy of mathematics concerned foundational questions. In the last quarter of a century, philosophers of mathematics began to return to basic questions concerning the philosophy of mathematics such as, what is the nature of mathematical knowledge and of mathematical objects, and how is mathematics related to science? Two new schools of philosophy of mathematics, social constructivism and structuralism, were added to the four traditional views (formalism, intuitionism, logicism, and platonism). The advent of the computer led to proofs and the development of mathematics assisted by computer, and to questions of the role of the computer in mathematics.

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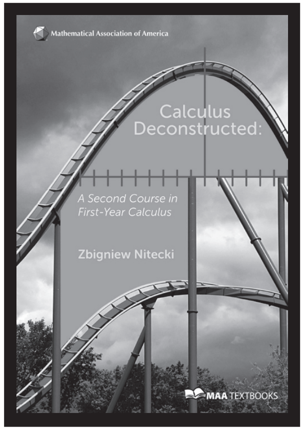

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NEW BOOK

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G Berzsenyi

George Berzsenyi sought to emulate KöMaL (the long-established Hungarian journal) in fostering a problem-solving program in talent development, first with the USA Mathematical Talent Search and then the International Mathematical Talent Search (IMTS).

He seeks to encourage problem solving as an intellectual habit and this book contains many interesting and some unusual problems, many with detailed backgrounds and insights. This collection contains the problems and solutions of the first five years (1991-1996) of the IMTS, plus an appendix of earlier problems and solutions of the USAMTS.

This 250-page book is aimed at advanced, senior students at Year 10 level and above.

Bundles of Past AMC Papers

Past Australian Mathematics Competition papers are packaged into bundles of ten identical papers in each of the Junior, Intermediate and Senior divisions of the Competition. Schools find these sets extremely valuable in setting their students miscellaneous exercises.

AMC Solutions and Statistics
Edited by PJ Taylor

This book provides, each year, a record of the AMC questions and solutions, and details of medallists and prize winners. It also provides a unique source of information for teachers and students alike, with items such as levels of Australian response rates and analyses including discriminatory powers and difficulty factors.

Australian Mathematics Competition Book 1 1978–1984
Edited by W Atkins, J Edwards, D King, PJ O'Halloran & PJ Taylor

This 258-page book consists of over 500 questions, solutions and statistics from the AMC papers of 1978-84. The questions are grouped by topic and ranked in order of difficulty. The book is a powerful tool for motivating and challenging students of all levels. A must for every mathematics teacher and every school library.

Australian Mathematics Competition Book 2 1985–1991
Edited by PJ O'Halloran, G Pollard & PJ Taylor

Over 250 pages of challenging questions and solutions from the Australian Mathematics Competition papers from 1985-1991.

Australian Mathematics Competition Book 3 1992–1998
W Atkins, JE Munro & PJ Taylor

More challenging questions and solutions from the Australian Mathematics Competition papers from 1992-1998.

Australian Mathematics Competition Book 3 CD
Programmed by E Storozhev

This CD contains the same problems and solutions as in the corresponding book. The problems can be accessed in topics as in the book and in this mode is ideal to help students practice particular skills. In another mode students can simulate writing one of the actual papers and determine the score that they would have gained. The CD runs on Windows platform only.

Australian Mathematics Competition Book 4 1999–2005
W Atkins & PJ Taylor

More challenging questions and solutions from the Australian Mathematics Competition papers from 1999-2005.

Australian Mathematics Competition Primary Problems & Solutions Book 1 2004–2008
W Atkins & PJ Taylor
This book consists of questions and full solutions from past AMC papers and is designed for use with students in Middle and Upper Primary. The questions are arranged in papers of 10 and are presented ready to be photocopied for classroom use.

Problem Solving via the AMC
Edited by Warren Atkins

This 210-page book consists of a development of techniques for solving approximately 150 problems that have been set in the Australian Mathematics Competition. These problems have been selected from topics such as Geometry, Motion, Diophantine Equations and Counting Techniques.

Methods of Problem Solving, Book 1
Edited by JB Tabov & PJ Taylor

This book introduces the student aspiring to Olympiad competition to particular mathematical problem solving techniques. The book contains formal treatments of methods which may be familiar or introduce the student to new, sometimes powerful techniques.

Methods of Problem Solving, Book 2
JB Tabov & PJ Taylor

After the success of Book 1, the authors have written Book 2 with the same format but five new topics. These are the Pigeon-Hole Principle, Discrete Optimisation, Homothety, the AM-GM Inequality and the Extremal Element Principle.

Mathematical Toolchest
Edited by AW Plank & N Williams

This 120-page book is intended for talented or interested secondary school students, who are keen to develop their mathematical knowledge and to acquire new skills. Most of the topics are enrichment material outside the normal school syllabus, and are accessible to enthusiastic year 10 students.

International Mathematics — Tournament of Towns (1980–1984)
Edited by PJ Taylor

The International Mathematics Tournament of the Towns is a problem-solving competition in which teams from different cities are handicapped according to the population of the city. Ranking only behind the International Mathematical Olympiad, this competition had its origins in Eastern Europe (as did the Olympiad) but is now open to cities throughout the world. This 115-page book contains problems and solutions from past papers for 1980-1984.

International Mathematics — Tournament of Towns (1984–1989)
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More challenging questions and solutions from the International Mathematics Tournament of the Towns competitions. This 180-page book contains problems and solutions from 1984-1989.

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This 200-page book contains problems and solutions from the 1989-1993 Tournaments.

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International Mathematics — Tournament of Towns (1997–2002)
Edited by AM Storozhev

This 214-page book contains problems and solutions from the 1997-2002 Tournaments.

Challenge! 1991 – 1998
Edited by JB Henry, J Dowsey, AR Edwards, L Mottershead, A Nakos, G Vardaro & PJ Taylor

This book is a major reprint of the original Challenge! (1991-1995) published in 1997. It contains the problems and full solutions to all Junior and Intermediate problems set in the Mathematics Challenge for Young Australians, exactly as they were proposed at the time. It is expanded to cover the years up to 1998, has more advanced typography and makes use of colour. It is highly recommended as a resource book for classes from Years 7 to 10 and also for students who wish to develop their problem-solving skills. Most of the problems are graded within to allow students to access an easier idea before developing through a few levels.

Challenge! 1999—2006 Book 2
JB Henry & PJ Taylor
This is the second book of the series and contains the problems and full solutions to all Junior and Intermediate problems set in the Mathematics Challenge for Young Australians, exactly as they were proposed at the time. They are highly recommended as a resource book for classes from Years 7 to 10 and also for students who wish to develop their problem-solving skills. Most of the problems are graded within to allow students to access an easier idea before developing through a few levels.

USSR Mathematical Olympiads 1989 – 1992
Edited by AM Slinko

Arkadii Slinko, now at the University of Auckland, was one of the leading figures of the USSR Mathematical Olympiad Committee during the last years before democratisation. This book brings together the problems and solutions of the last four years of the All-Union Mathematics Olympiads. Not only are the problems and solutions highly expository but the book is worth reading alone for the fascinating history of mathematics competitions to be found in the introduction.

Australian Mathematical Olympiads 1979 – 1995
H Lausch & PJ Taylor

This book is a complete collection of all Australian Mathematical Olympiad papers from the first competition in 1979. Solutions to all problems are included and in a number of cases alternative solutions are offered.

Chinese Mathematics Competitions and Olympiads Book 1 1981–1993
A Liu

This book contains the papers and solutions of two contests, the Chinese National High School Competition and the Chinese Mathematical Olympiad. China has an outstanding record in the IMO and this book contains the problems that were used in identifying the team candidates and selecting the Chinese team. The problems are meticulously constructed, many with distinctive flavour. They come in all levels of difficulty, from the relatively basic to the most challenging.

Asian Pacific Mathematics Olympiads 1989–2000
H Lausch & C Bosch-Giral

With innovative regulations and procedures, the APMO has become a model for regional competitions around the world where costs and logistics are serious considerations. This 159 page book reports the first twelve years of this competition, including sections on its early history, problems, solutions and statistics.

Polish and Austrian Mathematical Olympiads 1981–1995
ME Kuczma & E Windischbacher

Poland and Austria hold some of the strongest traditions of Mathematical Olympiads in Europe even holding a joint Olympiad of high quality. This book contains some of the best problems from the national Olympiads. All problems have two or more independent solutions, indicating their richness as mathematical problems.

Seeking Solutions
JC Burns

Professor John Burns, formerly Professor of Mathematics at the Royal Military College, Duntroon and Foundation Member of the Australian Mathematical Olympiad Committee, solves the problems of the 1988, 1989 and 1990 International Mathematical Olympiads. Unlike other books in which only complete solutions are given, John Burns describes the complete thought processes he went through when solving the problems from scratch. Written in an inimitable and sensitive style, this book is a must for a student planning on developing the ability to solve advanced mathematics problems.

101 Problems in Algebra from the Training of the USA IMO Team
Edited by T Andreescu & Z Feng

This book contains one hundred and one highly rated problems used in training and testing the USA International Mathematical Olympiad team. The problems are carefully graded, ranging from quite accessible towards quite challenging. The problems have been well developed and are highly recommended to any student aspiring to participate at National or International Mathematical Olympiads.

Hungary Israel Mathematics Competition
S Gueron

The Hungary Israel Mathematics Competition commenced in 1990 when diplomatic relations between the two countries were in their infancy. This 181-page book summarizes the first 12 years of the competition (1990 to 2001) and includes the problems and complete solutions. The book is directed at mathematics lovers, problem solving enthusiasts and students who wish to improve their competition skills. No special or advanced knowledge is required beyond that of the typical IMO contestant and the book includes a glossary explaining the terms and theorems which are not standard that have been used in the book.

Chinese Mathematics Competitions and Olympiads Book 2 1993–2001
A Liu
This book is a continuation of the earlier volume and covers the years 1993 to 2001.

Bulgarian Mathematics Competition 1992–2001
BJ Lazarov, JB Tabov, PJ Taylor & A Storozhev

The Bulgarian Mathematics Competition has become one of the most difficult and interesting competitions in the world. It is unique in structure combining mathematics and informatics problems in a multi-choice format. This book covers the first ten years of the competition complete with answers and solutions. Students of average ability and with an interest in the subject should be able to access this book and find a challenge.

Mathematical Contests – Australian Scene
Edited by PJ Brown, A Di Pasquale & K McAvaney

These books provide an annual record of the Australian Mathematical Olympiad Committee's identification, testing and selection procedures for the Australian team at each International Mathematical Olympiad. The books consist of the questions, solutions, results and statistics for: Australian Intermediate Mathematics

Olympiad (formerly AMOC Intermediate Olympiad), AMOC Senior Mathematics Contest, Australian Mathematics Olympiad, Asian-Pacific Mathematics Olympiad, International Mathematical Olympiad, and Maths Challenge Stage of the Mathematical Challenge for Young Australians.

Mathematics Competitions
Edited by J Švrcek

This bi-annual journal is published by AMT Publishing on behalf of the World Federation of National Mathematics Competitions. It contains articles of interest to academics and teachers around the world who run mathematics competitions, including articles on actual competitions, results from competitions, and mathematical and historical articles which may be of interest to those associated with competitions.

Problems to Solve in Middle School Mathematics
B Henry, L Mottershead, A Edwards, J McIntosh, A Nakos, K Sims, A Thomas & G Vardaro

This collection of problems is designed for use with students in years 5 to 8. Each of the 65 problems is presented ready to be photocopied for classroom use. With each problem there are teacher's notes and fully worked solutions. Some problems have extension problems presented with the teacher's notes. The problems are arranged in topics (Number, Counting, Space and Number, Space, Measurement, Time, Logic) and are roughly in order of difficulty within each topic. There is a chart suggesting which problem-solving strategies could be used with each problem.

Teaching and Assessing Working Mathematically Book 1 & Book 2
Elena Stoyanova

These books present ready-to-use materials that challenge students understanding of mathematics. In exercises and short assessments, working mathematically processes are linked with curriculum content and problem solving strategies. The books contain complete solutions and are suitable for mathematically able students in Years 3 to 4 (Book 1) and Years 5 to 8 (Book 2).

A Mathematical Olympiad Primer
G Smith

This accessible text will enable enthusiastic students to enter the world of secondary school mathematics competitions with confidence. This is an ideal book for senior high school students who aspire to advance from school mathematics to solving olympiad-style problems. The author is the leader of the British IMO team.

ENRICHMENT STUDENT NOTES

The Enrichment Stage of the Mathematics Challenge for Young Australians (sponsored by the Dept of Innovation, Industry, Science and Research) contains formal course work as part of a structured, in-school program. The Student Notes are supplied to students enrolled in the program along with other materials provided to their teacher. We are making these Notes available as a text book to interested parties for whom the program is not available.

Newton Enrichment Student Notes
JB Henry

Recommended for mathematics students of about Year 5 and 6 as extension material. Topics include polyominoes, arithmetricks, polyhedra, patterns and divisibility.

Dirichlet Enrichment Student Notes
JB Henry

This series has chapters on some problem solving techniques, tessellations, base five arithmetic, pattern seeking, rates and number theory. It is designed for students in Years 6 or 7.

Euler Enrichment Student Notes
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Emmy Noether T–shirt

The Emmy Noether t-shirts show a schematic representation of her work on algebraic structures in the form of a brightly coloured cartoon.

George Pólya T–shirt

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Peter Gustav Lejeune Dirichlet T–shirt

Dirichlet formulated the Pigeonhole Principle, often known as Dirichlet's Principle, which states: "If there are p pigeons placed in h holes and $p>h$ then there must be at least one pigeonhole containing at least 2 pigeons." The t-shirt has a bright cartoon representation of this principle.

Alan Mathison Turing T-shirt

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